The Mathematics of Porous Media

Abstract Existence Theorem for Parabolic Systems

Hans Wilhelm Alt

Abstract Existence Theorem for Parabolic Systems

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Plan of the talk:

- Motivation: Formal calculation
- Examples, historical papers, etc.
- New result: The abstract theorem
- The proof

PDE system given by:

$$\partial_t v_k + \mathrm{div} q_k = r_k + g_k \qquad \text{for } k = 1, \ldots, N$$

Multiply by λ_k:

$$(*)\ \underbrace{\sum_k \lambda_k \partial_t v_k}_{= \partial_t w} + \underbrace{\sum_k \lambda_k \mathrm{div} q_k}_{= \mathrm{div}(\sum_k \lambda_k q_k) - \sum_k \nabla \lambda_k \bullet q_k} - \sum_k \lambda_k r_k = \sum_k \lambda_k g_k$$

If $(*)$, then in $]t_0, t_1[\times\Omega$ the "free energy identity" is

$$\partial_t w + \mathrm{div}(\sum_k \lambda_k q_k)$$
$$+ \Big(\underbrace{\sum_k \nabla \lambda_k \bullet (-q_k) + \sum_k \lambda_k (-r_k)}_{=: D \geq 0} \Big) = \sum_k \lambda_k g_k$$

$$\partial_t w + \mathrm{div}(\sum_k \lambda_k q_k) + \Big(\underbrace{\sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k(-r_k)}_{=:\, D \geq 0} \Big) = \sum_k \lambda_k g_k$$

Integral version of the "free energy equality"

$$\int_\Omega \underbrace{\int_{t_0}^{t_1} \partial_t w \, \mathrm{dL}_1}_{\text{integrate in } t} \, \mathrm{dL}_n + \int_{t_0}^{t_1}\int_\Omega \underbrace{\sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k(-r_k)}_{= D \geq 0} \, \mathrm{dL}_n \, \mathrm{dL}_1$$

$$= -\int_{t_0}^{t_1} \underbrace{\int_\Omega \mathrm{div}(\sum_k \lambda_k q_k) \, \mathrm{dL}_n}_{\text{boundary data}} \, \mathrm{dL}_1 + \underbrace{\int_{t_0}^{t_1}\int_\Omega \sum_k \lambda_k g_k \, \mathrm{dL}_n \, \mathrm{dL}_1}_{\text{external terms}}$$

leads to

$$\int_\Omega w(t_1,x)\, \mathrm{d}x + \int_{t_0}^{t_1}\int_\Omega \underbrace{\sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k(-r_k)}_{= D \geq 0} \, \mathrm{dL}_n \, \mathrm{dL}_1$$

$$= \underbrace{\int_\Omega w(t_0,x)\, \mathrm{d}x}_{\text{initial data}} + \underbrace{\int_{t_0}^{t_1}\int_{\partial\Omega} (\sum_k \lambda_k q_k) \bullet \nu \, \mathrm{dH}_{n-1} \, \mathrm{dL}_1}_{\text{boundary data}} + (\text{external terms})$$

$$\partial_t v_k + \mathrm{div} q_k = r_k + g_k \quad \text{for } k = 1,\dots,N, \qquad (*)\ \partial_t w = \sum_k \lambda_k \partial_t v_k$$

$$\partial_t w + \mathrm{div}(\sum_k \lambda_k q_k) + D = \sum_k \lambda_k g_k, \qquad D = \sum_k \nabla\lambda_k \bullet (-q_k) + \sum_k \lambda_k(-r_k) \geq 0$$

Independent variable $u = (u_k)_{k=1,\dots,N}$

$\lambda_k = \lambda_k(u)$, $v_k = \beta_k(u)$, $w = f(u)$ free energy

$$\partial_t f(u) + \mathrm{div}\left(\sum_k \lambda_k(u) q_k\right) + D = g_0$$

$$D = \sum_k \nabla\lambda_k(u) \bullet (-q_k) + \sum_k \lambda_k(u)(-r_k) \geq 0$$

$$g_0 = \sum_k \lambda_k(u) g_k \text{ free energy production}$$

and $(*)$ is equivalent to

$$\partial_t w = \sum_k \lambda_k \partial_t v_k = \sum_k \lambda_k(u)\partial_t \beta_k(u) = \sum_{kl} \lambda_k(u) \beta_{k\,'u_l}(u) \partial_t u_l$$

$$= \partial_t f(u) = \sum_l f_{'u_l}(u) \partial_t u_l$$

that is

$$(*)\ f_{'u_l}(u) = \sum_l \lambda_k(u) \beta_{k\,'u_l}(u) \qquad \text{(integrability conditions on } \beta)$$

If multipliers are independent variables $\lambda_k = u_k$

$$(*)\ f_{'u_l}(u) = \sum_l u_k \beta_{k\,'u_l}(u) \qquad \text{(integrability conditions on } \beta)$$

Lemma Property $(*)$ is satisfied if and only if

$$\exists\ \psi\ :\ \beta_k(u) = \psi_{'u_k}(u) \qquad \text{for } k = 1,\dots,N$$

Proof:

$$\boxed{\begin{array}{l}\sum_k u_k \partial_t v_k = \partial_t w \\ v = \beta(u), \quad w = f(u)\end{array}} = B(u) + \text{const} \qquad\qquad \begin{array}{l}\sum_k u_k \partial_t v_k = \partial_t w \\ u = \beta^*(v), \quad w = \psi^*(v)\end{array}$$

$$\Longleftrightarrow \qquad\qquad \Longleftrightarrow$$

$$\sum_k u_k \partial_t \beta_k(u) = \partial_t f(u) \qquad\qquad \sum_k \beta_k^*(v_k)\partial_t v_k = \partial_t \psi^*(v)$$

$$\Longleftrightarrow \qquad\qquad \Longleftrightarrow$$

$$\sum_{kl} u_k \beta_{k\,'u_l}(u)\partial_t u_l = \sum_l f_{'u_l}(u)\partial_t u_l \qquad\qquad \sum_k \beta_k^*(v_k)\partial_t v_k = \sum_k \psi^*_{'v_k}(v)\partial_t v_k$$

$$\Longleftrightarrow \qquad\qquad \Longleftrightarrow$$

$$\sum_k u_k \beta_{k\,'u_l}(u) = f_{'u_l}(u) \qquad\qquad \beta_k^*(v_k) = \psi^*_{'v_k}(v)$$

$$\Longrightarrow$$

$$\sum_k u_k \beta_{k\,'u_l u_m}(u) + \underbrace{\sum_k \delta_{km}\beta_{k\,'u_l}(u)}_{= \beta_{m\,'u_l}(u)} = f_{'u_l u_m}(u)$$

$$\Longrightarrow$$

$\beta_{m\,'u_l}(u)$ symmetric in m, l

$$\Longrightarrow$$

$\beta_k(u) = \psi_{'u_k}(u)$ for $k = 1, \ldots, N$

$$\Longrightarrow$$

$$B(u) := u\bullet\beta(u) - \psi(u) \qquad \Big(B(u) = \psi^*(v) \quad \psi(u) + \psi^*(v) = u\bullet v \Big)$$

satisfies

$$B_{'u_l}(u) = (u\bullet\beta(u) - \psi(u))_{'u_l} = \sum_k u_k \beta_{k\,'u_l}(u)$$

$$\Longleftrightarrow \qquad\qquad \Longleftrightarrow$$

(top) (top)

$$\partial_t v_k + \mathrm{div} q_k = r_k + g_k \quad \text{for } k = 1, \dots, N$$

$$\partial_t w + \mathrm{div}(\sum_k \lambda_k q_k) + D = \sum_k \lambda_k g_k \quad \text{(free energy)}$$

Diffusion system $(v_k = u_k,\ \lambda_k = u_k,\ r_k = 0)$

$$\partial_t u_k + \mathrm{div} q_k = g_k, \quad q_k = -\sum_l a_{kl}(u)\nabla u_l, \quad \text{for } k = 1, \dots, N$$

$$(*)\ \partial_t w = \sum_k \lambda_k \partial_t v_k = \sum_k u_k \partial_t u_k = \partial_t(\tfrac{1}{2}\sum_k u_k^2), \quad w = \tfrac{1}{2}\sum_k u_k^2$$

$$\partial_t\big(\tfrac{|u|^2}{2}\big) + \mathrm{div}\big(\sum_k u_k q_k\big) + D = \sum_k u_k g_k$$

$$D = \sum_k \nabla u_k \bullet (-q_k) = \sum_{kl} a_{kl}(u)\nabla u_k \bullet \nabla u_l \geq 0, \quad \text{if } (a_{kl}(u))_{kl} \geq 0$$

Single equation $(\lambda = u,\ r = 0)$

- A. Visintin. *Models of Phase Transition*. 1996

$$\partial_t v + \mathrm{div}(-\nabla u) = g, \quad v = \beta(u), \quad \beta \text{ monotone increasing (containing a jump)}$$

$$(*)\ \partial_t w = \lambda \partial_t v = u\partial_t \beta(u) = u\beta'(u)\partial_t u = \partial_t \int_0^u s\beta'(s)\,\mathrm{d}s, \quad w = \int_0^u s\beta'(s)\,\mathrm{d}s$$

$$\partial_t w + \mathrm{div}\,(-u\nabla u) + D = ug$$

$$D = |\nabla u|^2 \geq 0$$

Chapter 4

L_1-semigroup Theory of Nonlinear Diffusion

C.J. van Duijn

- Ph. Clément, H.J.A.M. Heijmans, S. Angenent, C.J. van Duijn, B. de Pagter. *One-Parameter Semigroups.* CWI Centrum voor Wiskunde en Informatica, North-Holland 1987

$u_t = (u^m)_{xx}$, $u > 0$ density, $m > 1$

$f(u)_t + (-u(u^m)_x)_x + D = 0$, $f(u) = \frac{1}{2}|u|^2$ free energy

$D = u_x \cdot (u^m)_x = m u^{m-1}|u_x|^2 \geq 0$

1. Introduction

Consider the initial value problem

$$\text{(I)} \qquad \begin{cases} u_t = (\phi(u))_{xx}, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where ϕ is a maximal monotone graph in $\mathbb{R}^2$ with domain $\mathfrak{D}(\phi)$: see Section 2.1 and Appendix 1. Equations of this type arise in many problems of physical interest.

Example 4.1.

(a) The Porous Media Equation, where $\phi(u) = |u|^m \cdot \text{sign}(u)$, $m > 0$.

(b) The Stefan Problem, where ϕ is constant on an interval $I \subset \mathfrak{D}(\phi)$ and strictly increasing on $\mathfrak{D}(\phi) \setminus I$.

$v_t = (\varphi(v))_{xx}$, $\quad\varphi$ constant on interval, monotone increasing

$u = \varphi(v)$, $\quad v = \varphi^{-1}(u) = \beta(u)$, $\quad\beta$ has a jump, monotone increasing

$\beta(u)_t + (-u_x)_x = 0$ (see example above)

3. THE CASE $\mathcal{R}(\beta) \subseteq (0, \infty)$

This case arises when studying the nonlinear diffusion equation

$$u_t = (\log u)_{xx}, \quad u > 0.$$

Then $\phi(u) = \log u$ and $\beta(w) = e^w$. Consider again Problem (P) which we denote here by

$$-w'' + \beta(w) \ni f, \text{ on } \mathbb{R}. \qquad (\mathrm{P}_f)$$

Based on the knowledge obtained in the previous section we define

DEFINITION 4.16. A function $w : \mathbb{R} \to \mathbb{R}$ is a *solution* of (P_f) for $f \in L_1(\mathbb{R})$ if

(i) w, w' are locally absolutely continuous on $\mathbb{R}$,

(ii) $f(x) + w''(x) \in \beta(w(x))$ a.e. on $\mathbb{R}$,

(iii) $w'(\pm\infty) = 0$.

We also introduce

$v_t = (\varphi(v))_{xx}$, $\quad\varphi(v) = \log v$, $\quad v > 0$

$u = \varphi(v)$, $\quad v = \varphi^{-1}(u) = \beta(u) = \exp u$

$\beta(u)_t + (-u_x)_x = 0$

Let $u_0 \in \overline{\mathcal{D}(A_\phi)}$ and let $\epsilon > 0$. Consider the backward difference scheme

$$\begin{cases} u_\epsilon(t) - u_\epsilon(t-\epsilon) = \epsilon A_\phi u_\epsilon(t), & t > 0, \\ u_\epsilon(t) = u_0, & t \leq 0. \end{cases}$$

Then $u_\epsilon(t) = E_\epsilon^{A_\phi} u_\epsilon(t-\epsilon)$ for $t > 0$. We iterate to find:

$$u_\epsilon(t) = (E_\epsilon^{A_\phi})^{[t/\epsilon]+1} u_0, \text{ for } t > 0,$$

where $[t/\epsilon]$ is the largest integer in the interval $(-\infty, t/\epsilon]$. The semigroup generation Theorem 2.3 gives that

$$u(t) = T_{A_\phi}(t) u_0 = \lim_{\epsilon \downarrow 0} u_\epsilon(t)$$

exists uniformly on $[0,T]$, for any $T > 0$, i.e.

$$\limsup_{\epsilon \downarrow 0\ [0,T]} \| T_{A_\phi}(t) u_0 - u_\epsilon(t) \|_1 = 0.$$

It follows from the maximum principle for $E_\lambda^{A_\phi}$ that when $u_0 \in \overline{\mathcal{D}(A_\phi)} \cap L_\infty(\mathbb{R})$ such that $m \leq u_0(x) \leq M$, then $u(t,x) := T_{A_\phi}(t) u_0(x)$ satisfies

$$m \leq u(t,x) \leq M, \text{ a.e. in } [0,\infty) \times \mathbb{R}.$$

Moreover, since $E_\lambda^{A_\phi}$ is order preserving we have,

$$u_0(x) \geq v_0(x) \text{ a.e. on } \mathbb{R} \text{ implies } u(t,x) \geq v(t,x) \text{ a.e., in } [0,\infty) \times \mathbb{R},$$

where $v(t,x) := T_{A_\phi}(t) v_0(x)$ for $(t,x) \in (0,\infty) \times \mathbb{R}$. Finally, we have conservation of mass:

$$\int_{\mathbb{R}} u(t,x)\,dx = \int_{\mathbb{R}} u_0(x)\,dx, \text{ for all } t > 0.$$

THEOREM 4.24. *Let $u_0 \in \overline{\mathcal{D}(A_\phi)} \cap L_\infty(\mathbb{R})$ and $\phi : \mathcal{D}(\phi) \to \mathbb{R}$ both continuous and nondecreasing such that either $\phi(0) = 0$ and $0 \in \operatorname{int} \mathcal{D}(\phi)$ or $\mathcal{D}(\phi) \subset (0,\infty)$. Then $u(t,x)$ is a weak solution of Problem* (I) *and satisfies* $\lim_{t\downarrow 0} \|u(t) - u_0\|_1 = 0$.

PROOF. Clearly, $\beta = \phi^{-1}$ satisfies the conditions of Sections 4.2 and 4.3. Then define $u_\epsilon(t) = (E_\lambda^{A_\phi})^{[t/\epsilon]+1} u_0$ for $t > 0$ and $u_\epsilon(t) = u_0$ for $t \leq 0$. The function $u_\epsilon(t,x)$ satisfies for each $t > 0$,

$$\frac{1}{\epsilon}\{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)\} = A_\phi u_\epsilon(t,x), \text{ in } \mathcal{D}'(0,\infty),$$

or

$$\frac{1}{\epsilon}\{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)\} = w_\epsilon''(t,x), \text{ in } \mathcal{D}'(0,\infty),$$

or, with $\xi \in C_0^\infty(\mathbb{R})$:

Method for construction of a solution: backward difference scheme

[C.J. van Duijn]

$$\frac{1}{\epsilon}\int_{\mathbb{R}} \{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)\}\xi\,dx - \int_{\mathbb{R}} w_\epsilon(t,x)\xi''(x)\,dx = 0,$$

where $w_\epsilon = \phi(u_\epsilon)$. Next let $\zeta \in C_0^\infty(0,\infty)$. Multiply the equation by ζ and integrate. This gives

$$\int_0^\infty \int_{\mathbb{R}} \frac{1}{\epsilon}\{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)\}\zeta(t)\xi(x)\,dx = \int_0^\infty \int_{\mathbb{R}} \phi(u_\epsilon(t,x))\zeta(t)\xi''(x)\,dx.$$

Set $f(t,x) = \zeta(t)\xi(x)$. Then

$$\frac{1}{\epsilon}\int_0^\infty \int_{\mathbb{R}} \{u_\epsilon(t,x) - u_\epsilon(t-\epsilon,x)\} f(t,x) =$$

$$\frac{1}{\epsilon}\int_0^\infty \int_{\mathbb{R}} u_\epsilon(t,x) f(t,x) - \frac{1}{\epsilon}\int_{-\epsilon}^\infty \int_{\mathbb{R}} u_\epsilon(t,x) f(t+\epsilon,x) =$$

$$\int_0^\infty \int_{\mathbb{R}} u_\epsilon(t,x) \frac{f(t,x) - f(t+\epsilon,x)}{\epsilon},$$

since $f \in C_0^\infty((0,\infty)\times\mathbb{R})$. To take the limit for $\epsilon \downarrow 0$, we use that (i) $u_\epsilon \to u$ a.e., (ii) $\phi(u_\epsilon) \to \phi(u)$ a.e. (by the continuity of ϕ), (iii) $\|u_\epsilon\|_\infty \leq \|u_0\|_\infty$ for all $\epsilon > 0$, and (iv) the dominated convergence theorem. We obtain

$$\int_0^\infty \int_{\mathbb{R}} \{u(t,x) f_t(t,x) - \phi(u(x,t)) f_{xx}(t,x)\} = 0,$$

for all $f \in C_0^\infty((0,\infty)\times\mathbb{R})$. $\square$

REMARK 4.25. BREZIS & CRANDALL [1979] showed that if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and if $\phi(0) = 0$, then bounded weak solutions of Problem (I) which satisfy $\lim_{t\downarrow 0} \|u(t) - u_0\|_1 = 0$ are unique.

- G. Duvaut, J.L. Lions. *Inequalities in Mechanics and Physics*. Grundlehren der mathematischen Wissenschaften 219. Springer-Verlag 1976

- H.W. Alt, S. Luckhaus. *Elliptic-parabolic differential equations*. Math. Z. **183**, pp. 311-341 (1983).

1.4. Weak Solution. Assume 1.3. We call $u \in u^D + L^r(0, T; V)$ a *weak solution* of the initial boundary value problem (0.1)–(0.3), if the following two properties are fulfilled:

1) $b(u) \in L^\infty(0, T; L^1(\Omega))$ and $\partial_t b(u) \in L^{r^*}(0, T; V^*)$ with initial values b^0, that is,

$$\int_0^T \langle \partial_t b(u), \zeta \rangle + \int_0^T \int_\Omega (b(u) - b^0)\, \partial_t \zeta = 0$$

for every test function $\zeta \in L^r(0, T; V) \cap H^{1,1}(0, T; L^\infty(\Omega))$ with $\zeta(T) = 0$.

2) $a(b(u), \nabla u)$, $f(b(u)) \in L^{r^*}(]0, T[\times \Omega)$ and u satisfies the differential equation, that is,

$$\int_0^T \langle \partial_t b(u), \zeta \rangle + \int_0^T \int_\Omega a(b(u), \nabla u) \cdot \nabla \zeta = \int_0^T \int_\Omega f(b(u))\, \zeta$$

for every $\zeta \in L^r(0, T; V)$.

The main tool for proving the existence of a weak solution is to give an energy estimate, which follows, if we can justify the formula

$$\int_0^T \partial_t b(u) \cdot u = B(u(T)) - B(u^0).$$

This is done in the following lemma (see also Remark 1.6).

- S. Luckhaus, A. Visintin. *Phase transition in multicomponent systems*. manuscripta math. **43**, pp. 261-288 (1983).

• H.W. Alt, E. Di Benedetto. *Nonsteady flow of water and oil through inhomogeneous porous media*. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 12, 335-392 (1985).

The flow of two immiscible fluids through a porous medium is described by (see e.g. [2] (9.3.25) and [3] (6.36), (6.52))

$$\partial_t s_i - \nabla\cdot(k_i(\nabla p_i + e_i)) = 0\,, \qquad i = 1, 2 \tag{1.1}$$

with

$$s_i = s_i(x, p_1 - p_2) \qquad \text{and} \qquad k_i = k(x, s_i)\,,$$

and

$$s_1(x, p_1 - p_2) + s_2(x, p_1 - p_2) = s_0(x)\,. \tag{1.2}$$

and for all $(v_1, v_2) \in \mathcal{K}$ the inequality

$$\sum_{i=1,2} \int_0^T\!\!\int_\Omega \big(\partial_t s_i(p_1 - p_2)(v_i - p_i) + k_i(s_i(p_1 - p_2))(\nabla p_i + e_i)\nabla(v_i - p_i)\big) \geqslant 0 \tag{1.8}$$

is satisfied. This weak formulation may be inaccurate in two points. First $\partial_t s_i$ needs not to be a function, and secondly ∇p_i may explode near the set $\{k_i = 0\}$ and therefore it may be well defined in the sense of distribution. Because of this we did not specify the topology in the above defini-

• H.W. Alt, E. Di Benedetto. *Nonsteady flow of water and oil through inhomogeneous porous media*. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 12, 335-392 (1985).

$$
\begin{aligned}
(2.1)\qquad & \int_\Omega \big(\Psi(s_1(p_1-p_2)(t)) - \Psi(s_1^0)\big) \\
& + \int_0^t\int_\Omega \left(\sum_i \frac{\left|\sum_j k_{ij}(s_i(p_1-p_2))\nabla u_j\right|^2}{k_i(s_i(p_1-p_2))} + \sum_{ij} k_{ij}(s_i(p_1-p_2))\nabla u_j \cdot e_i \right) \\
& \leqslant \sum_i \left(\int_\Omega \big(s_i(p_1-p_2)(t) v_i(t_i^0) - s_i^0 v_i(0)\big) - \int_0^t\int_\Omega s_i(p_1-p_2)\,\partial_t v_i \right. \\
& \left. + \int_0^t\int_\Omega \nabla v_i \Big(\sum_j k_{ij}(s_i(p_1-p_2))\nabla u_j + k_i(s_i(p_1-p_2)) e_i \Big) \right).
\end{aligned}
$$

2.4. Definitions. We set

$$
\Psi(x,z) := \sup_{p_{\min} \leqslant \sigma \leqslant p_{\max}} \int_0^\sigma \big(z - s_1(x,\xi)\big)\, d\xi\,.
$$

Then

$$
\Psi(x, s_1(x,z)) = \int_0^z \big(s_1(x,z) - s_1(x,\xi)\big)\, d\xi\,,
$$

- Pseudo-monotone operators
(abstract elliptic existence theorems)

- H.W. Alt. *Ellptische Probleme mit freiem Rand*.
Vorlesungsreihe SFB 256, No. 21, Bonn, Juli 1991

Wir beweisen den folgenden abstrakten Existenzsatz:

2.1 Existenzsatz. Es sei V ein separabler, reflexiver Banachraum, $M \subset V$ konvex, abgeschlossen und nichtleer und $F : M \to V^*$ habe die folgenden Eigenschaften:

(2.1.1) F sei beschränkt auf beschränkten Teilmengen von M.

(2.1.2) F sei koerziv bezüglich eines Punktes $u_0 \in M$, d.h. es gilt

$$\frac{\langle u - u_0, F(u)\rangle}{\|u - u_0\|} \to \infty \qquad \text{für } u \in M \text{ mit } \|u\| \to \infty.$$

(2.1.3) F erfülle die folgende Stetigkeitsbedingung für $u \in M$, $u^* \in V^*$ und jede Folge $(u_m)_{m\in\mathbb{N}}$ in M:

$$\left.\begin{array}{l} u_m \to u \text{ schwach in } V \\ F(u_m) \to u^* \text{ schwach}^* \text{ in } V^* \\ \limsup\limits_{m\to\infty}\langle u_m, F(u_m)\rangle \le \langle u, u^*\rangle \end{array}\right\} \Rightarrow \begin{array}{l} \langle u - v, F(u) - u^*\rangle \le 0 \text{ für alle } v \in M \\ \text{und} \\ \limsup\limits_{m\to\infty}\langle u_m, F(u_m)\rangle = \langle u, u^*\rangle\ . \end{array}$$

Dann existiert eine Lösung $u \in M$ der Variationsungleichung

$$\langle u - v, F(u)\rangle \le 0 \qquad \text{für alle } v \in M$$

V separable reflexive Banach space

Constraint $M \subset V$ nonempty closed convex, $F : M \to V^*$

Assumptions:

(1) Boundedness

F maps bounded sets of M into bounded sets of V^*

(2) Continuity condition

Let $u_m, u \in V$ and $v^* \in V^*$ with

$$\left\{ \begin{array}{c} u_m, u \in M \text{ and } u_m \to u \text{ weakly in } V \text{ for } m \to \infty, \\ F(u_m) \to v^* \text{ weakly}^* \text{ in } V^* \text{ for } m \to \infty, \text{ and} \\ \limsup_{m\to\infty} \langle\, u_m \,,\, F(u_m) \,\rangle_V \leq \langle\, u \,,\, v^* \,\rangle_V \,, \end{array} \right\}$$

then

$$\left\{ \begin{array}{c} \langle\, u - v \,,\, F(u) - v^* \,\rangle_V \leq 0 \text{ for all } v \in M, \text{ and} \\ \limsup_{m\to\infty} \langle\, u_m \,,\, F(u_m) \,\rangle_V = \langle\, u \,,\, v^* \,\rangle_V \,. \end{array} \right\}$$

(3) Coerciveness

For some $\bar{u} \in M$ there holds

$$\frac{\langle\, u - \bar{u} \,,\, F(u) \,\rangle_V}{\|u - \bar{u}\|_V} \to \infty \qquad \text{for } u \in M,\ \|u - \bar{u}\|_V \to \infty.$$

Conclusion:

Under these assumptions there exists $u \in M$, so that

$$\langle\, u - v \,,\, F(u) \,\rangle_V \leq 0 \qquad \text{for all } v \in M.$$

Proof of (2) for monotone operators:

$F = A$ monotone

$$0 \le \langle u_m - v\,,\, A(u_m) - A(v)\,\rangle_V$$

$$= \underbrace{\langle u_m\,,\, A(u_m)\,\rangle_V}_{\text{limit}} - \underbrace{\langle v\,,\, A(u_m)\,\rangle_V}_{\to \langle v\,,\, v^*\,\rangle_V} - \underbrace{\langle u_m - v\,,\, A(v)\,\rangle_V}_{\to \langle u - v\,,\, A(v)\,\rangle_V}$$

$$\text{limit} \le \langle u\,,\, v^*\,\rangle_V$$

For $v = u$ it follows $\lim_{m\to\infty} \langle u_m\,,\, A(u_m)\,\rangle_V = \langle u\,,\, v^*\,\rangle_V$ and

$$0 \le \langle u - v\,,\, v^* - A(v)\,\rangle_V$$

Minty lemma ($v \rightsquigarrow (1-\varepsilon)u + \varepsilon v = u - \varepsilon(u-v) \in M$ and $\varepsilon \to 0$)

$$0 \le \langle u - v\,,\, v^* - A(u)\,\rangle_V$$

Proof of (2) for compact perturbations of monotone operators:

$F(u) = A(u,u)$, $\quad u \mapsto A(v,u)$ monotone

$$0 \le \langle u_m - v\,,\, A(u_m,u_m) - A(u_m,v)\,\rangle_V$$

$$= \underbrace{\langle u_m\,,\, F(u_m)\,\rangle_V}_{\text{limit}} - \underbrace{\langle v\,,\, F(u_m)\,\rangle_V}_{\to \langle v\,,\, v^*\,\rangle_V} - \underbrace{\langle u_m - v\,,\, A(u_m,v)\,\rangle_V}_{\to \langle u - v\,,\, A(u,v)\,\rangle_V}$$

$$\text{limit} \le \langle u\,,\, v^*\,\rangle_V$$

Rest of argumentation similar as above.

Parabolic problem

$$\text{“}\ \partial_t \underbrace{b(u)}_{\in H} + \underbrace{\mathcal{A}(u)}_{\in V^*} = 0\ \text{”}$$

$$\Big(u^i - v\,,\ \frac{1}{h}(b(u^i) - b(u^{i-1})) \Big)_H + \big\langle\, u^i - v\,,\ A^i(u^i)\,\big\rangle_V \le 0 \text{ for all } v \in M$$

In literature under
“doubly nonlinear”

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[**vDP**] van Duijn C. J. & Peletier L., A class of similarity solutions of the nonlinear diffusion equation, Non. Anal. T. M. & A. vol. 1, 223-233, 1977.

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$$\text{“}\ \partial_t \underbrace{b(u)}_{\in H} + \underbrace{\mathcal{A}(u)}_{\in V^*} = 0\ \text{”}$$

$$\Big(u^i - v\,,\, \frac{1}{h}(b(u^i) - b(u^{i-1})) \Big)_H + \big\langle\, u^i - v\,,\, A^i(u^i)\,\big\rangle_V \leq 0 \text{ for all } v \in M$$

- H.W. Alt. *Partielle Differentialgleichungen III*. Vorlesung Wintersemester 2003/04, Universität Bonn.
- H.W. Alt. *An abstract existence theorem for parabolic systems*. CPAA Communications Pure Applied Analysis Vol.10 pp.??-?? (2011)

General setting

Spaces: H Hilbert space, V separable reflexive Banach space
(V, H, V^*) Gelfand tripel: $V \subset H$ ($\Longrightarrow H^* \subset V^*$)
Constraint $M \subset V$ nonempty closed convex

Parabolic: $b : H \to H$ and $\psi : H \to \mathbb{R}$ convex continuously differentiable
$(*)$ $b = \nabla\psi$

Elliptic: $\mathcal{A} : \mathcal{M} \to L^{p^*}([0,T]; V^*)$
$\mathcal{M} := \{u \in L^p([0,T]; M)\ ;\ B(u) \in L^\infty([0,T])\}$
$\mathcal{A}(u)(t) = A(t, u(t))$

Assumptions:

(1) Boundedness

$\mathcal{A}$ maps “bounded” sets of $\mathcal{M}$ into bounded sets of $L^{p^*}([0,T];V^*)$
b maps bounded sets of H into bounded sets of H

(2) Continuity condition

Let $u_m, u \in L^p([0,T];V)$ and $v^* \in L^{p^*}([0,T];V^*)$ with

$$\left\{ \begin{array}{c} u_m(t), u(t) \in M \text{ and } u_m \to u \text{ weakly in } L^p([0,T];V) \text{ for } m \to \infty, \\ \{B(u_m)\,;\ m \in \mathbb{N}\} \text{ bounded in } L^\infty([0,T]) \text{ and} \\ b(u_m) \to b(u) \text{ strongly in } L^1([0,T];H), \\ \mathcal{A}(u_m) \to v^* \text{ weakly in } L^{p^*}([0,T];V^*) \text{ for } m \to \infty, \text{ and} \\ \limsup_{m\to\infty} \int_0^T \langle\, u_m(t)\,,\, \mathcal{A}(u_m)(t)\,\rangle_V \,\mathrm{d}t \le \int_0^T \langle\, u(t)\,,\, v^*(t)\,\rangle_V \,\mathrm{d}t, \end{array} \right\}$$

then

$$\left\{ \begin{array}{c} \int_0^T \langle\, u(t) - v(t)\,,\, \mathcal{A}(u)(t) - v^*(t)\,\rangle_V \,\mathrm{d}t \le 0 \text{ for all } v \in L^p([0,T];V), \text{ and} \\ \limsup_{m\to\infty} \int_0^T \langle\, u_m(t)\,,\, \mathcal{A}(u_m)(t)\,\rangle_V \,\mathrm{d}t = \int_0^T \langle\, u(t)\,,\, v^*(t)\,\rangle_V \,\mathrm{d}t. \end{array} \right\}$$

(3) Coerciveness There is an $\bar{u} \in M$ so, that for almost all $t \in]0,T[$

$$\langle\, u - \bar{u}\,,\, A(t,u)\,\rangle_V \ge c_0 \|u - \bar{u}\|_V^p - C_0 B_{\bar{u}}(u) - G_0(t)$$

for all $u \in M$. Here $G_0 \in L^1([0,T])$.

Conclusion:

Then there exist solutions of the "evolution equation", that is for given $u_0 \in \mathrm{clos}_H(M)$ there is an $u \in L^p([0,T];V)$ with

$$
\begin{gathered}
u(t) \in M \quad \text{for almost all } t, \\
\left\{
\begin{aligned}
& B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) + (\, \bar u - v(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\
& + \int_0^{\bar t} \Big(-(\, \partial_t(\bar u - v)(t)\,,\, b(u(t)) - b(u_0)\,)_H \\
& \qquad\qquad + \langle\, u(t) - v(t)\,,\, A(t,u(t))\,\rangle_V \Big)\, \mathrm{d}t \le 0 \\
& \text{for almost all } \bar t \in]0,T[,
\end{aligned}
\right\} \\
\text{for all } v \in C^\infty([0,T];V) \text{ with } v(t) \in M \text{ for almost all } t.
\end{gathered}
$$

Conclusion, if $M \subset V$ is a closed affine set:

There is an $u \in L^p([0,T];V)$ with

$$
\begin{gathered}
u(t) \in M \quad \text{for almost all } t, \\
\left\{ \int_0^T \Big(-(\, \partial_t\varphi(t)\,,\, b(u(t)) - b(u_0)\,)_H + \langle\, \varphi(t)\,,\, A(t,u(t))\,\rangle_V \Big)\, \mathrm{d}t = 0 \right\} \\
\text{for all } \varphi \in C_0^\infty([0,T[;V) \text{ with } \varphi(t) \in M_0 \text{ for almost all } t.
\end{gathered}
$$

Here $M_0 \subset V$ is the subspace, such that $M = u_1 + M_0$ for every $u_1 \in M$.

Time discrete case

Starting point:

$$u^i \in M \text{ with}$$
$$\left(u^i - v \,,\, \frac{1}{h}(b(u^i) - b(u^{i-1})) \right)_H + \left\langle u^i - v \,,\, A^i(u^i) \right\rangle_V \leq 0$$
$$\text{for all } v \in M$$

where

$$A^i(w) = A_h(t,w) := \frac{1}{h}\int_{(i-1)h}^{ih} A(s,w)\,\mathrm{d}s$$
$$u^i = u_h(t) \qquad \text{for } (i-1)h < t \leq ih$$

Plan of Proof

Basic estimates:

- Energy estimate
- Time compactness

Proof of theorem:

- Convergence proof using pseudo-monotonicity
- Special subspace case

Time compactness

$$\left(u^i - v\,,\, \frac{1}{h}(u^{*i} - u^{*i-1}) \right)_H + \left\langle\, u^i - v\,,\, w^{*i} \,\right\rangle_V \le 0$$

at time $t^i = ih$ where $u^{*i} := b(u^i),\ w^{*i} := A^i(u^i)$

It is assumed that t and s are multiple of h, say,

$$t = kh, \quad t + s = (k+j)h$$

We set $v = u^k$, sum over $i = k+1, \dots, k+j$. The result is

$$\sum_{i=k+1}^{k+j} \left(\, u^i - u^k\,,\, u^{*i} - u^{*i-1} \,\right)_H \le \sum_{i=k+1}^{k+j} h \left\langle\, u^k - u^i\,,\, w^{*i} \,\right\rangle_V$$

The left side is

$$\sum_{i=k+1}^{k+j} \left(\, u^i - u^k\,,\, u^{*i} - u^{*i-1} \,\right)_H$$

$$= \sum_{i=k+1}^{k+j} \left(\left(\, u^i\,,\, u^{*i} \,\right)_H - \left(\, u^i\,,\, u^{*i-1} \,\right)_H \right) - \sum_{i=k+1}^{k+j} \left(\, u^k\,,\, u^{*i} - u^{*i-1} \,\right)_H$$

Now by Young's inequality it holds

$$\left(\, u^i\,,\, u^{*i} \,\right)_H = \psi^*(u^{*i}) + \psi(u^i), \quad \left(\, u^i\,,\, u^{*i-1} \,\right)_H \le \psi^*(u^{*i-1}) + \psi(u^i)$$

we compute for the left side

$$
\begin{aligned}
&\sum_{i=k+1}^{k+j} \left(u^i - u^k \, , \, u^{*i} - u^{*i-1} \right)_H \\
&= \sum_{i=k+1}^{k+j} \left(\left(u^i \, , \, u^{*i} \right)_H - \left(u^i \, , \, u^{*i-1} \right)_H \right) - \sum_{i=k+1}^{k+j} \left(u^k \, , \, u^{*i} - u^{*i-1} \right)_H \\
&\geq \sum_{i=k+1}^{k+j} \left(\psi^*(u^{*i}) - \psi^*(u^{*i-1}) \right) - \left(u^k , \sum_{i=k+1}^{k+j} (u^{*i} - u^{*i-1}) \right)_H \\
&= \psi^*(u^{*k+j}) - \psi^*(u^{*k}) - \left(u^k \, , \, u^{*k+j} - u^{*k} \right)_H \\
&= E_{\psi^*}(u^{*k+j}, u^{*k}, u^k) = E_{\psi^*}(b(u^{k+j}), b(u^k), u^k) \\
&\geq 0
\end{aligned}
$$

Thus we have shown

$$
\begin{aligned}
&0 \leq E_{\psi^*}(b(u_h(t+s)), b(u_h(t)), u_h(t)) \\
&= E_{\psi^*}(u^{*k+j}, u^{*k}, u^k) \leq \sum_{i=k+1}^{k+j} \left(u^i - u^k \, , \, u^{*i} - u^{*i-1} \right)_H \\
&\leq \sum_{i=k+1}^{k+j} h \left\langle u^k - u^i \, , \, w^{*i} \right\rangle_V = s \cdot \frac{1}{j} \sum_{i=1}^{j} \left\langle u^k - u^{k+i} \, , \, w^{*k+i} \right\rangle_V
\end{aligned}
$$

$$\int_0^t (\, u(s) - v(s)\, ,\, \partial_t b(u(s))\,)_H \, \mathrm{d}s =$$

$$\Phi_{\bar u}(u,v)(t) := \quad B_{\bar u}(u(t)) - B_{\bar u}(u_0) + (\, \bar u - v(t)\, ,\, b(u(t)) - b(u_0)\,)_H \\ - \int_0^t (\, \partial_t(\bar u - v)(s)\, ,\, b(u(s)) - b(u_0)\,)_H \, \mathrm{d}s$$

We have proved the inequality

$$\Phi_{\bar u}(u,v)(\bar t) + \int_0^{\bar t} \langle\, u(t) - v(t)\, ,\, A(t,u(t)\, \rangle_V \, \mathrm{d}t \le 0$$

for all $v \in C^\infty([0,T];V)$ which satisfies $v(t) \in M$. Here M now is an affine subspace. It follows that this inequality then also holds for all $v \in W^{1,p}(]0,T[;M) \subset W^{1,1}(]0,T[;H) \cap L^p(]0,T[;M)$. Now $u_0 \in \mathrm{clos}_H\,(M)$, hence there is $u_{0\delta} \in M$ so that $u_{0\delta} \to u_0$ in H. Then define u_δ as

$$u_\delta(t) := \frac{1}{\delta}\int_{t-\delta}^t \widetilde{u}_\delta(s)\,\mathrm{d}s, \qquad \text{where} \qquad \widetilde{u}_\delta(t) := \begin{cases} u(t) & \text{for } t > 0, \\ u_{0\delta} & \text{for } t < 0, \end{cases}$$

and let $\quad v := u_\delta - \varphi \in W^{1,p}(]0,T[;M)$

with $\varphi \in W^{1,\infty}(]0,T[;V)$ and $\varphi(t) \in M_0$. Now (since $u_\delta \to u$ in $L^p(]0,T[;V)$)

$$\int_0^{\bar t} \langle\, u(t) - (u_\delta(t) - \varphi(t))\, ,\, A(t,u(t))\, \rangle_V \, \mathrm{d}t \longrightarrow \int_0^{\bar t} \langle\, \varphi(t)\, ,\, A(t,u(t))\, \rangle_V \, \mathrm{d}t$$

$$
\begin{aligned}
\Phi_{\bar u}(u,v)(\bar t) = \;& \Phi_{\bar u}(u,u_\delta-\varphi)(\bar t) \\
= \;& B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) \\
& - \int_0^{\bar t} (\, \partial_t(\bar u - u_\delta + \varphi)(t)\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
& + (\, \bar u - u_\delta(\bar t) + \varphi(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\
= \;& \Phi_{\bar u}(u,u_\delta)(\bar t) \\
& - \int_0^{\bar t} (\, \partial_t\varphi(t)\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
& + (\, \varphi(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H
\end{aligned}
$$

Since $\liminf_{\delta\to 0} \Phi_{\bar u}(u,u_\delta)(\bar t) \geq 0$ for almost all $\bar t > 0$ we obtain

$$
\begin{aligned}
& (\, \varphi(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\
& - \int_0^{\bar t} (\, \partial_t\varphi(t)\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t + \int_0^{\bar t} \langle\, \varphi(t)\,,\, A(t,u(t))\,\rangle_V \,\mathrm{d}t \leq 0
\end{aligned}
$$

This is obviously equivalent to the assertion. Now we can replace φ by $-\varphi$ to obtain that the left side equals zero. If we now restrict $\varphi \in C_0^\infty([0,T[;M)$, one chooses $\bar t$ close to T in order to get

$$
\int_0^T \Big((\, -\partial_t\varphi(t)\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t + \int_0^T \langle\, \varphi(t)\,,\, A(t,u(t))\,\rangle_V \Big) \,\mathrm{d}t = 0
$$

Sketch of proof of final theorem

Energy estimate is

$$B_{\bar{u}}(u_h(\bar{t})) + c_0 \int_0^{\bar{t}} \|u_h(t) - \bar{u}\|_V^p \, \mathrm{d}t \le C$$

The compactness lemma implies that

$$\{b(u_h)\,;\; 0 < h < h_0\} \text{ is compact in } L^1([0,T];H)$$

From energy estimate for a subsequence $h \to 0$

$$u_h \to u \text{ weakly in } L^p([0,T];V)$$

Then there is a convergent subsequence $h \to 0$ so that

$$b(u_h) \to b(u) \text{ strongly in } L^1([0,T];H)$$

By the boundedness condition (1) there is a subsequence $h \to 0$

$$\mathcal{A}(u_h) = A(t,u_h) \to u^* \text{ weakly}^* \text{ in } L^{p*}([0,T];V^*)$$

Hence all convergence properties of the "continuity condition" (2) are satisfied with one exception. Now the time discrete inequality reads

$$\Phi_{\bar{u}}^h(u_h,v)(\widetilde{t}) + \int_0^{\widetilde{t}} \langle\, u_h(t) - v(t)\,,\; A_h(t,u_h(t)\,\rangle_V \, \mathrm{d}t \le 0$$

$$\Phi_{\bar{u}}^h(u_h,v)(\widetilde{t}) := \int_0^{\widetilde{t}} \big(\, u_h(t) - v(t)\,,\; \partial_t^{-h} b(u_h(t))\,\big)_H \, \mathrm{d}t$$

We know

$$\Phi^h_{\bar u}(u_h,v)(\bar t_h) + \int_0^{\bar t_h} \langle\, u_h(t)\,,\, A(t,u_h(t)\,\rangle_V \,\mathrm{d}t \le \int_0^{\bar t_h} \langle\, v(t)\,,\, A_h(t,u_h(t)\,\rangle_V \,\mathrm{d}t$$

For the parabolic part

$$\begin{aligned}
\Phi^h_{\bar u}(u_h,v)(\bar t_h) &= \int_0^{\bar t_h} \Big(\, u_h(t) - v(t)\,,\, \partial_t^{-h} b(u_h(t))\,\Big)_H \,\mathrm{d}t \\
&\ge \int_0^{\bar t_h} \partial_t^{-h} B_{\bar u}(u_h(t))\,\mathrm{d}t + \frac{1}{h}\int_0^{\bar t_h} (\,\bar u - v(t)\,,\, b(u_h(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&\qquad - \frac{1}{h}\int_{-h}^{\bar t_h - h} (\,\bar u - v(t+h)\,,\, b(u_h(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&= \; B_{\bar u}(u_h(\bar t_h)) - B_{\bar u}(u_0) + (\,\bar u - v_h(\bar t_h)\,,\, b(u_h(\bar t_h)) - b(u_0)\,)_H \\
&\qquad - \int_0^{\bar t_h - h} \Big(\, \partial_t^{+h}(\bar u - v(t))\,,\, b(u_h(t)) - b(u_0)\,\Big)_H \,\mathrm{d}t \\
&\ge\!\longrightarrow \; B_{\bar u}(u(\bar t)) - B_{\bar u}(u_0) + (\,\bar u - v(\bar t)\,,\, b(u(\bar t)) - b(u_0)\,)_H \\
&\qquad - \int_0^{\bar t} (\,\partial_t(\bar u - v(t))\,,\, b(u(t)) - b(u_0)\,)_H \,\mathrm{d}t \\
&= \Phi_{\bar u}(u,v)(\bar t)
\end{aligned}$$

Therefore we have proved that

$$\liminf_{h\to 0} \Phi^h_{\bar u}(u_h,v)(\bar t_h) \ge \Phi_{\bar u}(u,v)(\bar t)$$

Since equation reads

$$\Phi^h_{\bar u}(u_h,v)(\bar t_h)+\int_0^{\bar t_h}\langle\, u_h(t)\,,\;A(t,u_h(t)\,\rangle_V\;\mathsf{d}t$$
$$\le\int_0^{\bar t_h}\langle\, v(t)\,,\;A_h(t,u_h(t)\,\rangle_V\;\mathsf{d}t=\int_0^{\bar t_h}\langle\, v_h(t)\,,\;A(t,u_h(t)\,\rangle_V\;\mathsf{d}t$$
$$\longrightarrow\int_0^{\bar t}\langle\, v(t)\,,\;u^*(t)\,\rangle_V\;\mathsf{d}t$$

for $h\to 0$, we obtain

$$\liminf_{h\to 0}\Phi^h_{\bar u}(u_h,v)(\bar t_h)+\limsup_{h\to 0}\int_0^{\bar t_h}\langle\, u_h(t)\,,\;A(t,u_h(t)\,\rangle_V\;\mathsf{d}t$$
$$\le\int_0^{\bar t}\langle\, v(t)\,,\;u^*(t)\,\rangle_V\;\mathsf{d}t,$$

that is

$$\Phi_{\bar u}(u,v)(\bar t)+\limsup_{h\to 0}\int_0^{\bar t_h}\langle\, u_h(t)\,,\;A(t,u_h(t)\,\rangle_V\;\mathsf{d}t\le\int_0^{\bar t}\langle\, v(t)\,,\;u^*(t)\,\rangle_V\;\mathsf{d}t$$

Now we come to the remaining property for the sequences in (2).
We set $v=u_\delta$:

$$\Phi_{\bar u}(u,u_\delta)(\bar t) + \limsup_{h\to 0}\int_0^{\bar t_h} \langle\, u_h(t)\,,\ A(t,u_h(t)\,\rangle_V \ \mathrm{d}t$$
$$\le \int_0^{\bar t} \langle\, u_\delta(t)\,,\ u^*(t)\,\rangle_V \ \mathrm{d}t \longrightarrow \int_0^{\bar t} \langle\, u(t)\,,\ u^*(t)\,\rangle_V \ \mathrm{d}t$$

as $\delta \to 0$. Since $\Phi_{\bar u}(u,u_\delta)(\bar t)$ in the limit $\delta \to 0$ was nonnegative, we arrive at

$$\limsup_{h\to 0}\int_0^{\bar t_h} \langle\, u_h(t)\,,\ A(t,u_h(t)\,\rangle_V \ \mathrm{d}t \le \int_0^{\bar t} \langle\, u(t)\,,\ u^*(t)\,\rangle_V \ \mathrm{d}t$$

This was the last property in the assumption of (2), and therefore

$$\left\{\begin{array}{c} \displaystyle\int_0^{\bar t} \langle\, u(t)-v(t)\,,\ A(t,u(t))-u^*(t)\,\rangle_V \ \mathrm{d}t \le 0, \qquad \text{and} \\ \displaystyle\limsup_{h\to 0}\int_0^{\bar t_h} \langle\, u_h(t)\,,\ A(t,u_h(t))\,\rangle_V \ \mathrm{d}t = \int_0^{\bar t} \langle\, u(t)\,,\ u^*(t)\,\rangle_V \ \mathrm{d}t \end{array}\right\}$$

Plugging in the equality one gets

$$\Phi_{\bar u}(u,v)(\bar t) + \int_0^{\bar t} \langle\, u(t)-v(t)\,,\ u^*(t)\,\rangle_V \ \mathrm{d}t \le 0$$

and then the inequality, therefore the assertion

$$\Phi_{\bar u}(u,v)(\bar t) + \int_0^{\bar t} \langle\, u(t)-v(t)\,,\ A(t,u(t)\,\rangle_V \ \mathrm{d}t \le 0$$

Collaboration with Hans van Duijn

Stationary problems in porous media:

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