Distributional equations and the entropy principle

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Plan of the talk:

- Entropy principle
- Example: Gravity
- Distributions in spacetime
- Limit of phase field models

Zeroth Law There exists for every thermodynamic system in equilibrium a property called temperature. Equality of temperature is a necessary and sufficient condition for thermal equilibrium.

First Law There exists for every thermodynamic system a property called the energy. The change of energy of a system is equal to the mechanical work done on the system in an adiabatic process. In a non-adiabatic process, the change in energy is equal to the heat added to the system minus the mechanical work done by the system.

Second Law There exists for every thermodynamic system in equilibrium an extensive scalar property called the entropy, S, such that in an infinitesimal reversible change of state of the system, $dS = dQ/T$, where T is the absolute temperature and dQ is the amount of heat received by the system. The entropy of a thermally insulated system cannot decrease and is constant if and only if all processes are reversible.

[MIT, Lecture on Thermodynamics (Spakovszky, Fall 2008)]
(This is the so-called "axiomatic formulation")

Zeroth Law There exists for every thermodynamic system in equilibrium a property called temperature. Equality of temperature is a necessary and sufficient condition for thermal equilibrium.

θ absolute temperature

First Law There exists for every thermodynamic system a property called the energy. ...

e (total) energy

$$\partial_t e + \mathrm{div}\varphi = ...$$

Second Law There exists for every thermodynamic system in equilibrium an extensive scalar property called the entropy, ... The entropy of a thermally insulated system cannot decrease and is constant if and only if all processes are reversible.

$$\partial_t \eta + \mathrm{div}\psi \geq 0 \qquad \eta \quad \text{entropy}$$

$$\eta'_\varepsilon = \frac{1}{\theta} \quad (dS = dQ/T), \qquad \varepsilon \text{ internal energy}$$

For a set $\mathcal{P}$ of processes: There exists (η, ψ) with

$$h := \partial_t \eta + \mathrm{div}\, \psi \geq 0 \qquad \text{(objective scalar equation)}$$

for each process in $\mathcal{P}$

(η the entropy, ψ the entropy flux, constitutive relations for (η, ψ))

Basic assumptions come from this principle

Example 1 : **System of hyperbolic conservation laws**

$$\partial_t u_k + \mathrm{div} q_k(u) = f_k(u) \quad (k = 1, \ldots, N)$$

Constitutive ansatz : $\eta = \widehat{\eta}(u)$, $\psi = \widehat{\psi}(u)$

$$\partial_t \eta = \sum_k \eta'_k \partial_t u_k$$

$$\eta'_k \partial_t u_k = \underbrace{\eta'_k}_{\lambda_k = \text{multiplicator}} \cdot \underbrace{(\partial_t u_k + \mathrm{div} q_k - f_k)}_{=0 \text{ for solutions}} - \eta'_k \mathrm{div} q_k + \eta'_k f_k$$

$$h := \partial_t \eta + \mathrm{div} \psi = \mathrm{div} \psi - \sum_k \eta'_k \mathrm{div} q_k + \sum_k \eta'_k f_k$$

$$h := \partial_t \eta + \mathrm{div}\psi \;= \mathrm{div}\psi - \sum_k \eta'_k \mathrm{div} q_k + \sum_k \eta'_k f_k$$

$$= \sum_l \Big(\psi'_l - \sum_k \eta'_k q_k{}'_l \Big) \bullet \nabla u_l + \sum_k \eta'_k f_k$$

Requirement: For all solutions of the system

$$h = \sum_l \Big(\psi'_l(u) - \sum_k \eta'_k(u) q_k{}'_l(u) \Big) \bullet \nabla u_l + \sum_k \eta'_k(u) f_k(u) \geq 0$$

Conclusion : Entropy principle satisfied, if

$$\psi'_l(u) = \sum_k \eta'_k(u) q_k{}'_l(u) \text{ for all } l$$

$$\sum_k \eta'_k(u) f_k(u) \geq 0$$

Special case of a single equation : ψ ergibt sich aus $\psi' = \eta' q'$

General case : ψ exists if and only if

$$\sum_k \eta'_{km} q_k{}'_l \qquad \text{symmetric in } l, m$$

Consequence : Strong connection between entropy terms and terms of the differential equations

Example 2 : **Compressible fluid**

$$\partial_t \varrho + \mathrm{div}(\varrho v) = 0$$

$$\partial_t(\varrho v) + \mathrm{div}(\varrho v v^{\mathsf{T}} + \Pi) = \mathbf{f} \qquad (\Pi \text{ pressure tensor})$$

$$\partial_t e + \mathrm{div}(e v + \Pi^{\mathsf{T}} v + q) = v \bullet \mathbf{f} \qquad (e \text{ total energy})$$

$$e = \varepsilon + \frac{\varrho}{2} |v|^2 \qquad (\varepsilon \text{ inner energy})$$

Outcome of entropy principle is :

$$\Pi = p\mathrm{Id} - S \qquad (p \text{ pressure, } S \text{ tension tensor})$$

Constitutive ansatz : $\eta = \widehat{\eta}(\varrho, \varepsilon) \qquad (\eta \text{ objective scalar})$

$$\partial_t \eta = \eta'_{\varrho} \partial_t \varrho + \eta'_{\varepsilon} \partial_t \varepsilon, \qquad \partial_t \varepsilon = \tfrac{|v|^2}{2} \partial_t \varrho - v \bullet \partial_t(\varrho v) + \partial_t e$$

$$\partial_t \eta = \underbrace{\left(\eta'_{\varrho} + \eta'_{\varepsilon} \frac{|v|^2}{2}\right)}_{\lambda_\varrho} \partial_t \varrho + \underbrace{(-\eta'_{\varepsilon} v)}_{\lambda_v} \partial_t(\varrho v) + \underbrace{\eta'_{\varepsilon}}_{\lambda_e} \partial_t e$$

$$\dot{\eta} = \eta'_{\varrho} \dot{\varrho} + \eta'_{\varepsilon} \dot{\varepsilon} \qquad (\dot{\ } = \partial_t + v \bullet \nabla \quad \text{substantial derivative}))$$

$$\dot{\varrho} + \varrho \mathrm{div} v = 0, \qquad \dot{\varepsilon} + \varepsilon \mathrm{div} v = -\mathrm{div} q - \mathrm{D}v \bullet \Pi \qquad (\text{no force})$$

Use the differential equations and obtain

$$
\begin{aligned}
h &:= \partial_t\eta + \mathrm{div}\psi = \dot\eta + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\
&= \eta'_\varrho\dot\varrho + \eta'_\varepsilon\dot\varepsilon + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\
&= \mathrm{div}(\psi - \eta v) - \eta'_\varepsilon \mathrm{div}q \\
&\quad + \mathrm{D}v\bullet\Big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi\Big) \\
&= \mathrm{div}(\psi - \eta v - \eta'_\varepsilon q) \\
&\quad + \mathrm{D}v\bullet\Big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi\Big) + \nabla\eta'_\varepsilon\bullet q
\end{aligned}
$$

hence $\Pi = p\mathrm{Id} - S$, if $\frac{1}{\theta} = \eta'_\varepsilon > 0$, with

$$\eta - \varrho\eta'_\varrho - (\varepsilon + p)\eta'_\varepsilon = 0$$

$$
\begin{aligned}
h &= \underbrace{\mathrm{div}(\psi - \eta v - \eta'_\varepsilon q)}_{\text{flux term}} \\
&\quad + \underbrace{\eta'_\varepsilon \mathrm{D}v\bullet S + \nabla\eta'_\varepsilon\bullet q}_{\text{dissipative}}
\end{aligned}
$$

Requirement : $h \geq 0$ for all solutions of the system

$$h = \underbrace{\mathrm{div}(\psi - \eta v - \eta'_\varepsilon q)}_{\text{flux term}} + \underbrace{\eta'_\varepsilon \mathrm{D}v \bullet S + \nabla \eta'_\varepsilon \bullet q}_{\text{dissipative}}$$

Requirement : $h \geq 0$ for all solutions of the system

Conclusion : Entropy principle satisfied, if $\Pi = p\mathrm{Id} - S$ and

$$\psi = \eta v + \eta'_\varepsilon q \qquad \text{(Entropy flux)}$$

$$\eta = \varrho \eta'_\varrho + (\varepsilon + p)\eta'_\varepsilon \qquad \text{(Gibbs relation)}$$

$$\eta'_\varepsilon \mathrm{D}v \bullet S + \nabla \eta'_\varepsilon \bullet q \geq 0 \qquad \text{(Dissipative terms)}$$

The entropy identity is

$$h = \partial_t \eta + \mathrm{div}(\eta v + \eta'_\varepsilon q) = \underbrace{\eta'_\varepsilon \mathrm{D}v \bullet S + \nabla \eta'_\varepsilon \bullet q}_{\text{entropy production}} \geq 0$$

$$\frac{1}{\theta} = \eta'_\varepsilon > 0$$

Consider Newton's law :

$$\partial_t \varrho + \mathrm{div}\,(\varrho v) = 0 \qquad (\varrho \text{ total mass})$$

$$\partial_t(\varrho v) + \mathrm{div}\,(\varrho v v^{\mathsf{T}} + \Pi) = \mathbf{f} \qquad (\Pi \text{ pressure tensor})$$

$$-\Delta\phi = \varrho \qquad (\phi \text{ ``gravitational potential''})$$

$$\mathbf{f} = g\varrho\nabla\phi + \widetilde{\mathbf{f}} \qquad (\mathbf{f} \text{ ``force''},\ g \text{ gravity constant})$$

$g = 4\pi G$, $G = 6.67384\frac{\mathrm{m}^3}{\mathrm{kg\cdot s}^2}$ for $n = 3$
($\Phi = -g\phi$ is the gravitational potential)

Lemma[2011]: Gravity equation implies

$$\mathrm{div}\Big(\nabla\phi(\nabla\phi)^{\mathsf{T}} - \frac{1}{2}|\nabla\phi|^2 \mathrm{Id}\Big) = -\varrho\nabla\phi$$

Momentum equation :

$$\partial_t(\varrho v) + \mathrm{div}\,(\varrho v v^{\mathsf{T}} + \widetilde{\Pi}) = \widetilde{\mathbf{f}} \qquad (\widetilde{\mathbf{f}} \text{ may be } 0)$$

$$\widetilde{\Pi} = g(\nabla\phi(\nabla\phi)^{\mathsf{T}} - \frac{1}{2}|\nabla\phi|^2 \mathrm{Id}) + \Pi$$

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0$$

$$\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathsf{T}} + \widetilde{\Pi}) = \widetilde{\mathbf{f}}, \qquad \widetilde{\Pi} = g(\nabla\phi(\nabla\phi)^{\mathsf{T}} - \tfrac{1}{2}|\nabla\phi|^2 \mathrm{Id}) + \Pi$$

$$-\Delta\phi = \varrho$$

Energy conservation (traditional) :

$$\partial_t e + \operatorname{div}(ev + \Pi^{\mathsf{T}} v + q) = v \bullet \mathbf{f}, \qquad e = \varepsilon + \frac{\varrho}{2}|v|^2$$

(ε internal energy)

Lemma[2011]: Gravity equation and mass equation implies

$$\partial_t\Big(\frac{|\nabla\phi|^2}{2} - \varrho\phi\Big) + \operatorname{div}\Big(\Big(\frac{|\nabla\phi|^2}{2} - \varrho\phi\Big)v - \dot{\phi}\nabla\phi + \Big(\nabla\phi(\nabla\phi)^{\mathsf{T}} - \frac{1}{2}|\nabla\phi|^2\mathrm{Id}\Big)v\Big) = -v\bullet(\varrho\nabla\phi)$$

Energy conservation :

$$\partial_t \widetilde{e} + \operatorname{div}(\widetilde{e} v + \widetilde{\Pi}^{\mathsf{T}} v + \widetilde{q}) = v \bullet \widetilde{\mathbf{f}} \qquad (\widetilde{\mathbf{f}} \text{ may be } 0)$$

$$\widetilde{e} = e + g\Big(\frac{|\nabla\phi|^2}{2} - \varrho\phi\Big), \qquad \widetilde{q} = q - g\dot{\phi}\nabla\phi$$

The traditional entropy principle :

$$\text{constitutive ansatz : } \eta = \widehat{\eta}(\varrho, \varepsilon), \quad \psi = \eta v + \frac{1}{\theta} q, \quad \frac{1}{\theta} = \eta'_{\varepsilon}$$

$$h := \partial_t \eta + \operatorname{div} \psi = -\mathrm{D}v \bullet S$$

$$\Pi = p\mathrm{Id} - S, \quad \eta = \varrho \eta'_{\varrho} + (\varepsilon + p)\eta'_{\varepsilon}$$

Conclusion : The gravity does not change the entropy principle, but it changes the energy identity

$$\partial_t \widetilde{e} + \operatorname{div}(\widetilde{e} v + \widetilde{\Pi}^{\mathsf{T}} v + \widetilde{q}) = v \bullet \widetilde{\mathbf{f}}$$

$$\widetilde{e} = \underbrace{\underbrace{\varepsilon + g\Big(\frac{|\nabla \phi|^2}{2} - \varrho \phi\Big)}_{\text{internal energy}} + \frac{\varrho}{2}|v|^2}_{\text{total energy}}$$

$$\widetilde{\Pi} = \nabla \phi (\nabla \phi)^{\mathsf{T}} - \frac{1}{2}|\nabla \phi|^2 \mathrm{Id} + \Pi, \quad \widetilde{q} = q - \dot{\phi} \nabla \phi$$

Isothermal limit

Equations with constant temperature θ
There is no equation for the energy

Free energy inequality

There is an (f, φ) with

$$g := \partial_t f + \operatorname{div} \varphi - g_0 \leq 0$$

g_0 contains external terms (objectivity)

(f the total free energy, φ the corresponding flux)

Derivation: In the situation with an energy equation take

$$f := e - \theta\eta \,, \qquad \eta'_{\varepsilon} = \frac{1}{\theta}$$

$$q \longrightarrow \infty$$

and compute, by eliminating q,

$$h := \partial_t \eta + \mathrm{div}\psi \geq 0 \quad \text{(Second law)}$$

Entropy production contains "heat flux": $\psi = \eta v + \frac{1}{\theta} q + \ \ldots$

$$\partial_t \eta + \mathrm{div}(\eta v + \frac{1}{\theta} q + \ \ldots\ ) = h \geq \nabla \left(\frac{1}{\theta}\right) \bullet q \geq 0$$

$$\partial_t e + \mathrm{div}(e v + q + \ \ldots\ ) = g_0 \quad \text{(First law)}$$

$$\Rightarrow$$

$$\partial_t \eta + \mathrm{div}(\eta v + \ \ldots\ ) + \frac{1}{\theta}\mathrm{div} q = \underbrace{h - \nabla \left(\frac{1}{\theta}\right) \bullet q}_{= h_1 \geq 0} \geq 0$$

$$\partial_t e + \mathrm{div}(e v + \ \ldots\ ) + \mathrm{div} q = g_0$$

$$\Rightarrow$$

$$(\partial_t e - \theta \partial_t \eta) + (\mathrm{div}(e v + \ \ldots\ ) - \theta \mathrm{div}(\eta v + \ \ldots\ ))$$

$$= g_0 - \theta \cdot h_1 \leq g_0 \quad (\text{since } \theta \geq 0)$$

Hence for the free energy $f = e - \theta \eta$ in the isothermal limit

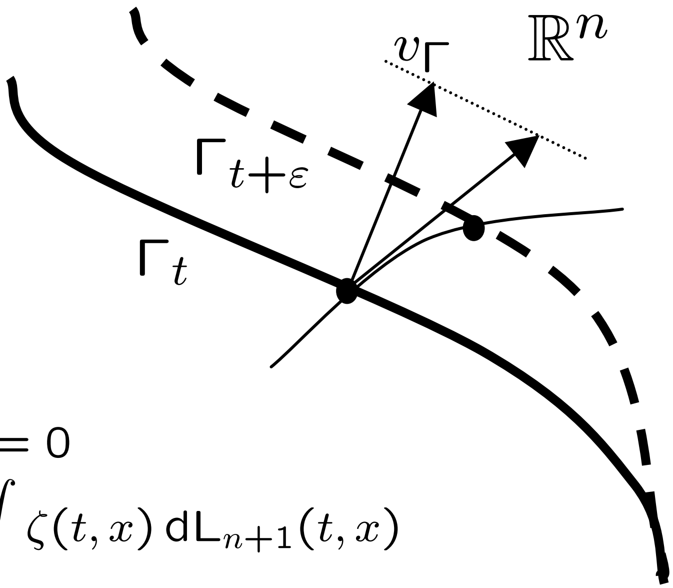
$$\partial_t f + \mathrm{div}(f v + \ \ldots\ ) - g_0 = -\theta h_1 \leq 0$$

Surfaces: $(t,x) \in \Gamma \iff x \in \Gamma_t$

$\Gamma \subset \mathbb{R} \times \mathbb{R}^n$ $(m+1)$-dimensional, $T_{(t,x)}\Gamma \neq \{0\} \times \mathbb{R}^n$
$\Gamma_t := \{x \in \mathbb{R}^n\,;\ (t,x) \in \Gamma\}$ m-dimensional, $0 \le m \le n$
v_Γ "surface velocity" , $T_{(t,x)}\Gamma = \mathrm{span}\,\{(1, v_\Gamma(t,x))\} \cup \{0\} \times T_x\Gamma_t$

Surface measures:

$$\langle \zeta\,,\boldsymbol{\mu}_\Gamma \rangle := \int_{\mathbb{R}} \int_{\Gamma_t} \zeta(t,x)\,\mathrm{dH}_m(x)\,\mathrm{dL}_1(t)$$
$$= \int_\Gamma \frac{\zeta(t,x)}{\sqrt{1+|v_\Gamma|^2}}\,\mathrm{dH}_{m+1}(t,x)$$



Extreme cases $m = n$ and $m = 0$:

Ω (Ω_t n-dimensional = open set) $v_\Omega = 0$

$$\langle \zeta\,,\boldsymbol{\mu}_\Omega \rangle = \int_{\mathbb{R}} \int_{\Omega_t} \zeta(t,x)\,\mathrm{dH}_n(x)\,\mathrm{dL}_1(t) = \int_\Omega \zeta(t,x)\,\mathrm{dL}_{n+1}(t,x)$$

Γ ($\Gamma_t = \{y_t\}$ 0-dimensional = point) $v_\Gamma = \dot{y}_t$

$$\langle \zeta\,,\boldsymbol{\mu}_\Gamma \rangle = \int_{\mathbb{R}} \int_{\{y_t\}} \zeta(t,x)\,\mathrm{dH}_0(x)\,\mathrm{dL}_1(t) = \int_{\mathbb{R}} \zeta(t,y_t)\,\mathrm{dL}_1(t)$$

Theorem (Distributional and strong version)

Let $t \mapsto \Gamma_t$ be m-dimensional, $0 \le m \le n$. The equation

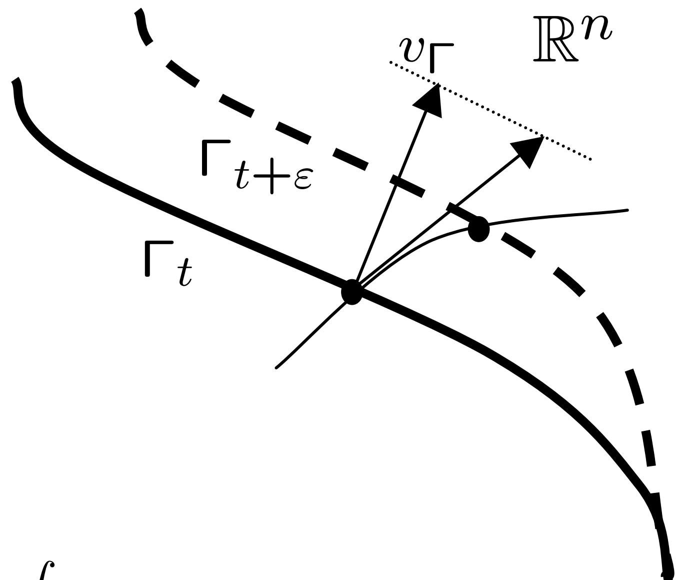
$$\partial_t(e\boldsymbol{\mu}_\Gamma) + \operatorname{div}(q\boldsymbol{\mu}_\Gamma) = f\boldsymbol{\mu}_\Gamma$$

is equivalent to

$q - ev_\Gamma$ tangential on Γ $\qquad (q - ev_\Gamma)(t,x) \in T_x\Gamma_t$

$\partial_t^\Gamma e - ev_\Gamma \bullet \kappa_\Gamma + \operatorname{div}^\Gamma(q - ev_\Gamma) = f$ on Γ $\qquad \partial_t^\Gamma e + \operatorname{div}^\Gamma q = f$

Here



$\Gamma_t := \{x \in \mathbb{R}^n \,;\, (t,x) \in \Gamma\}$

$v_\Gamma(t,x) \in T_x\Gamma_t{}^\perp$ "surface velocity" of Γ_t

$\kappa_\Gamma(t,x)$ m-times mean curvature of Γ_t

$\partial_t^\Gamma := \partial_t + v_\Gamma \bullet \nabla$

$\operatorname{div}^\Gamma := \sum_{i=1,\dots,m} \tau_k \bullet \partial_{\tau_k}$

$$\langle \zeta\,,\, -\partial_t(e\boldsymbol{\mu}_\Gamma) - \operatorname{div}(q\boldsymbol{\mu}_\Gamma) + f\boldsymbol{\mu}_\Gamma \rangle$$

$$= \langle \partial_t\zeta\,,\, e\boldsymbol{\mu}_\Gamma \rangle + \langle \nabla\zeta\,,\, q\boldsymbol{\mu}_\Gamma \rangle + \langle \zeta\,,\, f\boldsymbol{\mu}_\Gamma \rangle = \int \big(\partial_t\zeta \cdot e + \nabla\zeta \bullet q + \zeta \cdot f\big)\, \mathrm{d}\boldsymbol{\mu}_\Gamma$$

Examples of distributions in $\mathcal{D}'(U)$, $U \subset \mathbb{R} \times \mathbb{R}^n$:

Body Ω with mass exchange at the boundary $\Gamma = \partial\Omega$

$$\partial_t(\varrho\boldsymbol{\mu}_\Omega) + \operatorname{div}(\varrho v \boldsymbol{\mu}_\Omega) = \tau\boldsymbol{\mu}_\Gamma$$

(τ mass production at boundary)

$$\Longleftrightarrow$$

$$\partial_t\varrho + \operatorname{div}(\varrho v) = 0 \quad \text{in } \Omega$$

$$0 = \tau + \varrho(v - v_\Gamma)\bullet\nu_\Omega \quad \text{on } \Gamma$$

PDE as distribution Body $\Omega = U$

$$\partial_t(\varrho\boldsymbol{\mu}_U) + \operatorname{div}(\varrho v \boldsymbol{\mu}_U) = 0$$

$$\Longleftrightarrow$$

$$\partial_t\varrho + \operatorname{div}(\varrho v) = 0 \quad \text{in all of } U$$

Mass balance on a membrane Γ

with mass density $\varrho^s > 0$ and "particle velocity" v^s

$$\partial_t(\varrho^s\boldsymbol{\mu}_\Gamma) + \operatorname{div}(\varrho^s v^s \boldsymbol{\mu}_\Gamma) = 0$$

$\Longleftrightarrow$ on Γ:

$v^s - v_\Gamma$ tangential

$$\partial_t^\Gamma\varrho^s - \varrho^s\,\kappa_\Gamma\bullet v_\Gamma + \operatorname{div}^\Gamma(\varrho^s(v^s - v_\Gamma)) = 0$$

$\Longleftrightarrow$ on Γ:

$v^s - v_\Gamma$ tangential

$$\partial_t^\Gamma\varrho^s + \operatorname{div}^\Gamma(\varrho^s v^s) = 0$$

v_Γ normal velocity vector of Γ

κ_Γ $(n-1)$-times mean curvature vector of Γ

$\partial_t^\Gamma := \partial_t + v_\Gamma\bullet\nabla$ time derivative of Γ

An Allen-Cahn model for compressible fluids, φ phase field variable

Witterstein model [2010]:

(Conservation of two masses and conservation of momentum)

$$\partial_t \varrho + \mathrm{div}\,(\varrho v) = 0 \qquad \varrho^1_\delta = (1-\varphi)\varrho$$

$$\varrho(\partial_t \varphi + v \bullet \nabla \varphi) = -\mathbf{r}_\delta, \quad \mathbf{r}_\delta := c_\delta(\varrho,\varphi)\frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi} \qquad \varrho^2_\delta = \varphi\varrho$$

$$\partial_t(\varrho v) + \mathrm{div}\,(\varrho v v^{\mathsf{T}} + \Pi_\delta) = \mathbf{f}_\delta, \quad \Pi_\delta = P_\delta - S$$

with

$$P_\delta := (\varrho f_{\delta}{}'_{\varrho} - f_\delta)\mathrm{Id} + \nabla\varphi (f_{\delta}{}'_{\nabla\varphi})^{\mathsf{T}}$$

$$S := \lambda_1(\varrho,\varphi)\mathrm{div}\, v\ \mathrm{Id} + \lambda_2(\varrho,\varphi)\Big((\mathrm{D}v)^{\mathsf{S}} - \frac{1}{n}\mathrm{div}\, v\ \mathrm{Id}\Big)$$

$$f_\delta(\varrho,\varphi,\nabla\varphi) := \frac{1}{\delta}\varrho W(\varphi) + \delta\, h(\varrho)\frac{|\nabla\varphi|^2}{2} + U(\varrho,\varphi)$$

W has two local minima at 0 and 1, $U{}'_{\varphi}(\varrho,0) = 0$, $U{}'_{\varphi}(\varrho,1) = 0$

$$\frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi} = \frac{1}{\delta}\varrho W{}'_{\varphi}(\varphi) - \delta\,\mathrm{div}\,(h(\varrho)\nabla\varphi) + U{}'_{\varphi}(\varrho,\varphi)$$

Free energy inequality: $f = f(\varrho,\varphi,v,\nabla\varphi) = \frac{\varrho}{2}|v|^2 + f_\delta(\varrho,\varphi,\nabla\varphi)$

$$\partial_t f + \mathrm{div}\,(f v + \Pi_\delta^{\mathsf{T}} v - \dot\varphi f{}'_{\nabla\varphi}) - v \bullet \mathbf{f}_\delta = -\frac{1}{\varrho}\mathbf{r}_\delta \frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi} - \mathrm{D}v \bullet S \le 0$$

System written for the two masses ϱ^1 and ϱ^2

$$\varrho^1_\delta = (1-\varphi)\varrho\,, \quad \varrho^2_\delta = \varphi\varrho\,, \quad \text{or} \quad \varrho = \varrho^1_\delta + \varrho^2_\delta\,, \quad \varphi = \frac{\varrho^2_\delta}{\varrho^1 + \varrho^2}$$

becomes in $U \subset \mathbb{R} \times \mathbb{R}^n$

$$\partial_t(\varrho^1_\delta \mu_U) + \mathrm{div}\,(\varrho^1_\delta v_\delta \mu_U) = \mathbf{r}_\delta \mu_U$$

$$\partial_t(\varrho^2_\delta \mu_U) + \mathrm{div}\,(\varrho^2_\delta v_\delta \mu_U) = -\mathbf{r}_\delta \mu_U$$

$$\partial_t(\varrho v \mu_U) + \mathrm{div}\left((\varrho v v^\mathsf{T} + \Pi_\delta)\mu_U\right) = \mathbf{f}_\delta \mu_U$$

Theorem [Alt&Witterstein 2011]: In the limit $\delta \to 0$ this becomes

$$\partial_t(\varrho^1 \boldsymbol{\mu}_{\Omega^1}) + \mathrm{div}\,(\varrho^1 v^1 \boldsymbol{\mu}_{\Omega^1}) = \mathbf{r}\,\boldsymbol{\mu}_\Gamma$$

$$\partial_t(\varrho^2 \boldsymbol{\mu}_{\Omega^2}) + \mathrm{div}\,(\varrho^2 v^2 \boldsymbol{\mu}_{\Omega^2}) = -\mathbf{r}\,\boldsymbol{\mu}_\Gamma$$

$$\partial_t\Big(\sum_m \varrho^m v^m \boldsymbol{\mu}_{\Omega^m}\Big) + \mathrm{div}\left(\sum_m (\varrho^m v^m v^{m\mathsf{T}} + \Pi^m)\boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma\right) = \sum_m \mathbf{f}^m \boldsymbol{\mu}_{\Omega^m}$$

The convergence is in the sense of distributions, in particular

$$(\varrho v v^\mathsf{T} + \Pi_\delta)\boldsymbol{\mu}_U \quad \longrightarrow \quad \sum_m (\varrho^m v^m v^{m\mathsf{T}} + \Pi^m)\boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma$$

$$\Pi^s = -\gamma(\mathrm{Id} - \nu\nu^\mathsf{T})\,, \quad \gamma \text{ given by an integral over “local coordinates”}$$

where $U = \Omega^1 \cup \Gamma \cup \Omega^2$

Proof contains: For $\eta \in \mathcal{D}(U;\mathbb{R}^n \times \mathbb{R}^n)$, $U \subset \mathbb{R} \times \mathbb{R}^n$, $U = \Omega_\delta^1 \cup \Gamma_\delta \cup \Omega_\delta^2$

$$\int_U \eta \bullet (\varrho v v^\mathsf{T} + \Pi_\delta)\, \mathrm{dL}_{n+1} = \sum_m \int_{\Omega_\delta^m} \ldots\, \mathrm{dL}_{n+1} + \int_{\Gamma_\delta} \ldots\, \mathrm{dL}_{n+1}$$

In a small neighbourhood of Γ

$$\begin{aligned}
\Pi_\delta = {} & \mathsf{p}_U \,\mathrm{Id} + \frac{\delta}{2}\mathsf{p}_h |\nabla\varphi|^2 \,\mathrm{Id} + \delta h \nabla\varphi(\nabla\varphi)^\mathsf{T} - (\lambda_1 - \frac{\lambda_2}{n})\mathrm{div}\, v \,\mathrm{Id} - \lambda_2(\nabla v)^\mathsf{S} \\
= {} & \frac{1}{\delta}\Big(\frac{1}{2}\mathsf{p}_h |\partial_r \Phi^0|^2 \,\mathrm{Id} + h|\partial_r\Phi^0|^2 \nu\nu^\mathsf{T} \\
& \quad -(\lambda_1 - \frac{\lambda_2}{n})\nu \bullet \partial_r V^0 \,\mathrm{Id} - \frac{1}{2}\lambda_2(\nu(\partial_r V^0)^\mathsf{T} + \partial_r V^0 \nu^\mathsf{T})\Big) + \mathcal{O}(1)
\end{aligned}$$

$$\begin{aligned}
\delta\Pi_\delta \nu \to {} & \frac{1}{2}\mathsf{p}_h|\partial_r\Phi^0|^2\nu + h|\partial_r\Phi^0|^2\nu - (\lambda_1 - \frac{\lambda_2}{n})\nu\bullet\partial_r V^0 \nu - \lambda_2 \nu\bullet\partial_r V^0 \nu \\
= {} & (\frac{1}{2}\mathsf{p}_h + h)|\partial_r\Phi^0|^2\nu - (\lambda_1 + \frac{(n-1)\lambda_2}{n})\partial_r V^0 \;=\; 0
\end{aligned}$$

$$\begin{aligned}
(\delta\Pi_\delta)_{\mathrm{tan}} := {} & \delta\Pi_\delta - (\delta\Pi_\delta\nu)\nu \;=\; \delta\Pi_\delta(\mathrm{Id} - \nu\nu^\mathsf{T}) \\
\to {} & \Big(\frac{1}{2}\mathsf{p}_h|\partial_r\Phi^0|^2 - (\lambda_1 - \frac{\lambda_2}{n})\nu\bullet\partial_r V^0\Big)(\mathrm{Id} - \nu\nu^\mathsf{T}) \\
= {} & \Big(\frac{1}{2}\mathsf{p}_h(R^0)|\partial_r\Phi^0|^2 - \big(\lambda_1(R^0,\Phi^0) - \frac{\lambda_2(R^0,\Phi^0)}{n}\big)\nu\bullet\partial_r V^0\Big)(\mathrm{Id} - \nu\nu^\mathsf{T})
\end{aligned}$$

$$\gamma = \int_\mathbb{R} \Big(-\frac{1}{2}\mathsf{p}_h(R^0)|\partial_r\Phi^0|^2 + \big(\lambda_1(R^0,\Phi^0) - \frac{\lambda_2(R^0,\Phi^0)}{n}\big)\nu\bullet\partial_r V^0\Big)\, \mathrm{d}r$$

The limit behaviour of the free energy is given in the presentation of G. Witterstein

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Entropy principle involving interfaces

For a set $\mathcal{P}$ of processes: There exists (H, Ψ) with

$\partial_t H + \operatorname{div} \Psi \geq 0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$	objective scalar equation
for each process in $\mathcal{P}$	defined by rule for test function

(Constitutive relations for (H, Ψ))

Free energy inequality involving interfaces

For a set $\mathcal{P}$ of processes: There exists (F, Φ) with

$\partial_t F + \operatorname{div} \Phi + G_0 \leq 0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$	objectivity like energy equation
for each process in $\mathcal{P}$	

(G_0 contains external terms, constitutive relations for (F, Φ))