Relativistic equations for the generalized Chapman-Enskog hierarchy

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- Principles: conservation laws, observer transformations
- The Chapman-Enskog hierarchy and the reduction
- The matrix G and the dual basis
- The 4-velocity $\underline{v}$
- Reduction theorems

Chapman-Enskog hierarchy

The conservations laws for $T = \big(T_\beta\big)_{\beta\in\{0,\ldots,3\}^M}$, $M = N+1$, and $g = (g_\alpha)_{\alpha\in\{0,\ldots,3\}^N}$:

$$\int_{\mathbb{R}^4} \sum_\alpha \Big(\sum_{j\geq 0} \partial_{y_j}\zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha \cdot g_\alpha \Big)\, \mathrm{d}L^4 = 0 \qquad \text{for } \alpha \in \{0,\ldots,3\}^N$$

for test functions $\zeta = (\zeta_\alpha)_{\{0,\ldots,3\}^N}$ with $\zeta_\alpha \in C_0^\infty(\mathcal{U})$ where $\mathcal{U} \subset \mathbb{R}^4$

Example: Coming from Boltzmann equation for $f = f(t,x,c)$, $y = (t,x)$

$$\partial_t f + \sum_{i\geq 1} c_i \partial_{x_i} f + \sum_{i\geq 1} \mathbf{g}_i \partial_{c_i} f = r, \qquad r \text{ reaction rate}, \qquad \sum_{i\geq 1} \partial_{c_i}\mathbf{g}_i = 0$$

define

$$T_{k_1\cdots k_M}(t,x) := F_{k_1\cdots k_M}(t,x) = \int_{\mathbb{R}^3} m \underline{c}_{k_1}\cdots \underline{c}_{k_M} f(t,x,c)\,\mathrm{d}c \qquad \text{with } \underline{c} := \begin{bmatrix} 1 \\ c \end{bmatrix}$$

$$g_{k_1\cdots k_N}(t,x) := \int_{\mathbb{R}^3} m \underline{c}_{k_1}\cdots \underline{c}_{k_N} r(t,x,c)\,\mathrm{d}c + \sum_{i\geq 1} \int_{\mathbb{R}^3} m\mathbf{g}_i(t,x,c)\partial_{c_i}(\underline{c}_{k_1}\cdots\underline{c}_{k_N}) f(t,x,c)\,\mathrm{d}c$$

For different classical observers we have $f(t,x,c) = f^*(t^*,x^*,c^*)$ and

$$\begin{bmatrix} t \\ x \\ c \end{bmatrix} = \begin{bmatrix} t^* + a \\ X(t^*,x^*) \\ \dot{X}(t^*,x^*) + Q(t^*)c^* \end{bmatrix}, \qquad \begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} t^* + a \\ X(t^*,x^*) \end{bmatrix} := \begin{bmatrix} t^* + a \\ Q(t^*)x^* + b(t^*) \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ \dot{X} + Qc^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} 1 \\ c^* \end{bmatrix} = \underline{\mathrm{D}}Y \begin{bmatrix} 1 \\ c^* \end{bmatrix} = \underline{\mathrm{D}}Y\,\underline{c}^* \quad \text{hence} \quad \underline{c}_k = \sum_{\bar{k}\geq 1} Y_{k\,'\bar{k}}\,\underline{c}^*_{\bar{k}}$$

therefore

$$T_{k_1\cdots k_M}\circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_M\geq 0} Y_{k_1\,'\bar{k}_1}\cdots Y_{k_M\,'\bar{k}_M} T^*_{\bar{k}_1\cdots\bar{k}_M}$$

Generalized definition: The conservation laws

$$\int_{\mathbb{R}^4} \sum_{\alpha} \Big(\sum_{j \geq 0} \partial_{y_j} \zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha \cdot g_\alpha \Big) \, \mathrm{d}\mathrm{L}^4 = 0 \qquad \text{for } \alpha \in \{0, \ldots, 3\}^N$$

for test functions $\zeta = (\zeta_\alpha)_{\{0,\ldots,3\}^N}$ with $\zeta_\alpha \in C_0^\infty(\mathcal{U})$ where $\mathcal{U} \subset \mathbb{R}^4$

are called a ***N*-moments system** if the test functions ζ satisfy:

$$\zeta^*_{\bar{k}_1 \cdots \bar{k}_N} = \sum_{k_1,\ldots,k_N \geq 0} Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_N}{}'_{\bar{k}_N} \zeta_{k_1 \cdots k_N} \circ Y \qquad \text{(covariant } N\text{-tensor)}$$

By a general theorem [Alt: Mathematical Continuum Mechanics, I §5] this is satisfied if

$$T_{k_1 \cdots k_M} \circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_M \geq 0} Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_M}{}'_{\bar{k}_M} T^*_{\bar{k}_1 \cdots \bar{k}_M} \qquad \text{(contravariant } M\text{-tensor)}$$

$$g_{k_1 \cdots k_N} \circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_N, j \geq 0} \Big(Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_N}{}'_{\bar{k}_N} \Big){}_{,j}{}' \, T^*_{\bar{k}_1 \cdots \bar{k}_N j} + \sum_{\bar{k}_1,\ldots,\bar{k}_N \geq 0} Y_{k_1}{}'_{\bar{k}_1} \cdots Y_{k_N}{}'_{\bar{k}_N} g^*_{\bar{k}_1 \cdots \bar{k}_N}$$

This is true for all observer transformations (with $\det \underline{\mathrm{D}} Y = 1$).

Coriolis coefficients:

$$g_\alpha = \underline{\mathrm{f}}_\alpha + \sum_{\beta \in \{0,\ldots,3\}^{N+1}} \mathrm{C}_\alpha^\beta T_\beta \qquad \text{for } \alpha \in \{0, \ldots, 3\}^N$$

$\underline{\mathrm{f}}$ is a contravariant N-tensor. The Christoffel symbols are $\Gamma^k_{pq} = -\mathrm{C}_k^{pq}$ ($N = 1$).

Reduction

The generalized hierarchy for classical observers (indices $i,j,l,m \geq 1$, $k,k_* \geq 0$):

$$T_{k_1\cdots k_M} = \varrho\,\underline{v}_{k_1}\cdots\underline{v}_{k_M} + \underline{\Pi}_{k_1\cdots k_M}\,, \qquad M = N+1$$

$N=0:$ $\quad T_0 = \varrho, \quad T_j = \varrho v_j + \mathbf{J}_j \qquad T_k = \varrho\underline{v}_k + \underline{\mathbf{J}}_k, \qquad \underline{\mathbf{J}} = \begin{bmatrix} 0 \\ \mathbf{J} \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ v \end{bmatrix}$

$N=1:$ $\quad T_0 = \varrho, \quad T_j = \varrho v_j + \mathbf{J}_j \qquad (T_{0k} = T_k)$

$\quad T_{i0} = \varrho v_i, \quad T_{ij} = \varrho v_i v_j + \Pi_{ij} \qquad T_{k_1k_2} = \varrho\underline{v}_{k_1}\underline{v}_{k_2} + \underline{\Pi}_{k_1k_2}$

$$(\underline{\Pi}_{k_1k_2})_{k_1,k_2\geq 0} := \begin{bmatrix} 0 & \mathbf{J}^\top \\ 0 & \Pi \end{bmatrix}$$

$N=2:$ $\quad T_0 = \varrho, \quad T_j = \varrho v_j + \mathbf{J}_j \qquad (T_{0k} = T_k)$

$\quad T_{i0} = \varrho v_i, \quad T_{ij} = \varrho v_i v_j + \Pi_{ij} \qquad (T_{k0k_3} = T_{0kk_3} = T_{kk_3})$

$\quad T_{lm0} = \varrho v_l v_m + E_{lm}, \qquad T_{k_1k_2k_3} = \varrho\underline{v}_{k_1}\underline{v}_{k_2}\underline{v}_{k_3} + \underline{Q}_{k_1k_2k_3}$

$\quad T_{lmj} = \varrho v_l v_m v_j + Q_{lmj}$

(Energy and energy flux is trace divided by 2)

$$\underline{Q}_{k_1k_2k_3} = \left\{ \begin{array}{rl} k_1,k_2,k_3 \geq 1: & Q_{k_1k_2k_3} \\ k_1,k_2 \geq 1,\ k_3 = 0: & E_{k_1k_2} \\ k_1,k_3 \geq 1,\ k_2 = 0: & \Pi_{k_1k_3} \\ k_1,k_2 = 0,\ k_3 \geq 1: & \mathbf{J}_{k_3} \end{array} \right\}$$

Not: For $l,m \geq 1$: $\varrho v_l v_m + E_{lm} = T_{lm0} \overset{?}{=} T_{l0m} \overset{!}{=} T_{lm} = \varrho v_l v_m + \Pi_{lm}$

$T_{lm0} = T_{l0m}$ is true for the Chapman-Enskog example

Reduction in the classical case: The test functions

$$\zeta_\alpha \text{ with } \alpha \in \{0,\ldots,3\}^N \text{ for the } N\text{-moments system}$$

contain (with $\alpha = (k,\gamma)$ and $k \in \{0,\ldots,3\}$)

$$\eta_\gamma = \zeta_{0\gamma} \text{ with } \gamma \in \{0,\ldots,3\}^{N-1} \text{ for the } (N-1)\text{-moments system}$$

Reduction in the relativistic case:

Choose a vector $\boldsymbol{e}$ such that the test functions

$$\zeta_\alpha \text{ with } \alpha \in \{0,\ldots,3\}^N \text{ for the } N\text{-moments system}$$

contain (with $\alpha = (k,\gamma)$ and $k \in \{0,\ldots,3\}$, $\gamma \in \{0,\ldots,3\}^{N-1}$)

$$\zeta_{k\gamma} = \boldsymbol{e}_k \eta_\gamma \text{ where } \eta \text{ is a test function for the } (N-1)\text{-moments system}$$

Therefore $\boldsymbol{e}$ should be chosen as a covariant vector:

$$\boldsymbol{e}^*_{\bar{k}} = \sum_{k \geq 0} Y_{k'\bar{k}} \boldsymbol{e}_k \circ Y \text{ for } \bar{k} \geq 0 \qquad \text{or} \qquad \boldsymbol{e}^* = \underline{\mathrm{D}} Y^\top \boldsymbol{e} \circ Y$$

and in the classical limit

$$\boldsymbol{e} \longrightarrow \mathrm{e}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{ time component} \\ \leftarrow \text{ space components} \end{matrix} \qquad \text{for } c \to \infty \text{ (c speed of light)}$$

The following is devoted to this problem

- We deal with a generalized version of the Chapman-Enskog hierarchy
- The N-moments system contains as part the $(N-1)$-moments system
- This reduction is made possible with a "time" vector $\boldsymbol{e}$ which is a covariant vector: $\boldsymbol{e}^* = \underline{\mathsf{D}}Y^{\mathsf{T}}\boldsymbol{e}\circ Y$

To avoid misunderstandings:

- $c \in \mathbb{R}^3$ denotes the velocity as variable (in Boltzmann equation)
- $\mathsf{c} > 0$ denotes the constant speed of light in vacuum
- $\mathsf{c} = 2.99792458 \cdot 10^8\, m s^{-1}$ in experimental physics
- $\mathsf{c} \gg 1$ or $\mathsf{c} \to \infty$ denotes classical physics

Geometry G and time vector $\boldsymbol{e}$

Assumption: The matrix $y \in \mathbb{R}^4 \longmapsto \mathsf{G}(y) \in \mathbb{R}^{4\times 4}$ is invertible, symmetric, and describes the hyperbolic geometry:

one eigenvalue λ_0 negative   e.g. $\lambda_0 = -\frac{1}{\mathsf{c}^2}$ (c speed of light in vacuum)
other eigenvalues λ_i positive $(i \geq 1)$ e.g. $\lambda_i = 1$

For different observers it transforms as

$$\mathsf{G}_{kl}\circ Y = \sum_{\bar{k},\bar{l}} Y_{k\,'\bar{k}} Y_{l\,'\bar{l}} \mathsf{G}^*_{\bar{k}\,\bar{l}} \quad \text{(a contravariant tensor:} \quad \mathsf{G}\circ Y = \mathsf{D}Y\mathsf{G}^*\mathsf{D}Y^\mathsf{T})$$

Consequence: There is an orthonormal set $\{e_0^\perp, e_1^\perp, e_2^\perp, e_3^\perp\}$ with $\mathsf{G}e_k^\perp = \lambda_k e_k^\perp$

Example:

$$\mathsf{G} = \mathsf{G}_\mathsf{c} := \begin{bmatrix} -1/\mathsf{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \quad \text{as } \mathsf{c} \to \infty \text{ it converges to} \quad \mathsf{G}_\infty := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$$

Assumption: The "time" vector $y \in \mathbb{R}^4 \longmapsto \boldsymbol{e}(y) \in \mathbb{R}^4$ satifies

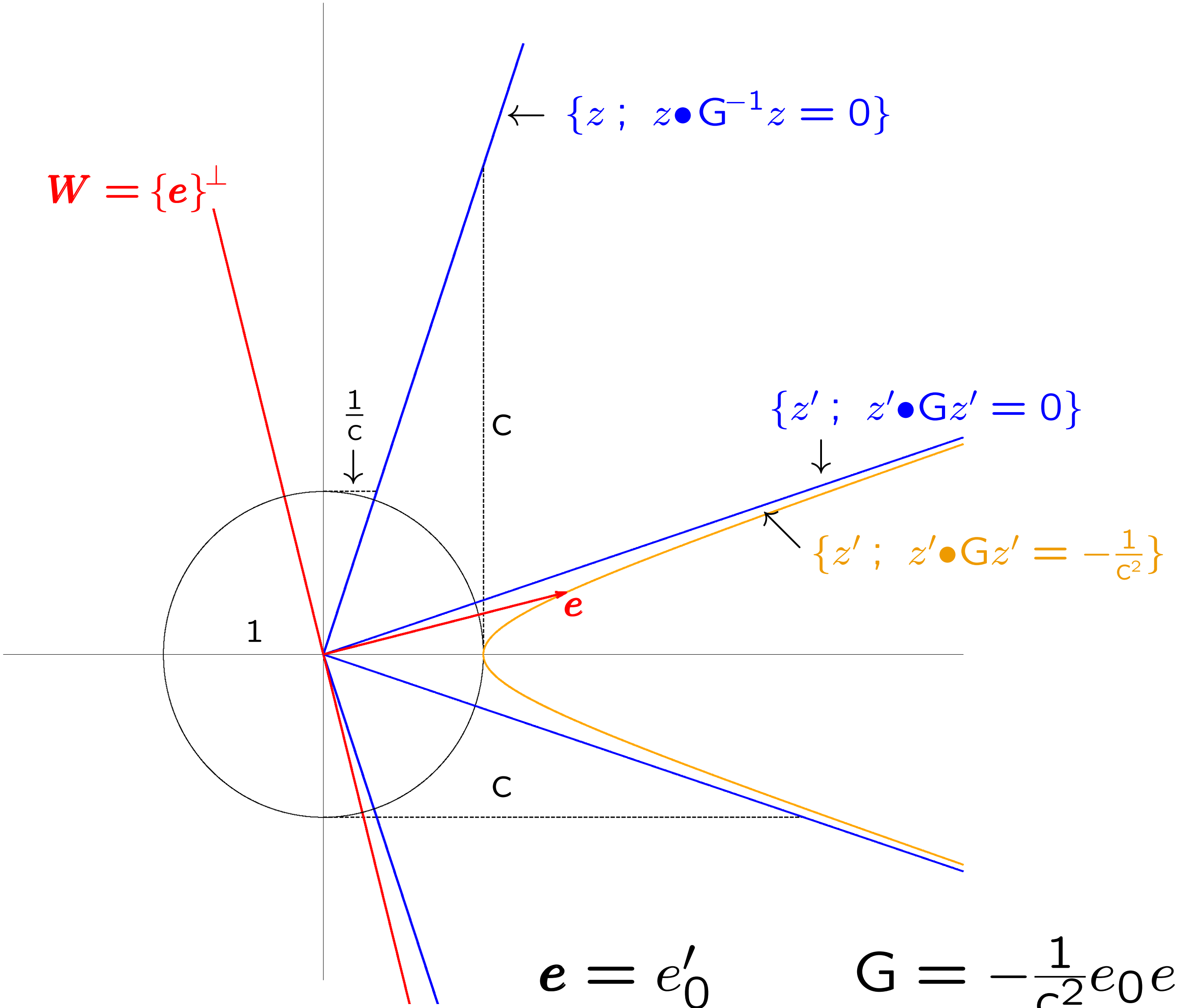
$\boldsymbol{e}^\mathsf{T}\mathsf{G}\boldsymbol{e} = -\frac{1}{\mathsf{c}^2}$ and is a covariant vector: $\boldsymbol{e}^* = \mathsf{D}Y^\mathsf{T}\boldsymbol{e}\circ Y$

One posssibility for one observer is to choose $\boldsymbol{e} = \pm e_0^\perp$ if $\lambda_0 = -1/\mathsf{c}^2$.

It is $\boldsymbol{e}^{*\mathsf{T}}\mathsf{G}^*\boldsymbol{e}^* = (\mathsf{D}Y^\mathsf{T}\boldsymbol{e}\circ Y)^\mathsf{T}\mathsf{G}^*\mathsf{D}Y^\mathsf{T}\boldsymbol{e}\circ Y = (\boldsymbol{e}\circ Y)^\mathsf{T}\mathsf{D}Y\mathsf{G}^*\mathsf{D}Y^\mathsf{T}\boldsymbol{e}\circ Y = (\boldsymbol{e}^\mathsf{T}\mathsf{G}\boldsymbol{e})\circ Y$.

Geometry G and time vector $\boldsymbol{e}$

$(w_1 \bullet w_2 := {w_1}^{\mathsf{T}} w_2)$



$$\boldsymbol{e} = e'_0 \qquad \mathsf{G} = -\frac{1}{\mathsf{c}^2} e_0 {e_0}^{\mathsf{T}} + \sum_{i \geq 1} e_i {e_i}^{\mathsf{T}}$$

Geometry G and time vector $\boldsymbol{e}$

Let $\boldsymbol{e}$ be as above and define the 3-dimensional subspace $\boldsymbol{W} := \{\boldsymbol{e}\}^\perp$

Construction of the dual Basis:

- Let $e'_0 := \boldsymbol{e}$ and define $e_0 := -\mathsf{c}^2\mathsf{G}\boldsymbol{e}$
- Theorem [Alt: Lemma 3.8]: $(z_1, z_2) \mapsto {z_1}^\mathsf{T}\mathsf{G}^{-1}z_2$ is a scalar product on $\boldsymbol{W}$
- On $\boldsymbol{W}$ choose a G^{-1}-orthonormal set $\{e_1, e_2, e_3\} \subset \boldsymbol{W}$: ${e_i}^\mathsf{T}\mathsf{G}^{-1}e_j = \boldsymbol{\delta}_{ij}$
- Define $e'_i := \mathsf{G}^{-1}e_i$ for $i \geq 1$

Theorem [Alt: Theorem 3.4]: It follows $\mathsf{G}e'_k = \lambda_k e_k$ with $\lambda_0 = -\frac{1}{\mathsf{c}^2}$, $\lambda_i = 1$ $(i \geq 1)$, and $\{e'_0, e'_1, e'_2, e'_3\}$ is the dual basis to $\{e_0, e_1, e_2, e_3\}$:

$$
{e'_k}^\mathsf{T} e_l = \boldsymbol{\delta}_{kl}
$$

moreover the non-unique representation

$$
\mathsf{G} = -\frac{1}{\mathsf{c}^2} e_0 {e_0}^\mathsf{T} + \sum_{i \geq 1} e_i {e_i}^\mathsf{T}
$$

e'_k are covariant and e_k contravariant (Proofs next slide)

Remark: $-\mathsf{G}^{-1} = (g_{ij})_{ij}$ is the usual considered matrix:
$(z_1, z_2) \mapsto -{z_1}^\mathsf{T}\mathsf{G}^{-1}z_2 = \sum_{i,j\geq 0} z_{1i}\, g_{ij}\, z_{2j}$ is the used “inner product” in $\mathbb{R}^4$

Lemma: G^{-1} is a covariant tensor

$$
\mathsf{G}^{*-1} = \mathsf{D}Y^\mathsf{T}\mathsf{G}^{-1} \circ Y \mathsf{D}Y
$$

Example: $-\mathsf{G}^{-1} = -{\mathsf{G}_\mathsf{c}}^{-1} := \begin{bmatrix} \mathsf{c}^2 & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}$

Proof of [Theorem 3.4]:

$$\mathrm{G} = -\frac{1}{\mathrm{c}^2} e_0 {e_0}^{\mathsf{T}} + \sum_{i\geq 1} e_i {e_i}^{\mathsf{T}} \quad \Longleftrightarrow \quad \mathrm{G} e_0' = -\frac{1}{\mathrm{c}^2} e_0 \,, \quad \mathrm{G} e_i' = e_i$$

Proof of [Lemma 3.8]: Let $\lambda_0 < 0$ and $\lambda_i > 0$ for $i \geq 1$.

$$z' \bullet \mathrm{G} z' = \textstyle\sum_{k\geq 0} \lambda_k |z_k'|^2 \quad \left(z_k' := z' \bullet e_k^{\perp}\right), \quad z \bullet \mathrm{G}^{-1} z = \textstyle\sum_{k\geq 0} \dfrac{|z_k|^2}{\lambda_k} \quad \left(z_k := z \bullet e_k^{\perp}\right)$$

Since $\boldsymbol{e} \bullet \mathrm{G} \boldsymbol{e} = -\frac{1}{\mathrm{c}^2}$ and hence $\bar{\boldsymbol{e}} \bullet \mathrm{G} \bar{\boldsymbol{e}} = -|\lambda_0| = \lambda_0$ with $\bar{\boldsymbol{e}} := \mathrm{c}\sqrt{|\lambda_0|}\, \boldsymbol{e}$
The first identity (with $z' = \bar{\boldsymbol{e}}$) gives

$$-|\lambda_0| = \bar{\boldsymbol{e}} \bullet \mathrm{G} \bar{\boldsymbol{e}} = -|\lambda_0| \cdot |\bar{\boldsymbol{e}}_0|^2 + \textstyle\sum_{i\geq 1} \lambda_i |\bar{\boldsymbol{e}}_i|^2 \,, \quad \bar{\boldsymbol{e}}_k := \bar{\boldsymbol{e}} \bullet e_k^{\perp}$$

$$\text{hence} \quad |\bar{\boldsymbol{e}}_0|^2 = 1 + \frac{1}{|\lambda_0|} \textstyle\sum_{i\geq 1} \lambda_i |\bar{\boldsymbol{e}}_i|^2 \geq 1 \,.$$

Then for $z \in \boldsymbol{W} \setminus \{0\} = \{\bar{\boldsymbol{e}}\}^{\perp} \setminus \{0\}$

$$0 = z \bullet \bar{\boldsymbol{e}} = z_0 \bar{\boldsymbol{e}}_0 + \textstyle\sum_{i\geq 1} z_i \bar{\boldsymbol{e}}_i \quad \text{hence} \quad - z_0 = \textstyle\sum_{i\geq 1} z_i \dfrac{\bar{\boldsymbol{e}}_i}{\bar{\boldsymbol{e}}_0} \quad \text{and this gives}$$

$$|z_0|^2 \leq \left(\textstyle\sum_{i\geq 1} |z_i| \dfrac{|\bar{\boldsymbol{e}}_i|}{|\bar{\boldsymbol{e}}_0|}\right)^2 \leq \textstyle\sum_{i\geq 1} \dfrac{|z_i|^2}{\lambda_i} \cdot \textstyle\sum_{i\geq 1} \lambda_i \dfrac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2} \quad \text{and} \quad \textstyle\sum_{i\geq 1} \lambda_i \dfrac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2} < |\lambda_0| \,.$$

Then from the second identity

$$z \bullet \mathrm{G}^{-1} z = -\frac{|z_0|^2}{|\lambda_0|} + \textstyle\sum_{i\geq 1} \dfrac{|z_i|^2}{\lambda_i} \geq \textstyle\sum_{i\geq 1} \dfrac{|z_i|^2}{\lambda_i} \cdot \left(1 - \dfrac{1}{|\lambda_0|} \textstyle\sum_{i\geq 1} \lambda_i \dfrac{|\bar{\boldsymbol{e}}_i|^2}{|\bar{\boldsymbol{e}}_0|^2}\right) > 0$$

Standard observer with $y^* = (t^*, x^*) \in \mathbb{R}^4$, matrix $\mathsf{G}^* = \mathsf{G_c}$ and vectors

$$\boldsymbol{e}^* = e_0'^* = e_0^* = \mathbf{e}_0\,, \quad e_i'^* = e_i^* = \mathbf{e}_i \text{ for } i \geq 1.$$

Actual observer with $y = (t, x) \in \mathbb{R}^4$ and Lorentz transformation $y = Y(y^*)$

$$\mathrm{D}Y = \begin{bmatrix} \gamma & \frac{\gamma}{\mathsf{c}^2} V^\mathsf{T} Q \\ \gamma V & \mathrm{B}(V) Q \end{bmatrix}, \quad \mathrm{B}(V) := \mathrm{Id} + \frac{\gamma^2}{\mathsf{c}^2(\gamma+1)} V V^\mathsf{T}, \quad \gamma := \frac{1}{\sqrt{1 - \frac{|V|^2}{\mathsf{c}^2}}} > 1.$$

Then it follows for the actual observer: $\mathsf{G} = \mathsf{G_c}$ and $\boldsymbol{e} = e_0'$ and if $Q = \mathrm{Id}$:

$$e_0' = \begin{bmatrix} \gamma \\ -\frac{\gamma}{\mathsf{c}^2} V \end{bmatrix}, \qquad e_0 = \begin{bmatrix} \gamma \\ \gamma V \end{bmatrix},$$

$$e_i = \begin{bmatrix} \frac{\gamma}{\mathsf{c}^2} V_i \\ \mathbf{e}_i + \frac{\gamma^2 V_i}{\mathsf{c}^2(\gamma+1)} V \end{bmatrix}, \qquad e_i' = \begin{bmatrix} -\gamma V_i \\ \mathbf{e}_i + \frac{\gamma^2 V_i}{\mathsf{c}^2(\gamma+1)} V \end{bmatrix} \qquad \text{for } i \geq 1.$$

In the classical limit $\mathsf{c} \to \infty$

$$e_0' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_0 = \begin{bmatrix} 1 \\ V \end{bmatrix}, \qquad \boldsymbol{W} = \{e_0'\}^\perp = \mathrm{span}\,\{e_i\,;\ i \geq 1\} = \{0\} \times \mathbb{R}^3,$$

$$e_i = \begin{bmatrix} 0 \\ \mathbf{e}_i \end{bmatrix}, \qquad e_i' = \begin{bmatrix} -V_i \\ \mathbf{e}_i \end{bmatrix} \qquad \text{for } i \geq 1.$$

- The geometry G and the time vector $\boldsymbol{e}$ are quantities showing up in the physical equations
- With a dual basis one has the not unique representation

$$\mathsf{G} = -\frac{1}{\mathsf{c}^2} e_0 {e_0}^\mathsf{T} + \sum_{i\geq 1} e_i {e_i}^\mathsf{T}, \quad \boldsymbol{e} = e'_0$$

- If $\mathsf{G} = \mathsf{G_c}$ the dual basis is given by a vector $y \mapsto V(y)$ with $|V(y)| < \mathsf{c}$ and for $Q(y) = \mathrm{Id}$

$$e'_0 = \begin{bmatrix} \gamma \\ -\frac{\gamma}{\mathsf{c}^2} V \end{bmatrix}, \qquad e_0 = \begin{bmatrix} \gamma \\ \gamma V \end{bmatrix}, \quad \gamma := \frac{1}{\sqrt{1 - \frac{|V|^2}{\mathsf{c}^2}}},$$

$$e_i = \begin{bmatrix} \frac{\gamma}{\mathsf{c}^2} V_i \\ \mathbf{e}_i + \frac{\gamma^2 V_i}{\mathsf{c}^2(\gamma + 1)} V \end{bmatrix}, \qquad e'_i = \begin{bmatrix} -\gamma V_i \\ \mathbf{e}_i + \frac{\gamma^2 V_i}{\mathsf{c}^2(\gamma + 1)} V \end{bmatrix} \quad \text{for } i \geq 1.$$

- $\boldsymbol{e} = e'_0$ is in general not a unit vector

The simplest example is a single equation in a domain $\mathcal{U} \subset \mathbb{R}^4$

$$\sum_{j\geq 0} \partial_{y_j} q_j = \mathbf{r} \qquad \text{for a 4-flux } q = (q_j)_{j\geq 0} \text{ and a scalar } \mathbf{r}$$

or the distributional version of this **scalar law** (it is $N = 0$ and $q_j = T_j$, $\mathbf{r} = g$)

$$\int_{\mathcal{U}} \Big(\sum_{j\geq 0} \partial_{y_j}\zeta \cdot q_j + \zeta \cdot \mathbf{r} \Big) \, \mathrm{d}\mathrm{L}^4 = 0 \qquad \text{for test functions } \zeta \in C_0^\infty(\mathcal{U};\mathbb{R})$$

with the definition that ζ is an objective scalar $\zeta^* = \zeta \circ Y$

This holds, if the quantities q and $\mathbf{r}$ satisfy the transformation rule

$$q \circ Y = \mathrm{D}Y q^* , \qquad \mathbf{r} \circ Y = \mathbf{r}^* \qquad (q \text{ contravariant vector, } \mathbf{r} \text{ objective scalar})$$

We call it a **mass equation** if $q = \varrho \underline{v} + \underline{\mathbf{J}}$, where:

Definition of velocity: $\underline{v}$ is called a **4-velocity** if

$$\boldsymbol{e}^\top \underline{v} = 1 \quad \text{and} \quad \underline{v} \circ Y = \mathrm{D}Y \underline{v}^* \qquad (\underline{v} \text{ contravariant vector})$$

Equation is objective since $(\boldsymbol{e}^\top \underline{v}) \circ Y = (\boldsymbol{e} \circ Y)^\top \mathrm{D}Y \underline{v}^* = \big(\mathrm{D}Y^\top (\boldsymbol{e} \circ Y)\big)^\top \underline{v}^* = \boldsymbol{e}^{*\top} \underline{v}^*$.

Definition of mass density: If ϱ is a **mass density** then it is an objective scalar $\varrho \circ Y = \varrho^*$.

The mass density is a mass density per volume and time.

Comparison with the usual definition:

A vector $\underline{u}$ is usually called a “4-velocity” if

$$\text{(u)} \qquad \underline{u}\bullet \mathrm{G}^{-1}\underline{u} = -\mathrm{c}^2 \quad \text{and} \quad \underline{u} \text{ is a contravariant vector}$$

Equation (u) is objective, that is, is the same for all observers.

(1) If $\underline{u}$ satisfies (u) then $\boldsymbol{e}\bullet\underline{u} \neq 0$ and

$$\underline{v} := \frac{\underline{u}}{\boldsymbol{e}\bullet\underline{u}}$$

defines a 4-velocity as in (v). It is $||\underline{v}|| < \mathrm{c}$.

$$\left(||\underline{w}|| := \sqrt{\textstyle\sum_{i\geq 1} e_i'\bullet\underline{w}} \right)$$

(2) Let $\underline{v}$ be a 4-velocity as in (v) with $||\underline{v}|| < \mathrm{c}$ then

$$\underline{u} := \gamma_{\underline{v}}\underline{v}\,, \qquad \gamma_{\underline{v}} := \frac{1}{\sqrt{1 - \frac{||\underline{v}||^2}{\mathrm{c}^2}}}$$

satisfies (u).

Our definition of a 4-velocity $\underline{v}$ was

$$\text{(v)} \qquad \boldsymbol{e}\bullet\underline{v} = 1 \quad \text{and} \quad \underline{v} \text{ is a contravariant vector}$$

This is equivalent to $\underline{v}\bullet\mathrm{G}^{-1}e_0 = -\mathrm{c}^2$

because $\mathrm{G}\boldsymbol{e} = -\frac{1}{\mathrm{c}^2}e_0$ hence $\mathrm{G}^{-1}e_0 = -\mathrm{c}^2\boldsymbol{e}$ and $\underline{v}\bullet\mathrm{G}^{-1}e_0 = -\mathrm{c}^2\boldsymbol{e}\bullet\underline{v} = -\mathrm{c}^2$

Proof (1) : It is $\boldsymbol{e} = e'_0$ and $\mathrm{G}^{-1} = -\mathrm{c}^2 e'_0 {e'_0}^\mathsf{T} + \sum_{i\geq 1} e'_i {e'_i}^\mathsf{T}$

Hence

$$-\mathrm{c}^2 = \underline{u} \bullet \mathrm{G}^{-1} \underline{u} = -\mathrm{c}^2 |e'_0 \bullet \underline{u}|^2 + \sum_{i\geq 1} |e'_i \bullet \underline{u}|^2,$$

$$|e'_0 \bullet \underline{u}|^2 = 1 + \frac{1}{\mathrm{c}^2} \sum_{i\geq 1} |e'_i \bullet \underline{u}|^2 = 1 + \frac{||\underline{u}||^2}{\mathrm{c}^2}$$

hence $|e'_0 \bullet \underline{u}| \geq 1 > 0$. If $e'_0 \bullet \underline{u} > 0$ this means $e'_0 \bullet \underline{u} = \sqrt{1 + \frac{||\underline{u}||^2}{\mathrm{c}^2}}$.

Then, with $\underline{u}_k := e'_k \bullet \underline{u}$ for $k \geq 0$ we can write $\underline{u} = \sum_{k\geq 0} \underline{u}_k e_k$ and we obtain

$$\underline{v} := \frac{1}{e'_0 \bullet \underline{u}} \underline{u} = \frac{\underline{u}}{\underline{u}_0} = e_0 + \sum_{i\geq 1} \frac{\underline{u}_i}{\underline{u}_0} e_i$$

From this it follows immediately that $e'_0 \bullet \underline{v} = 1$ and

$$||\underline{v}||^2 = \sum_{i\geq 1} \left| \frac{\underline{u}_i}{\underline{u}_0} \right|^2 = \frac{||\underline{u}||^2}{1 + \frac{||\underline{u}||^2}{\mathrm{c}^2}} < \mathrm{c}^2$$

Proof (2) : Let $\underline{u} = \mu \underline{v}$. It should be satisfied

$$-\mathrm{c}^2 = \underline{u} \bullet \mathrm{G}^{-1} \underline{u} = \mu^2 (-\mathrm{c}^2 |\underline{v}_0|^2 + \sum_{i\geq 1} |\underline{v}_i|^2) \quad \text{where } \underline{v}_k := e'_k \bullet \underline{v}$$

Since $\underline{v}_0 = 1$ this means $||\underline{v}|| < \mathrm{c}$ and

$$\mu^{-2} = 1 - \frac{||\underline{v}||^2}{\mathrm{c}^2}$$

Momentum equation

The simplest vector equation in a domain $\mathcal{U} \subset \mathbb{R}^4$ is

$$\sum_{j\geq 0} \partial_{y_j} T_{ij} = g_i \qquad \text{for a 4-tensor } T = (T_{ij})_{i,j\geq 0} \text{ and a 4-field } (g_i)_{i\geq 0}$$

or the distributional version of the **4-momentum system** (it is $N = 1$)

$$\int_{\mathcal{U}} \Big(\sum_{i,j\geq 0} \partial_{y_j}\zeta_i \cdot T_{ij} + \sum_{i\geq 0} \zeta_i\, g_i \Big)\, \mathrm{dL}^4 = 0 \ \text{ for test functions } \zeta$$

with the definition that ζ is a covariant vector $\zeta^* = (\underline{\mathsf{D}}Y)^{\mathsf{T}}\zeta\circ Y$

Reduction to mass equation: Take $\zeta := \eta\boldsymbol{e}$ as test function, where η is an objective scalar, hence ζ is a covariant vector

$$\zeta^* = \eta^*\boldsymbol{e}^* = \eta^*\underline{\mathsf{D}}Y^{\mathsf{T}}\boldsymbol{e}\circ Y = \eta\circ Y\,\underline{\mathsf{D}}Y^{\mathsf{T}}\boldsymbol{e}\circ Y = \underline{\mathsf{D}}Y^{\mathsf{T}}\zeta\circ Y,$$

and it follows

$$0 = \int_{\mathcal{U}} \Big(\sum_{i,j\geq 0} \partial_{y_j}\zeta_i T_{ij} + \sum_{i\geq 0} \zeta_i g_i \Big)\, \mathrm{dL}^4 = \int_{\mathcal{U}} \Big(\sum_{i,j\geq 0} \partial_{y_j}(\eta\boldsymbol{e}_i) T_{ij} + \eta \sum_{i\geq 0} \boldsymbol{e}_i g_i \Big)\, \mathrm{dL}^4$$

$$= \int_{\mathcal{U}} \Big(\sum_{j\geq 0} \partial_{y_j}\eta \underbrace{\sum_i \boldsymbol{e}_i T_{ij}}_{=:\, q_j = T_j} + \eta \Big(\underbrace{\sum_{i,j\geq 0} (\partial_{y_j}\boldsymbol{e}_i) T_{ij} + \sum_{i\geq 0} \boldsymbol{e}_i g_i}_{=:\, \mathbf{r} = g} \Big) \Big)\, \mathrm{dL}^4$$

Similar are the equations in a domain $\mathcal{U} \subset \mathbb{R}^4$ for $\alpha \in \{0,\ldots,3\}^N$

$$\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} = g_\alpha \quad \text{for } T = \big(T_\beta\big)_{\beta\in\{0,\ldots,3\}^{N+1}} \text{ and } g = (g_\alpha)_{\alpha\in\{0,\ldots,3\}^N}$$

or the distributional version of the **N-moments system**

$$\int_{\mathbb{R}^4} \sum_\alpha \Big(\sum_{j\geq 0} \partial_{y_j}\zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha \cdot g_\alpha \Big)\, \mathrm{dL}^4 = 0 \ \text{ for test functions } \zeta$$

with the definition that ζ is a covariant N-tensor

This is satisfied if

T is a contravariant $(N+1)$-tensor
$\underline{\mathbf{f}}$ is a contravariant N-tensor
$g_\alpha = \underline{\mathbf{f}}_\alpha + \sum_{\beta\in\{0,\ldots,3\}^{N+1}} \mathrm{C}_\alpha^\beta T_\beta$ with Coriolis coefficients C_α^β for all α

having the transformation rule

$$\sum_{m_1,\ldots,m_{N+1}\geq 0} Y_{m_1{}'\bar{m}_1} \cdots Y_{m_{N+1}{}'\bar{m}_{N+1}} \mathrm{C}_{k_1\cdots k_N}^{m_1\cdots m_{N+1}} \circ Y$$
$$= \sum_{\bar{k}_1,\ldots,\bar{k}_N\geq 0} Y_{k_1{}'\bar{k}_1} \cdots Y_{k_N{}'\bar{k}_N} \mathrm{C}_{\bar{k}_1\cdots\bar{k}_N}^{*\bar{m}_1\cdots\bar{m}_{N+1}} + \big(Y_{k_1{}'\bar{m}_1} \cdots Y_{k_N{}'\bar{m}_N}\big)_{{}'\bar{m}_{N+1}}$$

for all $k_1,\ldots,k_N$ and $\bar{m}_1,\ldots,\bar{m}_{N+1}$

Reduction to $(N-1)^{\text{th}}$-order equation: Take as test function

$\zeta_{\alpha_1\cdots\alpha_N} := \boldsymbol{e}_{\alpha_1}\eta_{\alpha_2\cdots\alpha_N}$ where η is a covariant $(N-1)$-tensor. Since

$$\zeta^*_{\bar\alpha_1\cdots\bar\alpha_N} = \boldsymbol{e}^*_{\bar\alpha_1}\eta^*_{\bar\alpha_2\cdots\bar\alpha_N} = \sum_{\alpha_1\geq 0} Y_{\alpha_1{}'\bar\alpha_1}\boldsymbol{e}_{\alpha_1}\circ Y \cdot \sum_{\alpha_2,\dots,\alpha_N\geq 0} Y_{\alpha_2{}'\bar\alpha_2}\cdots Y_{\alpha_N{}'\bar\alpha_N}\eta_{\alpha_2\cdots\alpha_N}\circ Y$$
$$= \sum_{\alpha_1,\dots,\alpha_N\geq 0} Y_{\alpha_1{}'\bar\alpha_1}\cdots Y_{\alpha_N{}'\bar\alpha_N}\boldsymbol{e}_{\alpha_1}\circ Y\,\eta_{\alpha_2\cdots\alpha_N}\circ Y$$
$$= \sum_{\alpha_1,\dots,\alpha_N\geq 0} Y_{\alpha_1{}'\bar\alpha_1}\cdots Y_{\alpha_N{}'\bar\alpha_N}\zeta_{\alpha_1\cdots\alpha_N}\circ Y$$

it means that ζ is a covariant N-tensor and writing $\alpha = (i,\gamma)$

$$0 = \int_{\mathbb{R}^4}\sum_\alpha\Big(\sum_j \partial_{y_j}\zeta_\alpha\cdot T_{\alpha j} + \zeta_\alpha\cdot g_\alpha\Big)\,\mathrm{dL}^4 = \int_{\mathbb{R}^4}\sum_{i\gamma}\Big(\sum_j \partial_{y_j}(\eta_\gamma\boldsymbol{e}_i)\cdot T_{i\gamma j} + \eta_\gamma\boldsymbol{e}_i\cdot g_{i\gamma}\Big)\,\mathrm{dL}^4$$
$$= \int_{\mathbb{R}^4}\sum_\gamma\Big(\sum_j \partial_{y_j}\eta_\gamma\cdot\underbrace{\textstyle\sum_i \boldsymbol{e}_i T_{i\gamma j}}_{=:\,T_{\gamma j}} + \eta_\gamma\underbrace{\sum_i\Big(\partial_{y_j}\boldsymbol{e}_i T_{i\gamma j} + \boldsymbol{e}_i g_{i\gamma}\Big)}_{=:\,g_\gamma}\Big)\,\mathrm{dL}^4$$
$$= \int_{\mathbb{R}^4}\sum_\gamma\Big(\sum_j \partial_{y_j}\eta_\gamma\cdot T_{\gamma j} + \eta_\gamma g_\gamma\Big)\,\mathrm{dL}^4$$

Special: Use $\boldsymbol{e}\bullet\underline{v} = 1$ and the identity $\underline{\Pi}_{\gamma j} = \sum_i \boldsymbol{e}_i\underline{\Pi}_{i\gamma j}$ if

$$T_{\beta_1\cdots\beta_N} = \varrho\underline{v}_{\beta_1}\cdots\underline{v}_{\beta_N} + \underline{\Pi}_{\beta_1\cdots\beta_N}$$

- We deal with a generalized version of the Chapman-Enskog hierarchy
- The hierarchy of differential equations is

$$\sum_{j\geq 0} T_{\alpha j} - \sum_{\beta\in\{0,\dots,3\}^{N+1}} \mathsf{C}_\alpha^\beta T_\beta = \underline{\mathbf{f}}_\alpha \qquad \text{for a flux } T = \big(T_\beta\big)_{\beta\in\{0,\dots,3\}^{N+1}}$$

- The N-moments system contains as part the $(N-1)$-moments system

$$T_\alpha = \sum_k \boldsymbol{e}_k T_{k\alpha}$$

- This reduction is made possible by the "time vector" $\boldsymbol{e}$ and the test functions $\zeta_\alpha = \boldsymbol{e}_k \eta_\gamma$ for $\alpha = (k,\gamma)$
- For $T_\beta = \varrho \underline{v}_{\beta_1} \cdots \underline{v}_{\beta_{N+1}} + \underline{\Pi}_\beta$ we use the fact that $\boldsymbol{e}\bullet\underline{v} = 1$

This talk is based on the paper

[H.W. Alt: *Relativistic equations for the generalized Chapman-Enskog hierarchy*]
www-m6.ma.tum.de/~alt/alt-enskog.pdf arXiv:1605.04573