Distributional equations and interfaces

Hans Wilhelm Alt

Distributions

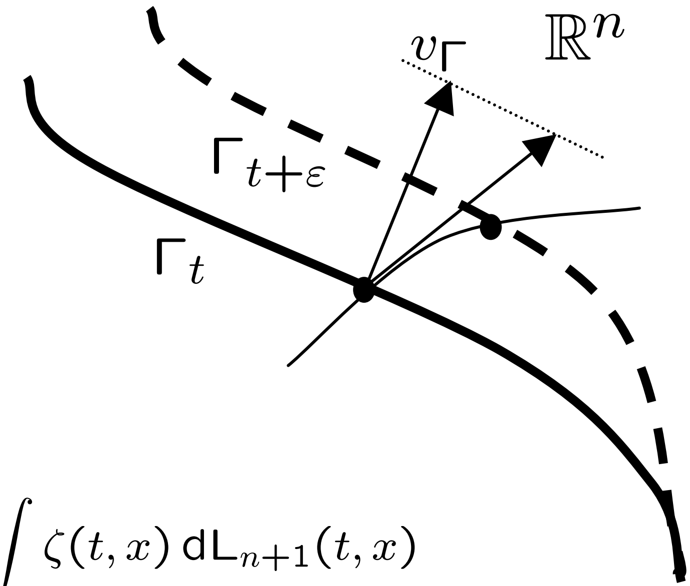
Surfaces: $(t,x) \in \Gamma \iff x \in \Gamma_t$

$$\Gamma \subset \mathbb{R} \times \mathbb{R}^n, \quad (m+1)\text{-dimensional}, \quad T_{(t,x)}\Gamma \neq \{0\} \times \mathbb{R}^n$$

$$\Gamma_t := \{x \in \mathbb{R}^n \,;\ (t,x) \in \Gamma\}, \quad T_{(t,x)}\Gamma = \operatorname{span}\{(1, v_\Gamma(t,x))\} \cup T_x\Gamma_t$$

Surface measures:

$$\langle \zeta , \boldsymbol{\mu}_\Gamma \rangle := \int_{\mathbb{R}} \int_{\Gamma_t} \zeta(t,x)\, \mathrm{dH}_m(x)\, \mathrm{dL}_1(t) = \int_{\Gamma} \frac{\zeta(t,x)}{\sqrt{1+|v_\Gamma|^2}}\, \mathrm{dH}_{m+1}(t,x)$$



Extreme cases $m = n$ and $m = 0$:

Ω (Ω_t n-dimensional = open set)

$$\langle \zeta , \boldsymbol{\mu}_\Omega \rangle = \int_{\mathbb{R}} \int_{\Omega_t} \zeta(t,x)\, \mathrm{dH}_n(x)\, \mathrm{dL}_1(t) = \int_{\Omega} \zeta(t,x)\, \mathrm{dL}_{n+1}(t,x)$$

Γ ($\Gamma_t = \{y_t\}$ 0-dimensional = point)

$$\langle \zeta , \boldsymbol{\mu}_\Gamma \rangle = \int_{\mathbb{R}} \int_{\{y_t\}} \zeta(t,x)\, \mathrm{dH}_0(x)\, \mathrm{dL}_1(t) = \int_{\mathbb{R}} \zeta(t,y_t)\, \mathrm{dL}_1(t)$$

Theorem (Distributional and strong version)

Let $t \mapsto \Gamma_t$ be m-dimensional, $0 \le m \le n$. The equation

$$\partial_t(e\boldsymbol{\mu}_\Gamma) + \operatorname{div}(q\boldsymbol{\mu}_\Gamma) = f\boldsymbol{\mu}_\Gamma$$

is equivalent to

$q - ev_\Gamma$ tangential on Γ $\qquad (q - ev_\Gamma)(t,x) \in T_x\Gamma_t$

$\partial_t^\Gamma e - ev_\Gamma \bullet \kappa_\Gamma + \operatorname{div}^\Gamma(q - ev_\Gamma) = f$ on Γ $\qquad \partial_t^\Gamma e + \operatorname{div}^\Gamma q = f$

Here

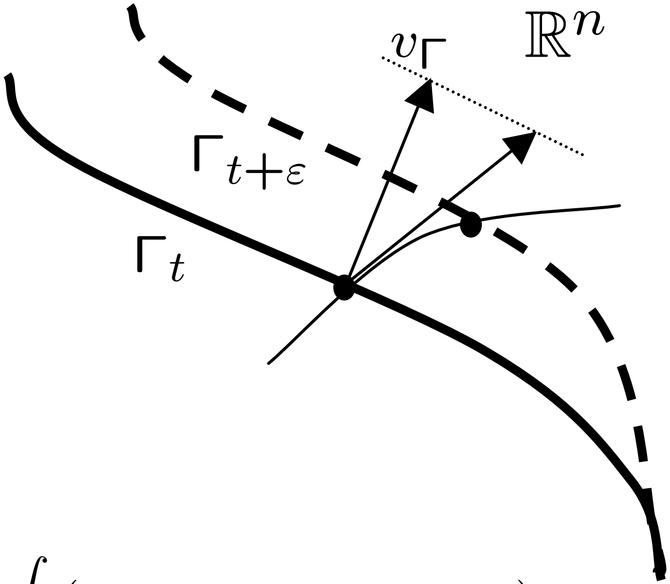
$\Gamma_t := \{x \in \mathbb{R}^n \,;\ (t,x) \in \Gamma\}$

$v_\Gamma(t,x) \in T_x\Gamma_t{}^\perp$ "surface velocity" of Γ_t

$\kappa_\Gamma(t,x)$ n-times mean curvature of Γ_t

$\partial_t^\Gamma := \partial_t + v_\Gamma \bullet \nabla$

$\operatorname{div}^\Gamma := \sum_{i=1,\ldots,m} \tau_k \bullet \partial_{\tau_k}$



$$\langle\, \zeta\,,\, -\partial_t(e\boldsymbol{\mu}_\Gamma) - \operatorname{div}(q\boldsymbol{\mu}_\Gamma) + f\boldsymbol{\mu}_\Gamma \,\rangle$$
$$= \langle\, \partial_t\zeta\,,\, e\boldsymbol{\mu}_\Gamma \,\rangle + \langle\, \nabla\zeta\,,\, q\boldsymbol{\mu}_\Gamma \,\rangle + \langle\, \zeta\,,\, f\boldsymbol{\mu}_\Gamma \,\rangle = \int \big(\partial_t\zeta \cdot e + \nabla\zeta \bullet q + \zeta \cdot f\big)\, \mathrm{d}\boldsymbol{\mu}_\Gamma$$

Examples of distributions in $\mathcal{D}'(U)$, $U \subset \mathbb{R} \times \mathbb{R}^n$:

Body Ω with mass exchange at the boundary $\Gamma = \partial\Omega$

$$\partial_t(\varrho \boldsymbol{\mu}_\Omega) + \operatorname{div}(\varrho v \boldsymbol{\mu}_\Omega) = \tau \boldsymbol{\mu}_\Gamma$$

(τ mass production at boundary)

$\Longleftrightarrow$

$\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \quad$ in Ω

$0 = \tau + \varrho(v - v_\Gamma)\bullet\nu_\Omega \quad$ on Γ

PDE as distribution Body $\Omega = U$

$$\partial_t(\varrho \boldsymbol{\mu}_U) + \operatorname{div}(\varrho v \boldsymbol{\mu}_U) = 0$$

$\Longleftrightarrow$

$\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \quad$ in all of U

Mass balance on a membrane Γ

with mass density $\varrho^s > 0$ and "particle velocity" v^s

$$\partial_t(\varrho^s \boldsymbol{\mu}_\Gamma) + \operatorname{div}(\varrho^s v^s \boldsymbol{\mu}_\Gamma) = 0$$

$\Longleftrightarrow \quad$ on Γ:

$v^s - v_\Gamma \quad$ tangential

$\partial_t^\Gamma \varrho^s - \varrho^s\, \kappa_\Gamma \bullet v_\Gamma + \operatorname{div}^\Gamma(\varrho^s(v^s - v_\Gamma)) = 0$

κ_Γ n-times mean curvature vector of Γ

$\partial_t^\Gamma := \partial_t + v_\Gamma \bullet \nabla$ time derivative of Γ

Strong differential equality $\Longleftrightarrow$ $\partial_t^\Gamma \varrho^s + \operatorname{div}^\Gamma(\varrho^s v^s) = 0$

Mass in a system of M curves Γ_t^k which meet at point P_t, everything is moving

$$\partial_t(\textstyle\sum_{k=1}^M \varrho^k \boldsymbol{\mu}_{\Gamma^k}) + \operatorname{div}(\textstyle\sum_{k=1}^M \varrho^k v^k \boldsymbol{\mu}_{\Gamma^k}) = a\,\boldsymbol{\mu}_P$$

$$\Longleftrightarrow$$

$$\left.\begin{array}{ll} v^k - v_{\Gamma^k} & \text{tangential} \\ \partial_t^{\Gamma^k}\varrho^k + \operatorname{div}^{\Gamma^k}(\varrho^k v^k) = 0 & \end{array}\right\} \quad \text{on } \Gamma^k,\ k = 1,2,3$$

$$\textstyle\sum_{k=1}^M \varrho^k \frac{(1, v^k)\bullet \underline{n}^k}{\sqrt{1+|v_\Gamma|^2}} = \frac{a}{\sqrt{1+|v_P|^2}} \quad \text{on } P$$

$\underline{n}^k$ tangent at γ, normal to P

System of hyperbolic conservation laws

$$\partial_t u_k + \operatorname{div} q_k(u) = f_k(u) \qquad \text{for } k = 1, \dots, N$$
with $u \in L^\infty(U; \mathbb{R}^N)$

Let u have a jump at Γ, smooth in remainder: $u = u^m$ in Ω^m, $m = 1, 2$
Then differential equation is equivalent to

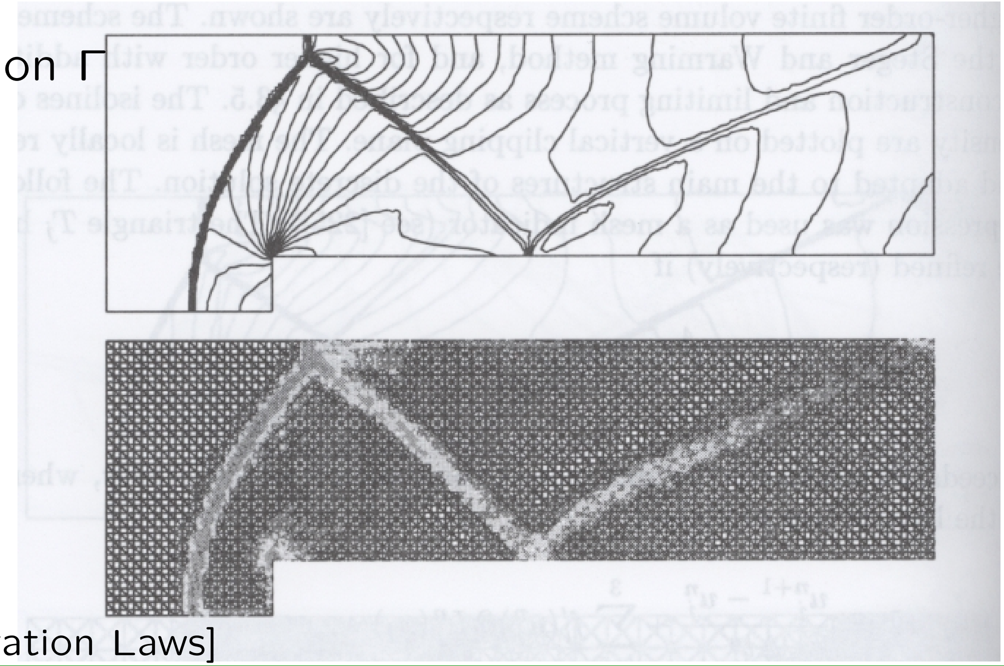
$$\partial_t \big(\textstyle\sum_m u_k^m \boldsymbol{\mu}_{\Omega^m} \big) + \operatorname{div} \big(\textstyle\sum_m q_k(u^m) \boldsymbol{\mu}_{\Omega^m} \big) = \textstyle\sum_m f_k(u^m) \boldsymbol{\mu}_{\Omega^m} \qquad \text{for } k = 1, \dots, N$$

This is equivalent to

$$\partial_t u_k^m + \operatorname{div} q_k(u^m) = f_k(u^m) \qquad \text{in } \Omega^m$$

$$\textstyle\sum_m (q_k(u^m) - u_k^m v_\Gamma) \bullet \nu_{\Omega^m} = 0 \qquad \text{on } \Gamma$$
(Rankine-Hugoniot condition)

for $k = 1, \dots, N$



[D. Kröner: Numerical Schemes for Conservation Laws]

Objectivity (Frame indifference)

- The value of physical quantities depend on the observer
- The type of a physical quantity is given by a transformation rule
- The description of physical processes has to be independent of the observer

Observer transformations (classical group = Newton's physics)

$$\begin{bmatrix} t \\ x \end{bmatrix} = y = Y(y^*) = Y(t^*, x^*) = \begin{bmatrix} T(t^*, x^*) \\ X(t^*, x^*) \end{bmatrix} = \begin{bmatrix} t^* + a \\ Q(t^*)x^* + b(t^*) \end{bmatrix}$$

where $a \in \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}^n$, $Q : \mathbb{R} \to \mathbb{R}^{n \times n}$ orthogonal transformation, $\det Q = 1$

$$\underline{\mathrm{D}}Y = (Y_{k\,'l})_{k,l=0,\dots,n} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dot{Q}x^* + \dot{c} & Q \end{bmatrix}$$

Examples of transformation rules:

ϱ objective scalar $\iff$ $\varrho \circ Y = \varrho^*$

that is $\varrho(t, x) = \varrho^*(t^*, x^*)$ for $(t, x) = Y(t^*, x^*)$

v velocity $\iff$ $v \circ Y = \dot{X} + Qv^*$

that is $v(t, x) = \dot{X}(t^*, x^*) + Q(t^*)v^*(t^*, x^*)$ for $(t, x) = Y(t^*, x^*)$

$\mathbf{f}$ force $\iff$ $\mathbf{f} \circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}J^* + r^*\dot{X} + Q\mathbf{f}^*$

(if mass conservation is $\partial_t \varrho + \operatorname{div}(\varrho v + J) = r$)

They come from objectivity of balance laws

Theorem (Objectivity for systems of balance laws)

$$\partial_t(e^k\mu_\Gamma) + \sum_{i=1}^n \partial_{x_i}(q_i^k\mu_\Gamma) = f^k\mu_\Gamma \qquad \text{(k=1,...,N)}$$

Physical type given by (linear) transformation rule for test functions

$$\zeta\circ Y = Z^{-\mathsf{T}}\zeta^* \qquad (\ \zeta^* = Z^{\mathsf{T}}\zeta\circ Y\ )$$

Then this system is objective, if

$$\begin{aligned} e\circ Y &= Ze^* \\ q_i\circ Y &= \dot{X}_i Z e^* + \sum_{j=1}^n Q_{ij} Z q_j^* \qquad \text{for } i = 1,\ldots,n \\ f\circ Y &= Z_{'t}e^* + \sum_{j=1}^n Z_{'j} q_j^* + Zf^* \end{aligned}$$

Examples:

mass | mass-momentum | mass-momentum-energy

$$Z = 1 \qquad Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} \qquad Z := \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & Q & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathsf{T}}Q & 1 \end{bmatrix}$$

($Z = Z(Y)$ is a differential operator in Y)

Proof: All μ_Γ have the same transformation behaviour

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Weak formulation in spacetime:

$$\partial_0 := \partial_t,\ q_0 := e,\ q_i = (q_i^k)_{k=1,\dots,N},\ f = (f^k)_{k=1,\dots,N}$$

$$\int \Big(\textstyle\sum_{i=0}^n \partial_i \zeta \bullet q_i + \zeta \bullet f \Big)\, \mathrm{d}\mu_\Gamma = 0 \text{ for test functions } \zeta = (\zeta^k)_{k=1,\dots,N}$$

$$\zeta^* = Z^\mathsf{T} \zeta \circ Y \quad \Rightarrow \quad \partial_j \zeta^* = Z_{',j}{}^\mathsf{T} \zeta \circ Y + \textstyle\sum_{i=0}^n Y_{i',j} Z^\mathsf{T} (\partial_i \zeta) \circ Y$$

$\Longrightarrow$

$$\int \Big(\textstyle\sum_{j=0}^n \partial_j \zeta^* \bullet q_j^* + \zeta^* \bullet f^* \Big)\, \mathrm{d}\mu_{\Gamma^*}$$

$$= \int \Big(\textstyle\sum_{i=0}^n (\partial_i \zeta \circ Y) \bullet (\sum_{j=0}^n Y_{i',j} Z q_j^*) + (\zeta \circ Y) \bullet (Z f^* + \sum_{j=0}^n Z_{',j} q_j^*) \Big)\, \mathrm{d}\mu_{\Gamma^*}$$

$$\overset{(\det = 1)}{=} \int \Big(\textstyle\sum_{i=0}^n \partial_i \zeta \bullet (\sum_{j=0}^n Y_{i',j} Z q_j^*) \circ Y^{-1} + \zeta \bullet (Z f^* + \sum_{j=0}^n Z_{',j} q_j^*) \circ Y^{-1} \Big)\, \mathrm{d}\mu_\Gamma$$

$$\overset{!}{=} \int \Big(\textstyle\sum_{i=0}^n \partial_i \zeta \bullet q_i + \zeta \bullet f \Big)\, \mathrm{d}\mu_\Gamma$$

$\Longleftarrow$

$$q_i \circ Y = \textstyle\sum_{j=0}^n Y_{i',j} Z q_j^* \text{ for } i = 0, \dots, n \qquad f \circ Y = \sum_{j=0}^n Z_{',j} q_j^* + Z f^*$$

Examples:

Mass-momentum system for compressible fluids

$$\partial_t(\varrho\mu_U) + \mathrm{div}(\varrho v\mu_U) = 0$$

$$\partial_t(\varrho v\mu_U) + \mathrm{div}((\varrho v\otimes v + \Pi)\mu_U) = \mathbf{f}\mu_U$$

The transformation matrix is

$$Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} = \underline{\mathrm{D}}Y$$

Objectivity says

$$\begin{bmatrix} \varrho & \varrho v^{\mathsf{T}} \\ \varrho v & \varrho v\otimes v + \Pi \end{bmatrix}\circ Y = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} \varrho^* & \varrho^* v^{*\mathsf{T}} \\ \varrho^* v^* & \varrho^* v^*\otimes v^* + \Pi^* \end{bmatrix}\begin{bmatrix} 1 & \dot{X}^{\mathsf{T}} \\ 0 & Q \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix}\circ Y = \begin{bmatrix} 0 & 0 \\ \ddot{X} & \dot{Q} \end{bmatrix}\begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \sum_{j=1}^{n}\begin{bmatrix} 0 & 0 \\ \dot{X}_{,j} & 0 \end{bmatrix}\begin{bmatrix} \varrho^* v_j^* \\ \cdots \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{f}^* \end{bmatrix}$$

$\Longleftrightarrow$

ϱ is an objective scalar, i.e. $\varrho\circ Y = \varrho^*$

v is a velocity, i.e. $v\circ Y = \dot{X} + Qv^*$

Π is an objective tensor, i.e. $\Pi\circ Y = Q\Pi^* Q^{\mathsf{T}}$

$\mathbf{f}$ is a force, i.e. $\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*$

Mass-momentum system for an interface problem

Let $\Gamma := \partial\Omega$

$$\partial_t(\varrho\boldsymbol{\mu}_\Omega) + \mathrm{div}(\varrho v\boldsymbol{\mu}_\Omega) = \mathbf{r}^s\boldsymbol{\mu}_\Gamma$$

$$\partial_t(\varrho v\boldsymbol{\mu}_\Omega) + \mathrm{div}((\varrho v\otimes v + \Pi)\boldsymbol{\mu}_\Omega + \Pi^s\boldsymbol{\mu}_\Gamma) = \mathbf{f}\boldsymbol{\mu}_\Omega + \mathbf{f}^s\boldsymbol{\mu}_\Gamma$$

The transformation matrix is still

$$Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} = \underline{\mathsf{D}}Y$$

Objectivity says on Γ

$$\begin{bmatrix} 0 & 0 \\ 0 & \Pi^s \end{bmatrix}\circ Y = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Pi^{s*} \end{bmatrix}\begin{bmatrix} 1 & \dot{X} \\ 0 & Q \end{bmatrix}^\mathsf{T}$$

$$\begin{bmatrix} \mathbf{r}^s \\ \mathbf{f}^s \end{bmatrix}\circ Y = \begin{bmatrix} 0 & 0 \\ \ddot{X} & \dot{Q} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sum_{j=1}^n \begin{bmatrix} 0 & 0 \\ \dot{X}_{,j} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \dots \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} \mathbf{r}^{s*} \\ \mathbf{f}^{s*} \end{bmatrix}$$

$\Longleftrightarrow$

$\mathbf{r}^s$ is an objective scalar, i.e. $\mathbf{r}^s\circ Y = \mathbf{r}^{s*}$

Π^s is an objective tensor, i.e. $\Pi^s\circ Y = Q\Pi^{s*}Q^\mathsf{T}$

$\mathbf{f}^s$ is surface force, i.e. $\mathbf{f}^s\circ Y = \mathbf{r}^{s*}\dot{X} + Q\mathbf{f}^{s*}$

Lemma A possible choice is $\mathbf{f}^s = \mathbf{r}^s v + \mathbf{f}_0^s$ with $\mathbf{f}_0^s\circ Y = Q\mathbf{f}_0^{s*}$

Drop given by distributional conservation laws for mass and momentum in Ω and on $\Gamma := \partial\Omega$

$$\partial_t(\varrho\boldsymbol{\mu}_\Omega) + \mathrm{div}(\varrho v\boldsymbol{\mu}_\Omega) = 0$$

$$\partial_t(\varrho v\boldsymbol{\mu}_\Omega) + \mathrm{div}\Big((\varrho v v^\top + \Pi)\boldsymbol{\mu}_\Omega + \Pi^s\boldsymbol{\mu}_\Gamma\Big) = \mathbf{f}\boldsymbol{\mu}_\Omega$$

System gives transformation rules

$\Longleftrightarrow$

mass and momentum equation in Ω

on Γ: $(v - v_\Gamma)\bullet\nu_\Omega = 0$, $\Pi^s\nu = 0$ for normal ν, $\mathrm{div}^\Gamma\Pi^s = \Pi\nu_\Omega =: \mathbf{f}^s$

Therefore it contains the

Surface tension law on Γ

$$\mathrm{div}(\Pi^s\boldsymbol{\mu}_\Gamma) = \mathbf{f}^s\boldsymbol{\mu}_\Gamma \qquad \left(\Longleftrightarrow \int (\mathrm{D}\zeta\bullet\Pi^s + \zeta\bullet\mathbf{f}^s)\,\mathrm{d}\boldsymbol{\mu}_\Gamma = 0 \text{ for } \zeta \in \mathcal{D}(U;\mathbb{R}^n)\right)$$

$\mathrm{D}\zeta = (\partial_j\zeta_i)_{i,j=1,\dots,n}$

$\Longleftrightarrow$

$\mathrm{div}^\Gamma\Pi^s = \mathbf{f}^s$,

$\Pi^s\nu = 0$ for $\nu = \pm\nu_\Omega$

Objectivity

v_Γ is "surface velocity":	$v_\Gamma \circ Y = \dot{X}\bullet(Q\nu^*)Q\nu^* + Qv_{\Gamma^*}$
Π^s objective tensor	ν_Ω objective vector
$\Pi^s\nu_\Omega = 0$ is objective	$\mathbf{f}^s = \Pi\nu_\Omega$ objective vector

Surface tension as function of normal

It is $\Pi^s \nu_\Omega = 0$, and if Π^s symmetric tensor, $n = 3$

Result: If $\Pi^s = \widehat{\Pi}^s(\nu)$ then

$$\Pi^s = -\gamma(\mathrm{Id} - \nu\nu^\mathsf{T})$$
$\gamma \in \mathbb{R}$ surface tension

Result: If Π is constant, then

$$\Pi = p\mathrm{Id}$$
$p \in \mathbb{R}$ pressure

General: Π^s (and Π) can depend also on objective scalars, like ϱ or p or θ

Laplace formula: If $\gamma : \Gamma \to \mathbb{R}$ and $\Pi^s = -\gamma(\mathrm{Id} - \nu\nu^\mathsf{T})$ then

$$\operatorname{div}^\Gamma \Pi^s = \operatorname{div}^\Gamma(-\gamma(\mathrm{Id} - \nu\nu^\mathsf{T})) = -\gamma\kappa_\Gamma - \nabla^\Gamma\gamma$$

Hence the surface law of the momentum equation is $\gamma\kappa_\Gamma + \nabla^\Gamma\gamma + \Pi\nu_\Omega = 0$

Liquid drop If $\Pi^s = -\gamma(\mathrm{Id} - \nu\nu^\mathsf{T})$ and $\Pi = p\mathrm{Id} - S$ then

mass and momentum equation in Ω

$$\left.\begin{array}{ll} (v - v_\Gamma)\bullet\nu = 0 & \\ \partial_\tau\gamma = \tau\bullet S\nu_\Omega & \text{for tangential vectors } \tau \\ \gamma\kappa_\Gamma\bullet\nu_\Omega + p = \nu\bullet S\nu & \end{array}\right\} \text{ on } \Gamma$$

Further constitutive constraints $\mathrm{D}v\bullet S \geq 0$, $f^s = \gamma = \mathrm{const}$
come from free energy inequality:

Free energy inequality

$$\partial_t F + \operatorname{div}\Phi - G_0 \le 0$$

that is

$$0 \ge \partial_t\big(\underbrace{f\boldsymbol{\mu}_\Omega + f^s\boldsymbol{\mu}_\Gamma}_{F} \big) + \operatorname{div}\big(\underbrace{(fv + \Pi^\top v)\boldsymbol{\mu}_\Omega + (f^s v + \Pi^{s\top} v)\boldsymbol{\mu}_\Gamma}_{\Phi} \big) - \underbrace{v\bullet\mathbf{f}\ \boldsymbol{\mu}_\Omega}_{G_0}$$

$$=: \ g\,\boldsymbol{\mu}_\Omega + g^s\,\boldsymbol{\mu}_\Gamma$$

that is

$$g \le 0 \text{ on } \Omega \quad \text{and} \quad g^s \le 0 \text{ on } \Gamma$$

that is on Ω:

$$0 \ge g = \partial_t f + \operatorname{div}(fv + \Pi^\top v) - v\bullet\mathbf{f} \qquad f = f(\varrho, v) = \frac{\varrho}{2}|v|^2 + f_0(\varrho)$$

$$= \partial_t f_0 + \operatorname{div}(f_0 v) + \mathrm{D}v\bullet\Pi \qquad \Pi = p\mathrm{Id} - S$$

$$= \mathrm{D}v\bullet((f_0 - \varrho f_0{}'_\varrho)\mathrm{Id} + \Pi) = -\mathrm{D}v\bullet S \qquad p = \varrho f_0{}'_\varrho - f_0$$

and on Γ: $\quad (f^s v + \Pi^{s\top} v)\bullet\nu_\Omega = f^s v_\Gamma\bullet\nu_\Omega, \quad (v - v_\Gamma)\bullet\nu_\Omega = 0$

$$0 \ge g^S = \partial_t^\Gamma f^s + \operatorname{div}^\Gamma(f^s v + \Pi^{s\top} v) - (f(v - v_\Gamma) + \Pi^\top v)\bullet\nu_\Omega$$

$$= (\partial_t^\Gamma + v\bullet\nabla^\Gamma) f^s + (\mathrm{D}^\Gamma v)\bullet(f^s\mathrm{Id} + \Pi^s) + v\bullet\big(\underbrace{\operatorname{div}^\Gamma \Pi^s - \Pi\nu_\Omega}_{=0} \big)$$

$$= (\partial_t + v\bullet\nabla) f^s + (\mathrm{D}^\Gamma v)\bullet(f^s\mathrm{Id} + \Pi^s) \qquad \mathrm{D}^\Gamma v = \textstyle\sum_\tau (\partial_\tau v)\otimes\tau$$

satisfied, if $\mathrm{D}v\bullet S \ge 0$, $f^s = \gamma = \mathrm{const}$

Liquid drop It is $\gamma = \text{const}$, and it follows

$$\left.\begin{array}{l} \partial_t \varrho + \operatorname{div}(\varrho v) = 0 \\ \partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathsf{T}} + p\mathrm{Id} - S) = \mathbf{f} \\ p = \varrho f_{0\,'\varrho} - f_0, \quad \mathrm{D}v \bullet S \geq 0 \end{array}\right\} \text{ on } \Omega$$

$$\left.\begin{array}{l} (v - v_\Gamma) \bullet \nu = 0 \\ \tau \bullet S\nu = 0 \quad \text{for tangential vectors } \tau \\ \gamma \kappa_\Gamma \bullet \nu_\Omega + p = \nu \bullet S\nu \\ f^s = \gamma \end{array}\right\} \text{ on } \Gamma$$

Remark If $v = 0$, $\varrho = \text{const}$, $\mathbf{f} = 0$ define ($\Gamma = \partial\Omega$)

$$E_\Gamma = \int_\Gamma \gamma \, \mathrm{dH}_{n-1} - \int_\Omega p \, \mathrm{dL}_n \quad \Big(= \int_\Gamma f^s \, \mathrm{dH}_{n-1} + \int_\Omega f \, \mathrm{dL}_n \Big)$$

and solution is stationary point of E_Γ

In general the solution is related to the free energy inequality

$$\partial_t (f \boldsymbol{\mu}_\Omega + f^s \boldsymbol{\mu}_\Gamma) + \operatorname{div}(\ldots) - v \bullet \mathbf{f} \, \boldsymbol{\mu}_\Omega \leq 0$$

An Allen-Cahn model for compressible fluids

Witterstein model

$$\partial_t\varrho + \operatorname{div}(\varrho v) = 0 \qquad f_\delta = f_\delta(\varrho,\varphi,\nabla\varphi)$$

$$\varrho(\partial_t\varphi + v\bullet\nabla\varphi) = -\mathbf{r}_\delta \qquad \mathbf{r}_\delta := c_\delta(\varrho,\varphi)\frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi}$$

$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\otimes v + \Pi_\delta) = \mathbf{f}_\delta \qquad \Pi_\delta = P_\delta - S$$

with

$$P_\delta := (\varrho f_{\delta}{}'_{\varrho} - f_\delta)\mathrm{Id} + \nabla\varphi\otimes f_{\delta}{}'_{\nabla\varphi}$$

$$S := \lambda_1(\varrho,\varphi)\operatorname{div} v\ \mathrm{Id} + \lambda_2(\varrho,\varphi)\Big((\mathrm{D}v)^{\mathsf{S}} - \frac{1}{n}\operatorname{div} v\ \mathrm{Id}\Big)$$

$$f_\delta(\varrho,\varphi,\nabla\varphi) := \frac{1}{\delta}\varrho W(\varphi) + \delta\, h(\varrho)\frac{|\nabla\varphi|^2}{2} + U(\varrho,\varphi)$$

W has two local minima at 0 and 1, $U'_\varphi(\varrho,0) = 0$, $U'_\varphi(\varrho,1) = 0$

$$\frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi} = \frac{1}{\delta}\varrho W'_\varphi(\varphi) - \delta\operatorname{div}(h(\varrho)\nabla\varphi) + U'_\varphi(\varrho,\varphi)$$

Free energy inequality $f = f(\varrho,\varphi,v,\nabla\varphi) = \frac{\varrho}{2}|v|^2 + f_\delta(\varrho,\varphi,\nabla\varphi)$

$$\partial_t f + \operatorname{div}(fv + \Pi_\delta^{\mathsf{T}} v - \dot\varphi f'_{\nabla\varphi}) - v\bullet\mathbf{f}_\delta = -\frac{1}{\varrho}\mathbf{r}_\delta\frac{\boldsymbol{\delta} f_\delta}{\boldsymbol{\delta}\varphi} - \mathrm{D}v\bullet S \le 0$$

System written for the two masses ϱ^1 and ϱ^2

$$\varrho^1_\delta = (1-\varphi)\varrho\,, \quad \varrho^2_\delta = \varphi\varrho\,, \quad \text{or} \quad \varrho = \varrho^1_\delta + \varrho^2_\delta\,, \quad \varphi = \frac{\varrho^2_\delta}{\varrho^1+\varrho^2}$$

becomes

$$\partial_t(\varrho^1_\delta \mu_U) + \operatorname{div}(\varrho^1_\delta v_\delta \mu_U) = \mathbf{r}_\delta \mu_U$$

$$\partial_t(\varrho^2_\delta \mu_U) + \operatorname{div}(\varrho^2_\delta v_\delta \mu_U) = -\mathbf{r}_\delta \mu_U$$

$$\partial_t(\varrho v \mu_U) + \operatorname{div}\big((\varrho v v^{\mathsf{T}} + \Pi_\delta)\mu_U\big) = \mathbf{f}\mu_U$$

Theorem In the limit $\delta \to 0$ this becomes

$$\partial_t(\varrho^1 \boldsymbol{\mu}_{\Omega^1}) + \operatorname{div}(\varrho^1 v^1 \boldsymbol{\mu}_{\Omega^1}) = \mathbf{r}\,\boldsymbol{\mu}_\Gamma$$

$$\partial_t(\varrho^2 \boldsymbol{\mu}_{\Omega^2}) + \operatorname{div}(\varrho^2 v^2 \boldsymbol{\mu}_{\Omega^2}) = -\mathbf{r}\,\boldsymbol{\mu}_\Gamma$$

$$\partial_t\big(\textstyle\sum_m \varrho^m v^m \boldsymbol{\mu}_{\Omega^m}\big) + \operatorname{div}\Big(\textstyle\sum_m (\varrho^m v^m \otimes v^m + \Pi^m)\boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma\Big) = \textstyle\sum_m \mathbf{f}^m \boldsymbol{\mu}_{\Omega^m}$$

The convergence is in the sense of distributions, in particular

$$(\varrho v v^{\mathsf{T}} + \Pi_\delta)\boldsymbol{\mu}_U \quad \longrightarrow \quad \textstyle\sum_m (\varrho^m v^m \otimes v^m + \Pi^m)\boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma$$

$\Pi^s = -\gamma(\mathrm{Id} - \nu\otimes\nu)\,,$ $\quad\gamma$ given by an integral over “local coordinates”

where $U = \Omega^1 \cup \Gamma \cup \Omega^2$

For $\eta \in \mathcal{D}(U;\mathbb{R}^n \times \mathbb{R}^n)$, $U \subset \mathbb{R} \times \mathbb{R}^n$, $U = \Omega_\delta^1 \cup \Gamma_\delta \cup \Omega_\delta^2$

$$\int_U \eta \bullet (\varrho v v^{\mathsf{T}} + \Pi_\delta)\, \mathrm{dL}_{n+1} = \sum\nolimits_m \int_{\Omega_\delta^m} \dots\, \mathrm{dL}_{n+1} + \int_{\Gamma_\delta} \dots\, \mathrm{dL}_{n+1}$$

In a small neighbourhood of Γ

$$\begin{aligned}
\Pi_\delta = {} & \mathsf{p}_U \,\mathrm{Id} + \frac{\delta}{2}\mathsf{p}_h |\nabla\varphi|^2 \,\mathrm{Id} + \delta h \nabla\varphi \otimes \nabla\varphi - (\lambda_1 - \frac{\lambda_2}{n})\mathrm{div}\, v \,\mathrm{Id} - \lambda_2 (\nabla v)^{\mathsf{S}} \\
= {} & \frac{1}{\delta}\Big(\frac{1}{2}\mathsf{p}_h |\partial_r \Phi^0|^2 \,\mathrm{Id} + h |\partial_r \Phi^0|^2 \nu \otimes \nu \\
& \quad -(\lambda_1 - \frac{\lambda_2}{n})\nu \bullet \partial_r V^0 \,\mathrm{Id} - \frac{1}{2}\lambda_2 (\nu \otimes \partial_r V^0 + \partial_r V^0 \otimes \nu)\Big) + \mathcal{O}(1) \\
\delta \Pi_\delta \nu \to {} & \frac{1}{2}\mathsf{p}_h |\partial_r \Phi^0|^2 \nu + h |\partial_r \Phi^0|^2 \nu - (\lambda_1 - \frac{\lambda_2}{n})\nu \bullet \partial_r V^0 \nu - \lambda_2 \nu \bullet \partial_r V^0 \nu \\
= {} & (\frac{1}{2}\mathsf{p}_h + h)|\partial_r \Phi^0|^2 \nu - (\lambda_1 + \frac{(n-1)\lambda_2}{n})\partial_r V^0 \;=\; 0 \\
(\delta \Pi_\delta)_{\mathsf{tan}} := {} & \delta \Pi_\delta - (\delta \Pi_\delta \nu)\nu \;=\; \delta \Pi_\delta (\mathrm{Id} - \nu \otimes \nu) \\
\to {} & \Big(\frac{1}{2}\mathsf{p}_h |\partial_r \Phi^0|^2 - (\lambda_1 - \frac{\lambda_2}{n})\nu \bullet \partial_r V^0\Big)(\mathrm{Id} - \nu \otimes \nu) \\
= {} & \Big(\frac{1}{2}\mathsf{p}_h(R^0) |\partial_r \Phi^0|^2 - \big(\lambda_1(R^0,\Phi^0) - \frac{\lambda_2(R^0,\Phi^0)}{n}\big)\nu \bullet \partial_r V^0\Big)(\mathrm{Id} - \nu \otimes \nu)
\end{aligned}$$

$$\gamma = \int_{\mathbb{R}} \Big(-\frac{1}{2}\mathsf{p}_h(R^0)|\partial_r \Phi^0|^2 + \big(\lambda_1(R^0,\Phi^0) - \frac{\lambda_2(R^0,\Phi^0)}{n}\big)\nu \bullet \partial_r V^0\Big)\, \mathrm{d}r$$

Conclusion

There are universal principles
which are applied to physical situations
and give mathematical rigorous assumptions

All the best for Dietmar Kröner