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FREE ENERGY INEQUALITY IN THE LIMIT OF PHASE TRANSITION

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Abstract. We study the convergence of diffusive interface models to sharp interface models. These diffusive phase field models consist of the conservation of mass and momentum, where the mass undergoes a phase transition. The model is based on a free energy and scaled by a small parameter $\delta > 0$ representing the thickness of the interface. For diffusive models a fundamental property is the free energy inequality. Since we consider the isothermal case, this inequality is equivalent to the well known entropy principle.

We go to the limit with this inequality depending on the scaling limit we consider. This contains cases where the entropy principle in the limit is not a standard situation. Since there is no physical constraint for the sequence approximating such limit, we are legitimized to consider the limit entropy principle, whatever it is, as a suitable inequality. Therefore we insist that this modified entropy principle is the appropriate choice.

The paper complements the results in [4]. All models deal with compressible fluids.

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1 Introduction

This paper deals with the limit of certain phase field approximations to problems with sharp interface. For the approximations a free energy inequality is satisfied and we go to the limit with this inequality. The model equations contain mass and momentum conservation. We use the notion of distributions to formulate this limit, and also the convergence is formulated in the sense of distributions.

We concentrate in this paper on examples, where the free energy is scaled by $\frac{1}{\delta}$, see (2.1) and (1.4). Here δ is the parameter, which describes the thickness of the interface. The $\frac{1}{\delta}$-scaling will have consequences for the appearance of curvature in the sharp interface limit, and it has consequences for the role of the viscosity in the formula for the surface tension. It was the achievement in the papers [17] and [18] to see this connection. An approach with the incompressible Navier-Stokes equation can be found in [1]. We mention that the first paper with a $\frac{1}{\delta}$-scaling of the free energy was [14], where the stationary problem was considered and the limit $\delta\to 0$ for the free energy functional was treated from the mathematical point of view.

In connection with the incompressible Navier-Stokes system a variety of papers deals with phase field models of Cahn-Hilliard type, see [12], [13], [1], [16], [8]. We consider a compressible Navier-Stokes system combined also with the Allen-Cahn equation, a system which has been considered in [17] and [18]. There it has been shown that such system is equivalent to the mass equation for two fluids with mass densities ϱ_δ^1 and ϱ_δ^2 combined with the momentum equation for the total mass $\varrho_\delta=\varrho_\delta^1+\varrho_\delta^2$ of this mixture,

$$\begin{aligned}
&\partial_t\varrho_\delta^1+\operatorname{div}(\varrho_\delta^1 v_\delta+J_\delta)=\boldsymbol{\tau}_\delta\,,\\
&\partial_t\varrho_\delta^2+\operatorname{div}(\varrho_\delta^2 v_\delta-J_\delta)=-\boldsymbol{\tau}_\delta\,,\\
&\partial_t(\varrho_\delta v_\delta)+\operatorname{div}(\varrho_\delta v_\delta\otimes v_\delta+\mathit{\Pi}_\delta)=\mathbf{f}_\delta\,,
\end{aligned}\tag{1.1}$$

where $\boldsymbol{\tau}_\delta$ is a reaction rate, J_δ a mass flux, and $\mathbf{f}_\delta$ a force acting on the fluid. The tensor $\mathit{\Pi}_\delta$ contains pressures and stress tensor. Here and in the entire paper all mass densities are positive. We also account for a Cahn-Hilliard type equation if $J_\delta\neq 0$, whereas $\boldsymbol{\tau}_\delta\neq 0$ is the Allen-Cahn case. On the interface there is a mass exchange between ϱ_δ^1 and ϱ_δ^2 with a rate $\boldsymbol{\tau}_\delta-\operatorname{div}J_\delta$. If we define

$$\varphi:=\frac{\varrho_\delta^2}{\varrho_\delta},\quad 1-\varphi=\frac{\varrho_\delta^1}{\varrho_\delta},\quad \varrho_\delta=\varrho_\delta^1+\varrho_\delta^2,\tag{1.2}$$

then the first two equations in (1.1) are equivalent to the mass conservation of the total mass ϱ_δ and an Allen-Cahn, resp. a Cahn-Hilliard equation for the mass fraction φ,

$$\begin{aligned}
&\partial_t\varrho_\delta+\operatorname{div}(\varrho_\delta v_\delta)=0\,,\\
&\varrho_\delta(\partial_t\varphi+v_\delta\bullet\nabla\varphi)=\operatorname{div}J_\delta-\boldsymbol{\tau}_\delta\,,\\
&\partial_t(\varrho_\delta v_\delta)+\operatorname{div}(\varrho_\delta v_\delta\otimes v_\delta+\mathit{\Pi}_\delta)=\mathbf{f}_\delta\,.
\end{aligned}\tag{1.3}$$

Here one has to add constitutive equations for $\mathit{\Pi}_\delta$ and $\boldsymbol{\tau}_\delta$ resp. J_δ, which have to satisfy the free energy inequality, which is the entropy principle in the isothermal situation. As mentioned above the total mass ϱ_δ is conserved, since the J_δ-terms and the $\boldsymbol{\tau}_\delta$-terms have different signs in the single mass equations. Therefore these terms don't arise in the momentum equation (see [2, Section 8] and [3, Section 3]). For details see section 2, where we also state the constitutive equations for the internal free energy

$$f_\delta := \frac{1}{\delta}\varrho_\delta W(\varphi) + \delta h(\varrho_\delta)\frac{|\nabla\varphi|^2}{2} + U(\varrho_\delta,\varphi), \tag{1.4}$$

which is the same in all examples 2.5, 2.6, 2.7, and 2.8, that is in all examples we use such a free energy and we assume that a mass transition occurs on the interface, that is, it is a general assumption in this paper that

$$\mathbf{m}^0 \not\equiv 0 \text{ on } \varGamma, \tag{1.5}$$

a definition of $\mathbf{m}^0$ one finds in (4.6). There is another case where the two media do not interact at the free boundary ($\mathbf{m}^0=0$), a case which we do not consider in this paper. Here we treat the case of a phase change, that is generically $\mathbf{m}^0$ is nonzero and accidentally there might be points at which it is zero. Also we assume that all arising mass densities are positive, see (4.2) and (4.4).

It is the common knowledge that models for concrete materials satisfy the basic conservation laws (1.1), that is the mass-momentum system, and the free energy inequality. This is the isothermal version of the entropy principle. As stated in 2.1 this principle consists of a single inequality. It is the purpose of this paper to go with the free energy inequality to the limit $\delta\to 0$, and this way we will arrive at free energy inequalities for the sharp interface problems.

To make this procedure transparent, we work with distributions (see [2] for a mathematical and [5] for a physical introduction to the problems considered here). We think that this is the adequate formulation for the convergence procedure and also for the final physical equations. In section 4 we summarize these limit equations. For the examples 2.5 and 2.6 this already has been shown in [4], and for example 2.7 we refer to the appendix 10, and for example 2.8 to appendix 11.

In this paper all phase-field models satisfy the free energy inequality (see 2.1 for the definition and 2.2 for the result), which is a special case of the distributional version

$$\partial_t F + \operatorname{div}\varPhi - G_0 \le 0 \tag{1.6}$$

in $\mathcal{D}'(\mathcal{U})$, where F is the distributional free energy and $\varPhi$ the distributional free energy flux. The term G_0 is such that the left hand side of the inequality is an objective scalar.

If now $(F_k,\varPhi_k)$ and G_{0k} are such distributions, that is (1.6) for $k\in\mathbb{N}$ holds, which is $\partial_t F_k + \operatorname{div}\varPhi_k - G_{0k} \le 0$, and these distributions converge to F, that is $F_k\to F$ pointwise in $\mathcal{D}'(\mathcal{U})$, and similarly $\varPhi_k\to\varPhi$, $G_{0k}\to G_0$, then obviously $\partial_t F + \operatorname{div}\varPhi - G_0 \le 0$, that is the free energy inequality (1.6) is satisfied for the limit objects. Also if $\varepsilon_k F_k\to F$, $\varepsilon_k\varPhi_k\to\varPhi$, $\varepsilon_k G_{0k}\to G_0$, where ε_k are positive constants, we arrive at the inequality $\partial_t F + \operatorname{div}\varPhi - G_0 \le 0$. We only have to multiply the k-version of the inequality by the positive number ε_k.

Here we consider δ as index for a sequence, where δ describes the interface thickness. We show in sections 6, 7, 8, 9, that the terms of the free energy inequality converge for $\delta\to 0$, see the theorems 7.1 and 7.4 for example 2.5, theorem 6.2 for example 2.6, theorem 8.1 for example 2.7, and theorem 9.1 for example 2.8. These convergencies give rise to define the limit of the free energy inequality. In the examples 2.6 and 2.7 it is an equation (1.6) in the entire domain, whereas for example 2.5 one has to multiply the free energy inequality by $\frac{1}{\delta}$ to obtain the free energy inequality in a neighbourhood of the surface Γ and one obtains the usual limit for the normalized free energy in the bulks $\mathcal{U}^\alpha$, see 7.5.

Therefore our considerations give a definition of the free energy inequality in the limit case, and this definition is consistent with the free energy inequality in the diffuse case. The paper is organized as follows:

- The phase-field models (Section 2)
- Limit equations (Section 4)
- Equipartition of energy (Section 5)
- Jump in the density (Example 2.5, see Section 7)
- Allen-Cahn continuous case (Example 2.6, see Section 6)
- Cahn-Hilliard example (Example 2.7, see Section 8)
- Cahn-Hilliard example with degenerate mobility
  (Example 2.8, see Section 9)

The last four sections contain the free energy inequalities in the limit. For the Cahn-Hilliard cases the limit equations (see 4.9 and 4.11) are proved in the appendix (section 10 and 11). For the other cases the proofs are contained in [17] and [18]. Our convergence results do not contain estimates of the remainder terms.

2 Phase field models

We consider phase field models which are given by system (1.1) which is equivalent to (1.3). For the mass transitions $\boldsymbol{\tau}_\delta$ and J_δ and for the momentum matrix Π_δ we have special constitutive equations which are motivated by concrete materials. These constitutive relations are given by a free energy density, which is assumed to be a function of the densities $(\varrho_\delta^1, \varrho_\delta^2)$, or equivalently $(\varrho_\delta, \varphi)$, and the derivative $\nabla\varphi$ of the mass fraction φ, that is, the total free energy is

$$f_\delta^{tot} = f_\delta(\varrho_\delta, \varphi, \nabla\varphi) + \frac{\varrho_\delta}{2}|v_\delta|^2. \tag{2.1}$$

Connected to the internal free energy f_δ is the chemical potential μ_δ given by

$$\mu_\delta := \frac{\delta f_\delta}{\delta\varphi} = f_{\delta'\varphi} - \operatorname{div} f_{\delta'\nabla\varphi}, \tag{2.2}$$

which is defined with respect to the mass fraction φ. The free energy is subject to the main principle. This consists in the following free energy inequality, which is the entropy principle in the isothermal case.

2.1 Postulate of the free energy inequality. With the total free energy f_δ^{tot} as in (2.1) and a total free energy flux Φ_δ^{tot} we postulate the free energy inequality

$$g_\delta := \partial_t f_\delta^{tot} + \operatorname{div}\Phi_\delta^{tot} - v_\delta \bullet \mathbf{f}_\delta \leq 0 \tag{2.3}$$

for solutions of system (1.1) or the equivalent system (1.3), that is, the free energy production g_δ is nonpositive.

Since $g_\delta \leq 0$ for all observers, it follows that g_δ has to be an objective scalar. We mention that this postulate has consequences for the system. The representation of Π_δ, see (2.5), which has as part the pressure tensor $\mathbb{P}_\delta$ and the stress tensor $\mathbb{S}_\delta$, is forced by the energy inequality. And, of course, there is the residual inequality for g_δ, see (2.6).

2.2 Theorem. For the free energy flux we assume

$$\Phi_\delta^{tot} = f_\delta^{tot} v_\delta + {\Pi_\delta}^{\mathrm{T}} v_\delta - \dot{\varphi} f_{\delta'\nabla\varphi} - \frac{\mu_\delta}{\varrho_\delta} J_\delta . \tag{2.4}$$

Then the postulate for solutions of system (1.1) is satisfied, if

$$\begin{aligned} \Pi_\delta &= \mathbb{P}_\delta - \mathbb{S}_\delta , \\ \mathbb{P}_\delta &= \mathrm{p}_{f_\delta} \mathbb{I} + \nabla\varphi \otimes f_{\delta'\nabla\varphi} \\ &= \mathrm{p}_{f_\delta}(\varrho_\delta, \varphi, \nabla\varphi)\mathbb{I} + \nabla\varphi \otimes f_{\delta'\nabla\varphi}(\varrho_\delta, \varphi, \nabla\varphi), \end{aligned} \tag{2.5}$$

and if the right-hand side of (2.6) is nonpositive, that is the residual inequality

$$g_\delta = -\mathrm{D}v_\delta : \mathbb{S}_\delta - \nabla\left(\frac{\mu_\delta}{\varrho_\delta}\right) \bullet J_\delta - \frac{\mu_\delta}{\varrho_\delta} \cdot \boldsymbol{\tau}_\delta \leq 0 \tag{2.6}$$

is satisfied.

Here $\dot{\xi} := (\partial_t + v_\delta \bullet \nabla)\xi$ for any function ξ. If ξ is an objective scalar also $\dot{\xi}$ is. Also

$$\begin{aligned} \mathrm{p}_\xi(\varrho) &:= \varrho \xi_{'\varrho}(\varrho) - \xi(\varrho), \\ \mathrm{e}_\xi(\varrho) &:= \varrho \xi_{'\varrho}(\varrho) + \xi(\varrho) \end{aligned} \tag{2.7}$$

for any function $\varrho \mapsto \xi(\varrho)$ and similar for functions also depending in addition on other arguments.

Proof. This has been shown in the appendix of [17]. We repeat the proof here. Denoting the kinetic free energy by

$$f_\delta^{kin} = \frac{\varrho_\delta}{2} |v_\delta|^2 \tag{2.8}$$

a simple but lengthy computation shows for solutions of (1.1)

$$\partial_t f_\delta^{kin} + \mathrm{div}(f_\delta^{kin} v_\delta + {\Pi_\delta}^{\mathrm{T}} v_\delta) - v_\delta \bullet \mathbf{f}_\delta = \mathrm{D}v_\delta : \Pi_\delta ,$$

therefore we obtain for arbitrary flux $\Phi_\delta^{tot} = f_\delta^{tot} v_\delta + \Phi_\delta$

$$\begin{aligned} g_\delta &:= \partial_t f_\delta^{tot} + \mathrm{div}\Phi_\delta^{tot} - v_\delta \bullet \mathbf{f}_\delta \\ &= \partial_t f_\delta^{kin} + \mathrm{div}(f_\delta^{kin} v_\delta + \Phi_\delta) - v_\delta \bullet \mathbf{f}_\delta + \partial_t f_\delta + \mathrm{div}(f_\delta v_\delta) \\ &= \mathrm{div}(\Phi_\delta - {\Pi_\delta}^{\mathrm{T}} v_\delta) + \mathrm{D}v_\delta : \Pi_\delta + (\partial_t + v_\delta \bullet \nabla) f_\delta + f_\delta \mathrm{div} v_\delta \\ &= \mathrm{div}(\Phi_\delta - {\Pi_\delta}^{\mathrm{T}} v_\delta) + \mathrm{D}v_\delta : (f_\delta \mathbb{I} + \Pi_\delta) + \dot{f}_\delta . \end{aligned}$$

Now

$$\begin{aligned}\dot{f_\delta}&=f_{\delta'\varrho_\delta}\dot\varrho_\delta+f_{\delta'\varphi}\dot\varphi+f_{\delta'\nabla\varphi}\bullet(\nabla\varphi)^{\cdot}\\&=f_{\delta'\varrho_\delta}\dot\varrho_\delta+f_{\delta'\varphi}\dot\varphi+f_{\delta'\nabla\varphi}\bullet(\nabla\dot\varphi-(\mathrm{D}v_\delta)^{\mathrm{T}}\nabla\varphi)\\&=\mathrm{div}(\dot\varphi f_{\delta'\nabla\varphi})+f_{\delta'\varrho_\delta}\dot\varrho_\delta+(f_{\delta'\varphi}-\mathrm{div}f_{\delta'\nabla\varphi})\dot\varphi-\mathrm{D}v_\delta\colon(\nabla\varphi\otimes f_{\delta'\nabla\varphi})\end{aligned}$$

and for the terms with time derivative we obtain from the differential equations

$$f_{\delta'\varrho_\delta}\dot\varrho_\delta=-f_{\delta'\varrho_\delta}\varrho_\delta\mathrm{div}v_\delta=-\mathrm{D}v_\delta\colon(\varrho_\delta f_{\delta'\varrho_\delta}\mathbb{I})$$

and with μ_δ as in the statement and (1.3) writing

$$\varrho_\delta\dot\varphi=\mathrm{div}J_\delta-\boldsymbol{\tau}_\delta \tag{2.9}$$

we calculate

$$\begin{aligned}(f_{\delta'\varphi}-\mathrm{div}f_{\delta'\nabla\varphi})\dot\varphi&=\mu_\delta\dot\varphi=\frac{\mu_\delta}{\varrho_\delta}(\mathrm{div}J_\delta-\boldsymbol{\tau}_\delta)\\&=\mathrm{div}(\frac{\mu_\delta}{\varrho_\delta}J_\delta)-\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta-\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta.\end{aligned}$$

Altogether

$$\begin{aligned}g_\delta&=\mathrm{div}(\varPhi_\delta-{\varPi_\delta}^{\mathrm{T}}v_\delta)+\mathrm{D}v_\delta\colon(f_\delta\mathbb{I}+\varPi_\delta)\\&+\mathrm{div}(\dot\varphi f_{\delta'\nabla\varphi})-\mathrm{D}v_\delta\colon(\varrho_\delta f_{\delta'\varrho_\delta}\mathbb{I})\\&+\mathrm{div}(\frac{\mu_\delta}{\varrho_\delta}J_\delta)-\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta-\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta-\mathrm{D}v_\delta\colon(\nabla\varphi\otimes f_{\delta'\nabla\varphi})\\&=\mathrm{div}\Big(\varPhi_\delta-{\varPi_\delta}^{\mathrm{T}}v_\delta+\dot\varphi f_{\delta'\nabla\varphi}+\frac{\mu_\delta}{\varrho_\delta}J_\delta\Big)\\&+\mathrm{D}v_\delta\colon\Big((f_\delta-\varrho_\delta f_{\delta'\varrho_\delta})\mathbb{I}-\nabla\varphi\otimes f_{\delta'\nabla\varphi}+\varPi_\delta\Big)-\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta-\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta.\end{aligned}$$

To make the divergence term disappear, we define

$$\varPhi_\delta={\varPi_\delta}^{\mathrm{T}}v_\delta-\dot\varphi f_{\delta'\nabla\varphi}-\frac{\mu_\delta}{\varrho_\delta}J_\delta.$$

The remaining should be nonpositive. Therefore let $\varPi_\delta=\mathbb{P}_\delta-\mathbb{S}_\delta$, where the pressure term is

$$\mathbb{P}_\delta:=(\varrho_\delta f_{\delta'\varrho_\delta}-f_\delta)\mathbb{I}+\nabla\varphi\otimes f_{\delta'\nabla\varphi},$$

so that we obtain

$$g_\delta=-\mathrm{D}v_\delta\colon\mathbb{S}_\delta-\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta-\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta,$$

which is the residual free energy inequality. □

The following remark is from [3]. It states that the free energy can also be considered as a function of $(\varrho_\delta^1,\varrho_\delta^2)$ instead of (ϱ_δ,φ), and a certain algebraic linear combination of $\nabla\varrho_\delta^1$ and $\nabla\varrho_\delta^2$.

2.3 Remark. Let $\widetilde{f}_\delta$ be the function

$$\bar{f}_\delta(\varrho_\delta^1,\varrho_\delta^2,\nabla\varrho_\delta^1,\nabla\varrho_\delta^2):=f_\delta(\varrho_\delta^1+\varrho_\delta^2,\frac{\varrho_\delta^2}{\varrho_\delta^1+\varrho_\delta^2},\frac{\varrho_\delta^1}{(\varrho_\delta^1+\varrho_\delta^2)^2}\nabla\varrho_\delta^2-\frac{\varrho_\delta^2}{(\varrho_\delta^1+\varrho_\delta^2)^2}\nabla\varrho_\delta^1)$$

The definition is consistent, since $\varphi=\frac{\varrho_\delta^2}{\varrho_\delta^1+\varrho_\delta^2}$ therefore

$$\nabla\varphi=\frac{\varrho_\delta^1}{(\varrho_\delta^1+\varrho_\delta^2)^2}\nabla\varrho_\delta^2-\frac{\varrho_\delta^2}{(\varrho_\delta^1+\varrho_\delta^2)^2}\nabla\varrho_\delta^1.$$

If we define the chemical potentials by $\mu_k:=\frac{\delta\bar{f}_\delta}{\delta\varrho_\delta^k}=\bar{f}_{\delta'\varrho_\delta^k}-\operatorname{div}\bar{f}_{\delta'\nabla\varrho_\delta^k}$ we see after a simple computation that

$$\mu_1=f_{\delta'\varrho_\delta}-\frac{\mu_\delta}{\varrho_\delta}\varphi,\quad \mu_2=f_{\delta'\varrho_\delta}+\frac{\mu_\delta}{\varrho_\delta}(1-\varphi),\quad \mu_2-\mu_1=\frac{\mu_\delta}{\varrho_\delta},$$

hence in the free energy production

$$\frac{\mu_\delta}{\varrho_\delta}\boldsymbol{\tau}_\delta=\mu_2\boldsymbol{\tau}_\delta-\mu_1\boldsymbol{\tau}_\delta=-\sum_k\mu_k\boldsymbol{\tau}_\delta^k,$$

$$\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta=\nabla\mu_2\bullet J_\delta-\nabla\mu_1\bullet J_\delta=-\sum_k\nabla\mu_k\bullet J_\delta^k,$$

if $\boldsymbol{\tau}_\delta^1=\boldsymbol{\tau}_\delta$, $\boldsymbol{\tau}_\delta^2=-\boldsymbol{\tau}_\delta$ and $J_\delta^1=J_\delta$, $J_\delta^2=-J_\delta$ are the productions and the fluxes in the mass equation for ϱ_δ^1 and ϱ_δ^2.

We now consider models for concrete materials satisfy the basic conservation laws (1.1), that is the mass-momentum system, and the entropy principle, that is in the isothermal case the free energy inequality. As stated in the above theorem 2.2, this principle consists of a single inequality $g_\delta\leq 0$. We split this inequality into several terms and assuming the nonnegativity of each single contribution. This reflects the property of the material under consideration. Here we assume that

$$\mathrm{D}v_\delta{:}\mathbb{S}_\delta\geq 0, \tag{2.10}$$

$$\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta+\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta\geq 0. \tag{2.11}$$

This entropy principle influences also the terms in the conservation laws, which are the essential equations describing the material. The models discussed in this paper satisfy the following assumptions.

2.4 General model assumptions. The free energy density is given by

$$f_\delta(\varrho,\varphi,\nabla\varphi):=\frac{1}{\delta}W_0(\varrho,\varphi)+\delta h(\varrho)\frac{|\nabla\varphi|^2}{2}+U(\varrho,\varphi), \tag{2.12}$$

with a positive function h and where W_0 satisfies

$$\begin{aligned}&W_0(\varrho,\varphi):=\varrho W(\varphi),\quad W\text{ has two local minima at 0 and 1,}\\&U_{'\varphi}(\varrho,0)=0,\quad U_{'\varphi}(\varrho,1)=0.\end{aligned} \tag{2.13}$$

There is no general assumption on the value of the minima of W. The tensor $\varPi_\delta$ has the structure $\varPi_\delta = \mathbb{P}_\delta - \mathbb{S}_\delta$ with a pressure $\mathbb{P}_\delta$ and a stress tensor $\mathbb{S}_\delta$ as in (2.5), where $\mathbb{P}_\delta = \mathrm{p}_{f_\delta}\mathbb{I} + \nabla\varphi \otimes f_{\delta' \nabla\varphi}$ has because of (2.13) now the structure

$$\mathbb{P}_\delta = (\varrho_\delta U'_\varrho - U)\mathbb{I} + \delta(\varrho_\delta h'_\varrho - h)\frac{|\nabla\varphi|^2}{2}\mathbb{I} + \delta h \nabla\varphi \otimes \nabla\varphi. \tag{2.14}$$

The stress tensor $\mathbb{S}_\delta$ in (2.5) is given as

$$\mathbb{S}_\delta \equiv \mathbb{S}_\delta(\varrho_\delta, \varphi, (\mathrm{D}v)^{\mathrm{S}}) = a_{1\delta}(\mathrm{div} v)\mathbb{I} + a_{2\delta}\Big((\mathrm{D}v)^{\mathrm{S}} - \frac{1}{n}(\mathrm{div} v)\mathbb{I}\Big), \tag{2.15}$$

where the inequality (2.10) holds, if the Lamé coefficients satisfy

$$a_{1\delta} \equiv a_{1\delta}(\varrho_\delta, \varphi) > 0, \quad a_{2\delta} \equiv a_{2\delta}(\varrho_\delta, \varphi) > 0. \tag{2.16}$$

Therefore in the following examples one has only to verify the inequality (2.11).

The first example is taken from [18], and serves for a jump in the densities at the interface.

2.5 Example (Jump case). Besides the general assumptions in 2.4 one assumes that

$$\begin{aligned} &W \text{ has two local minima at 0 and 1,} \\ &W(0) \neq W(1). \end{aligned} \tag{2.17}$$

The inequality (2.11) is satisfied, if

$$\boldsymbol{\tau}_\delta := \eta_\delta \mu_\delta, \quad \eta_\delta \equiv \eta_\delta(\varrho_\delta, \varphi) = \frac{\eta_0(\varrho_\delta)}{\delta} > 0, \quad J_\delta := 0. \tag{2.18}$$

There is a strong connection between this case $W(0) \neq W(1)$ and the function $\mathrm{e}_h(s) := h(s) + sh'(s)$, which follows from the equipartition of energy, see (5.9) and [18, Lemma 31].

The second example is the same as the above except that the minima of the double well function have the same height. This case was considered also in [17]. It turns out, that the densities are continuous at the interface provided there is a real mass transition, see [18, Theorem 1 and Remark 2]. We remark that in the other physical case, that is, when there is no mass transition at all, the densities at the interface are arbitrarily. By (1.5) this case is not considered here.

2.6 Example (Continuous case). Besides the general assumptions in 2.4 one assumes that

$$\begin{aligned} &W \text{ has two local minima at 0 and 1,} \\ &W(0) = W(1). \end{aligned} \tag{2.19}$$

The inequality (2.11) is satisfied, if

$$\boldsymbol{\tau}_\delta := \eta_\delta \mu_\delta, \quad \eta_\delta \equiv \eta_\delta(\varrho_\delta, \varphi) = \frac{\eta_0(\varrho_\delta)}{\delta} > 0, \quad J_\delta := 0, \tag{2.20}$$

where for simplicity we set $\eta_0{=}1$. We assume further, that $\mathrm{e}_h(s){=}0$ for all s (see also the proof of 5.2), an assumption which implies $W(0){=}W(1)$, see [18, Lemma 31] and [17, Lemma 16]. It is then $s{\mapsto}sh(s)$ constant, that is,

$$h(s){=}\frac{h_0}{s} \text{ with } h_0{=}\text{const}{>}0\text{, for simplicity we set } h_0{=}1. \tag{2.21}$$

This implies that (2.14) becomes

$$\mathbb{P}_\delta{=}(\varrho_\delta U'_\varrho - U)\mathbb{I}+\frac{\delta}{\varrho}\big(\nabla\varphi\otimes\nabla\varphi-\left|\nabla\varphi\right|^2\mathbb{I}\big), \tag{2.22}$$

where the last term is trace free.

This example, treated in [17], contains the Allen-Cahn equation, which has been widely considered in literature but without the compressible fluid equations. Also the Cahn-Hilliard equation with the free energy (2.12) has been studied in literature, see [1]. Here we combine as third example the Cahn-Hilliard equation with the compressible fluid system. Hence we arrive at an example with $J_\delta{\neq}0$.

2.7 Cahn-Hilliard example. Besides the general assumptions in 2.4 one assumes that

$$\begin{aligned} &W \text{ has two local minima at 0 and 1,}\\ &W(0){=}W(1). \end{aligned} \tag{2.23}$$

The inequality (2.11) is satisfied, if

$$\begin{aligned} &J_\delta{:=}m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big), \quad m_\delta{:=}m_0(\varrho_\delta,\varphi){>}0, \quad \boldsymbol{\tau}_\delta{:=}0,\\ &\text{in particular } s{\mapsto}m_0(s,0){>}0,\ s{\mapsto}m_0(s,1){>}0. \end{aligned} \tag{2.24}$$

We also assume here, that h satisfies (2.21) as above in example 2.6, that is $\mathrm{e}_h(s){=}0$ for all s.

The Cahn-Hilliard equation has also been studied in [1, Section 4.1 case III] with a mobility of the form (2.26). It seems to be more realistic, since in this case in the limit $\delta{\to}0$ the mass equation and the momentum equation are the only bulk equations, see the papers [6] and [9].

2.8 Cahn-Hilliard example with degenerate mobility. Besides the general assumptions in 2.4 one assumes that

$$\begin{aligned} &W \text{ has two local minima at 0 and 1,}\\ &W(0){=}W(1). \end{aligned} \tag{2.25}$$

The inequality (2.11) is satisfied, if

$$\begin{aligned} &J_\delta{:=}m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big), \quad \boldsymbol{\tau}_\delta{:=}0,\\ &m_\delta{\equiv}m_\delta(\varrho_\delta,\varphi){=}\frac{1}{\delta}m_0(\varrho_\delta)V(\varphi) \text{ with } V(\varphi){:=}\varphi^2(1-\varphi)^2 \end{aligned} \tag{2.26}$$

We also assume here, that h satifies (2.21) as above in example 2.6, that is $\mathrm{e}_h(s){=}0$ for all s.

3 General inner and outer expansion

Let $\Gamma\subset\mathcal{U}$ be a smooth evolving surface without boundary in the open set $\mathcal{U}\subset\mathbb{R}\times\mathbb{R}^n$. We write the coordinates (t,x) near the set Γ as coordinates (t,y,r), where r is the signed distance from Γ_t (say, positive in direction of the normal), that is

$$x=y+r\nu(t,y),\quad y\in\Gamma_t \text{ the orthogonal projection of } x,$$
$$\Gamma_t:=\{y\in\mathbb{R}^n\ ;\ (t,y)\in\Gamma\},\quad \nu(t,y)\in\mathbb{R}^n \text{ a unit normal vector of } \Gamma_t.$$

A small number $\delta>0$ is given. It is connected with a phase field approximation and describes the thickness of the interface. This means, that the quantities of the diffusive interface make it main changes in the domain

$$\Gamma_\delta:=\{(t,y+r\nu(t,y))\ ;\ (t,y)\in\Gamma,|r|\le\varepsilon_\delta\}. \tag{3.1}$$

We want to define the inner and outer expansion of a given function $u{:}\mathcal{U}\to\mathbb{R}^N$. We write $(t,x)\mapsto u(t,x)$ in terms of the local coordinates (t,y,z) as a function U (here: U is the inner variable belonging to u and not a part of the free energy f_δ), therefore $(t,y,z)\mapsto U(t,y,z)$ is given by

$$u(t,x)=U(t,y,z) \text{ for } x=y+\delta z\nu(t,y) \tag{3.2}$$

with

$$z:=\frac{r}{\delta},\quad |r|\le\varepsilon_\delta\to 0,\quad |z|\le z_\delta:=\frac{\varepsilon_\delta}{\delta}\to\infty,$$

that is,

$$\varepsilon_\delta\to 0 \text{ and } z_\delta=\frac{\varepsilon_\delta}{\delta}\to\infty \text{ as } \delta\to 0. \tag{3.3}$$

(It should be noted: In this section U is the inner variable belonging to u and not a part of the free energy f_δ.) The function U is called the inner variable with an δ expansion, that is, with an integer $j\ge 0$

$$\begin{aligned}&U=U_\delta=\frac{1}{\delta^j}U^{-j}+\cdots+\frac{1}{\delta}U^{-1}+U^0+\delta U^1+\cdots+\delta^m U^m+o(\delta^m)\\&\text{in } \{(t,y,z)\ ;\ (t,y)\in\Gamma,|z|\le\frac{\varepsilon_\delta}{\delta}\},\end{aligned} \tag{3.4}$$

and the function u is called the outer variable, that is (we do not need here higher expansions in δ),

$$\begin{aligned}&u=u_\delta=u^0+o(1)\\&\text{in } \{(t,x)\ ;\ x=y+r\nu(t,y),|r|\ge\frac{\varepsilon_\delta}{2}\}.\end{aligned} \tag{3.5}$$

We mean this in the following strong sense.

3.1 Error term. The functions $u=u_\delta$ and $U=U_\delta$ depend on δ in the way that for every k as $\delta\to 0$

$$\left\|u_\delta-u^0\right\|_{C^k(\{\mathrm{dist}((t,x),\Gamma)\ge\frac{\varepsilon_\delta}{2}\})}\to 0,$$

$$\frac{1}{\delta^m}\left\|U_\delta-(\delta^{-j}U^{-j}+\ldots+U^0+\delta U^1+\ldots+\delta^m U^m)\right\|_{C^k(\{|z|\le\frac{\varepsilon_\delta}{\delta}\})}\to 0.$$

Here the function u^0 is defined in $\mathcal{U}\setminus\Gamma$, that is outside Γ, and $U^{-j},\ldots,U^0,\ldots,U^m$ are defined in $\Gamma\times\mathbb{R}=\{(t,y,z)\ ;\ (t,y)\in\Gamma, z\in\mathbb{R}\}$.

Therefore we have a set, where the expansion of u_δ and U_δ can be compared, and this common domain is characterized by

$$\frac{\varepsilon_\delta}{2}\le|r|\le\varepsilon_\delta \quad\text{or}\quad \frac{z_\delta}{2}\le|z|\le z_\delta=\frac{\varepsilon_\delta}{\delta}.$$

In this common domain we can compare u_δ and U_δ, see the identity (3.2), and this identity can also be written as

$$u_\delta(t,y+\delta z\nu(t,y))=U_\delta(t,y,z) \text{ for } \frac{z_\delta}{2}\le|z|\le z_\delta. \tag{3.6}$$

We now assume $j=0$. Letting $\delta\to 0$ (then $z_\delta\to\infty$, $\varepsilon_\delta\to 0$) and using 3.1 we get the following boundary conditions

$$\begin{aligned}U^0(t,y,+\infty)&=u^0_+(t,y),\\ U^0(t,y,-\infty)&=u^0_-(t,y).\end{aligned} \tag{3.7}$$

Here u^0_+ is u^0 restricted to $\{r>0\}$ and u^0_- is u^0 restricted to $\{r<0\}$. Both u^0_+ and u^0_- are evaluated at $\{r=0\}$. If we perform the derivative with respect to z of the above formula (3.6) we derive

$$\delta\partial_{\nu(t,y)}u_\delta(t,y+\delta z\nu(t,y))=\partial_z U_\delta(t,y,z) \text{ for } \frac{z_\delta}{2}\le|z|\le z_\delta, \tag{3.8}$$

where $\partial_e v(t,x):=e\bullet\nabla v(t,x)$ denotes the directional derivative in direction e. Inserting $\partial_e u_\delta=\partial_e u^0+o(1)$ and $\partial_z U_\delta=\partial_z U^0+\delta\partial_z U^1+o(\delta)$ and taking for example $z=z_\delta$ we get from (3.8) that

$$\begin{aligned}&\delta\partial_{\nu(t,y)}u^0(t,y+\delta z_\delta\nu(t,y))+o(\delta)\\ &=\partial_z U^0(t,y,z_\delta)+\delta\partial_z U^1(t,y,z_\delta)+o(\delta).\end{aligned} \tag{3.9}$$

Letting $\delta\to 0$ we derive $\partial_z U^0(t,y,+\infty)=0$. Similar by taking $z=-z_\delta$ we derive $\partial_z U^0(t,y,-\infty)=0$. Hence

$$\partial_z U^0(t,y,+\infty)=0,\quad \partial_z U^0(t,y,-\infty)=0. \tag{3.10}$$

To proceed further we make the

3.2 Assumption. We consider a function $V=\partial_z^i U^l$, $i\ge 0$, such that the limit $V(t,y,\pm\infty)$ exists. The assumption then is, that V satisfies for every k

$$V(t,y,z)-V(t,y,+\infty)=\mathcal{O}(z^{-k}) \text{ as } z\to\infty,$$

and similarly for $z\to-\infty$. This is satisfied, if $z\mapsto V(t,y,z)-V(t,y,+\infty)$ for example decays exponentially at $+\infty$ and similarly at $-\infty$.

We want to use this assumption in equation (3.9). Therefore we make another

3.3 Assumption. We assume that we can choose, for example,

$$\varepsilon_\delta=\delta^\alpha \text{ with } 0<\alpha<1. \tag{3.11}$$

Then it is satisfied that $\varepsilon_\delta=\delta^\alpha\to 0$ and $z_\delta=\delta^{\alpha-1}\to\infty$ as $\delta\to 0$.

We use these assumptions. By identity (3.10) we can set $V:=\partial_z U^0$ and obtain $\partial_z U^0(t,y,z_\delta)=\mathcal{O}(z_\delta^{-k})=\mathcal{O}(\delta^{k(1-\alpha)})$. Now choosing k with $k(1-\alpha)>1$ this implies $\partial_z U^0(t,y,z_\delta)=o(\delta)$ and we obtain for $z=z_\delta$ from (3.9) that

$$\delta\partial_{\nu(t,y)}u^0(t,y+\delta z_\delta\nu(t,y))=\delta\partial_z U^1(t,y,z_\delta)+o(\delta)$$

and dividing the identity by δ

$$\partial_{\nu(t,y)}u^0(t,y+\delta z_\delta\nu(t,y))=\partial_z U^1(t,y,z_\delta)+o(1).$$

Similar formula we get for $z=-z_\delta\to-\infty$. Thus by letting $\delta\to 0$ we derive

$$\begin{aligned}\partial_z U^1(t,y,+\infty)&=\partial_{\nu(t,y)}u^0_+(t,y),\\ \partial_z U^1(t,y,-\infty)&=\partial_{\nu(t,y)}u^0_-(t,y).\end{aligned}\tag{3.12}$$

The boundary conditions shown so far, that is (3.7) and (3.12), are the boundary conditions which are used in this paper. Now let j be arbitrary. We mention that the choice of z_δ in 3.3 has the following consequence.

3.4 Remark. Let $j>0$. If $U=U_\delta$ satisfies (3.4) and $U^l(\pm\infty)=0$ for $l=-j,\dots,-1$, then with z_δ as in 3.3

$$\left\|\delta^{-j}U^{-j}\right\|_{C^0(\{\frac{z_\delta}{2}\le|z|\le z_\delta\})}\to 0$$

and

$$\left\|u_\delta-u^0\right\|_{C^0(\{\varepsilon_\delta\ge\operatorname{dist}((t,x),\Gamma)\ge\frac{\varepsilon_\delta}{2}\})}\to 0$$

as $\delta\to 0$. Therefore, if we assume that u_δ satisfies reasonable differential equations, the conclusion (3.5) is satisfied (for $k=0$).

Proof. If U_δ satisfies (3.4), then

$$U_\delta-(\delta^{-j}U^{-j}+\dots+\delta^{-1}U^{-1}+U^0)\to 0$$

uniformly in $\{|z|\le z_\delta\}$. Since $U^l(\pm\infty)=0$ for $l<0$, we conclude from assumption 3.2 for $V=U^l$ that $|U^l(z)|\le C|z|^{-k}$ for all k, hence for $\frac{z_\delta}{2}\le|z|\le z_\delta$ and $z_\delta=\delta^{\alpha-1}$ as in 3.3

$$\delta^l|U^l(z)|\le C\delta^l|z_\delta|^{-k}\le 2^kC\delta^{l+k(1-\alpha)}\to 0$$

as $\delta\to 0$, if k is large enough. Consequently $U_\delta-U^0\to 0$ in $C^0(\{\frac{z_\delta}{2}\le|z|\le z_\delta\})$, therefore it follows again that U^0 has boundary data given by u^0 and that the assertion is true. □

One can get more results on boundary data by performing similar to (3.8) the m-th derivative with respect to z of the identity (3.6). One obtains

$$\delta^m\partial^m_{\nu(t,y)}u_\delta(t,y+\delta z\nu(t,y))=\partial^m_z U_\delta(t,y,z),\tag{3.13}$$

where $\partial^m_e v(t,x):=\mathrm{D}^m v(t,x)(e,\dots,e)$. Note, that this identity is valid only in a z-strip depending on δ. By inserting the outer and inner expansion one gets

$$\begin{aligned}&\delta^m\partial^m_{\nu(t,y)}u^0(t,y+\delta z\nu(t,y))+o(\delta^m)\\ &=\partial^m_z U^0(t,y,z)+\delta\partial^m_z U^1(t,y,z)+\dots+\delta^m\partial^m_z U^m(t,y,z)+o(\delta^m).\end{aligned}$$

By taking inductively for V the functions $\partial_z^m U^j$ for $j{=}0,\ldots,m-1$, and choosing k large enough, one arrives similar as in the above procedure for $z{=}z_\delta$ at

$$\partial^m_{\nu(t,y)} u^0(t,y+\delta z_\delta \nu(t,y)) = \partial_z^m U^m(t,y,z_\delta) + \mathcal{O}(1), \tag{3.14}$$

and similar for $z{=}-z_\delta$. Therefore

3.5 Boundary conditions. For each $m \geq 1$

$$\begin{aligned}
&\partial_z^m U^j(t,y,\pm\infty)=0 \text{ for } 0\leq j<m,\\
&\partial_z^m U^m(t,y,+\infty)=\partial^m_{\nu(t,y)} u^0_+(t,y),\\
&\partial_z^m U^m(t,y,-\infty)=\partial^m_{\nu(t,y)} u^0_-(t,y).
\end{aligned}$$

Therefore, the boundary conditions for the inner expansions depend only on the zeroth order of the outer expansion. We mention, that the boundary conditions can be integrated to obtain a polynomial growth of the function U^m. For $m{=}1$ one gets

3.6 Lemma.

$$\begin{aligned}
&U^1(t,y,z)=U^1(t,y,0)+z\partial_{\nu(t,y)} u^0_+(t,y)+\mathcal{O}(z) \text{ as } z\to\infty,\\
&U^1(t,y,z)=U^1(t,y,0)+z\partial_{\nu(t,y)} u^0_-(t,y)+\mathcal{O}(|z|) \text{ as } z\to-\infty.
\end{aligned}$$

Proof. For $z>0$

$$U^1(t,y,z)-U^1(t,y,0)=\int_0^z \partial_z U^1(t,y,s)\mathrm{d}s = z\Big(\frac{1}{z}\int_0^z f(s)\mathrm{d}s\Big),$$

if $f(s){:=}\partial_z U^1(t,y,s)$. Now we have the well known result

$$\frac{1}{z}\int_0^z f(s)\mathrm{d}s \to f(+\infty) \text{ as } z\to\infty, \tag{3.15}$$

if the limit $f(+\infty)$ exists. Clearly, from the fact that $f(s)\to f(+\infty)$ as $s\to\infty$ (we assume that this converges uniformly), we conclude that

$$|f(s)-f(+\infty)|\leq\varepsilon \text{ for } s\geq\kappa_\varepsilon.$$

Hence for $z\geq\kappa_\varepsilon$

$$\left|\frac{1}{z}\int_0^z (f(s)-f(+\infty))\mathrm{d}s\right| \leq \frac{1}{z}\int_0^{\kappa_\varepsilon} |f(s)-f(+\infty)|\mathrm{d}s + \frac{1}{z}\int_{\kappa_\varepsilon}^z |f(s)-f(+\infty)|\mathrm{d}s.$$

The first term is bounded by $\frac{2}{z}\kappa_\varepsilon \sup|f|$, which tends to 0 as $z\to\infty$, and the second term is estimated by ε. □

4 Limit equations

The basis for the pointwise convergence of the phase-field approximation is the distributional formulation of the limit equation, which here requires the measures $\boldsymbol{\mu}_{\mathcal{U}^\alpha}$ and $\boldsymbol{\mu}_\Gamma$ in a domain $\mathcal{U}{=}\mathcal{U}^1\cup\Gamma\cup\mathcal{U}^2\subset\mathbb{R}\times\mathbb{R}^n$, where Γ is an interface of the surrounding domains $\mathcal{U}^1$ and $\mathcal{U}^2$, which are defined by

$$\begin{aligned}
&\boldsymbol{\mu}_{\mathcal{U}^\alpha}(E){:=}\mathrm{L}_{n+1}(E\cap\mathcal{U}^\alpha) \text{ for } \alpha{=}1{,}2,\\
&\boldsymbol{\mu}_\Gamma(E){:=}\int_{\mathbb{R}}\mathrm{H}_{n-1}(\{x{\in}\Gamma_t\ ;\ (t{,}x){\in}E\})\mathrm{dL}_1(t)
\end{aligned}\tag{4.1}$$

for Borel sets $E\subset\mathcal{U}$. Therefore

$$\boldsymbol{\mu}_{\mathcal{U}^\alpha}{=}\mathrm{L}_{n+1}\llcorner\mathcal{U}^\alpha \quad\text{and}\quad \boldsymbol{\mu}_\Gamma{=}(\mathrm{L}_1\times\mathrm{H}_{n-1})\llcorner\Gamma.$$

To these measures there correspond in a unique way distributions, which are defined for test functions $\zeta{\in}C_0^\infty(\mathcal{U})$ by

$$\begin{aligned}
&\langle\zeta{,}\boldsymbol{\mu}_{\mathcal{U}^\alpha}\rangle{=}\int_{\mathcal{U}^\alpha}\zeta\,\mathrm{dL}_{n+1}{=}\int_{\mathbb{R}}\int_{\mathcal{U}_t^\alpha}\zeta(t{,}x)\mathrm{dL}_n(x)\mathrm{dL}_1(t),\\
&\langle\zeta{,}\boldsymbol{\mu}_\Gamma\rangle{=}\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t{,}x)\mathrm{dH}_{n-1}(x)\mathrm{dL}_1(t).
\end{aligned}$$

Here $\Gamma_t{:=}\{x\ ;\ (t{,}x){\in}\Gamma\}$ and $\mathcal{U}_t^\alpha{:=}\{x\ ;\ (t{,}x){\in}\mathcal{U}^\alpha\}$.

With respect to these distributions we formulate the limit equations for the examples 2.5, 2.6, 2.7, and 2.8.

For all models

The equation for the total mass and for the momentum

$$\begin{aligned}
&\partial_t\varrho_\delta+\mathrm{div}(\varrho_\delta v_\delta){=}0,\\
&\partial_t(\varrho_\delta v_\delta)+\mathrm{div}(\varrho_\delta v_\delta\otimes v_\delta+\Pi_\delta){=}\mathbf{f}_\delta
\end{aligned}$$

are the same in all models considered in this paper. They are the first and third equation of (1.3). The equation of the total mass is the sum of the two mass equations of system (1.1). We assume that as $\delta{\to}0$, see section 3, that

$$\begin{aligned}
&\varrho_\delta{=}\varrho^\alpha+o(1) \text{ locally in } \mathcal{U}^\alpha \text{ with } \varrho^\alpha{>}0,\\
&v_\delta{=}v^\alpha+o(1) \text{ locally in } \mathcal{U}^\alpha,\\
&\mu_\delta{=}\mu^\alpha+o(1) \text{ locally in } \mathcal{U}^\alpha,\\
&\mathbf{f}_\delta{=}\mathbf{f}^\alpha+o(1) \text{ locally in } \mathcal{U}^\alpha,\\
&a_{k\delta}{=}a_k+o(1) \text{ in arguments in } \mathbb{R}^2 \text{ for } k{=}1{,}2,
\end{aligned}\tag{4.2}$$

where a_k are assumed to be smooth. Moreover, we define

$$\begin{aligned}\widehat{f}^1(s)&:=U(s,0),\quad \widehat{f}^2(s):=U(s,1),\\ f^1&=\widehat{f}^1(\varrho^1),\quad f^2=\widehat{f}^2(\varrho^2).\end{aligned}\tag{4.3}$$

the free energy in the bulk regions. For the inner quantities we assume

$$\begin{aligned}&\varphi=\Phi^0+\delta\Phi^1+o(\delta) \text{ locally in } \Gamma\times\mathbb{R},\\ &\varrho_\delta=R^0+\delta R^1+o(\delta) \text{ locally in } \Gamma\times\mathbb{R} \text{ with } R^0>0,\\ &v_\delta=V^0+\delta V^1+o(\delta) \text{ locally in } \Gamma\times\mathbb{R},\\ &\mu_\delta=M^0+\delta M^1+o(\delta) \text{ locally in } \Gamma\times\mathbb{R}.\end{aligned}\tag{4.4}$$

With this we obtain the following two basic properties.

4.1 Limit mass equation. For all models the equation for the total mass converges as $\delta\to 0$ to the following distributional equation

$$\partial_t\Big(\sum_\alpha \varrho^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big)+\operatorname{div}\Big(\sum_\alpha \varrho^\alpha v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big)=0.\tag{4.5}$$

This equation is equivalent to the strong version

$$\begin{aligned}&\partial_t\varrho^\alpha+\operatorname{div}(\varrho^\alpha v^\alpha)=0 \text{ in } \mathcal{U}^\alpha,\\ &\mathbf{m}^0:=\varrho^1(v_\Gamma-v^1)\bullet\nu=\varrho^2(v_\Gamma-v^2)\bullet\nu \text{ on } \Gamma,\end{aligned}\tag{4.6}$$

where $\nu=\nu_{\mathcal{U}^1}=-\nu_{\mathcal{U}^2}$ for $\alpha=1,2$ on Γ.

Proof. The equation for the total mass of the phase field problem reads in the weak sense

$$\int_{\mathcal{U}}\Big(\partial_t\zeta\cdot\varrho_\delta+\nabla\zeta\bullet(\varrho_\delta v_\delta)\Big)\mathrm{dL}_{n+1}=0\tag{4.7}$$

for $\zeta\in C_0^\infty(\mathcal{U})$. First we take local versions of the test function, that is with a C_0^∞-function ξ around the free boundary Γ we set $\zeta(t,x)=\xi(t,y,z)$. For the definition of the local coordinates see section 3. One obtains from (4.7) that

$$\begin{aligned}0&=\int_{\mathcal{U}}\Big((\partial_t^\Gamma\xi-\frac{1}{\delta}v_\Gamma\bullet\nu\partial_z\xi)\varrho+(\nabla^\Gamma\xi+\frac{1}{\delta}\partial_z\xi\nu)\bullet(\varrho v)\Big)\mathrm{dL}_{n+1}\\ &=\frac{1}{\delta}\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\Big\{\int_{\Gamma_t}\Big(-v_\Gamma\bullet\nu\partial_z\xi\cdot R^0+\partial_z\xi\nu\bullet(R^0V^0)+\mathcal{O}(\delta)\Big)\\ &\qquad\cdot(1+\mathcal{O}(|r|))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}r\,\mathrm{d}t\\ &=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\ \partial_z\xi\cdot R^0\cdot(V^0-v_\Gamma)\bullet\nu+\mathcal{O}(\delta)\Big)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t\,.\end{aligned}$$

Letting $\delta\to 0$ one gets

$$0=\int_{\mathbb{R}}\int_{-\infty}^{+\infty}\int_{\Gamma_t}\partial_z\xi\cdot R^0\cdot(V^0-v_\Gamma)\bullet\nu\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t$$

and it follows $\partial_z(R^0(V^0-v_\Gamma)\bullet\nu)=0$. This is a consequence in local coordinates. Assume (4.2) and (4.4). Then for $(t,y)\in\Gamma$ we have in local coordinates z

$$\partial_z(R^0\Lambda^0)=0\quad\text{for all } z\in\mathbb{R}\,, \tag{4.8}$$

where

$$\Lambda^0:=(v_\Gamma-V^0)\bullet\nu\,. \tag{4.9}$$

The boundary conditions for are (not writing the arguments (t,y))

$$\Lambda^0(-\infty)=(v_\Gamma-v^1)\bullet\nu\,,\quad \Lambda^0(+\infty)=(v_\Gamma-v^2)\bullet\nu\,.$$

□

4.2 Limit momentum equation. Also for all models the momentum equation converges as $\delta\to 0$ to the following distributional equation

$$\partial_t\Big(\sum_\alpha\varrho^\alpha v^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big)+\mathrm{div}\Big(\sum_\alpha\big(\varrho^\alpha v^\alpha v^{\alpha\,\mathrm{T}}+\Pi^\alpha\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha}+\Pi^s\boldsymbol{\mu}_\Gamma\Big) =\sum_\alpha\mathbf{f}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}\,, \tag{4.10}$$

where

$$\begin{aligned}
&\Pi^\alpha:=p^\alpha\mathbb{I}-\mathbb{S}^\alpha,\quad p^\alpha=\varrho^\alpha\widehat{f}^\alpha_{\prime\varrho}(\varrho^\alpha)-\widehat{f}^\alpha(\varrho^\alpha),\\
&\mathbb{S}^\alpha:=a_1^\alpha(\varrho^\alpha)(\mathrm{div}v^\alpha)\mathbb{I}+a_2^\alpha(\varrho^\alpha)\Big((\mathrm{D}v^\alpha)^{\mathrm{S}}-\frac{1}{n}(\mathrm{div}v^\alpha)\mathbb{I}\Big),\\
&a_k^1(s)=a_k(s,0),\quad a_k^2(s)=a_k(s,1)\quad\text{for } k=1,2,\\
&\Pi^s=-\boldsymbol{\gamma}(\mathbb{I}-\nu\otimes\nu),
\end{aligned} \tag{4.11}$$

where $\boldsymbol{\gamma}$ is given by (4.13). The equation (4.10) is equivalent to the strong version

$$\begin{aligned}
&\partial_t\big(\varrho^\alpha v^\alpha\big)+\mathrm{div}\big(\varrho^\alpha v^\alpha v^{\alpha\,\mathrm{T}}+\Pi^\alpha\big)=\mathbf{f}^\alpha\quad\text{in }\mathcal{U}^\alpha,\\
&\mathrm{div}^\Gamma\Pi^s=-\sum_\alpha\varrho^\alpha(v_\Gamma-v^\alpha)\bullet\nu_{\mathcal{U}^\alpha}v^\alpha+\sum_\alpha\Pi^\alpha\nu_{\mathcal{U}^\alpha}\quad\text{on }\Gamma.
\end{aligned} \tag{4.12}$$

Proof. This is proved in [4, Section 6, in particular (50)] for the Allen-Cahn cases, in particular the representation

$$\boldsymbol{\gamma}:=\int_{-\infty}^{\infty}\big(h(R^0)|\partial_z\Phi^0|^2-a_2(R^0,\Phi^0)\partial_zV^0\bullet\nu\big)\,\mathrm{d}z. \tag{4.13}$$

However, it follows from theorem 2.2, that in all four cases the tensor Π_δ is the same (see (2.5)). Besides this the general free energy (1.4) has been used to derive for Π_δ the representation in (2.14) and (2.15). Hence in all four examples the momentum equation is the same, therefore the statement of the theorem holds also for the Cahn-Hilliard cases.

□

After these general statements we now state the special properties of the models 2.5, 2.6, 2.7, and 2.8. In particular, the surface tension $\boldsymbol{\gamma}$ is given by (4.13) and as we shall see, that $\boldsymbol{\gamma}$ has in different applications different representations, see (4.19), and see (4.27) in the cases with continuous velocity at the interface. Both formulas follow from (4.13). We mention that we assume that $\mathbf{m}^0 \neq 0$ on a dense set in Γ_t for each t (see (1.5)). This holds for the following examples.

Example 2.5

First we consider example 2.5. It is shown in [4] that under certain assumptions the quantities under the derivatives in (1.1) converge pointwise to quantities in $\mathcal{U}^\alpha$ and Γ, so that the following set of equations is satisfied.

4.3 Limit equation for the jump case 2.5. In the limit $\delta \to 0$ the distributional equations are

$$\begin{aligned} &\partial_t(\varrho^1 \boldsymbol{\mu}_{\mathcal{U}^1} + \varrho^2 \boldsymbol{\mu}_{\mathcal{U}^2}) + \operatorname{div}(\varrho^1 v^1 \boldsymbol{\mu}_{\mathcal{U}^1} + \varrho^2 v^2 \boldsymbol{\mu}_{\mathcal{U}^2}) = 0, \\ &\partial_t\Big(\sum_\alpha \varrho^\alpha v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big) + \operatorname{div}\Big(\sum_\alpha (\varrho^\alpha v^\alpha v^{\alpha\,\mathrm{T}} + \Pi^\alpha) \boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^s \boldsymbol{\mu}_\Gamma\Big) \\ &\qquad\qquad = \sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}, \end{aligned} \tag{4.14}$$

and the kinetic equations read

$$\left.\begin{aligned} v^1_{\tan} &= v^2_{\tan} \\ \varrho^1 &= \mathbf{g}_1(\mathbf{m}^0) \\ \varrho^2 &= \mathbf{g}_2(\mathbf{m}^0) \end{aligned}\right\} \text{ on } \Gamma. \tag{4.15}$$

Here the quantity Π^s is defined through the surface tension $\boldsymbol{\gamma}$ by

$$\Pi^s := -\boldsymbol{\gamma}(\mathbb{I} - \nu \otimes \nu), \quad \boldsymbol{\gamma} \equiv \widehat{\boldsymbol{\gamma}}(\mathbf{m}^0), \tag{4.16}$$

where $\nu := \nu_{\mathcal{U}^1} = -\nu_{\mathcal{U}^2}$ and the functions $\widehat{\boldsymbol{\gamma}}$, $\mathbf{g}_1$ and $\mathbf{g}_2$ are defined in 4.4. Moreover, the mass flux $\mathbf{m}^0$ at the interface is

$$\mathbf{m}^0 := \varrho^1(v_\Gamma - v^1) \bullet \nu = \varrho^2(v_\Gamma - v^2) \bullet \nu, \tag{4.17}$$

where the equality is satisfied by the distributional mass equation. Here v_Γ is the velocity vector of Γ_t.

This theorem has been proved in [4] and the free energy inequality in the limit case is considered in section 7. We remark, that one can also write

$$\begin{aligned} \partial_t(\varrho^1 \boldsymbol{\mu}_{\mathcal{U}^1}) + \operatorname{div}(\varrho^1 v^1 \boldsymbol{\mu}_{\mathcal{U}^1}) &= \boldsymbol{\tau} \boldsymbol{\mu}_\Gamma, \\ \partial_t(\varrho^2 \boldsymbol{\mu}_{\mathcal{U}^2}) + \operatorname{div}(\varrho^2 v^2 \boldsymbol{\mu}_{\mathcal{U}^2}) &= -\boldsymbol{\tau} \boldsymbol{\mu}_\Gamma \end{aligned} \tag{4.18}$$

with a mass flux $\boldsymbol{\tau}$. This does not give any additional information compared to the equation in (4.14) except that $\boldsymbol{\tau} = \mathbf{m}^0$.

We mention, that the global quantities are induced by the approximation of a diffusive interface model, where the essential step is an existence theorem for the transition profile, see [18, Lemma 49]. In detail, the equations (4.21) below can be formulated in the phase space [18, Section 6.2]. In this formulation the existence proof for a transition profile was performed, see [18, Problem 45]. The proof of [18, Lemma 49] requires that $\mathrm{e}_h(s){:=}h(s)+sh'(s)\neq 0$ for all $s>0$, an assumption, which is a bit stronger than $W_0(0)\neq W_0(1)$, see [18, Lemma 31]. Therefore we are sure that in this case the following functions $(R_{\mathbf{m}},\Phi_{\mathbf{m}})$ exist.

4.4 Quantities for example 2.5. The function $\mathbf{m}\mapsto\widehat{\boldsymbol{\gamma}}(\mathbf{m})$, which gives the surface tension, is given by

$$\begin{aligned}
&\widehat{\boldsymbol{\gamma}}(\mathbf{m}){:=}\int_{-\infty}^{\infty}\Big(h(R_{\mathbf{m}})-\frac{\mathrm{e}_h(R_{\mathbf{m}})a_2(R_{\mathbf{m}},\Phi_{\mathbf{m}})}{2\widetilde{a}(R_{\mathbf{m}},\Phi_{\mathbf{m}})}\Big)|\partial_z\Phi_{\mathbf{m}}|^2\,\mathrm{d}z,\\
&\widetilde{a}{:=}a_1+\frac{n-1}{n}a_2,\quad \mathrm{e}_h(s){:=}sh'(s)+h(s)\ \text{for}\ s>0,
\end{aligned}\tag{4.19}$$

and the functions $\mathbf{m}\mapsto\mathbf{g}_1(\mathbf{m})$ and $\mathbf{m}\mapsto\mathbf{g}_2(\mathbf{m})$, which enter in the kinetic relations, are given by

$$\mathbf{g}_1(\mathbf{m}){:=}R_{\mathbf{m}}(-\infty),\quad \mathbf{g}_2(\mathbf{m}){:=}R_{\mathbf{m}}(+\infty).\tag{4.20}$$

Here $\mathbf{m}\mapsto(R_{\mathbf{m}},\Phi_{\mathbf{m}})$ is the solution of the following boundary value problem on $\mathbb{R}$ (compare the equations in [4, (7.4)]):

$$\begin{aligned}
&-\partial_z(h(R_{\mathbf{m}})\partial_z\Phi_{\mathbf{m}})+R_{\mathbf{m}}W'(\Phi_{\mathbf{m}})=0,\\
&\mathbf{m}\partial_z(\frac{1}{R_{\mathbf{m}}})+\frac{\mathrm{e}_h(R_{\mathbf{m}})}{2\widetilde{a}(R_{\mathbf{m}},\Phi_{\mathbf{m}})}|\partial_z\Phi_{\mathbf{m}}|^2=0,\\
&\Phi_{\mathbf{m}}(-\infty)=0,\quad \Phi_{\mathbf{m}}(+\infty)=1,\\
&\partial_z R_{\mathbf{m}}(-\infty)=0,\quad \partial_z R_{\mathbf{m}}(+\infty)=0.
\end{aligned}\tag{4.21}$$

4.5 Remark. Obviously the functions in 4.4 are part of the zeroth order approximation of the inner expansion (see (4.4)), that is, it is $(R_{\mathbf{m}^0},\Phi_{\mathbf{m}^0})=(R^0,\Phi^0)$. In [4, Theorem 5.3 and Theorem 5.1 and Theorem 6.1] besides the fact that $\mathbf{m}^0=R^0\Lambda^0$ is a constant one finds the equations

$$\frac{\mathrm{e}_h(R^0)}{2}|\partial_z\Phi^0|^2-\widetilde{a}\partial_z V^0\bullet\nu=0,\tag{4.22}$$

$$R^0W'(\Phi^0)-\partial_z(h(R^0)\partial_z\Phi^0)=0.\tag{4.23}$$

These two equations are equivalent to the equations in (4.21) and we mention that they are true in all four cases we consider, that is, they hold for example 2.5 and 2.6, for example 2.7 (see 10.1) and 2.8 (see 11.1).

For completeness we add the strong mass equations.

4.6 Equivalent strong equations for example 2.5. The mass equations in (4.14) together with the kinetic relations (4.15) are equivalent to the strong equations

$$\begin{gathered}\partial_t\varrho^\alpha+\operatorname{div}(\varrho^\alpha v^\alpha)=0 \quad \text{in } \mathcal{U}^\alpha \text{ for } \alpha=1,2,\\ \mathbf{m}^0:=\varrho^1(v_\Gamma-v^1)\bullet\nu=\varrho^2(v_\Gamma-v^2)\bullet\nu \quad \text{on } \Gamma,\\ v^1_{\tan}=v^2_{\tan} \quad \text{on } \Gamma,\\ \varrho^1=\mathbf{g}_1(\mathbf{m}^0),\quad \varrho^2=\mathbf{g}_2(\mathbf{m}^0) \quad \text{on } \Gamma.\end{gathered} \tag{4.24}$$

Clearly the momentum equation also has a strong version.

Example 2.6

Next we consider example 2.6, where we have a mass transition given by a $\boldsymbol{\tau}$-term, for which a Gibbs-Thompson law holds. Like here, as also in the following examples, we assume that $\mathbf{m}^0\neq 0$ on a dense set in Γ_t for each t (see (1.5)).

4.7 Limit equation for the continuous case 2.6. In the limit $\delta\to 0$ the distributional equations are

$$\begin{gathered}\partial_t(\varrho^1\boldsymbol{\mu}_{\mathcal{U}^1})+\operatorname{div}(\varrho^1 v^1\boldsymbol{\mu}_{\mathcal{U}^1})=\boldsymbol{\tau}\boldsymbol{\mu}_\Gamma,\\ \partial_t(\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2})+\operatorname{div}(\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2})=-\boldsymbol{\tau}\boldsymbol{\mu}_\Gamma,\\ \partial_t\Big(\sum_\alpha\varrho^\alpha v^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big)+\operatorname{div}\bigg(\sum_\alpha\big(\varrho^\alpha v^\alpha v^{\alpha\mathrm{T}}+\Pi^\alpha\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha}+\Pi^s\boldsymbol{\mu}_\Gamma\bigg)\\ =\sum_\alpha\mathbf{f}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha},\end{gathered} \tag{4.25}$$

and the kinetic equations read

$$\left.\begin{aligned}v^1_{\tan}&=v^2_{\tan}\\ \varrho:=\varrho^1&=\varrho^2\\ \boldsymbol{\gamma}\varrho\boldsymbol{\tau}&=\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu+f^2-f^1\end{aligned}\right\}\ \text{on } \Gamma, \tag{4.26}$$

where the free energies on the interface are $f^\alpha=\widehat{f^\alpha}(\varrho)$, see the definition in (4.3). The distributional mass equations together with the first two kinetic equations imply $v^1=v^2$ on Γ. In the differential equation the quantity Π^s is defined through the constant surface tension $\boldsymbol{\gamma}$ by

$$\begin{aligned}&\Pi^s:=-\boldsymbol{\gamma}(\mathbb{I}-\nu\otimes\nu),\quad \boldsymbol{\gamma}>0,\\ &\boldsymbol{\gamma}:=\int_0^1\sqrt{2(W(s)-W(0))}\,\mathrm{d}s=\int_{-\infty}^{+\infty}h(R^0)|\partial_z\Phi^0|^2\mathrm{dL}_1,\end{aligned} \tag{4.27}$$

where $\nu:=\nu_{\mathcal{U}^1}=-\nu_{\mathcal{U}^2}$. Moreover, κ_Γ is the curvature vector of Γ_t.

This theorem, as far as the strong version of the equations are considered, has been proved in [17], see [17, Section 9 and Theorem 39]. Because of space limits in this paper we do not carry out the proof of the distributional version here, this is left to the reader. But we add the strong version of the mass equations.

4.8 Equivalent strong equations for example 2.6. The two mass equations in (4.25) together with the kinetic relations (4.26) are equivalent to the strong equations

$$\begin{aligned} \partial_t\varrho^\alpha + \operatorname{div}(\varrho^\alpha v^\alpha) = 0 \quad &\text{in } \mathcal{U}^\alpha \text{ for } \alpha = 1,2, \\ \varrho := \varrho^1 = \varrho^2, \quad v := v^1 = v^2 \quad &\text{on } \Gamma, \\ \boldsymbol{\tau} = \varrho(v_\Gamma - v)\bullet\nu \quad &\text{on } \Gamma, \\ \boldsymbol{\gamma}\varrho\boldsymbol{\tau} = \boldsymbol{\gamma}\kappa_\Gamma\bullet\nu + f^2 - f^1 \quad &\text{on } \Gamma. \end{aligned} \tag{4.28}$$

The last equation is the well known Gibbs-Thompson law. Clearly the momentum equation also has a strong version.

Example 2.7

The example 2.7 has the following limit equations and this will be proved in the appendix, section 10. The free energy inequality in the limit case is considered in section 8.

4.9 Limit equation for the Cahn-Hilliard case 2.7. In the limit $\delta \to 0$ the distributional equations are

$$\begin{aligned} &\partial_t(\varrho^1\boldsymbol{\mu}_{\mathcal{U}^1}) + \operatorname{div}(\varrho^1 v^1 \boldsymbol{\mu}_{\mathcal{U}^1} + \mathbf{J}) = 0, \\ &\partial_t(\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2}) + \operatorname{div}(\varrho^2 v^2 \boldsymbol{\mu}_{\mathcal{U}^2} - \mathbf{J}) = 0, \\ &\mathbf{J} = J^1\boldsymbol{\mu}_{\mathcal{U}^1} + J^2\boldsymbol{\mu}_{\mathcal{U}^2}, \quad J^\alpha = m_\alpha(\varrho^\alpha)\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big), \\ &\partial_t\Big(\sum_\alpha \varrho^\alpha v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big) + \operatorname{div}\Big(\sum_\alpha (\varrho^\alpha v^\alpha v^{\alpha\,\mathrm{T}} + \Pi^\alpha)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^s \boldsymbol{\mu}_\Gamma\Big) \\ &\qquad\qquad = \sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}, \end{aligned} \tag{4.29}$$

and the kinetic equations read (for a proof see the end of section 8)

$$\left.\begin{aligned} v^1_{\tan} &= v^2_{\tan} \\ \varrho &:= \varrho^1 = \varrho^2 \\ \mu &:= \mu^1 = \mu^2 = \boldsymbol{\gamma}\kappa_\Gamma\bullet\nu + f^2 - f^1 \end{aligned}\right\} \text{ on } \Gamma. \tag{4.30}$$

Here

$$m_1(s) := m_0(s,0), \quad m_2(s) := m_0(s,1). \tag{4.31}$$

The quantity Π^s and the surface tension $\boldsymbol{\gamma}$ is defined in the same way as in (4.27). It follows that $v{:=}v^1{=}v^2$ on Γ, see the proof of the following statement. The free energies on Γ are defined as in 4.7. As above $\nu{:=}\nu_{\mathcal{U}^1}{=}{-}\nu_{\mathcal{U}^2}$, and κ_Γ is the curvature vector of Γ_t.

4.10 Equivalent strong equations for example 2.7. Under the kinetic relations (4.30) the two mass equations in (4.29) are equivalent to the strong equations

$$\begin{aligned} &\partial_t\varrho^\alpha + \operatorname{div}(\varrho^\alpha v^\alpha) = 0 \quad \text{in } \mathcal{U}^\alpha \text{ for } \alpha=1,2,\\ &\operatorname{div} J^\alpha = 0 \quad \text{in } \mathcal{U}^\alpha \text{ for } \alpha=1,2,\\ &\varrho := \varrho^1 = \varrho^2, \quad v := v^1 = v^2 \quad \text{on } \Gamma,\\ &\mu := \mu^1 = \mu^2 = \boldsymbol{\gamma}\kappa_\Gamma \bullet \nu + f^2 - f^1 \quad \text{on } \Gamma,\\ &(J^2 - J^1)\bullet\nu = -\varrho(v_\Gamma - v)\bullet\nu \quad \text{on } \Gamma. \end{aligned} \tag{4.32}$$

Clearly the momentum equation also has a strong version.

Proof. The strong differential equations in $\mathcal{U}^1$ are as consequence of the distributional version

$$\partial_t\varrho^1 + \operatorname{div}(\varrho^1 v^1 + J^1) = 0, \quad \operatorname{div}(-J^1) = 0,$$

and similar the equations in $\mathcal{U}^2$ are

$$\operatorname{div} J^2 = 0, \quad \partial_t\varrho^2 + \operatorname{div}(\varrho^2 v^2 - J^2) = 0.$$

On Γ, by [2, Theorem 2.8], the distributional equations imply

$$\begin{aligned} &(J^1 + \varrho^1(v^1 - v_\Gamma))\bullet\nu_{\mathcal{U}^1} + J^2\bullet\nu_{\mathcal{U}^2} = 0,\\ &J^1\bullet\nu_{\mathcal{U}^1} + (J^2 - \varrho^2(v^2 - v_\Gamma))\bullet\nu_{\mathcal{U}^2} = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\varrho^2(v^2 - v_\Gamma)\bullet\nu = \varrho^1(v^1 - v_\Gamma)\bullet\nu,\\ &(J^2 - J^1)\bullet\nu = \varrho^1(v^1 - v_\Gamma)\bullet\nu. \end{aligned}$$

Since $\varrho^1{=}\varrho^2$ on Γ by the kinetic equations, the first inequality immediate implies $v^1\bullet\nu{=}v^2\bullet\nu$. Since the tangential components of the velocities also coincide by the kinetic equations we conclude that $v^1{=}v^2$ on Γ. The Gibbs-Thompson like law, that is $\mu^1{=}\mu^2{=}\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu + f^2 - f^1$, is given as a kinetic equation. □

Example 2.8

The example 2.8 has the following limit equations and this will be proved in the appendix, section 11. The free energy inequality in the limit case is considered in section 9.

4.11 Limit equation for the Cahn-Hilliard case 2.8. In the limit $\delta \to 0$ the distributional equations are

$$
\begin{aligned}
&\partial_t(\varrho^1 \boldsymbol{\mu}_{\mathcal{U}^1}) + \operatorname{div}(\varrho^1 v^1 \boldsymbol{\mu}_{\mathcal{U}^1} + J^s \boldsymbol{\mu}_\Gamma) = 0,\\
&\partial_t(\varrho^2 \boldsymbol{\mu}_{\mathcal{U}^2}) + \operatorname{div}(\varrho^2 v^2 \boldsymbol{\mu}_{\mathcal{U}^2} - J^s \boldsymbol{\mu}_\Gamma) = 0 \quad \text{with } J^s := \widehat{m}(\varrho) \nabla^\Gamma \Big(\frac{\mu}{\varrho}\Big),\\
&\partial_t\Big(\sum_\alpha \varrho^\alpha v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big) + \operatorname{div}\Big(\sum_\alpha (\varrho^\alpha v^\alpha v^{\alpha\,\mathrm{T}} + \Pi^\alpha) \boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^s \boldsymbol{\mu}_\Gamma\Big)\\
&\qquad\qquad = \sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha},
\end{aligned}
\tag{4.33}
$$

and the kinetic equations are

$$
\left.
\begin{aligned}
v^1_{\tan} &= v^2_{\tan}\\
\varrho &:= \varrho^1 = \varrho^2\\
\mu &:= \mu^1 = \mu^2 = \boldsymbol{\gamma} \kappa_\Gamma \bullet \nu + f^2 - f^1
\end{aligned}
\right\} \text{ on } \Gamma.
\tag{4.34}
$$

where again f^1 and f^2 on the interface are as in 4.7. The quantity Π^s and the surface tension $\boldsymbol{\gamma}$ is defined in the same way as in (4.27), and κ_Γ as usual. Also as above $\nu := \nu_{\mathcal{U}^1} = -\nu_{\mathcal{U}^2}$. The function $\widehat{m}$ is given by

$$
\widehat{m}(\varrho) := \frac{m_0(\varrho)}{\varrho} \int_0^1 \frac{V(s)}{\sqrt{2(W(s) - W(0))}} \, \mathrm{d}s
\tag{4.35}
$$

4.12 Equivalent strong equations for example 2.8. The two mass equations in (4.33) and the kinetic relations are equivalent to the strong equations

$$
\begin{gathered}
\partial_t \varrho^\alpha + \operatorname{div}(\varrho^\alpha v^\alpha) = 0 \quad \text{in } \mathcal{U}^\alpha \text{ for } \alpha = 1,2,\\
\varrho := \varrho^1 = \varrho^2, \quad v := v^1 = v^2 \quad \text{on } \Gamma,\\
-\varrho(v_\Gamma - v) \bullet \nu = \operatorname{div}^\Gamma J^s \quad \text{on } \Gamma, \quad J^s := \widehat{m}(\varrho) \nabla^\Gamma \Big(\frac{\mu}{\varrho}\Big),\\
\mu := \mu^1 = \mu^2 = \boldsymbol{\gamma} \kappa_\Gamma \bullet \nu + f^2 - f^1 \quad \text{on } \Gamma.
\end{gathered}
\tag{4.36}
$$

Clearly the momentum equation also has a strong version.

Proof. The strong differential equations in $\mathcal{U}^\alpha$ are a direct consequence of the distributional equation. On Γ, by [2, Theorem 2.8], the distributional equations imply

$$
\begin{aligned}
\varrho^1 (v^1 - v_\Gamma) \bullet \nu_{\mathcal{U}^1} &= \operatorname{div}^\Gamma J^s,\\
\varrho^2 (v^2 - v_\Gamma) \bullet \nu_{\mathcal{U}^2} &= -\operatorname{div}^\Gamma J^s,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\varrho^1 (v^1 - v_\Gamma) \bullet \nu &= \varrho^2 (v^2 - v_\Gamma) \bullet \nu,\\
\varrho^1 (v^1 - v_\Gamma) \bullet \nu &= \operatorname{div}^\Gamma J^s.
\end{aligned}
$$

Since $\varrho^1\!=\!\varrho^2$ on Γ by the kinetic equations, the first inequality immediate implies $v^1\bullet\nu\!=\!v^2\bullet\nu$. Since the tangential components of the velocities also coincide by the kinetic equations we conclude that $v^1\!=\!v^2$ on Γ. The Gibbs-Thompson like law, that is $\mu^1\!=\!\mu^2\!=\!\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu+f^2-f^1$, is given as a kinetic equation. □

5 Equipartition of energy

For the phase field model the distributional formulation of the free energy inequality (2.3) with (2.1) and (2.4) reads for $\zeta\in C_0^\infty(\mathcal{U};\mathbb{R})$ and $\zeta\geq 0$

$$\begin{aligned}\int_{\mathcal{U}}\Big(&-\partial_t\zeta\big(f_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2\big)\\ &-\nabla\zeta\bullet\big(f_\delta v_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2 v_\delta+\Pi_\delta^{\mathrm{T}}v_\delta-\dot{\varphi}\delta h\nabla\varphi-\frac{\mu_\delta}{\varrho_\delta}J_\delta\big)\\ &-\zeta v_\delta\bullet\mathbf{f}_\delta\Big)\,\mathrm{d}x\mathrm{d}t=\int_{\mathcal{U}}\zeta g_\delta\,\mathrm{d}x\mathrm{d}t\leq 0\,,\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}f_\delta&=\frac{1}{\delta}\varrho_\delta W(\varphi)+\delta h(\varrho_\delta)\frac{|\nabla\varphi|^2}{2}+U(\varrho_\delta,\varphi)\,,\\ \mu_\delta&=\frac{\delta f_\delta}{\delta\varphi}=\frac{1}{\delta}\varrho_\delta W'(\varphi)-\delta\operatorname{div}(h(\varrho_\delta)\nabla\varphi)+U'_\varphi(\varrho_\delta,\varphi)\,,\end{aligned}\tag{5.2}$$

$$g_\delta=-\mathrm{D}v_\delta\mathbin{:}\mathbb{S}_\delta-\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta-\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta\,,\tag{5.3}$$

$$\begin{aligned}&\boldsymbol{\tau}_\delta:=\frac{\eta_0(\varrho_\delta)}{\delta}\mu_\delta\text{ and }J_\delta=0\text{ in cases 2.5 and 2.6,}\\ &J_\delta:=m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\text{ and }\boldsymbol{\tau}_\delta=0\text{ in cases 2.7 and 2.8,}\\ &\Pi_\delta=\mathbb{P}_\delta-\mathbb{S}_\delta\text{ in all cases.}\end{aligned}\tag{5.4}$$

In the δ-problem we consider in this section local test functions ζ and obtain the well known equipartition of energy. We then consider, see the following four sections 6, 7, 8, and 9, global test functions ζ.

Local test functions

Consider the special case of a test function $\zeta\!=\!\xi$, which has compact support in a δ-neighbourhood of Γ, that is $\zeta(t,x)\!=\!\xi(t,y,z)$ with $(t,y)\in\Gamma$, $z\in\mathbb{R}$, where $x\!=\!y+\delta z\nu(t,y)$, $\nu\!=\!\nu_{\mathcal{U}^1}$. Therefore $\operatorname{supp}(\zeta)\subset\Gamma_\delta$, where Γ_δ is defined in (3.1). The support of $z\mapsto\xi(t,y,z)$ is contained in a fixed interval $[-z_\xi,z_\xi]\subset[-z_\delta,z_\delta]$. For the derivatives of the test function we compute

$$\begin{aligned}\partial_t\zeta&=\partial_t^\Gamma\xi-\frac{1}{\delta}v_\Gamma\bullet\nu\partial_z\xi+\mathcal{O}(\delta)\,,\\ \nabla\zeta&=\nabla^\Gamma\xi+\frac{1}{\delta}\partial_z\xi\nu+\mathcal{O}(\delta)\,.\end{aligned}\tag{5.5}$$

Then we obtain, if δ is small,

$$
\begin{aligned}
&-\int_{\mathcal{U}}\Big(\partial_t\zeta\big(f_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2\big)\\
&\qquad+\nabla\zeta\bullet\big(f_\delta v_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2 v_\delta+\Pi_\delta^{\mathrm{T}}v_\delta-\dot\varphi\delta h\nabla\varphi-\frac{\mu_\delta}{\varrho_\delta}J_\delta\big)+\zeta v_\delta\bullet\mathbf{f}_\delta\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&=-\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\int_{\Gamma_t}\ \Big\{\Big(\partial_t^\Gamma\xi-\frac{1}{\delta}v_\Gamma\bullet\nu\partial_z\xi\Big)f_\delta\\
&\qquad\qquad+\Big(\nabla^\Gamma\xi+\frac{1}{\delta}\partial_z\xi\nu\Big)\bullet\big(f_\delta v_\delta+\Pi_\delta^{\mathrm{T}}v_\delta-\dot\varphi\delta h\nabla\varphi-\frac{\mu_\delta}{\varrho_\delta}J_\delta\big)\\
&\qquad\qquad+\mathcal{O}(\frac{1}{\delta})\mathcal{X}_{\mathrm{supp}\xi}\Big\}(1+\mathcal{O}(|r|))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}r\,\mathrm{d}t\\
&=-\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\ \Big\{\partial_z\xi\big(f_\delta(v_\delta-v_\Gamma)\bullet\nu+\big(\Pi_\delta^{\mathrm{T}}v_\delta-\dot\varphi\delta h\nabla\varphi-\frac{\mu_\delta}{\varrho_\delta}J_\delta\big)\bullet\nu\big)\\
&\qquad\qquad+\mathcal{O}(1)\mathcal{X}_{\mathrm{supp}\xi}\Big\}\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t,
\end{aligned}
$$

where we have used that $f_\delta=\mathcal{O}(\delta^{-1})$, which is a consequence of the calculations from [4, Section 5 and 6] presented here as a list:

$$
\begin{aligned}
v_\delta&=V^0+\mathcal{O}(\delta)\\
\lambda_\delta&:=(v_\Gamma-v_\delta)\bullet\nu=\Lambda^0+\mathcal{O}(\delta)\\
f_\delta&=\frac{1}{\delta}\varrho_\delta W(\varphi)+\delta h(\varrho_\delta)\frac{|\nabla\varphi|^2}{2}+U(\varrho_\delta,\varphi)\\
&=\frac{1}{\delta}\Big(R^0W(\Phi^0)+h(R^0)\frac{|\partial_z\Phi^0|^2}{2}\Big)+\mathcal{O}(1)\\
\frac{\delta f_\delta}{\delta\varphi}&=\frac{1}{\delta}\varrho_\delta W'(\varphi)-\delta\mathrm{div}\big(h(\varrho_\delta)\nabla\varphi\big)+U\prime_\varrho(\varrho_\delta,\varphi)\\
&=\frac{1}{\delta}\Big(R^0W'(\Phi^0)-\partial_z\big(h(R^0)\partial_z\Phi^0\big)\Big)+\mathcal{O}(1)\\
\dot\varphi&=(\partial_t+v_\delta\bullet\nabla)\varphi=\frac{1}{\delta}\Big(-\partial_z\Phi^0\nu\bullet v_\Gamma+\partial_z\Phi^0 v_\delta\bullet\nu\Big)+\mathcal{O}(1)\\
&=-\frac{1}{\delta}\partial_z\Phi^0\cdot\Lambda^0+\mathcal{O}(1)\\
\dot\varphi\delta h\nabla\varphi\bullet\nu&=\dot\varphi h\partial_z\Phi=-\frac{1}{\delta}h(R^0)|\partial_z\Phi^0|^2\Lambda^0+\mathcal{O}(1)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{D}v_\delta =& \frac{1}{\delta}\partial_z V^0 \otimes \nu + \mathcal{O}(1)\\
\varPi_\delta =& \mathbb{P}_\delta - \mathbb{S}_\delta\\
=& \mathrm{p}_U \mathbb{I} + \frac{\delta}{2}\mathrm{p}_h |\nabla\varphi|^2 \mathbb{I} + \delta h \nabla\varphi \otimes \nabla\varphi - (a_1 - \frac{a_2}{n})\mathrm{div}\, v_\delta \mathbb{I} - a_2 (\nabla v_\delta)^{\mathrm{S}}\\
=& \frac{1}{\delta}\Big(\frac{1}{2}\mathrm{p}_h |\partial_z \varPhi^0|^2 \mathbb{I} + h |\partial_z \varPhi^0|^2 \nu \otimes \nu\\
& - (a_1 - \frac{a_2}{n})\nu \bullet \partial_z V^0 \mathbb{I} - a_2 \nu \bullet \partial_z V^0 \nu \otimes \nu \Big) + \mathcal{O}(1)\\
\varPi_\delta \nu =& \mathrm{p}_U \nu + \frac{\delta}{2}\mathrm{p}_h |\nabla\varphi|^2 \nu + \delta h \nabla\varphi \bullet \nu \nabla\varphi\\
& - (a_1 - \frac{a_2}{n})\mathrm{div}\, v_\delta \nu - a_2 (\nabla v_\delta)^{\mathrm{S}} \nu + \mathcal{O}(1)\\
=& \frac{1}{\delta}\Big(\frac{1}{2}\mathrm{e}_h(R^0)|\partial_z \varPhi^0|^2 - \widetilde{a}(R^0, \varPhi^0)\partial_z V^0 \bullet \nu\Big)\nu + \mathcal{O}(1),
\end{aligned}
$$

where we have used that $\partial_z V^0 \in \mathrm{span}\{\nu\}$, see [4, Lemma 6.3], and the definitions

$$
\mathrm{e}_h(s) := s h'(s) + h(s) \text{ for } s \in \mathbb{R}, \quad \widetilde{a} := a_1 + \frac{n-1}{n} a_2,
$$

see (4.19) and [4, Theorem 6.1]. We obtain that the above integral equals

$$
\begin{aligned}
= \frac{1}{\delta}\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\varGamma_t} & \Big\{\partial_z \xi \Big(\big(R^0 W(\varPhi^0) - \frac{h}{2}|\partial_z \varPhi^0|^2\big)\varLambda^0 - \big(\frac{\mathrm{e}_h}{2}|\partial_z \varPhi^0|^2\\
& - \widetilde{a}\partial_z V^0 \bullet \nu\big) V^0 \bullet \nu\Big) + \mathcal{O}(\delta)\mathcal{X}_{\mathrm{supp}\xi}\Big\} \mathrm{dH}_{n-1}(y)\mathrm{d}z\mathrm{d}t.
\end{aligned}
$$

Multiplying this left-hand side with δ we derive that it converges as $\delta \to 0$ to

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{-\infty}^{+\infty}\int_{\varGamma_t} \partial_z \xi \; & \Big(\big(R^0 W(\varPhi^0) - \frac{h}{2}|\partial_z \varPhi^0|^2\big)\varLambda^0\\
& - \big(\frac{\mathrm{e}_h}{2}|\partial_z \varPhi^0|^2 - \widetilde{a}\partial_z V^0 \bullet \nu\big) V^0 \bullet \nu\Big) \mathrm{dH}_{n-1}(y)\mathrm{d}z\mathrm{d}t.
\end{aligned} \tag{5.6}
$$

The first term will give the left-hand side of the equipartition of energy in 5.1, and the second term vanishes because of (4.22).

Allen-Cahn cases 2.5 and 2.6

We now turn to the right-hand side of (5.1). In the Allen-Cahn cases we compute using (2.9)

$$
\boldsymbol{\tau}_\delta = -\varrho_\delta \dot{\varphi} = +\frac{1}{\delta} R^0 \varLambda^0 \partial_z \varPhi^0 + \mathcal{O}(1),
$$

hence, taking μ_δ from the above list and keeping in mind that $\zeta \geq 0$ has support in Γ_δ,

$$\int_{\Gamma_\delta} \zeta\Big(-\boldsymbol{\tau}_\delta \frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \xi\Big(-\Lambda^0 \partial_z \Phi^0 \frac{1}{\delta}\Big(R^0 W'(\Phi^0) - \partial_z(h(R^0)\partial_z \Phi^0)\Big)\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \;+\; \mathcal{O}(1),$$

which by (4.23) is $\mathcal{O}(1)$.

Cahn-Hilliard case 2.7
In the Cahn-Hilliard case we compute

$$J_\delta = m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big) = \frac{m_0(R^0,\Phi^0)}{\delta} \partial_z\Big(\frac{M^0}{R^0}\Big)\nu + \mathcal{O}(1)$$

hence

$$\int_{\Gamma_\delta} \zeta\Big(-\nabla\big(\frac{\mu_\delta}{\varrho_\delta}\big) \bullet J_\delta\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \xi\Big(-\frac{m_0(R^0,\Phi^0)}{\delta}\Big|\partial_z\Big(\frac{M^0}{R^0}\Big)\Big|^2\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t + \mathcal{O}(1),$$

which by 10.1 is $\mathcal{O}(1)$.

Cahn-Hilliard case with degenerate mobility 2.8
In the other Cahn-Hilliard case we compute

$$J_\delta = m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big) = \frac{m_0(R^0)V(\Phi^0)}{\delta} \nabla^\Gamma\Big(\frac{M^0}{R^0}\Big) + \mathcal{O}(1),$$

hence

$$\int_{\Gamma_\delta} \zeta\Big(-\nabla\big(\frac{\mu_\delta}{\varrho_\delta}\big) \bullet J_\delta\Big) = \mathcal{O}(1),$$

since by 11.1 is $\partial_z\big(\frac{M^0}{R^0}\big) = 0$.

By (5.3) and making use of the above list we obtain for the right-hand side of (5.1)

(with $\zeta\geq 0$ having support in Γ_δ)

$$
\begin{aligned}
0\geq &\int_{\mathcal U}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t=\int_{\Gamma_\delta}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t\\
=&-\int_{\mathbb R}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\int_{\Gamma_t}\xi\Big(\mathrm{D}v_\delta\mathbin{:}\mathbb{S}_\delta+\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta+\frac{\mu_\delta}{\varrho_\delta}\cdot\boldsymbol{\tau}_\delta\Big)(1+\mathcal{O}(|r|))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}r\,\mathrm{d}t\\
=&-\int_{\mathbb R}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\xi\frac{1}{\delta}\partial_z V^0\otimes\nu\mathbin{:}\Big((a_1-\frac{1}{n}a_2)\partial_z V^0\bullet\nu\ \mathbb{I}\\
&\qquad\qquad+\frac{a_2}{2}\big(\partial_z V^0\otimes\nu+\nu\otimes\partial_z V^0\big)\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t\ +\ \mathcal{O}(1).\\
=&-\frac{1}{\delta}\int_{\mathbb R}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\xi\cdot\widetilde{a}|\partial_z V^0\bullet\nu|^2\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t+\mathcal{O}(1).
\end{aligned}
$$

We multiply this expression by δ and obtain in the limit $\delta\to 0$

$$
0\geq-\int_{\mathbb R}\int_{-\infty}^{+\infty}\int_{\Gamma_t}\xi\cdot\widetilde{a}|\partial_z V^0\bullet\nu|^2\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t. \tag{5.7}
$$

Thus with (5.6) and (5.7) we have derived the

5.1 Equipartition of energy. We have in local coordinates on $\Gamma\times\mathbb R$

$$
\partial_z\Big(\big(\frac{h(R^0)}{2}|\partial_z\Phi^0|^2-R^0W(\Phi^0)\big)\Lambda^0\Big)=-\widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\leq 0\,.
$$

We mention that this statement is the free energy identity on the surface and it has a classical version. In fact, integrating the above formula from $-\infty$ to z one obtains

$$
\begin{aligned}
0\geq&-\int_{-\infty}^{z}\widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\,\mathrm{dL}_1\\
=&\Big(\frac{h(R^0(z))}{2}|\partial_z\Phi^0(z)|^2-R^0(z)W(\Phi^0(z))\Big)\Lambda^0(z)+R^0(-\infty)W(0)\Lambda^0(-\infty)\\
=&\mathbf{m}^0\Big(\frac{h(R^0(z))}{2R^0(z)}|\partial_z\Phi^0(z)|^2-(W(\Phi^0(z))-W(0))\Big).
\end{aligned}\tag{5.8}
$$

Now choose $z=\infty$ and obtain

$$
\mathbf{m}^0\big(W(0)-W(1)\big)=-\int_{-\infty}^{\infty}\widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\,\mathrm{dL}_1\leq 0. \tag{5.9}
$$

This will be the free energy inequality on the surface Γ in section 7. For sections 6, 8, and 9, the following theorem is important. It has already been stated in [17, Lemma 20],

5.2 Equipartition of energy (classical). In the continuous cases 2.6, 2.7, and 2.8 it follows

$$
\partial_z V^0\bullet\nu=0
$$

in $\varGamma\times\mathbb{R}$, and if $\mathbf{m}^0$ is nonzero on a dense set of $\varGamma\times\mathbb{R}$

$$R^0\big(W(\varPhi^0)-W(0)\big)=\frac{h(R^0)}{2}|\partial_z\varPhi^0|^2. \tag{5.10}$$

Proof. The proof is taken from [17, Lemma 20]. From (5.9) and $W(0)=W(1)$ one obtains

$$\int_{-\infty}^{\infty}\widetilde{a}(R^0,\varPhi^0)|\partial_z V^0\bullet\nu|^2\,\mathrm{dL}_1=0.$$

Now $\widetilde{a}(R^0,\varPhi^0)>0$ and therefore one concludes from this identity that $\partial_z V^0\bullet\nu=0$. Inserting this in the above integral (5.8) we obtain

$$0=\mathbf{m}^0\Big(W(\varPhi^0(z))-W(0)-\frac{h(R^0(z))}{2R^0(z)}|\partial_z\varPhi^0(z)|^2\Big)$$

for all z. □

6 Continuous density

Here we are dealing with example 2.6 and start with the δ-version of the free energy inequality (5.1). Due to the fact, that in this case (as shown in 5.2)

$$\partial_z V^0\bullet\nu=0, \tag{6.1}$$

the right-hand side of the equipartition of energy 5.1 vanishes. We assume that $\mathbf{m}^0$ is nonzero on a dense set of $\varGamma\times\mathbb{R}$. Then the classical equipartition of energy 5.2 holds.

This has an important consequence, namely that the order $\frac{1}{\delta}$-term of the free energy production is zero (a different behaviour than in section 7). This implies that the order of the highest nonzero term in the free energy identity on the surface is the same as the order of the highest nonzero term in the surrounding fluids, resulting in a single distributional free energy inequality.

But before we prove this, we state a theorem, which also is needed in the proof of the limiting equations.

6.1 Theorem. Consider a solution $(\varrho_\delta,\varphi,v_\delta)$ of (1.3) for example 2.6 assuming (4.2) and (4.4). Then in $\varGamma_\delta$ we have for the first variation for δ small

$$\mu_\delta:=\frac{\delta f_\delta}{\delta\varphi}=U\prime_\varphi(R^0,\varPhi^0)+\kappa_\varGamma\bullet\nu h(R^0)\partial_z\varPhi^0+\varUpsilon_\delta+\mathcal{O}(\delta),$$

$$\varUpsilon_\delta:=\frac{1}{\delta}\big(W_0(\widehat{R},\widehat{\varPhi})-\partial_z(h(\widehat{R})\partial_z\widehat{\varPhi})\big)=\mathcal{O}(1),$$

$$\widehat{R}:=R^0+\delta R^1,\quad \widehat{\varPhi}:=\varPhi^0+\delta\varPhi^1,$$

$$\int_{-z_\delta}^{+z_\delta}\varUpsilon_\delta\partial_z\varPhi^0\,\mathrm{d}z\longrightarrow 0 \text{ as } \delta\to 0.$$

Here $\kappa_\varGamma$ is the curvature vector of $\varGamma_t$, the density is continuous on $\varGamma$, that is $\varrho:=\varrho^1=\varrho^2$, and we have used the assumptions $W_0(\varrho,\varphi)=\varrho W(\varphi)$ and $h(s)=\frac{1}{s}$.

Proof. From (4.23) it follows that $\Upsilon_\delta = \mathcal{O}(1)$. Then we obtain, since R^0 is independent of z,

$$
\begin{aligned}
\Upsilon_\delta &= \frac{1}{\delta}\big(W_0(\widehat{R},\widehat{\Phi}) - \partial_z(h(\widehat{R})\partial_z\widehat{\Phi})\big)\\
&= R^1 W'(\Phi^0) - \partial_z(h'(R^0)R^1\partial_z\Phi^0) + R^0 W''(\Phi^0)\Phi^1 - \partial_z(h(R^0)\partial_z\Phi^1) + \mathcal{O}(\delta)\\
&= R^1 W'(\Phi^0) + \frac{1}{|R^0|^2}\partial_z(R^1\partial_z\Phi^0) + R^0 W''(\Phi^0)\Phi^1 - \frac{1}{R^0}\partial_z^2\Phi^1 + \mathcal{O}(\delta)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{-z_\delta}^{z_\delta}\Upsilon_\delta\partial_z\Phi^0 dz &= \int_{-z_\delta}^{z_\delta}\Big(R^1\partial_z(W(\Phi^0)) + \frac{1}{|R^0|^2}\partial_z(R^1\partial_z\Phi^0)\partial_z\Phi^0\Big)\mathrm{d}z\\
&\quad + \int_{-z_\delta}^{z_\delta}\Big(R^0\partial_z(W'(\Phi^0))\Phi^1 - \frac{1}{R^0}\partial_{zz}\Phi^1\partial_z\Phi^0\Big)\mathrm{d}z\\
&= \int_{-z_\delta}^{z_\delta} R^1\partial_z\Big(W(\Phi^0) - \frac{1}{|R^0|^2}\frac{|\partial_z\Phi^0|^2}{2}\Big)\mathrm{d}z\\
&\quad + \Big[\frac{R^1}{|R^0|^2}|\partial_z\Phi^0|^2\Big]_{-z_\delta}^{z_\delta} + \Big[R^0 W'(\Phi^0)\Phi^1 - \frac{1}{R^0}\partial_z\Phi^1\partial_z\Phi^0\Big]_{-z_\delta}^{z_\delta}\\
&\quad + \int_{-z_\delta}^{z_\delta}\partial_z\Phi^1\Big(-R^0 W'(\Phi^0) + \partial_z\big(\frac{1}{R^0}\partial_z\Phi^0\big)\Big)\mathrm{d}z\\
&= \Big[\frac{R^1}{|R^0|^2}|\partial_z\Phi^0|^2\Big]_{-z_\delta}^{z_\delta} + \Big[R^0 W'(\Phi^0)\Phi^1 - \frac{1}{R^0}\partial_z\Phi^1\partial_z\Phi^0\Big]_{-z_\delta}^{z_\delta}\\
&\longrightarrow 0 \text{ as } \delta \to 0.
\end{aligned}
$$

Here we have used the identities (5.10) and (4.23). □

In order to show the convergence of the energy identity we have to normalize the free energy f_δ. We introduce

$$
\begin{aligned}
&\overline{W} := W(0) = W(1) \text{ (in example 2.6)},\\
&\widetilde{f}_\delta := f_\delta - \frac{1}{\delta}\varrho_\delta\overline{W} = \frac{1}{\delta}\varrho_\delta(W(\varphi) - \overline{W}) + \delta h(\varrho_\delta)\frac{|\nabla\varphi|^2}{2} + U(\varrho_\delta,\varphi)
\end{aligned}
\tag{6.2}
$$

Since from the mass conservation

$$
-\int_{\mathcal{U}}\Big(\partial_t\zeta\frac{1}{\delta}\varrho_\delta\overline{W} + \nabla\zeta\bullet\big(\frac{1}{\delta}\varrho_\delta\overline{W}v_\delta\big)\Big)\mathrm{d}x\,\mathrm{d}t = 0
$$

we can rewrite the energy identity (5.1) (it is $J_\delta = 0$ in example 2.6) by

$$
\begin{aligned}
&\int_{\mathcal{U}}\Big(-\partial_t\zeta\big(\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2\big)\\
&\quad -\nabla\zeta\bullet\big(\widetilde{f}_\delta v_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2 v_\delta + \Pi_\delta^{\mathrm{T}}v_\delta - \dot{\varphi}\delta h\nabla\varphi\big)\\
&\quad -\zeta v_\delta\bullet\mathbf{f}_\delta\Big)\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{U}}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t \le 0,
\end{aligned}
\tag{6.3}
$$

Of course, then

$$\frac{\delta\widetilde{f_\delta}}{\delta\varphi}=\frac{\delta f_\delta}{\delta\varphi}=\mu_\delta\,,\quad \mathrm{p}_{\widetilde{f_\delta}}=\mathrm{p}_{f_\delta}\,,$$

therefore the differential equations (1.1), which describe the material under consideration, that is the mass and momentum equations with the constitutive relations in 2.4 are not changed by the normalization of the free energy. We mention, that this normalization of the free energy is quite standard, see also 7.3.

Global test functions

Let us now choose test functions ζ as function of (t,x). We prove that the three terms in the free energy identity converge in the sense of distributions. By this we mean that the term under the time derivative, the term under the divergence, and the right side converge to a limit as $\delta\to 0$ as stated in 6.2. Therefore for each term we have to take a different test function. For the energy production term we compute with a test function $\zeta\in C_0^\infty(\mathcal{U};\mathbb{R})$

$$\int_{\mathcal{U}}\zeta g_\delta\,\mathrm{dL}_{n+1}=\sum_\alpha\int_{\mathcal{U}_\delta^\alpha}\zeta g_\delta\,\mathrm{dL}_{n+1}+\int_{\Gamma_\delta}\zeta g_\delta\,\mathrm{dL}_{n+1}\,,$$

$$\text{with}\quad g_\delta=\dot{\varphi}\frac{\delta f_\delta}{\delta\varphi}-\mathrm{D}v_\delta:\mathbb{S}_\delta\,.$$

Now $v_\delta\to v^\alpha$ as $\delta\to 0$ locally in $\mathcal{U}^\alpha$, hence $\mathrm{D}v_\delta\to\mathrm{D}v^\alpha$. Since $\mathrm{D}v_\delta=\mathcal{O}(\delta^{-1})$ near the interface by the list in section 5, it follows (see 3.4) that

$$\|\mathrm{D}v_\delta-\mathrm{D}v^\alpha\|_{C^0(\mathcal{U}_\delta^\alpha)}\to 0,$$

hence

$$\int_{\mathcal{U}_\delta^\alpha}\zeta\mathrm{D}v_\delta:\mathbb{S}_\delta\,\mathrm{dL}_{n+1}\to\int_{\mathcal{U}^\alpha}\zeta\mathrm{D}v^\alpha:\mathbb{S}^\alpha\,\mathrm{dL}_{n+1}\,.$$

Similarly, $\dot{\varphi}\to 0$ locally in $\mathcal{U}^\alpha$ and $\dot{\varphi}=\mathcal{O}(\delta^{-1})$ near the interface by the list in section 5, and

$$\frac{\delta f_\delta}{\delta\varphi}-U_{\prime\varphi}(\varrho_\delta,\varphi)=\frac{1}{\delta}\big(W_{0\prime\varphi}(\varrho_\delta,\varphi)-\delta^2\mathrm{div}(h(\varrho_\delta)\nabla\varphi)\big)$$

near the interface. It follows (see 3.4) that

$$\left\|\frac{\delta f_\delta}{\delta\varphi}-U_{\prime\varphi}(\varrho_\delta,\varphi)\right\|_{C^0(\mathcal{U}_\delta^\alpha)}\to 0,\quad \|\dot{\varphi}\|_{C^0(\mathcal{U}_\delta^\alpha)}\to 0.$$

We conclude that

$$\int_{\mathcal{U}_\delta^\alpha}\zeta\dot{\varphi}\frac{\delta f_\delta}{\delta\varphi}\,\mathrm{dL}_{n+1}\to 0.$$

We now come to the integral over Γ_δ. Let us start with the f_δ-term. With $\dot{\varphi}$ from the list and the statement about f_δ in 6.1 we compute

$$\int_{\Gamma_\delta}\zeta\dot{\varphi}\frac{\delta f_\delta}{\delta\varphi}\,\mathrm{dL}_{n+1}=\int_{\mathbb{R}}\int_{\Gamma_t}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\zeta\dot{\varphi}\frac{\delta f_\delta}{\delta\varphi}(1+\mathcal{O}(|r|))\,\mathrm{d}r\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{\Gamma_t}\int_{-z_\delta}^{+z_\delta}\zeta\delta\dot{\varphi}\frac{\delta f_\delta}{\delta\varphi}(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=-\int_{\mathbb{R}}\int_{\Gamma_t}\int_{-z_\delta}^{+z_\delta}\zeta(t,x)\big(\Lambda^0\partial_z\Phi^0(U_{,\varphi}(R^0,\Phi^0)+\kappa_\Gamma\bullet\nu h(R^0)\partial_z\Phi^0+\Upsilon_\delta)+\mathcal{O}(\delta)\big)$$
$$\cdot(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t,y)\int_{-z_\delta}^{+z_\delta}\big(\Lambda^0\partial_z\Phi^0(U_{,\varphi}(R^0,\Phi^0)+\kappa_\Gamma\bullet\nu h(R^0)\partial_z\Phi^0+\Upsilon_\delta)+\mathcal{O}(\delta)\big)$$
$$\cdot(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t+\mathcal{O}(\varepsilon_\delta),$$

where we have used, that for $x=y+\delta z\nu(t,y)$ the test function, because it is a global test function, satisfies $\zeta(t,x)=\zeta(t,y)+\mathcal{O}(\varepsilon_\delta)$, and therefore the difference of the integrals, since $\Upsilon_\delta=\mathcal{O}(1)$, is estimated by ($C$ and C' are constants)

$$\int_{\mathbb{R}}\int_{\Gamma_t}\int_{-z_\delta}^{+z_\delta}\mathcal{O}(\varepsilon_\delta)\big(C|\partial_z\Phi^0|+\mathcal{O}(\delta)\big)(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{\Gamma_t}\mathcal{O}(\varepsilon_\delta)\big(C'+\mathcal{O}(\delta z_\delta)\big)(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t=\mathcal{O}(\varepsilon_\delta).$$

We have for the Υ_δ-term in the above integral by 6.1, using that Λ^0 is independent of z,

$$\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t,y)\int_{-z_\delta}^{+z_\delta}\Lambda^0\partial_z\Phi^0\Upsilon_\delta\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t\to 0.$$

Thus the entire integral converges to

$$\to-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t,y)\int_{-\infty}^{+\infty}\Lambda^0\partial_z\Phi^0\big(U_{,\varphi}(R^0,\Phi^0)+\kappa_\Gamma\bullet\nu h(R^0)\partial_z\Phi^0\big)\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t,y)\Lambda^0\int_{-\infty}^{+\infty}\big(\partial_z(U(R^0,\Phi^0))+\kappa_\Gamma\bullet\nu h(R^0)|\partial_z\Phi^0|^2\big)\,\mathrm{d}z\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t,$$

where also R^0 is independent of z, therefore $\partial_z(U(R^0,\Phi^0))=U_{,\varphi}(R^0,\Phi^0)\partial_z\Phi^0$. Therefore we can integrate this and obtain that it is equal to

$$=-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta(t,y)\Lambda^0\Big(U(\varrho^2,1)-U(\varrho^1,0)+\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t$$
$$=-\int_{\mathbb{R}^{n+1}}\zeta\Lambda^0\Big(U(\varrho^2,1)-U(\varrho^1,0)+\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu\Big)\,\mathrm{d}\boldsymbol{\mu}_\Gamma$$
$$=-\big\langle\zeta,\Lambda^0\big(U(\varrho^2,1)-U(\varrho^1,0)+\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu\big)\boldsymbol{\mu}_\Gamma\big\rangle.$$

And in this case by (6.1)

$$\Lambda^0 = (v_\Gamma - V^0)\bullet\nu = (v_\Gamma - v^1)\bullet\nu = (v_\Gamma - v^2)\bullet\nu, \quad \nu = \nu_{\mathcal{U}^1} = -\nu_{\mathcal{U}^2}. \tag{6.4}$$

Thus we have shown the convergence of the production term in the following

6.2 Theorem. Consider a solution $(\varrho_\delta, \varphi, v_\delta)$ of (1.3) for example 2.6 assuming (4.2) and (4.4). Then as $\delta \to 0$ the solution converges pointwise in the sense of distributions in $\mathcal{U}$:

$$\begin{aligned}
&\big(\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2\big)\mathrm{L}_{n+1} \longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s\boldsymbol{\mu}_\Gamma, \\
&\big(\big(\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2\big)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{L}_{n+1} \\
&\qquad \longrightarrow \sum_\alpha \Big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\Big)v^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s(v_\Gamma + v_{\tan})\boldsymbol{\mu}_\Gamma, \\
&{\Pi_\delta}^{\mathrm{T}} v_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha {\Pi^\alpha}^{\mathrm{T}} v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + {\Pi^s}^{\mathrm{T}} v_{\tan}\boldsymbol{\mu}_\Gamma, \\
&v_\delta\bullet\mathbf{f}_\delta\mathrm{L}_{n+1} \longrightarrow \sum_\alpha v^\alpha\bullet\mathbf{f}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}, \\
&g_\delta\mathrm{L}_{n+1} \longrightarrow -\mathrm{D}v^\alpha{:}\mathbb{S}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha} - \lambda\big(\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu + f^2 - f^1\big)\boldsymbol{\mu}_\Gamma \\
&\qquad = -\mathrm{D}v^\alpha{:}\mathbb{S}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha} - \sum_\alpha \lambda^\alpha\big(\frac{1}{2}\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu_{\mathcal{U}^\alpha} - f^\alpha\big)\boldsymbol{\mu}_\Gamma.
\end{aligned} \tag{6.5}$$

Here

$$\begin{aligned}
&f^s = \boldsymbol{\gamma} := \int_{-\infty}^{+\infty} h(R^0)|\partial_z\Phi^0|^2\,\mathrm{d}z > 0, \\
&\Pi^s = -\boldsymbol{\gamma}(\mathbb{I} - \nu\otimes\nu),
\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}
&f^1 = \widehat{f}^1(\varrho^1) = U(\varrho^1, 0) \text{ and } f^2 = \widehat{f}^2(\varrho^2) = U(\varrho^2, 0), \\
&v := v^1 = v^2, \quad \varrho := \varrho^1 = \varrho^2 \text{ on } \Gamma, \\
&\lambda = \lambda^1 = -\lambda^2, \quad \nu = \nu_{\mathcal{U}^1} = -\nu_{\mathcal{U}^2}, \\
&\lambda^\alpha := (v_\Gamma - v)\bullet\nu_{\mathcal{U}^\alpha} \text{ for } \alpha = 1,2.
\end{aligned} \tag{6.7}$$

Let us continue the proof of the theorem by showing the convergence of the other terms under derivatives in the free energy inequality (5.1).

First we show that

$$\widetilde{f}^\delta\mathrm{L}_{n+1} \longrightarrow \sum_\alpha f^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s\boldsymbol{\mu}_\Gamma, \tag{6.8}$$

where f^s is given by (6.6). Here we use test functions $\zeta_1 \in C_0^\infty(\mathcal{U};\mathbb{R})$ and write

$$\int_{\mathcal{U}} \zeta_1\widetilde{f}_\delta\,\mathrm{d}x\,\mathrm{d}t = \sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_1\widetilde{f}_\delta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_\delta} \zeta_1\widetilde{f}_\delta\,\mathrm{d}x\,\mathrm{d}t.$$

Since W is a smooth function, hence $W(\varphi)-\overline{W}\to 0$ in $\mathcal{U}^\alpha$, it converges

$$\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_1 \widetilde{f}_\delta \,\mathrm{d}x\mathrm{d}t \longrightarrow \sum_\alpha \int_{\mathcal{U}^\alpha} \zeta_1 f^\alpha \,\mathrm{d}x\mathrm{d}t \text{ as } \delta\to 0,$$

where f^α in $\mathcal{U}^\alpha$ is given by (6.7). For the integral near the surface we compute

$$\begin{aligned}
&\int_{\Gamma_\delta} \zeta_1 \widetilde{f}_\delta \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}} \delta \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \zeta_1 \frac{1}{\delta}\Big(R^0(W(\Phi^0)-\overline{W}) + h(R^0)\frac{|\partial_z \Phi^0|^2}{2} + \mathcal{O}(\delta)\Big) \\
&\qquad\qquad \cdot(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\mathrm{d}z\mathrm{d}t \\
&\longrightarrow \int_{\mathbb{R}}\int_{\Gamma_t} \zeta_1 f^s \mathrm{dH}_{n-1}(y)\mathrm{d}t,
\end{aligned}$$

since $\overline{W}=W(0)$ and if

$$\begin{aligned}
f^s &:= \int_{-\infty}^{+\infty} \Big(R^0(W(\Phi^0)-W(0)) + h(R^0)\frac{|\partial_z\Phi^0|^2}{2}\Big)\mathrm{d}z \\
&= \int_{-\infty}^{+\infty} h(R^0)|\partial_z\Phi^0|^2 \mathrm{d}z,
\end{aligned} \tag{6.9}$$

where the last identity follows from the equipartition of energy 5.2. Thus (6.8) is proved.

Next we show that

$$\begin{aligned}
&\big((\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{L}_{n+1} \\
&\longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s(v_\Gamma + v_{\tan})\boldsymbol{\mu}_\Gamma\,.
\end{aligned} \tag{6.10}$$

For this we use test functions $\zeta_2 \in C_0^\infty(\mathcal{U};\mathbb{R}^n)$. Then

$$\begin{aligned}
&\int_{\mathcal{U}} \zeta_2 \bullet \big((\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{d}x\mathrm{d}t \\
&= \sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_2 \bullet \big((\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{\Gamma_\delta} \zeta_2 \bullet \big((\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{d}x\mathrm{d}t\,.
\end{aligned}$$

Since W is a smooth function, we obtain in the bulk regions

$$\begin{aligned}
&\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_2 \bullet \big((\widetilde{f}_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{d}x\mathrm{d}t \\
&\longrightarrow \sum_\alpha \int_{\mathcal{U}^\alpha} \zeta_2 \bullet (f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)v^\alpha \mathrm{d}x\mathrm{d}t \text{ as } \delta\to 0.
\end{aligned}$$

Now we evaluate the interface region. We write

$$v_\delta = v_\Gamma + v_{\tan} - \lambda_\delta \nu \quad \text{where} \quad \lambda_\delta := (v_\Gamma - v_\delta)\bullet\nu$$

and $\nu = \nu_{\mathcal{U}^1} = -\nu_{\mathcal{U}^2}$. We compute

$$\begin{aligned}
&\int_{\Gamma_\delta} \zeta_2\bullet\big(\widetilde{f}_\delta(v_\Gamma + v_{\tan})\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \delta \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \zeta_2\bullet\Big(\big(\tfrac{1}{\delta}\big(R^0(W(\Phi^0) - \overline{W}) + h(R^0)\frac{|\partial_z \Phi^0|^2}{2}\big) + \mathcal{O}(1)\big)(v_\Gamma + v_{\tan})\Big) \\
&\qquad\qquad \cdot(1 + \mathcal{O}(\varepsilon_\delta))\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
&\longrightarrow \int_{\mathbb{R}} \int_{\Gamma_t} \zeta_2\bullet\big(f^s(v_\Gamma + v_{\tan})\big)\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}t
\end{aligned}$$

whith f^s defined in (6.9). The term that remains is

$$\begin{aligned}
&\int_{\Gamma_\delta} \zeta_2\bullet(-\widetilde{f}_\delta \lambda_\delta \nu - \dot{\varphi}\delta h \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \delta \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \zeta_2\bullet\Big(-\frac{1}{\delta}\Big(R^0(W(\Phi^0) - \overline{W}) + h(R^0)\frac{|\partial_z \Phi^0|^2}{2}\Big)\Lambda^0 \nu \\
&\qquad + \frac{1}{\delta}\Lambda^0 \partial_z \Phi^0 h(R^0) \partial_z \Phi^0 \nu + \mathcal{O}(1)\Big)\cdot(1 + \mathcal{O}(\varepsilon_\delta))\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \delta \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \zeta_2\bullet\Big(-\frac{1}{\delta}\Big(R^0(W(\Phi^0) - \overline{W}) - h(R^0)\frac{|\partial_z \Phi^0|^2}{2}\Big)\Lambda^0 \nu \\
&\qquad\qquad + \mathcal{O}(1)\Big)\cdot(1 + \mathcal{O}(\varepsilon_\delta))\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
&\longrightarrow 0 \text{ as } \delta \to 0
\end{aligned}$$

again by the equipartition of energy 5.2. Thus (6.10) is proved.

The last property which we have to show is

$$\Pi_\delta^{\mathrm{T}} v_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha (\Pi^\alpha)^{\mathrm{T}} v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^s v_{\tan} \boldsymbol{\mu}_\Gamma, \tag{6.11}$$

where Π^s is given by (6.6). We use again test functions $\zeta_2 \in C_0^\infty(\mathcal{U};\mathbb{R}^n)$ and write

$$\int_{\mathcal{U}} \zeta_2\bullet\big(\Pi_\delta^{\mathrm{T}} v_\delta\big)\,\mathrm{d}x\,\mathrm{d}t = \sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_2\bullet\big(\Pi_\delta^{\mathrm{T}} v_\delta\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_\delta} \zeta_2\bullet\big(\Pi_\delta^{\mathrm{T}} v_\delta\big)\,\mathrm{d}x\,\mathrm{d}t.$$

We obtain

$$\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_2\bullet\big(\Pi_\delta^{\mathrm{T}} v_\delta\big)\,\mathrm{d}x\,\mathrm{d}t \longrightarrow \sum_\alpha \int_{\mathcal{U}^\alpha} \zeta_2\bullet\big((\Pi^\alpha)^{\mathrm{T}} v^\alpha\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Since in this case (4.22) holds and $\partial_z V^0=0$, hence $\mathrm{e}_h=0$, we obtain

$$
\begin{aligned}
\Pi_\delta^{\mathrm{T}} &= \mathrm{p}_U \mathbb{I} + \frac{\delta}{2}\mathrm{p}_h |\nabla\varphi|^2 \mathbb{I} + \delta h \nabla\varphi\otimes\nabla\varphi - (a_1 - \frac{a_2}{n})\operatorname{div} v_\delta \mathbb{I} - a_2 (\nabla v_\delta)^{\mathrm{S}} \\
&= \frac{1}{\delta}\Big(\frac{\mathrm{p}_h}{2}|\partial_z \Phi^0|^2\mathbb{I} + h|\partial_z\Phi^0|^2 \nu\otimes\nu - (a_1 - \frac{a_2}{n})\nu\bullet\partial_z V^0 \mathbb{I} \\
&\qquad\qquad - a_2(\nu\bullet\partial_z V^0)\nu\otimes\nu\Big) + \mathcal{O}(1) \\
&= \frac{1}{\delta}\big(\frac{\mathrm{p}_h}{2}|\partial_z\Phi^0|^2\mathbb{I} + h|\partial_z\Phi^0|^2\nu\otimes\nu\big) + \mathcal{O}(1) \\
&= \frac{1}{\delta}\big(\frac{\mathrm{e}_h}{2}|\partial_z\Phi^0|^2\mathbb{I} - h|\partial_z\Phi^0|^2(\mathbb{I} - \nu\otimes\nu)\big) + \mathcal{O}(1) \\
&= -\frac{1}{\delta}h(R^0)|\partial_z\Phi^0|^2(\mathbb{I} - \nu\otimes\nu) + \mathcal{O}(1),
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_{\Gamma_\delta} \zeta_2\bullet\big(\Pi_\delta^{\mathrm{T}} v_\delta\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t} \zeta_2\bullet\big(\delta\Pi_\delta^{\mathrm{T}} v_\delta\big)(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t} \zeta_2\bullet\Big(-h(R^0)|\partial_z\Phi^0|^2(\mathbb{I}-\nu\otimes\nu) + \mathcal{O}(\delta)\Big)\big(V^0 + \mathcal{O}(\delta)\big) \\
&\qquad\qquad \cdot(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
&\longrightarrow \int_{\mathbb{R}}\int_{\Gamma_t} \zeta_2\bullet\Big(-\int_{-\infty}^{+\infty} h(R^0)|\partial_z\Phi^0|^2\,\mathrm{d}z\Big)(\mathbb{I}-\nu\otimes\nu)v_{\tan}\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t
\end{aligned}
$$

as $\delta\to 0$. Thus also (6.11) is proved.

Theorem 6.2 in connection with inequality 2.2 implies the following free energy inequality.

6.3 Theorem. For example 2.6 the limit free energy inequality in the distributional sense is

$$
\begin{aligned}
&\partial_t\Big(\sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s\boldsymbol{\mu}_\Gamma\Big) \\
&+\operatorname{div}\Big(\sum_\alpha\Big(\big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha + \Pi^{\alpha\,\mathrm{T}} v^\alpha\Big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s v_\Gamma \boldsymbol{\mu}_\Gamma\Big) - \sum_\alpha v^\alpha\bullet\mathbf{f}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha} \\
&= -\sum_\alpha \mathrm{D}v^\alpha : \mathbb{S}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} - \sum_\alpha \lambda^\alpha\big(\frac{1}{2}\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu_{\mathcal{U}^\alpha} - f^\alpha\big)\boldsymbol{\mu}_\Gamma \le 0.
\end{aligned}
$$

Proof. In 6.2 we make the following observations. The definition of λ and λ^α in (6.7) gives

$$
\lambda\big(\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu + f^2 - f^1\big) = \sum_\alpha \lambda^\alpha\big(\frac{1}{2}\boldsymbol{\gamma}\kappa_\Gamma\bullet\nu_{\mathcal{U}^\alpha} - f^\alpha\big) \tag{6.12}
$$

and since f^s and Π^s are given by the surface tension $\boldsymbol{\gamma}$ we obtain

$$f^s v_{\tan} = \boldsymbol{\gamma} v_{\tan}, \quad {\Pi^s}^{\mathrm{T}} v_{\tan} = -\boldsymbol{\gamma} v_{\tan},$$

that is $f^s v_{\tan} + {\Pi^s}^{\mathrm{T}} v_{\tan} = 0$. □

From [2, Theorem 2.8] one obtains that this distributional equation is equivalent to differential equations in $\mathcal{U}^\alpha$ and a differential equation on Γ, where we use (6.12).

6.4 Theorem. For example 2.6 the distributional free energy inequality in the limit is equivalent to the following strong equations and inequalities:

$$\begin{aligned}\partial_t\big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big) + \operatorname{div}\big(\big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha + {\Pi^\alpha}^{\mathrm{T}} v^\alpha\big) - v^\alpha \bullet \mathbf{f}^\alpha \\ = -\mathrm{D}v^\alpha : \mathbb{S}^\alpha \leq 0\end{aligned} \tag{6.13}$$

in the domain $\mathcal{U}^\alpha$, and

$$\begin{aligned}\partial_t^\Gamma f^s - f^s \kappa_\Gamma \bullet v_\Gamma - \sum_{\alpha=1}^{2}\Big(v^\alpha \bullet \Pi^\alpha \nu_{\mathcal{U}^\alpha} - \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\lambda^\alpha\Big) \\ = -\lambda\big(\boldsymbol{\gamma}\kappa_\Gamma \bullet \nu + f^2 - f^1\big) \leq 0.\end{aligned} \tag{6.14}$$

on the surface Γ.

The equalities in this theorem can be proved by the following considerations. On the interface the velocity is continuous, that is $v := v^1 = v^2$, and $\varrho := \varrho^1 = \varrho^2$. The definitions of λ^α and λ imply, that in (6.14)

$$\sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\lambda^\alpha = \sum_\alpha f^\alpha \lambda^\alpha = -\lambda(f^2 - f^1)$$

and

$$\begin{aligned}f^s \kappa_\Gamma \bullet v_\Gamma + \sum_{\alpha=1}^{2} v^\alpha \bullet \Pi^\alpha \nu_{\mathcal{U}^\alpha} &= \boldsymbol{\gamma}\kappa_\Gamma \bullet v_\Gamma + v \bullet \sum_{\alpha=1}^{2} \Pi^\alpha \nu_{\mathcal{U}^\alpha} \\ &= \boldsymbol{\gamma}\kappa_\Gamma \bullet v_\Gamma - v \bullet (\boldsymbol{\gamma}\kappa_\Gamma \bullet \nu \nu) = \lambda \boldsymbol{\gamma}\kappa_\Gamma \bullet \nu.\end{aligned}$$

And $\partial_t^\Gamma f^s = 0$ (see [17, Lemma 28]).

7 Jump in the density

Here we are dealing with example 2.5. Now in this case $\partial_z V^0 \bullet \nu$ is nonzero in general. Therefore the right-hand side of the equipartition of energy 5.1 does not vanish (in contrast to section 6). This has, as we shall see, important consequences. So it follows from (5.9) that

$$\mathbf{m}^0\big(W(0) - W(1)\big) < 0,$$

hence $\mathbf{m}^0$ has the opposite sign of $W(0)-W(1)\neq 0$. More important is the fact, at the first look a surprising circumstance, that the δ-free energy identity (5.1) has a leading nonzero term of order $\frac{1}{\delta}$. As a consequence the δ-free energy identity has to be multiplied by δ. This procedure keeps the inequality untouched, because the dissipation term is multiplied by a positive number. Hence the sign of the free energy production is preserved.

Global test functions and multiplication by δ

We choose test functions ζ as function of (t,x) in (5.1) and multiply the entire equation by δ. We obtain in the limit $\delta\to 0$

7.1 Theorem. Consider the solution of the δ-problem. As $\delta\to 0$ the terms in the free energy identity multiplied by δ converge pointwise in the sense of distributions in $\mathcal{U}$:

$$\begin{gathered}\delta\big(f_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2\big)\,\mathrm{L}_{n+1}\longrightarrow\sum_\alpha \varrho^\alpha W(\varphi^\alpha)\boldsymbol{\mu}_{\mathcal{U}^\alpha},\\ \delta\big(\big(f_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2\big)v_\delta-\dot\varphi\delta h(\varrho_\delta)\nabla\varphi\big)\,\mathrm{L}_{n+1}\longrightarrow\sum_\alpha \varrho^\alpha W(\varphi^\alpha)v^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha},\\ \delta\,{\Pi_\delta}^{\mathrm{T}}v_\delta\,\mathrm{L}_{n+1}\longrightarrow 0,\qquad \delta v_\delta\bullet\mathbf{f}_\delta\,\mathrm{L}_{n+1}\longrightarrow 0,\\ \delta g_\delta\,\mathrm{L}_{n+1}\longrightarrow -\Big(\int_{-\infty}^{+\infty}\widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\,\mathrm{d}z\Big)\boldsymbol{\mu}_\Gamma.\end{gathered}\tag{7.1}$$

Here

$$\varphi^\alpha:=\begin{cases}0\text{ in }\mathcal{U}^1\text{ for }\alpha=1,\\ 1\text{ in }\mathcal{U}^2\text{ for }\alpha=2.\end{cases}$$

Proof. We multiply the free energy inequality (5.1) by $\delta>0$. The right-hand side of (5.1) becomes, since $\mathrm{D}v_\delta$ is bounded in $\mathcal{U}^\alpha$ and since in $\mathcal{U}^\alpha$ the functions approximate their limit exponentially in z,

$$\delta\int_{\mathcal{U}}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t=-\delta\int_{\Gamma_\delta}\zeta\mathrm{D}v\mathbin{\vdots}\mathbb{S}\,\mathrm{d}x\,\mathrm{d}t-\delta\int_{\Gamma_\delta}\zeta\boldsymbol{\tau}_\delta\frac{\mu_\delta}{\varrho_\delta}\,\mathrm{d}x\,\mathrm{d}t+\mathcal{O}(\delta).$$

Since $-\varrho_\delta\dot\varphi=\boldsymbol{\tau}_\delta=\eta_\delta\mu_\delta$ we obtain with $\dot\varphi$ from the above list

$$\begin{aligned}\frac{\boldsymbol{\tau}_\delta\mu_\delta}{\varrho_\delta}&=\frac{\varrho_\delta}{\eta_\delta}\dot\varphi^2=\frac{R^0+\mathcal{O}(\delta)}{\eta_0(R^0)+o(\delta)}\delta\Big(-\frac{1}{\delta}\Lambda^0\partial_z\Phi^0+\mathcal{O}(1)\Big)^2,\\ &=\frac{1}{\delta}\frac{R^0}{\eta_0(R^0)}\big(\Lambda^0\partial_z\Phi^0\big)^2+\mathcal{O}(1)\end{aligned}$$

and this gives

$$\begin{aligned}\int_{\Gamma_\delta}\zeta\boldsymbol{\tau}_\delta\frac{\mu_\delta}{\varrho_\delta}\,\mathrm{d}x\,\mathrm{d}t&=\int_{\Gamma_\delta}\Big(\zeta\frac{1}{\delta}\frac{R^0}{\eta_0(R^0)}\big(\Lambda^0\partial_z\Phi^0\big)^2+\mathcal{O}(1)\Big)\mathrm{d}x\,\mathrm{d}t\\ &=\int_{\mathbb{R}}\int_{\Gamma_t}\int_{-z_\delta}^{+z_\delta}\zeta\frac{R^0}{\eta_0(R^0)}\big(\Lambda^0\partial_z\Phi^0\big)^2\mathrm{d}z\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}t+\mathcal{O}(\varepsilon_\delta)=\mathcal{O}(1).\end{aligned}$$

Therefore the right-hand side is

$$
\begin{aligned}
&= -\delta \int_{\Gamma_\delta} \zeta \mathrm{D}v : \mathbb{S} \,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(\delta) \\
&= -\int_{\Gamma_\delta} \zeta \Big(\frac{1}{\delta} \widetilde{a}(R^0, \Phi^0) |\partial_z V^0|^2 + \mathcal{O}(1) \Big) \mathrm{d}x\,\mathrm{d}t + \mathcal{O}(\delta) \\
&\to -\int_{\mathbb{R}} \int_{\Gamma_t} \zeta \int_{-\infty}^{+\infty} \widetilde{a}(R^0, \Phi^0) |\partial_z V^0 \bullet \nu|^2 \,\mathrm{d}z\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}t
\end{aligned}
$$

as $\delta \to 0$. And for the left-hand side we obtain for test functions $\zeta_1 \in C_0^\infty(\mathcal{U};\mathbb{R})$ and $\zeta_2 \in C_0^\infty(\mathcal{U};\mathbb{R}^n)$, since locally in $\mathcal{U}^\alpha$ the functions approximate their limit exponentially in z,

$$
\begin{aligned}
&-\delta \int_{\mathcal{U}} \Big\{ \zeta_1 \big(f_\delta + \frac{1}{2} \varrho |v|^2 \big) + \zeta_2 \bullet \big(f_\delta v + \frac{1}{2} \varrho |v|^2 v + \Pi_\delta^{\mathrm{T}} v - \dot{\varphi} \delta h \nabla \varphi \big) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&= -\sum_{\alpha=1}^{2} \int_{\mathcal{U}_\delta^\alpha} \Big\{ \zeta_1 \big(\delta f_\delta \big) + \zeta_2 \bullet \big(\delta f_\delta v_\delta \big) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&- \int_{\Gamma_\delta} \Big\{ \zeta_1 \big(\varrho_\delta W(\varphi) \big) + \zeta_2 \bullet \big(\varrho_\delta W(\varphi) v_\delta \big) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&-\delta \int_{\Gamma_\delta} \Big\{ \zeta_1 \frac{\delta h}{2} |\nabla \varphi|^2 + \zeta_2 \bullet \big(\frac{\delta h}{2} |\nabla \varphi|^2 v_\delta - \dot{\varphi} \delta h \nabla \varphi \big) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&-\delta \int_{\Gamma_\delta} \zeta_2 \bullet (\Pi_\delta^{\mathrm{T}} v) \,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(\delta).
\end{aligned}
$$

Now

$$
\int_{\Gamma_\delta} \big| \zeta_1 \big(\varrho_\delta W(\varphi) \big) + \zeta_2 \bullet \big(\varrho_\delta W(\varphi) v_\delta \big) \big| \,\mathrm{d}x\,\mathrm{d}t \le C \int_{\Gamma_\delta} |\zeta_1| \,\mathrm{d}x\,\mathrm{d}t = \mathcal{O}(\varepsilon_\delta),
$$

and using the formula for $\dot{\varphi}$ from the above list

$$
\begin{aligned}
&\int_{\Gamma_\delta} \Big| \zeta_1 \frac{\delta h}{2} |\nabla \varphi|^2 + \zeta_2 \bullet \big(\frac{\delta h}{2} |\nabla \varphi|^2 v_\delta - \dot{\varphi} \delta h \nabla \varphi \big) \Big| \,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{C}{\delta} \int_{\Gamma_\delta} (|\zeta_1| + |\zeta_2|) \big(|\partial_z \Phi^0|^2 + (|\Lambda^0 \partial_z \Phi^0| + \mathcal{O}(\delta)) |\partial_z \Phi^0| \big) \mathrm{d}x\,\mathrm{d}t \\
&\le C \int_{\mathbb{R}} \int_{\Gamma_t} \int_{-z_\delta}^{+z_\delta} (|\zeta_1| + |\zeta_2|) \big(|\partial_z \Phi^0|^2 + \mathcal{O}(\delta^2) \big) \mathrm{d}z\,\mathrm{d}\mathrm{H}_{n-1}(y)\,\mathrm{d}t = \mathcal{O}(1).
\end{aligned}
$$

Further, since $\partial_z V^0$ satisfies $\partial_z V^0 \in \mathrm{span}\{\nu\}$ and (4.22) holds,

$$\begin{aligned}
&\int_{\Gamma_\delta} \big|\zeta_2 \bullet \Pi_\delta^{\mathrm{T}} v_\delta\big| \,\mathrm{d}x\mathrm{d}t \\
&= \int_{\Gamma_\delta} \left| \zeta_2 \bullet \Big(\frac{\delta \mathrm{p}_h}{2} |\nabla\varphi|^2 \mathbb{I} + \delta h \nabla\varphi \otimes \nabla\varphi - \mathbb{S}(\varrho_\delta, \varphi, (\nabla v_\delta)^{\mathrm{S}}) \Big) v_\delta \right| \mathrm{d}x\mathrm{d}t \\
&\le \frac{C}{\delta} \int_{\Gamma_\delta} |\zeta_2| \big(|\partial_z \Phi^0|^2 + |\partial_z V^0| + \mathcal{O}(\delta) \big) \,\mathrm{d}x\mathrm{d}t \\
&\le C \int_{\mathbb{R}} \int_{\Gamma_t} \int_{-z_\delta}^{+z_\delta} |\zeta_2| \big(|\partial_z \Phi^0|^2 + \mathcal{O}(\delta) \big) \,\mathrm{d}z \mathrm{d}\mathrm{H}_{n-1}(y) \mathrm{d}t = \mathcal{O}(1).
\end{aligned}$$

Hence the above integral equals

$$\begin{aligned}
&= -\sum_{\alpha=1}^{2} \int_{\mathcal{U}_\delta^\alpha} \Big\{ \zeta_1 \big(\delta f_\delta\big) + \zeta_2 \bullet \big(\delta f_\delta v_\delta\big) \Big\} \mathrm{d}x\mathrm{d}t + \mathcal{O}(\varepsilon_\delta) + \mathcal{O}(\delta) \\
&= -\sum_{\alpha=1}^{2} \int_{\mathcal{U}_\delta^\alpha} \Big\{ \zeta_1 \big(\varrho_\delta W(\varphi)\big) + \zeta_2 \bullet \big(\varrho_\delta W(\varphi) v_\delta\big) \Big\} \mathrm{d}x\mathrm{d}t + \mathcal{O}(\varepsilon_\delta) \\
&\to -\sum_{\alpha=1}^{2} \int_{\mathcal{U}^\alpha} \Big\{ \zeta_1 \big(\varrho^\alpha W(\varphi^\alpha)\big) + \zeta_2 \bullet \big(\varrho^\alpha W(\varphi^\alpha) v^\alpha\big) \Big\} \mathrm{d}x\mathrm{d}t
\end{aligned}$$

as $\delta \to 0$. □

Consequently the δ-free energy inequality (5.1) multiplied by δ converges to the following equality and inequality.

7.2 Theorem. The following is satisfied

$$\begin{aligned}
&\partial_t \Big(\sum_{\alpha=1}^{2} \varrho^\alpha W(\varphi^\alpha) \boldsymbol{\mu}_{\mathcal{U}^\alpha} \Big) + \mathrm{div} \Big(\sum_{\alpha=1}^{2} \varrho^\alpha W(\varphi^\alpha) v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} \Big) \\
&\qquad = -\Big(\int_{-\infty}^{+\infty} \widetilde{a}(R^0, \Phi^0) |\partial_z V^0 \bullet \nu|^2 \,\mathrm{d}z \Big) \boldsymbol{\mu}_\Gamma \le 0.
\end{aligned}$$

There is a strong version of this identity.

This is essential part of the free energy inequality in the limit. To derive the strong version of this (in)equality, we refer to [2, Theorem 2.8]. We carry out this procedure: We obtain for test functions $\zeta \in C_0^\infty(\mathcal{U};\mathbb{R})$

$$
\begin{aligned}
&-\Big\langle \partial_t\zeta, \sum_{\alpha=1}^{2} \varrho^\alpha W(\varphi^\alpha)\boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big\rangle - \Big\langle \nabla\zeta, \sum_{\alpha=1}^{2} \varrho^\alpha W(\varphi^\alpha) v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\Big\rangle \\
&= -\sum_{\alpha=1}^{2} \int_{\mathcal{U}^\alpha} W(\varphi^\alpha)\Big\{\partial_t\zeta\varrho^\alpha + \nabla\zeta\bullet(\varrho^\alpha v^\alpha)\Big\}\mathrm{d}x\,\mathrm{d}t \\
&= -\sum_{\alpha=1}^{2} W(\varphi^\alpha)\big(\langle\partial_t\zeta, \varrho^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}\rangle + \langle\nabla\zeta, \varrho^\alpha v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\rangle\big) \\
&= \sum_{\alpha=1}^{2} W(\varphi^\alpha)\langle\zeta, \partial_t(\varrho^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}) + \operatorname{div}(\varrho^\alpha v^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha})\rangle \\
&= (W(\varphi^1) - W(\varphi^2))\langle\zeta, \boldsymbol{\tau}\boldsymbol{\mu}_\Gamma\rangle = (W(0) - W(1))\langle\zeta, \boldsymbol{\tau}\boldsymbol{\mu}_\Gamma\rangle
\end{aligned}
$$

by the first two mass equations of (4.14). Thus the equation in 7.2 becomes

$$
(W(0) - W(1))\boldsymbol{\tau}\boldsymbol{\mu}_\Gamma = -\Big(\int_{-\infty}^{+\infty} \widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\mathrm{d}z\Big)\boldsymbol{\mu}_\Gamma \le 0,
$$

or

$$
(W(0) - W(1))\boldsymbol{\tau} = -\int_{\mathbb{R}} \widetilde{a}(R^0,\Phi^0)|\partial_z V^0\bullet\nu|^2\mathrm{d}z \le 0,
$$

an equation which is familiar to us from (5.9), since $\boldsymbol{\tau} = \mathbf{m}^0$. Hence one can say that the free energy inequality in its main part is the equipartition of energy integrated over $\mathbb{R}$.

Adding functions to normalize

We mention that we have derived the free energy inequality on the surface. What is left is the free energy inequality in the bulk regions. In these regions the free energy density f_δ for the δ-problem contains the term

$$
\frac{1}{\delta}\varrho_\delta W(\varphi) \text{ where } W(\varphi) \to W(\varphi^\alpha) \text{ as } \delta \to 0.
$$

Since $W(0) \neq W(1)$ the value of the free energy becomes unbounded at least in one phase. Therefore in order to proceed one has to normalize the free energy. A normalization in the bulk regions by a constant factor times the total mass ϱ_δ is a common procedure and it does not change the relevant mass and momentum balance. This is because the term

$$
\varrho_\delta f_{\delta'\varrho_\delta} - f_\delta,
$$

which enters the pressure, is invariant under a linear expression in ϱ_δ. Therefore the terms $\mathbb{P}_\delta$ and μ_δ in the mass and momentum equation are not changed. Note, that this is true in the bulk regions. Therefore we define

7.3 Definition. Consider locally in each set $\mathcal{U}^\alpha$

$$\widetilde{W}^\alpha(\varphi):=W(\varphi)-W(\varphi^\alpha)\,,$$

$$\widetilde{f}_\delta^\alpha:=\frac{1}{\delta}\varrho_\delta\widetilde{W}^\alpha(\varphi)+\delta h(\varrho_\delta)\frac{|\nabla\varphi|^2}{2}+U(\varrho_\delta,\varphi)=f_\delta-\frac{1}{\delta}\varrho_\delta W(\varphi^\alpha)\,.$$

Thus also the new function $\widetilde{W}^\alpha$ has a minimum at φ^α. Note that $\widetilde{f}_\delta^\alpha$ is defined only locally in $\mathcal{U}^\alpha$.

Global test functions and multiplication by 1

For the new free energy $\widetilde{f}_\delta^\alpha$ one gets the following inequality locally in $\mathcal{U}^\alpha$. For $\zeta\in C_0^\infty(\mathcal{U}^\alpha)$, $\mathcal{U}^\alpha$ fixed, there holds

$$\begin{aligned}\int_{\mathcal{U}^\alpha}\Big(&-\partial_t\zeta\big(\widetilde{f}_\delta^\alpha+\frac{\varrho_\delta}{2}|v_\delta|^2\big)\\ &-\nabla\zeta\bullet\big(\widetilde{f}_\delta^\alpha v_\delta+\frac{\varrho_\delta}{2}|v_\delta|^2v_\delta+\mathit{\Pi}_\delta^{\mathrm{T}}v_\delta-\dot{\varphi}\delta h\nabla\varphi\big)\\ &-\zeta v_\delta\bullet\mathbf{f}_\delta\Big)\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathcal{U}^\alpha}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t\le 0\quad\text{provided }\zeta\ge 0.\end{aligned}\tag{7.2}$$

This equation one obtains also, if one subtracts from (5.1) in $\mathcal{U}^\alpha$ the $W(\varphi^\alpha)$-multiple of the total mass balance. Since this total mass has no production term, the original free energy inequality is saved in the distributional sense in the domain $\mathcal{U}^\alpha$, that is, on the right-hand side we have the production term $g_\delta\le 0$. It is now clear, that we obtain the following theorem in the limit $\delta\to 0$.

7.4 Theorem. As $\delta\to 0$ the solution converges pointwise in $\mathcal{U}^\alpha$:

$$\begin{gathered}\big(\widetilde{f}_\delta^\alpha+\frac{\varrho_\delta}{2}|v_\delta|^2\big)\longrightarrow f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2,\\ \big(\widetilde{f}_\delta^\alpha+\frac{\varrho_\delta}{2}|v_\delta|^2\big)v_\delta-\dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\longrightarrow\big(f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha,\\ {\mathit{\Pi}_\delta}^{\mathrm{T}}v_\delta\longrightarrow{\mathit{\Pi}^\alpha}^{\mathrm{T}}v^\alpha\qquad v_\delta\bullet\mathbf{f}_\delta\longrightarrow v^\alpha\bullet\mathbf{f}^\alpha,\\ g_\delta\longrightarrow-\mathrm{D}v^\alpha:\mathbb{S}^\alpha.\end{gathered}\tag{7.3}$$

Here

$$f^1=\widehat{f}^1(\varrho^1)=U(\varrho^1,0)\text{ and }f^2=\widehat{f}^2(\varrho^2)=U(\varrho^2,0).\tag{7.4}$$

Proof. This follows by the pointwise convergence of terms and

$$\frac{1}{\delta}\big(W(\varphi)-W(\varphi^\alpha)\big)\to 0\text{ as }\delta\to 0$$

since $\varphi^\alpha=0,1$. □

As a consequence one obtains the identity in $\mathcal{U}^\alpha$

$$\partial_t(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2) + \operatorname{div}\big((f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)v^\alpha + \Pi^{\alpha\,\mathrm{T}} v^\alpha\big) - v^\alpha \bullet \mathbf{f}^\alpha \\ = -\mathrm{D}v^\alpha : \mathbb{S}^\alpha \leq 0. \tag{7.5}$$

This is the usual free energy inequality in domains for compressible fluids.

Altogether, multiplying the equation by δ and letting $\delta \to 0$ one obtains the inequality 7.2 essentially near Γ, and on the other hand by multiplying the equation by 1 and letting $\delta \to 0$ one obtains the inequality 7.5 in $\mathcal{U}^\alpha$. Therefore in the limit $\delta \to 0$ we have two free energy inequalities, that is, the following is true.

7.5 Theorem. In the limit $\delta \to 0$ we obtain the following free energy inequality on $\mathcal{U}^\alpha$:

$$\partial_t(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2) + \operatorname{div}\big((f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)v^\alpha + \Pi^{\alpha\,\mathrm{T}} v^\alpha\big) - v^\alpha \bullet \mathbf{f}^\alpha \\ = -\mathrm{D}v^\alpha : \mathbb{S}^\alpha \leq 0$$

and the following free energy inequality on Γ:

$$\sum_{\alpha=1}^{2} \varrho^\alpha W(\varphi^\alpha)(v_\Gamma - v^\alpha) \bullet \nu_{\mathcal{U}^\alpha} = -\int_{\mathbb{R}} \widetilde{a}(R^0, \Phi^0) |\partial_z V^0 \bullet \nu|^2 \, \mathrm{d}z \leq 0.$$

8 Cahn-Hilliard example

We start with the δ-version of the free energy inequality, that is (5.1), where the constitutive equations of 2.7 are inserted.

Global test functions

We choose test functions in the global variables (t,x). We prove that in the free energy identity (5.1) each term converges to a limit.

8.1 Theorem. Consider a solution $(\varrho_\delta, \varphi, v_\delta)$ of (1.3) for example 2.7 assuming (4.2) and (4.4). Then as $\delta \to 0$ the solution converges pointwise in the sense of distributions in $\mathcal{U}$:

$$\big(f_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2\big)\mathrm{L}_{n+1} \longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s \boldsymbol{\mu}_\Gamma\,,$$

$$\big((f_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot{\varphi}\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{L}_{n+1} \\ \longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s(v_\Gamma + v_{\tan})\boldsymbol{\mu}_\Gamma\,,$$

$$-\frac{\mu_\delta}{\varrho_\delta}J_\delta \mathrm{L}_{n+1} \longrightarrow -\sum_\alpha \frac{\mu^\alpha}{\varrho^\alpha} J^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\,, \qquad J^\alpha{:=}m_\alpha(\varrho^\alpha)\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big),$$

$$\Pi_\delta{}^{\mathrm{T}} v_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha \Pi^{\alpha\mathrm{T}} v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^{s\mathrm{T}} v_{\tan}\boldsymbol{\mu}_\Gamma\,,$$

$$v_\delta \bullet \mathbf{f}_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha v^\alpha \bullet \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\,,$$

$$g_\delta \mathrm{L}_{n+1} \longrightarrow -\sum_\alpha \mathrm{D}v^\alpha{:}\mathbb{S}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} - \sum_\alpha m_\alpha(\varrho^\alpha)\Big|\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big)\Big|^2 \boldsymbol{\mu}_{\mathcal{U}^\alpha}.$$

Here f^s and Π^s are as in (6.6).

Proof. We concentrate on the convergence of two terms, the flux term $-\frac{\mu_\delta}{\varrho_\delta}J_\delta$ (see (2.4)) and the dissipative term g_δ (see (2.6)). The first term we write for $\zeta_2{\in}C_0^\infty(\mathcal{U};\mathbb{R}^n)$ in distributional formulation

$$-\int_{\mathcal{U}} \zeta_2 \bullet \Big(\frac{\mu_\delta}{\varrho_\delta}J_\delta\Big)\mathrm{d}x\mathrm{d}t{=}\ -\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta_2 \bullet \Big(\frac{m_0(\varrho_\delta,\varphi)\mu_\delta}{\varrho_\delta}\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\Big)\mathrm{d}x\mathrm{d}t$$
$$-\int_{\Gamma_\delta} \zeta_2 \bullet \Big(\frac{m_0(\varrho_\delta,\varphi)\mu_\delta}{\varrho_\delta}\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\Big)\mathrm{d}x\mathrm{d}t.$$

We obtain in $\mathcal{U}^\alpha$ as $\delta{\to}0$

$$-\int_{\mathcal{U}_\delta^\alpha} \zeta_2 \bullet \Big(\frac{m_0(\varrho_\delta,\varphi)\mu_\delta}{\varrho_\delta}\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\Big)\mathrm{d}x\mathrm{d}t$$
$$\longrightarrow -\int_{\mathcal{U}^\alpha} \zeta_2 \bullet \Big(\frac{m_0(\varrho^\alpha,\varphi^\alpha)\mu^\alpha}{\varrho^\alpha}\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big)\Big)\mathrm{d}x\mathrm{d}t.$$

Considering in Γ_δ the inner expansion $\mu_\delta{=}M^0+\delta M^1+o(\delta)$, we obtain

$$-\int_{\Gamma_\delta} \zeta_2 \bullet \Big(\frac{m_0(\varrho_\delta,\varphi)\mu_\delta}{\varrho_\delta}\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\Big)\mathrm{d}x\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\int_{\Gamma_t} \zeta_2 \bullet \Big(\frac{m_0(R^0,\Phi^0)M^0}{R^0}\frac{1}{\delta}\partial_z\Big(\frac{M^0}{R^0}\Big)\nu+\mathcal{O}(1)\Big)$$
$$\cdot(1+\mathcal{O}(\varepsilon_\delta))\mathrm{d}\mathrm{H}_{n-1}(y)\mathrm{d}r\mathrm{d}t.$$

Since $\partial_z\big(\frac{M^0}{R^0}\big){=}0$ by 10.1, this is

$$=\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\int_{\Gamma_t} \zeta_2 \bullet \mathcal{O}(1)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{d}\mathrm{H}_{n-1}(y)\mathrm{d}r\mathrm{d}t$$
$$\longrightarrow 0 \text{ as } \delta{\to}0.$$

The second term is for $\zeta \in C_0^\infty(\mathcal{U};\mathbb{R})$

$$
\begin{aligned}
-\int_{\Gamma_\delta} \zeta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big) \bullet J_\delta \mathrm{d}x\mathrm{d}t &= -\int_{\Gamma_\delta} \zeta \cdot m_0(\varrho_\delta,\varphi)\Big|\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\Big|^2 \mathrm{d}x\mathrm{d}t \\
&= -\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t} \zeta \delta m_0(R^0,\Phi^0)\Big|\frac{1}{\delta}\partial_z\big(\frac{M^0}{R^0}\big) + \mathcal{O}(1)\Big|^2 (1+\mathcal{O}(\varepsilon_\delta)) d\mathrm{H}_{n-1}(y)\mathrm{d}z\mathrm{d}t \\
&\longrightarrow 0 \text{ as } \delta \to 0 .
\end{aligned}
$$

by using again 10.1. It follows by standard arguments

$$
\begin{aligned}
\int_{\mathcal{U}} \zeta g_\delta \mathrm{d}x\mathrm{d}t \longrightarrow \; & -\sum_\alpha \int_{\mathcal{U}^\alpha} \zeta \mathrm{D}v^\alpha \colon \mathbb{S}^\alpha \mathrm{d}x\mathrm{d}t \\
& -\sum_\alpha \int_{\mathcal{U}^\alpha} \zeta \cdot m_\alpha(\varrho^\alpha)\Big|\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big)\Big|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

□

From 8.1 one obtains the following immediate consequence of 2.3.

8.2 Theorem. For example 2.7 the limit free energy inequality in the distributional sense is

$$
\begin{aligned}
&\partial_t\Big(\sum_\alpha (f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s \boldsymbol{\mu}_\Gamma\Big) \\
&+\mathrm{div}\Big(\sum_\alpha \big((f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)v^\alpha + \Pi^{\alpha\mathrm{T}} v^\alpha - \frac{\mu^\alpha}{\varrho^\alpha} J^\alpha\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s v_\Gamma \boldsymbol{\mu}_\Gamma\Big) \\
&-\sum_\alpha v^\alpha \bullet \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} \\
&= -\sum_\alpha \mathrm{D}v^\alpha \colon \mathbb{S}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} - \sum_\alpha m_\alpha(\varrho^\alpha)\Big|\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big)\Big|^2 \boldsymbol{\mu}_{\mathcal{U}^\alpha} \le 0
\end{aligned}
$$

From [2, Theorem 2.8] one obtains that this distributional equation is equivalent to differential equations in $\mathcal{U}^\alpha$ and a differential equation on Γ.

8.3 Theorem. For example 2.7 the distributional free energy inequality in the limit is equivalent to the following strong equations and inequalities:

$$
\begin{aligned}
&\partial_t\big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big) + \mathrm{div}\Big(\big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha + \Pi^{\alpha\mathrm{T}} v^\alpha - \frac{\mu^\alpha}{\varrho^\alpha}J^\alpha\Big) - v^\alpha \bullet \mathbf{f}^\alpha \\
&= -\mathrm{D}v^\alpha \colon \mathbb{S}^\alpha - m_\alpha(\varrho^\alpha)\Big|\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big)\Big|^2 \le 0
\end{aligned}
$$

in the domain $\mathcal{U}^\alpha$ for $\alpha = 1,2$, and

$$
\partial_t^\Gamma f^s - f^s \kappa_\Gamma \bullet v_\Gamma - \sum_{\alpha=1}^2 \Big(v^\alpha \bullet \Pi^\alpha \nu_{\mathcal{U}^\alpha} - (f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)\lambda^\alpha - \frac{\mu^\alpha}{\varrho^\alpha} J^\alpha \bullet \nu_{\mathcal{U}^\alpha}\Big) = 0 \tag{8.1}
$$

on the surface $\mathit{\Gamma}$, where λ^α has to be inserted from (6.7).

From the strong interface equation (8.1) we show the last kinetic equation in (4.30). The definitions in (6.7) concerning ϱ, v, λ^α, and λ imply

$$\sum_\alpha (f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)\lambda^\alpha = -\lambda(f^2 - f^1).$$

As at the end of section 6,

$$f^s \kappa_\Gamma \bullet v_\Gamma + \sum_\alpha v^\alpha \bullet \mathit{\Pi}^\alpha \nu_{\mathcal{U}^\alpha} = \lambda \boldsymbol{\gamma} \kappa_\Gamma \bullet \nu,$$

and $\partial_t^\Gamma f^s = 0$, since $f^s = \boldsymbol{\gamma} = \text{const}$ (see [17, Lemma 28]). Plugging this into (8.1) we obtain

$$\sum_{\alpha=1}^2 \frac{\mu^\alpha}{\varrho^\alpha} J^\alpha \bullet \nu_{\mathcal{U}^\alpha} = \lambda\big(\boldsymbol{\gamma}\kappa_\Gamma \bullet \nu + (f^2 - f^1)\big).$$

Further, using 10.1 and the last identity in (4.32)

$$\sum_{\alpha=1}^2 \frac{\mu^\alpha}{\varrho^\alpha} J^\alpha \bullet \nu_{\mathcal{U}^\alpha} = -\frac{\mu}{\varrho}\big(J^2 - J^1\big)\bullet \nu = \mu\lambda,$$

so that $\mu = \boldsymbol{\gamma}\kappa_\Gamma \bullet \nu + (f^2 - f^1)$, see (4.30).

9 Cahn-Hilliard example with degenerate mobility

Here we are dealing with example 2.8. As in the previous section we consider the constitutive equations of 2.8 and identify the limit equations.

Global test functions

We choose test functions in the global variables (t,x). We prove that in the free energy identity (5.1) each term converges to a limit.

9.1 Theorem. Consider a solution $(\varrho_\delta, \varphi, v_\delta)$ of (1.3) for example 2.8 assuming (4.2) and (4.4). Then as $\delta \to 0$ the solution converges pointwise in the sense of distributions in $\mathcal{U}$:

$$\big(f_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2\big)\mathrm{L}_{n+1} \longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s \boldsymbol{\mu}_\Gamma,$$

$$\big((f_\delta + \frac{\varrho_\delta}{2}|v_\delta|^2)v_\delta - \dot\varphi\delta h(\varrho_\delta)\nabla\varphi\big)\mathrm{L}_{n+1}$$

$$\longrightarrow \sum_\alpha \big(f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + f^s(v_\Gamma + v_{\tan})\boldsymbol{\mu}_\Gamma,$$

$$-\frac{\mu_\delta}{\varrho_\delta}J_\delta \mathrm{L}_{n+1} \longrightarrow -\frac{\mu}{\varrho}J^s\boldsymbol{\mu}_\Gamma\,, \qquad J^s{:=}\widehat{m}(\varrho)\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\,,$$

$$\Pi_\delta{}^{\mathrm{T}} v_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha \Pi^{\alpha\mathrm{T}} v^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} + \Pi^{s\mathrm{T}} v_{\tan}\boldsymbol{\mu}_\Gamma\,,$$

$$v_\delta \bullet \mathbf{f}_\delta \mathrm{L}_{n+1} \longrightarrow \sum_\alpha v^\alpha \bullet \mathbf{f}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha}\,,$$

$$g_\delta \mathrm{L}_{n+1} \longrightarrow -\sum_\alpha \mathrm{D}v^\alpha{:}\mathbb{S}^\alpha \boldsymbol{\mu}_{\mathcal{U}^\alpha} - \widehat{m}(\varrho)\Big|\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\Big|^2 \boldsymbol{\mu}_\Gamma\,.$$

Here ϱ and μ are defined in (4.34) and $\widehat{m}$ in (4.35).

Proof. We concentrate on the convergence of two terms, that are $-(\mu_\delta/\varrho_\delta)J_\delta$ and g_δ.

$$-\int_{\mathcal{U}} \zeta \frac{\mu_\delta}{\varrho_\delta} J_\delta \,\mathrm{d}x\,\mathrm{d}t {=} -\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta \frac{\mu_\delta}{\varrho_\delta} m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Gamma_\delta} \zeta\frac{\mu_\delta}{\varrho_\delta} m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

It holds

$$-\sum_\alpha \int_{\mathcal{U}_\delta^\alpha} \zeta\frac{\mu_\delta}{\varrho_\delta} m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0$$

and considering $\mu_\delta{=}M^0+\delta M^1+o(\delta)$ (see (4.4)) and $\partial_z\big(\frac{M^0}{R^0}\big){=}0$ by theorem 11.1,

$$\begin{aligned}
-\int_{\Gamma_\delta} \zeta\frac{\mu_\delta}{\varrho_\delta} m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t &{=} -\int_{\Gamma_\delta} \zeta \frac{\mu_\delta}{\varrho_\delta}\frac{1}{\delta} m_0(\varrho_\delta)V(\varphi)\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
{=}-\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t} \zeta\delta\Big(&\frac{M^0}{R^0}\frac{1}{\delta}m_0(R^0)V(\Phi^0)\frac{1}{\delta}\partial_z\Big(\frac{M^0}{R^0}\Big) \\
&+\frac{M^0}{R^0}\frac{1}{\delta}m_0(R^0)V(\Phi^0)\nabla^\Gamma\Big(\frac{M^0}{R^0}\Big)+\mathcal{O}(1)\Big)\cdot(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
{=}-\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t} \zeta\Big(&\frac{M^0}{R^0}m_0(R^0)V(\Phi^0)\nabla^\Gamma\Big(\frac{M^0}{R^0}\Big)+\mathcal{O}(\delta)\Big) \\
&\cdot(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t \\
\longrightarrow -\int_{\mathbb{R}}\int_{\Gamma_t}\zeta\frac{\mu}{\varrho}m_0(\varrho)&\Big(\int_{-\infty}^{+\infty}V(\Phi^0)dz\Big)\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t \qquad \text{as} \quad \delta{\to}0 \\
{=}-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta\frac{\mu}{\varrho}\widehat{m}(\varrho)&\,\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t\,.
\end{aligned}$$

Considering

$$
\begin{aligned}
-\int_{\Gamma_\delta}\zeta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet J_\delta\,\mathrm{d}x\,\mathrm{d}t &= -\int_{\Gamma_\delta}\zeta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\bullet\frac{1}{\delta}m_0(\varrho_\delta)V(\varphi)\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\zeta\delta\,\big(\frac{1}{\delta}m_0(R^0)V(\Phi^0)+\mathcal{O}(1)\big)\left|\frac{1}{\delta}\partial_z\big(\frac{M^0}{R^0}\big)+\nabla^\Gamma\big(\frac{M^0}{R^0}\big)+\mathcal{O}(1)\right|^2\\
&\qquad\cdot(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t\\
&\longrightarrow -\int_{\mathbb{R}}\int_{\Gamma_t}\zeta m_0(\varrho)\Big(\int_{-\infty}^{+\infty}V(\Phi^0)dz\Big)\left|\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\right|^2\mathrm{dH}_{n-1}(y)\,\mathrm{d}t \qquad \text{as} \quad \delta\to 0\\
&= -\int_{\mathbb{R}}\int_{\Gamma_t}\zeta\widehat{m}(\varrho)\left|\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\right|^2\mathrm{dH}_{n-1}(y)\,\mathrm{d}t,
\end{aligned}
$$

it follows

$$
\int_{\mathcal{U}}\zeta g_\delta\,\mathrm{d}x\,\mathrm{d}t \longrightarrow -\sum_\alpha\int_{\mathcal{U}^\alpha}\zeta Dv^\alpha{:}\mathbb{S}^\alpha\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}\int_{\Gamma_t}\zeta\widehat{m}(\varrho)\left|\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\right|^2\mathrm{dH}_{n-1}(y)\,\mathrm{d}t.
$$

□

From 9.1 one obtains the following

9.2 Theorem. For example 2.8 the limit free energy inequality in the distributional sense is

$$
\begin{aligned}
&\partial_t\Big(\sum_\alpha\big(f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha}+f^s\boldsymbol{\mu}_\Gamma\Big)\\
&+\mathrm{div}\Big(\sum_\alpha\big(\big(f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha+\Pi^{\alpha\mathrm{T}}v^\alpha\big)\boldsymbol{\mu}_{\mathcal{U}^\alpha}+\big(f^s v_\Gamma-\frac{\mu}{\varrho}J^s\big)\boldsymbol{\mu}_\Gamma\Big)\\
&-\sum_\alpha v^\alpha\bullet\mathbf{f}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}\\
&=-\sum_\alpha \mathrm{D}v^\alpha{:}\mathbb{S}^\alpha\boldsymbol{\mu}_{\mathcal{U}^\alpha}-\widehat{m}(\varrho)\left|\nabla\Big(\frac{\mu}{\varrho}\Big)\right|^2\boldsymbol{\mu}_\Gamma\le 0
\end{aligned}
$$

From [2, Theorem 2.8] one obtains that this distributional equation is equivalent to differential equations in $\mathcal{U}^\alpha$ and a differential equation on Γ.

9.3 Theorem. For example 2.8 the distributional free energy inequality in the limit is equivalent to the following strong equations and inequalities:

$$
\begin{aligned}
\partial_t\big(f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2\big)+\mathrm{div}\Big(\big(f^\alpha+\frac{\varrho^\alpha}{2}|v^\alpha|^2\big)v^\alpha+\Pi^{\alpha\mathrm{T}}v^\alpha\Big)-v^\alpha\bullet\mathbf{f}^\alpha&\\
=-\mathrm{D}v^\alpha{:}\mathbb{S}^\alpha\le 0 \quad \text{in } \mathcal{U}^\alpha \text{ for } \alpha=1,2,&
\end{aligned}
$$

and

$$\partial_t^\Gamma f^s - f^s \kappa_\Gamma \bullet v_\Gamma - \sum_{\alpha=1}^{2}\Big(v^\alpha \bullet \Pi^\alpha \nu_{\mathcal{U}^\alpha} - (f^\alpha + \frac{\varrho^\alpha}{2}|v^\alpha|^2)\lambda^\alpha\Big)$$
$$-\mathrm{div}^\Gamma\Big(\frac{\mu}{\varrho}J^s\Big) = -J^s \bullet \nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big) \le 0 \quad \text{on the surface } \Gamma,$$

where λ^α are as in (6.7).

That these identities are indeed satisfied, follows from the properties of f^s and J^s.

10 Appendix (Proof of Theorem 4.9)

It is the purpose of this section to give a proof of the limit equations for example 2.7. Here we prove the statement about the mass conservation, the momentum equation has already been proven in the general part, see 4.2. The mass conservation for $(\varrho_\delta^1, \varrho_\delta^2)$ or equivalently for $(\varrho_\delta, \varphi)$ in (1.3) is

$$\partial_t \varrho_\delta + \mathrm{div}(\varrho_\delta v_\delta) = 0\,,$$
$$\partial_t(\varrho_\delta \varphi) + \mathrm{div}(\varrho_\delta \varphi v_\delta - J_\delta) = 0\,, \quad J_\delta = m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,,$$
$$\mu_\delta = \frac{\delta f_\delta}{\delta\varphi} = \frac{1}{\delta}\varrho_\delta W'(\varphi) - \delta\,\mathrm{div}(h(\varrho_\delta)\nabla\varphi) + U'_\varphi(\varrho_\delta,\varphi)\,.$$

The distributional formulation of this reads for $\zeta \in C_0^\infty(\mathcal{U};\mathbb{R})$

$$\int_{\mathcal{U}} \Big(\partial_t\zeta\,\varrho_\delta + \nabla\zeta \bullet (\varrho_\delta v_\delta)\Big)\,\mathrm{dL}_{n+1} = 0\,,$$
$$\int_{\mathcal{U}} \Big(\partial_t\zeta(\varrho_\delta\varphi) + \nabla\zeta \bullet (\varrho_\delta\varphi v_\delta - J_\delta)\Big)\,\mathrm{dL}_{n+1} = 0\,, \tag{10.1}$$
$$\int_{\mathcal{U}} \zeta\Big(\mu_\delta - \frac{1}{\delta}\varrho_\delta W'(\varphi) + \delta\,\mathrm{div}(h(\varrho_\delta)\nabla\varphi) - U'_\varphi(\varrho_\delta,\varphi)\Big)\,\mathrm{dL}_{n+1} = 0\,.$$

We consider two classes of test functions in (10.1). The first choice gives as a result an ordinary differential equation which one has to solve in the inner variables. This result is then used in the second choice of the test functions. These test functions are functions of the global variables. Therefore one gets the equations for the outer expansion, and in addition a distributional equation across the interface. We show the following results when $\delta \to 0$. The first result is the $\frac{1}{\delta}$-term at the boundary.

10.1 Theorem. Assume (4.2) and (4.4). Then for $(t,y) \in \Gamma$ we have in local coordinates

$$\partial_z\Big(\frac{M^0}{R^0}\Big) = 0 \quad \text{for all } z \in \mathbb{R}\,,$$
$$R^0 W'(\Phi^0) - \partial_z(h(R^0)\partial_z\Phi^0) = 0 \quad \text{for all } z \in \mathbb{R}\,.$$

This theorem is the version of the usual theorem on the zeroth order Φ^0 of the phase field. It is necessary to show the following result.

10.2 Theorem. Assume (4.2) and (4.4). Then as $\delta\to0$ in the sense of distributions the solution (as distribution) converges pointwise to

$$\varrho_\delta\varphi \mathrm{L}_{n+1}\longrightarrow\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2}, \quad \varrho_\delta\varphi v_\delta \mathrm{L}_{n+1}\longrightarrow\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2},$$
$$\varrho_\delta \mathrm{L}_{n+1}\longrightarrow\varrho^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2}, \quad \varrho_\delta v_\delta \mathrm{L}_{n+1}\longrightarrow\varrho^1 v^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2},$$
$$J_\delta \mathrm{L}_{n+1}\longrightarrow J^1\boldsymbol{\mu}_{\mathcal{U}^1}+J^2\boldsymbol{\mu}_{\mathcal{U}^2}, \quad J^\alpha:=m_\alpha(\varrho^\alpha)\nabla\Big(\frac{\mu^\alpha}{\varrho^\alpha}\Big).$$

Therefore the limit equations are

$$\partial_t(\varrho^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2})+\operatorname{div}(\varrho^1 v^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2})=0,$$
$$\partial_t(\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2})+\operatorname{div}\Big(\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2}-m_1(\varrho^1)\nabla\Big(\frac{\mu^1}{\varrho^1}\Big)\boldsymbol{\mu}_{\mathcal{U}^1}-m_2(\varrho^2)\nabla\Big(\frac{\mu^2}{\varrho^2}\Big)\boldsymbol{\mu}_{\mathcal{U}^2}\Big)=0.$$

Local test functions

With the choice of a local test function $\zeta=\xi$ with a C_0^∞-function ξ around the free boundary we derive the well known first equation of the inner expansion, see 10.1. This is the $\frac{1}{\delta}$-term in (10.1). Explicitly we choose

$$\zeta(t,x)=\xi(t,y,z), \quad x=y+\delta z\nu(t,y), \tag{10.2}$$

where $(t,y)\in\Gamma$, $z\in\mathbb{R}$, and $\nu=\nu_{\mathcal{U}^1}$. The support of $z\mapsto\xi(t,y,z)$ is contained in a fixed interval $[-z_\xi,z_\xi]$, so that $[-z_\xi,z_\xi]\subset[-z_\delta,z_\delta]$ for small $\delta>0$. We compute for the derivatives

$$\begin{aligned}\partial_t\zeta&=\partial_t^\Gamma\xi-\frac{1}{\delta}v_\Gamma\bullet\nu\partial_z\xi+\mathcal{O}(\delta),\\ \nabla\zeta&=\nabla^\Gamma\xi+\frac{1}{\delta}\partial_z\xi\nu+\mathcal{O}(\delta),\end{aligned} \tag{10.3}$$

and we get, if δ is small,

$$\begin{aligned}&\int_{\mathcal{U}}\Big(\partial_t\zeta\cdot(\varrho_\delta\varphi)+\nabla\zeta\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\Big)\mathrm{d}x\,\mathrm{d}t\\ &=\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_t^\Gamma\xi\cdot\varrho_\delta\varphi+\nabla^\Gamma\xi\bullet(\varrho_\delta\varphi v_\delta-J_\delta)+\frac{1}{\delta}\partial_z\xi\cdot\big(\varrho_\delta\varphi(v_\delta-v_\Gamma)-J_\delta\big)\bullet\nu\\ &\qquad+\mathcal{O}(\delta)\mathcal{X}_{\operatorname{supp}\xi}\Big)(1+\mathcal{O}(|r|))\mathrm{d}\mathrm{H}_{n-1}(y)\Big\}\mathrm{d}r\,\mathrm{d}t\end{aligned}$$

$$=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\delta\big(\partial_t^\Gamma\xi\cdot\varrho_\delta\varphi+\nabla^\Gamma\xi\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)+\partial_z\xi\cdot\big(\varrho_\delta\varphi(v_\delta-v_\Gamma)-J_\delta\big)\bullet\nu$$
$$+\mathcal{O}(\delta^2)\mathcal{X}_{\mathrm{supp}\xi}\Big)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t$$

$$=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_z\xi\big(-m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\big)\bullet\nu+\mathcal{O}(1)\mathcal{X}_{\mathrm{supp}\xi}\Big)$$
$$(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t$$

$$=-\frac{1}{\delta}\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_z\xi\cdot m_0(R^0,\Phi^0)\partial_z\Big(\frac{M^0}{R^0}\Big)+\mathcal{O}(\delta)\mathcal{X}_{\mathrm{supp}\xi}\Big)$$
$$(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t$$

$$=-\frac{1}{\delta}\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\partial_z\xi\, m_0(R^0,\Phi^0)\partial_z\Big(\frac{M^0}{R^0}\Big)\mathrm{dH}_{n-1}(y)\mathrm{d}z\mathrm{d}t\ +\ o(1)\,.$$

Then it follows for $\delta\searrow 0$ that the $\frac{1}{\delta}$-term vanishes. Since ξ is arbitrary one gets the first identity in theorem 10.1. Further

$$0=\int_{\mathcal{U}}\zeta\big(\mu_\delta-\frac{1}{\delta}\varrho_\delta W'(\varphi)+\delta\mathrm{div}(h(\varrho_\delta)\nabla\varphi)-U_{'\varphi}(\varrho_\delta,\varphi)\big)\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\Big\{\int_{\Gamma_t}\xi\Big(\mu_\delta-\frac{1}{\delta}\varrho_\delta W'(\varphi)+\delta\mathrm{div}(h(\varrho_\delta)\nabla\varphi)-U_{'\varphi}(\varrho_\delta,\varphi)\Big)$$
$$(1+\mathcal{O}(|r|))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}r\,\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\xi\big(-R^0W'(\Phi^0)+\partial_z(h(R^0)\partial_z\Phi^0)+\mathcal{O}(\delta)\chi_{\mathrm{supp}\xi}\Big)$$
$$(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t$$
$$=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\xi\big(-R^0W'(\Phi^0)+\partial_z(h(R^0)\partial_z\Phi^0)\mathrm{dH}_{n-1}(y)\mathrm{d}z\mathrm{d}t+o(\delta)$$

Since ξ is arbitrary one gets the second identity in theorem 10.1.

Global test functions

We now choose test functions as function of (t,x). Since we claim that the terms converge in the sense of distributions, we have to choose independent test functions $\zeta_1\in C_0^\infty(\mathcal{U};\mathbb{R})$ and $\zeta_2\in C_0^\infty(\mathcal{U};\mathbb{R}^n)$. We obtain

$$\int_{\mathcal{U}}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\mathcal{U}_\delta^2}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{U}_\delta^1}\zeta_2\bullet(-J_\delta)\mathrm{d}x\,\mathrm{d}t \tag{10.4}$$
$$+\int_{\Gamma_\delta}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t\ +\ o(1)\,.$$

Since ϱ_δ and φ are bounded and pointwise convergent with respect to the Lebesgue measure, we obtain further

$$\int_{\mathcal{U}_\delta^2} \big(\zeta_1 \varrho_\delta \varphi + \zeta_2 \bullet (\varrho_\delta \varphi v_\delta - J_\delta)\big) \,\mathrm{d}x\,\mathrm{d}t$$
$$\longrightarrow \int_{\mathcal{U}^2} \Big(\zeta_1 \varrho^2 + \zeta_2 \bullet \Big(\varrho^2 v^2 - m_2(\varrho^2) \nabla\Big(\frac{\mu^2}{\varrho^2}\Big)\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$\int_{\mathcal{U}_\delta^1} \zeta_2 \bullet (-J_\delta) \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_{\mathcal{U}^1} \zeta_2 \bullet \Big(-m_1(\varrho^1) \nabla\Big(\frac{\mu^1}{\varrho^1}\Big)\Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$\int_{\Gamma_\delta} \big(\zeta_1 \varrho_\delta \varphi + \zeta_2 \bullet (\varrho_\delta \varphi v_\delta)\big) \,\mathrm{d}x\,\mathrm{d}t = \mathcal{O}(\varepsilon_\delta) \longrightarrow 0$$

for $\delta \searrow 0$. And the J_δ-term converges, due to the first identity of theorem 10.1, to

$$\int_{\Gamma_\delta} \zeta_2 J_\delta \,\mathrm{d}x\,\mathrm{d}t = \int_{\Gamma_\delta} \zeta_2 m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathbb{R}} \int_{-z_\delta}^{+z_\delta} \int_{\Gamma_t} \zeta_2 \Big(m_0(R^0, \Phi^0) \partial_z \Big(\frac{M^0}{R^0}\Big) + \mathcal{O}(\delta)\Big) \cdot (1 + \mathcal{O}(\varepsilon_\delta)) \,\mathrm{dH}_{n-1}(y) \,\mathrm{d}z\,\mathrm{d}t$$
$$\longrightarrow 0$$

which is the result of theorem 10.2.

11 Appendix (Proof of Theorem 4.11)

It is the purpose of this section to give a proof of the limit equations for example 2.8. Here we prove the statement about the mass conservation, the momentum equation has already been proven in the general part, see 4.2. The mass conservation for $(\varrho_\delta^1, \varrho_\delta^2)$ or equivalent for $(\varrho_\delta, \varphi)$ in (1.3) is

$$\partial_t \varrho_\delta + \mathrm{div}(\varrho_\delta v_\delta) = 0,$$
$$\partial_t (\varrho_\delta \varphi) + \mathrm{div}(\varrho_\delta \varphi v_\delta - J_\delta) = 0,$$
$$J_\delta := m_\delta \nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big), \; m_\delta = \frac{1}{\delta} m_0(\varrho_\delta) V(\varphi) \text{ with } V(\varphi) = \varphi^2 (1-\varphi)^2,$$
$$\mu_\delta = \frac{\delta f_\delta}{\delta \varphi} = \frac{1}{\delta} \varrho_\delta W'(\varphi) - \delta \mathrm{div}(h(\varrho_\delta) \nabla \varphi) + U'_\varphi(\varrho_\delta, \varphi).$$

The distributional formulation of this reads for $\zeta \in C_0^\infty(\mathcal{U}; \mathbb{R})$

$$\int_{\mathcal{U}} \Big(\partial_t \zeta \varrho_\delta + \nabla \zeta \bullet (\varrho_\delta v_\delta)\Big) \,\mathrm{dL}_{n+1} = 0,$$
$$\int_{\mathcal{U}} \Big(\partial_t \zeta (\varrho_\delta \varphi) + \nabla \zeta \bullet (\varrho_\delta \varphi v_\delta - J_\delta)\Big) \,\mathrm{dL}_{n+1} = 0, \tag{11.1}$$
$$\int_{\mathcal{U}} \zeta \Big(\mu_\delta - \frac{1}{\delta} \varrho_\delta W'(\varphi) + \delta \mathrm{div}(h(\varrho_\delta) \nabla \varphi) - U'_\varphi(\varrho_\delta, \varphi)\Big) \,\mathrm{dL}_{n+1} = 0.$$

11.1 Theorem. Assume (4.2) and (4.4). Then for $(t,y)\in\Gamma$ we have in local coordinates

$$\partial_z\Big(\frac{M^0}{R^0}\Big)=0 \quad \text{for all } z\in\mathbb{R}\,,$$
$$R^0W'(\Phi^0)-\partial_z(h(R^0)\partial_z\Phi^0)=0 \quad \text{for all } z\in\mathbb{R}\,.$$

This theorem is the version of the usual theorem on the zeroth order Φ^0 of the phase field. It is necessary to show the following result.

11.2 Theorem. Assume (4.2) and (4.4). Then as $\delta\to 0$ in the sense of distributions the solution (as distribution) converges pointwise to

$$\varrho_\delta\varphi \mathrm{L}_{n+1}\longrightarrow \varrho^2\boldsymbol{\mu}_{\mathcal{U}^2}\,,\quad \varrho_\delta\varphi v_\delta \mathrm{L}_{n+1}\longrightarrow \varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2}\,,$$
$$\varrho_\delta \mathrm{L}_{n+1}\longrightarrow \varrho^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2}\,,\quad \varrho_\delta v_\delta \mathrm{L}_{n+1}\longrightarrow \varrho^1 v^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2 v^2\boldsymbol{\mu}_{\mathcal{U}^2}\,,$$
$$J_\delta \mathrm{L}_{n+1}\longrightarrow J^s\boldsymbol{\mu}_\Gamma\,,\quad J^s:=\widehat{m}(\varrho)\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\,,$$

where

$$\widehat{m}(\varrho)=\int_{-\infty}^{+\infty} m_0(R^0)V(\Phi^0)\,\mathrm{d}z=\frac{m_0(\varrho)}{\varrho}\int_0^1\frac{V(s)}{\sqrt{2(W(s)-W(0))}}\,\mathrm{d}s\,.$$

Therefore the limit equations are

$$\partial_t(\varrho^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2})+\mathrm{div}(\varrho^1v^1\boldsymbol{\mu}_{\mathcal{U}^1}+\varrho^2v^2\boldsymbol{\mu}_{\mathcal{U}^2})=0\,,$$
$$\partial_t(\varrho^2\boldsymbol{\mu}_{\mathcal{U}^2})+\mathrm{div}\Big(\varrho^2v^2\boldsymbol{\mu}_{\mathcal{U}^2}-\widehat{m}(\varrho)\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\boldsymbol{\mu}_\Gamma\Big)=0\,.$$

Local test functions

With the choice of a local test function $\zeta=\xi$ with a C_0^∞-function ξ around the free boundary we derive the well known first equation of the inner expansion, see 11.1. This is the $\frac{1}{\delta}$-term in (11.1). Explicitly we choose

$$\zeta(t,x)=\xi(t,y,z)\,,\qquad x=y+\delta z\nu(t,y)\,,$$

where $(t,y)\in\Gamma$, $z\in\mathbb{R}$, and $\nu=\nu_{\mathcal{U}^1}$. The support of $z\mapsto\xi(t,y,z)$ is contained in a fixed interval $[-z_\xi,z_\xi]$, so that $[-z_\xi,z_\xi]\subset[-z_\delta,z_\delta]$ for small $\delta>0$. We compute for the derivations

$$\partial_t\zeta=\partial_t^\Gamma\xi-\frac{1}{\delta}v_\Gamma\bullet\nu\partial_z\xi+\mathcal{O}(\delta)\,,$$
$$\nabla\zeta=\nabla^\Gamma\xi+\frac{1}{\delta}\partial_z\xi\nu+\mathcal{O}(\delta)\,.$$

and we get, if δ is small,

$$
\begin{aligned}
&\int_{\mathcal{U}}\Big(\partial_t\zeta(\varrho_\delta\varphi)+\nabla\zeta\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\Big)\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\mathbb{R}}\int_{-\varepsilon_\delta}^{+\varepsilon_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_t^\Gamma\xi\cdot\varrho_\delta\varphi+\nabla^\Gamma\xi\bullet(\varrho_\delta\varphi v_\delta-J_\delta)+\frac{1}{\delta}\partial_z\xi\cdot(\varrho_\delta\varphi(v_\delta-v_\Gamma)-J_\delta)\bullet\nu\\
&\qquad\qquad+\mathcal{O}(\delta)\chi_{\mathrm{supp}\xi}\Big)(1+\mathcal{O}(|r|))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}r\,\mathrm{d}t\\
&=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\delta(\partial_t^\Gamma\xi\cdot\varrho_\delta\varphi+\nabla^\Gamma\xi\bullet(\varrho_\delta\varphi v_\delta-J_\delta))+\partial_z\xi\cdot(\varrho_\delta\varphi(v_\delta-v_\Gamma)-J_\delta)\bullet\nu\\
&\qquad\qquad+\mathcal{O}(\delta^2)\chi_{\mathrm{supp}\xi}\Big)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t\\
&=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_z\xi(-m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big))\bullet\nu\\
&\qquad\qquad+\mathcal{O}(1)\chi_{\mathrm{supp}\xi}\Big)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t\\
&=-\frac{1}{\delta^2}\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\Big\{\int_{\Gamma_t}\Big(\partial_z\xi\cdot m_0(R^0)V(\Phi^0)\partial_z\Big(\frac{M^0}{R^0}\Big)\\
&\qquad\qquad+\mathcal{O}(\delta)\chi_{\mathrm{supp}\xi}\Big)(1+\mathcal{O}(\varepsilon_\delta))\mathrm{dH}_{n-1}(y)\Big\}\mathrm{d}z\,\mathrm{d}t\\
&=-\frac{1}{\delta^2}\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\partial_z\xi\,m_0(R^0)V(\Phi^0)\partial_z\Big(\frac{M^0}{R^0}\Big)\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t+o(1)\,.
\end{aligned}
$$

Then it follows for $\delta\searrow 0$ that the $\frac{1}{\delta}$-term vanishes. Since ξ is arbitrary one gets the first identity in theorem 11.1. The second identity of 11.1 follows as in theorem 10.1.

Global test functions

We now choose test functions as function of (t,x). Since we claim that the terms converge in the sense of distributions, we have to choose independent test functions $\zeta_1\in C_0^\infty(\mathcal{U};\mathbb{R})$ and $\zeta_2\in C_0^\infty(\mathcal{U};\mathbb{R})$. We obtain

$$
\begin{aligned}
&\int_{\mathcal{U}}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t\\
&\int_{\mathcal{U}_\delta^2}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t+\int_{\mathcal{U}_\delta^1}\zeta_2\bullet(-J_\delta)\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\Gamma_\delta}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\mathrm{d}x\,\mathrm{d}t+o(1)\,.
\end{aligned}
\tag{11.2}
$$

Since ϱ_δ and φ are bounded and pointwise convergent with respect to the Lebesgue measure, we obtain further

$$\int_{\mathcal{U}_\delta^2}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta-J_\delta)\big)\,\mathrm{d}x\,\mathrm{d}t\longrightarrow\int_{\mathcal{U}^2}\big(\zeta_1\varrho^2+\zeta_2\bullet(\varrho^2v^2)\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\int_{\mathcal{U}_\delta^1}\zeta_2\bullet(-J_\delta)\,\mathrm{d}x\,\mathrm{d}t=\mathcal{O}(\varepsilon_\delta)\longrightarrow 0$$

$$\int_{\Gamma_\delta}\big(\zeta_1\varrho_\delta\varphi+\zeta_2\bullet(\varrho_\delta\varphi v_\delta)\big)\,\mathrm{d}x\,\mathrm{d}t=\mathcal{O}(\varepsilon_\delta)\longrightarrow 0$$

for $\delta\to 0$. And the J_δ-term converges, due to identity $\partial_z\big(\frac{M^0}{R^0}\big)=0$ (the first identity of theorem 11.1),

$$\begin{aligned}\int_{\Gamma_\delta}\zeta_2\bullet J_\delta\,\mathrm{d}x\,\mathrm{d}t&=\int_{\Gamma_\delta}\zeta_2 m_\delta\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t=\int_{\Gamma_\delta}\zeta_2\frac{1}{\delta}m_0(\varrho_\delta)V(\varphi)\nabla\Big(\frac{\mu_\delta}{\varrho_\delta}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{\mathbb{R}}\int_{-z_\delta}^{+z_\delta}\int_{\Gamma_t}\zeta_2\Big(m_0(R^0)V(\Phi^0)\nabla^\Gamma\Big(\frac{M^0}{R^0}\Big)+\mathcal{O}(\delta)\Big)\cdot(1+\mathcal{O}(\varepsilon_\delta))\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}z\,\mathrm{d}t\\&\longrightarrow\int_{\mathbb{R}}\int_{\Gamma_t}\zeta_2\Big(\int_{-\infty}^{+\infty}m_0(R^0)V(\Phi^0)\nabla^\Gamma\Big(\frac{M^0}{R^0}\Big)\,\mathrm{d}z\Big)\,\mathrm{dH}_{n-1}(y)\,\mathrm{d}t\,.\end{aligned}$$

That means, because the first identity of theorem 11.1,

$$\int_{-\infty}^{+\infty}m_0(R^0)V(\Phi^0)\nabla^\Gamma\Big(\frac{M^0}{R^0}\Big)\,\mathrm{d}z=\Big(\int_{-\infty}^{+\infty}m_0(R^0)V(\Phi^0)\,\mathrm{d}z\Big)\;\nabla^\Gamma\Big(\frac{\mu}{\varrho}\Big)\,.$$

The factor we can rewrite

$$\begin{aligned}\int_{-\infty}^{+\infty}m_0(R^0)V(\Phi^0)\,\mathrm{d}z\;&=\int_{-\infty}^{+\infty}\frac{m_0(R^0)V(\Phi^0)}{\partial_z\Phi^0}\partial_z\Phi^0\,\mathrm{d}z\\&=\frac{m_0(\varrho)}{\varrho}\int_0^1\frac{V(s)}{\sqrt{2(W(s)-W(0))}}\,\mathrm{d}s\,,\end{aligned}$$

by transformation $s=\Phi^0(z)$ and replacing

$$\partial_z\Phi^0=\sqrt{2\frac{R^0}{h(R^0)}(W(\Phi^0)-W(0))}$$

coming from (5.10).

References

[1] H. Abels, H. Garcke, G. Grün. *Thermodynamically Consistent, Frame Indifferent Diffuse Interface Models for Incompressible Two-Phase Flows with Different Densities.* Math. Models Methods Appl. Sci. 22 (2012)

[2] H.W. Alt. *The entropy principle for interfaces. Solids and fluids.* AMSA 19, pp. 585-663 (2009)

[3] H.W. Alt, W. Alt. *Phase boundary dynamics: transition between ordered and disordered lipid monolayers.* IFB 11, pp. 1-36 (2009)

[4] H.W. Alt, G. Witterstein. *Distributional equation in the limit of phase transition.* IFB 13, pp. 531-554 (2011)

[5] D. Bedeaux. *Nonequilibrium Thermodynamics and Statistical Physics of Surfaces.* Advance in Chemical Physics, Vol. LXIV, Wiley 1986

[6] J.W. Cahn, C.M. Elliott, A. Novick-Cohen. *The Cahn-Hilliard equation with a concentration dependent mobility: Motion by minus the Laplacian of the mean curvature.* European J. Appl. Math. 7, pp. 287-301 (1996)

[7] J.M. Delhaye. *Jump conditions and entropy sources in two-phase systems. Local instant formulation.* Int. J. Multiphase Flow 1, pp. 395-409 (1974)

[8] C. M. Elliott, B. Stinner, V. M. Styles, and R. Welford. *Numerical computation of advection and diffusion on evolving diffuse interfaces.* IMAJ Num. Anal. 31, pp. 786-812 (2011)

[9] C.M. Elliott, H. Garcke. *On the Cahn-Hilliard equation with degenerate mobility.* SIAM J. Math. Anal. 27, pp. 404-423 (1996)

[10] H. Garcke, B. Nestler, B. Stoth. *On anisotropic order parameter models for multiphase systems and their sharp interface limit.* Physica D, Vol. 115, pp. 87-108 (1998)

[11] H. Garcke, B. Nestler, B. Stoth. *Anisotropy in multiphase systems: a phase-field approach.* Interfaces and Free Boundaries 1, pp. 175-198 (1999)

[12] P.C. Hohenberg, B.I. Halperin. *Theory of dynamic critical phenomena.* Rev. Mod. Phys. 49, pp. 435-479 (1977)

[13] J. Lowengrub, L. Truskinovsky. *Quasi-incompressible Cahn-Hilliard fluids and topological transitions.* Proc. R. Soc. Lond. Ser. A 454, pp. 2617-2654 (1998)

[14] S. Luckhaus, L. Modica. *The Gibbs-Thompson relation within the gradient theory of phase transition.* Arch. Rat. Mech. Anal. 107, pp. 71-83 (1989)

[15] L. Modica. *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rat. Mech. Anal. 98, pp. 123-142 (1987)

[16] J. Prüss, S. Shimizu, M. Wilke. *Qualitative behaviour of incompressible two-phase flows with phase transitions: The case of non-equal densities.* Preprint arXiv:1304.3302v1 (2013)

[17] G. Witterstein. *Sharp interface limit of phase change flows.* AMSA 20, pp. 558-629 (2010)

[18] G. Witterstein. *Phase change flows with mass exchange.* AMSA 21, pp. 559-611 (2011)