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NONSYMMETRIC PRESSURE TENSORS AND THE SPIN EQUATION

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Abstract. In this paper we are concerned with the dynamics of liquid crystals with a non-symmetric part of the pressure tensor. The non-symmetric form we have already treated in the paper [2]. Here we are dealing with the fact that the liquid crystal is embedded in a fluid with non-symmetric velocity gradient. This has the effect that the molecules are turned by the antisymmetric part $(\mathrm{D}v)^{\mathrm{A}}$, and this in addition to the movement induced by the director d. Therefore there are two reasons for the general dynamics, one reason from the outside behaviour of the velocity v and another reason is by the near neighbour behaviour done by the form of the molecules, caused by the director d. Hence we are able to combine the model of Grad and the theory of Ericksen & Leslie. We think that this paper gives the framework for other treatments of a system of spin equations.

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1 Introduction

Mathematically it is clear that models for liquid crystals have the property that the pressure tensor Π is non-symmetric. The angular momentum played an essential role from the very beginning, it is the natural way for mathematical models of liquid crystals. With the spin equation it provides the additional equation which describes the standard features of the material. In this paper we are concerned with several effects of the spin which will determine the microscopic behaviour of the molecules. They come from the surrounding and from the inside of the material, so they lead to different contributions of the antisymmetric spin.

That is, we assume that the mass-momentum system for (ϱ, v),

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho v) &= 0\,, \\ \partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) &= \mathbf{f}\,, \end{aligned} \tag{1.1}$$

is the basis for our final system (1.7). It assumes that the pressure tensor Π has non-symmetric contributions. The mass-momentum system implies for the orbital angular momentum $\mathscr{L} := (x-\xi) \wedge \varrho(v-\dot{\xi})$ the equation

$$\partial_t \mathscr{L} + \operatorname{div}\big(\mathscr{L} v^{\mathrm{T}} + (x-\xi)\wedge \Pi\big) = -2\Pi^{\mathrm{A}} + (x-\xi)\wedge\big(\mathbf{f} - \varrho\ddot{\xi}\big)\,, \tag{1.2}$$

where this is true for an observer which is at a position $t \mapsto \xi(t)$, see Section 2. Therefore, this equation is satisfied for all observers. A form which writes this equation as a general rule is the equation of ***angular momentum*** for the antisymmetric tensor $\mathscr{J}$

$$\partial_t \mathscr{J} + \operatorname{div}\big(\mathscr{J} v^{\mathrm{T}} + (x-\xi)\wedge\Pi + \Sigma\big) = (x-\xi)\wedge(\mathbf{f} - \varrho\ddot{\xi}) + \Gamma \tag{1.3}$$

where Σ is the "couple stress density" and Γ the "intrinsic body couple density", see the book of DeGroot & Mazur [4 : Chap.XII §1(3)] and also the paper of H.Grad [13 : (4.13)] ($\mathscr{L} \rightsquigarrow M$, $\Pi \rightsquigarrow P$), for more see Alt [1 : Sec.II.6]. In applications of ferrofluids in a magnetic field Rinaldi & Zahn [20 : (1.3), (1.9)] ($\Gamma \rightsquigarrow \mathbf{I}$, magnetization $\mathbf{M}$, magnetic field $\mathbf{H}$) call $\mathbf{I}$ the "body-couple density field" which they set to $\mathbf{I} = \mu_o \mathbf{M}\times\mathbf{H}$.

Now, the spin is defined by $\mathscr{S} := \mathscr{J} - \mathscr{L}$ and it satisfies the difference of (1.3) and (1.2), which is the total ***spin balance equation***

$$\partial_t \mathscr{S} + \operatorname{div}(\mathscr{S} v^{\mathrm{T}} + \Sigma) = 2\Pi^{\mathrm{A}} + \Gamma\,. \tag{1.4}$$

Since $\mathscr{J}$ and $\mathscr{L}$ have the same transformation rule, see e.g. [1 : Sec.II.6], the spin $\mathscr{S}$ is an antisymmetric objective tensor. (And if $\mathscr{S} = 0$ in (1.4) without couple terms, then Cauchy's second law of motion is satisfied, that is, Π is symmetric.)

We can consider, quite general, a situation with several spins $\mathscr{S}_m$ satisfying

$$\begin{aligned} &\partial_t \mathscr{S}_m + \operatorname{div}(\mathscr{S}_m v^{\mathrm{T}} + \Sigma_m) = 2\Pi^{\mathrm{A}}_m + \Gamma_m \\ &\text{with} \quad \mathscr{S} = \textstyle\sum_m \mathscr{S}_m\,, \quad \Pi^{\mathrm{A}} = \sum_m \Pi^{\mathrm{A}}_m\,, \end{aligned} \tag{1.5}$$

and similar equations for Σ_m and Γ_m. Now, the specific spins $\mathscr{S}^{sp}_m$ with $\mathscr{S}_m = \varrho\mathscr{S}^{sp}_m$ are given by the rotation axis $\omega_m \in \mathbb{R}^3$, that is, $\mathscr{S}^{sp}_m z = \omega_m \times z$ for $z \in \mathbb{R}^3$, or in other notation

ω_m is defined by $\omega_m := \big(\mathbf{e}_3\bullet\mathscr{S}^{sp}_m\mathbf{e}_2, \mathbf{e}_1\bullet\mathscr{S}^{sp}_m\mathbf{e}_3, \mathbf{e}_2\bullet\mathscr{S}^{sp}_m\mathbf{e}_1\big)$, see [1 : IV.17.2], and we take the sum of these rotations ω_m. Then the rotation ω of the antisymmetric matrix $\mathscr{S}^{sp}$ is given by the sum $\omega = \sum_m \omega_m$. So it makes clear what consequences a sum of the spin $\mathscr{S}^{sp} = \sum_m \mathscr{S}^{sp}_m$ has.

In this paper we consider a fluid (ϱ, v) and we assume that the spin consists of two subspins

$$\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1 \tag{1.6}$$

which are related to a microscopic small object at the point (t, x). The first spin is stimulated by the speed v, i.e. more precisely by $(\mathrm{D}v)^{\mathrm{A}}$ hence $\mathscr{S}^{sp}_0 = \mu((\mathrm{D}v)^{\mathrm{A}} - \Omega)$, like Grad did (see [2 : under (9.9)]), here Ω is the externally viewed quantity that produces the internal objective spin $\mathscr{S}_0 = \varrho\mathscr{S}^{sp}_0$. The second spin is due to a characteristic director d with length ℓ, that is $\mathscr{S}_1 = \varrho\, d \wedge Ad$ with an antisymmetric objective matrix A, like Leslie & Ericksen did, for more info we refer to the paper [2].

Therefore, we consider the following system which contains the mass and momentum equation with the balance laws for the two subspins, and it reads, together with the total energy equation,

$$\begin{gathered}
\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,, \\
\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f}\,, \quad \Pi^{\mathrm{A}} = \Pi^{\mathrm{A}}_0 + \Pi^{\mathrm{A}}_1\,, \\
\partial_t\mathscr{S}_m + \operatorname{div}(\mathscr{S}_m v^{\mathrm{T}} + \Sigma_m) = 2\,\Pi^{\mathrm{A}}_m + \Gamma_m =: \mathrm{H}_m\,, \quad m = 0, 1, \\
\partial_t e + \operatorname{div}\widetilde{q} = v\bullet\mathbf{f} + \mathrm{D}v : \Pi^{\mathrm{A}} =: \widetilde{\mathbf{g}}\,, \\
e = \frac{\varrho}{2}|v|^2 + \frac{\varrho}{2}\sum_m \tau_m|\mathscr{S}^{sp}_m|^2 + \varepsilon\,, \quad \widetilde{q} = \Pi^{\mathrm{T}} v + \sum_m \tau_m\mathscr{S}^{sp}_m : \Sigma_m + q\,.
\end{gathered} \tag{1.7}$$

With (1.6) together with $\Sigma := \Sigma_0 + \Sigma_1$ and $\Gamma := \Gamma_0 + \Gamma_1$ we get the equation of the total spin (1.4). The total energy equation deserves a special remark. The contributions of the total energy except the kinetic energy are objective scalars. Therefore the transformation rule for e is the same rule of the kinetic energy and it is $e\circ Y = \frac{1}{2}|\dot{X}|^2\varrho^* + \varrho^*\dot{X}\bullet(Qv^*) + e^*$, if Y is the observer transformation, see [1 : (II 3.32)]. This determines the rules for the flux $\widetilde{q}$ and the right side $\widetilde{g}$ of the equation for e, see [1 : (II 3.5)]. By [1 : (II 3.38)] we have the identity $\widetilde{\mathbf{g}} = v\bullet\mathbf{f} + \mathrm{D}v : \Pi^{\mathrm{A}} + \mathbf{g}$ with an objective scalar $\mathbf{g}$. We set $\mathbf{g} = 0$ because we have to satisfy the energy principle. The total energy could be generalized to have a term $\mathscr{S}^{sp}_m : \boldsymbol{\tau}_m\mathscr{S}^{sp}_m$ with a symmetric positive definite 4-tensor $\boldsymbol{\tau}_m$, but we restrict ourselves with a scalar τ_m and $\tau_m|\mathscr{S}^{sp}_m|^2$.

There is a property which solutions of system (1.7) have to satisfy, it is the entropy inequality, which is an encoding of the local behavior of the molecules in consideration. It means that for the entropy production

$$\sigma := \partial_t\eta + \operatorname{div}\psi \geq 0 \tag{1.8}$$

has to hold, where η is the entropy and ψ the entropy flux. We mention that this inequality is the reason why η has to be an objective scalar, that is $\eta\circ Y = \eta^*$, where $Y : \mathbb{R}^4 \to \mathbb{R}^4$ is the observer transformation and η and η^* are the entropies for the two observers. In this paper we are devoted to liquid crystals, which are defined by a local director $d\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ and we assume that η depends on the following assumption

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d)\,, \tag{1.9}$$

where $\widehat{\eta}$ is an objective constitutive function, that is $\widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) = \widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*)$. Further d is an objective vector and consequently $\mathrm{D}d$ an objective matrix. Now, that η is an objective scalar has a consequence:

1.1 Condition. That η is an objective scalar with (1.9) leads to the fact that

$$\eta'_d \otimes d + \eta'_{\mathrm{D}d}(\mathrm{D}d)^{\mathrm{T}} + (\eta'_{\mathrm{D}d})^{\mathrm{T}}\,\mathrm{D}d \quad \text{is symmetric.}$$

This is equivalent to

$$0 = \eta'_d \bullet Bd + \eta'_{\mathrm{D}d} : (B\mathrm{D}d + \mathrm{D}d\,B^{\mathrm{T}})$$

for every antisymmetric matrix B.

Proof. Since η in an objective scalar $\eta^* = \eta\circ Y$ it has the consequence that

$$\begin{aligned}&\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \eta^* = \eta\circ Y\\ &= \widehat{\eta}(\varrho\circ Y, \varepsilon\circ Y, d\circ Y, \mathrm{D}d\circ Y) = \widehat{\eta}(\varrho^*, \varepsilon^*, Qd^*, Q\mathrm{D}d^*\,Q^{\mathrm{T}})\,.\end{aligned}$$

Here ϱ and ε are objective scalars. Hence

$$\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \widehat{\eta}(\varrho^*, \varepsilon^*, Qd^*, Q\mathrm{D}d^*\,Q^{\mathrm{T}}) \tag{1.10}$$

for every value ϱ^*, ε^*, d^*, $\mathrm{D}d^*$. This is true for all orthogonal matrices Q with determinant 1. For this matrix take $s \mapsto Q_s$, where s is a real variable and

$$\frac{\mathrm{d}}{\mathrm{d}s}Q_s = A_sQ_s \text{ for } s \geq 0 \text{ and } Q_0 = \mathrm{Id}$$

with a given antisymmetric matrix A_s. Then by (1.10)

$$\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \widehat{\eta}(\varrho^*, \varepsilon^*, Q_sd^*, Q_s\mathrm{D}d^*\,{Q_s}^{\mathrm{T}})$$

that is

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}s}\widehat{\eta}(\varrho^*, \varepsilon^*, Q_sd^*, Q_s\mathrm{D}d^*\,{Q_s}^{\mathrm{T}})\\ &= \widehat{\eta}'_d(\ldots)\bullet(A_sQ_sd^*) + \widehat{\eta}'_{\mathrm{D}d}(\ldots) : \big(A_sQ_s\mathrm{D}d^*\,{Q_s}^{\mathrm{T}} + Q_s\mathrm{D}d^*\,(A_sQ_s)^{\mathrm{T}}\big)\,.\end{aligned}$$

In particular for $s = 0$

$$\begin{aligned}0 &= \widehat{\eta}'_d(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*)\bullet(A_0d^*) + \widehat{\eta}'_{\mathrm{D}d}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) : \big(A_0\mathrm{D}d^* + \mathrm{D}d^*\,A_0^{\mathrm{T}}\big)\\ &= A_0 : \big(\eta'_d \otimes d + \eta'_{\mathrm{D}d}(\mathrm{D}d)^{\mathrm{T}} + (\eta'_{\mathrm{D}d})^{\mathrm{T}}\,\mathrm{D}d\big)\,.\end{aligned}$$

Since A_0 is an arbitrary antisymmetric matrix the assertion follows. □

With these frameworks we are able to prove the Main theorem 3.2, where we show that the entropy principle is true and that the entropy production σ satisfies the following residual inequality (3.16)

$$\begin{aligned}
&0 \le \sigma = \eta'_\varepsilon \mathrm{D}(v - v_\xi) \mathbin{:} (P - \Pi^{\mathrm{S}}) \\
&+\nabla\eta'_\varepsilon \bullet q - \eta'_\varepsilon \tau_0 \big(\mathrm{D}\mathscr{S}_0^{sp} \mathbin{:} \Sigma_0 + \mathscr{S}_0^{sp} \mathbin{:} (2\,\Pi_0^{\mathrm{A}} + \overline{\Gamma}_0)\big) \\
&+d' \bullet \Big(\frac{\delta\eta}{\delta d} + 2\tau_1 \big(\eta'_\varepsilon(2\,\Pi_1^{\mathrm{A}} + \overline{\Gamma}_1) - \operatorname{div}(\eta'_\varepsilon \Sigma_1)\big) d\Big) \,,
\end{aligned} \tag{1.11}$$

where the pressure P is defined in (3.15), which is a tensor. For $\overline{\Gamma}_m$ see (2.5).

In this paper we then concentrate on the description of this two spin case, in which we have combined the Grad theory with the theory of Leslie & Ericksen. The general residual inequality (1.11) is applied to the case that, see (3.18), for Σ_1 and $\overline{\mathrm{H}}_1 := 2\,\Pi_1^{\mathrm{A}} + \overline{\Gamma}_1$ the tensor π and the vector g are introduced, and for $\overline{\mathrm{H}}_0$ the equation (2.6) is used, which yield the Final theorem 3.5. In this situation we obtain in system (1.7)

$$\begin{gathered}
\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,, \\
\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f} \,, \quad \Pi^{\mathrm{A}} = \Pi_0^{\mathrm{A}} + \Pi_1^{\mathrm{A}} \,, \\
2\,\Pi_m^{\mathrm{A}} = \overline{\mathrm{H}}_m - \overline{\Gamma}_m \quad \text{for } m = 0, 1,
\end{gathered} \tag{1.12}$$

where

$$\begin{gathered}
\overline{\mathrm{H}}_0 = \big(\varrho(\mathscr{S}_0^{sp})^\circ - \mathrm{H}_0^0\big) + \operatorname{div}\Sigma_0 \,, \\
\overline{\mathrm{H}}_1 = d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j} \,, \quad d \wedge (\varrho d'' + \operatorname{div}\pi - g) = 0 \,,
\end{gathered} \tag{1.13}$$

and $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ have to be inserted by concrete applications, where additional equations corresponding to the application are necessary, and therefore the entropy has to be generalized to more variables. In particular, it is thought about an application with Maxwell equations.

2 The virtual body

We define the orbital angular momentum $\mathscr{L} = r \wedge p$ by a matrix consisting of the relative position $r = x - \xi$ and the relative momentum $p = \varrho(v - \dot{\xi})$

$$\mathscr{L} = (x - \xi) \wedge \varrho(v - \dot{\xi}) \,,$$

where $t \mapsto \xi(t)$ is the reference orbit, hence this is an observer independent formulation. Therefore one says that the angular momentum is done by a virtual body at the point $t \mapsto \xi(t)$. At the point $\xi(t)$ a special observer can be. Hence we have to take a relative movement, that is if necessary, we subtract from the quantities given by the measurements the quantities of the virtual body.

For the reference orbit not only the position $\xi(t)$ is required, but also how the virtual observer turns around its body, that means its rotation $A_\xi(t)$:

2.1 The virtual body. The trajectory of the virtual body is given by $t \mapsto \xi(t)$, and the antisymmetric matrix $t \mapsto A_\xi(t)$ describes the rotation part of the velocity of the virtual body, and is given by

$$v_\xi(t,x) := \dot{\xi}(t) + A_\xi(t)(x - \xi(t))\,. \tag{2.1}$$

This means $\mathrm{D}v_\xi = A_\xi$ and that A_ξ satisfies the transformation formula

$$A_\xi \circ Y = \dot{Q}Q^{\mathrm{T}} + QA_\xi^* Q^{\mathrm{T}} \tag{2.2}$$

as derivative of a velocity. The speed of this body in space is $t \mapsto \dot{\xi}(t) = v_\xi(t,\xi(t))$.

This virtual body has the advantage that with the antisymmetric matrix A_ξ the quantities can be written in an objective manner.

This is especially true for the quantities around the spin equation, which is for each $m = 0,1$

$$\partial_t \mathscr{S}_m + \operatorname{div}(\mathscr{S}_m v^{\mathrm{T}} + \Sigma_m) = 2\Pi_m^{\mathrm{A}} + \Gamma_m =: \mathrm{H}_m\,, \tag{2.3}$$

where the spin $\mathscr{S}_m$ is an antisymmetric objective tensor, i.e. $\mathscr{S}_m \circ Y = Q\mathscr{S}_m^* Q^{\mathrm{T}}$. The spin equation should be an invariant system, see [1 : (I.5.13) Invariance of the divergence system] and [1 : IV.17.5 Lemma]. The spin equation is an invariant system if Σ_m is an objective 3-tensor and H_m satisfies the transformation rule

$$\mathrm{H}_m \circ Y = \dot{Q}\mathscr{S}_m^* Q^{\mathrm{T}} + Q\mathscr{S}_m^* \dot{Q}^{\mathrm{T}} + Q\mathrm{H}_m^* Q^{\mathrm{T}}\,. \tag{2.4}$$

If we define

$$\begin{gathered} \mathrm{H}_m^0 := A_\xi \mathscr{S}_m + \mathscr{S}_m A_\xi^{\mathrm{T}}\,, \\ \mathrm{H}_m = \mathrm{H}_m^0 + \overline{\mathrm{H}}_m\,, \quad \Gamma_m = \mathrm{H}_m^0 + \overline{\Gamma}_m\,, \quad \overline{\mathrm{H}}_m = 2\Pi_m^{\mathrm{A}} + \overline{\Gamma}_m\,, \end{gathered} \tag{2.5}$$

then H_m^0 has the property (2.4), so that $\overline{\mathrm{H}}_m$ is an objective tensor.

If we define the specific spin $\mathscr{S}_m^{sp}$ by $\mathscr{S}_m = \varrho\mathscr{S}_m^{sp}$ then we get for the spin equation (2.3) by using the mass equation

$$\begin{gathered} \varrho(\mathscr{S}_m^{sp})^{\circ} + \operatorname{div}\Sigma_m = 2\Pi_m^{\mathrm{A}} + \Gamma_m = \mathrm{H}_m \\ \text{or} \quad \left(\varrho(\mathscr{S}_m^{sp})^{\circ} - \mathrm{H}_m^0\right) + \operatorname{div}\Sigma_m = \overline{\mathrm{H}}_m = 2\Pi_m^{\mathrm{A}} + \overline{\Gamma}_m\,. \end{gathered} \tag{2.6}$$

We finally show the advantage of A_ξ with a computation of the constant τ_1.

2.2 Lemma. If the fluid particles are given by a bar B of length ℓ we get for the relative velocity and for the spin

$$|u_B|^2 = \frac{1}{12}|A_{mic}d|^2\,, \quad |\mathscr{S}_{1B}^{sp}|^2 = \frac{2\ell^2}{12^2}|A_{mic}d|^2\,,$$

where A_{mic} is the microscopic antisymmetric matrix, resulting in

$$\tau_1 = \frac{6}{\ell^2} \quad \text{if} \quad |u_B|^2 = \tau_1|\mathscr{S}_{1B}^{sp}|^2\,.$$

Proof. The representation for the specific spin for a 3-dimensional bar (see e.g. [1 : IV.17.4])

$$\bar{B}_t := \{x_B(t) + x' \in \mathbb{R}^3\,;\; |x'\bullet e_3| < \tfrac{\ell}{2}, |x'\bullet e_1| < \tfrac{r}{2}, |x'\bullet e_2| < \tfrac{r}{2}\}$$

with position depending basis $\{e_1, e_2, e_3\}$, $\ell e_3 = d$, is given by

$$\mathscr{S}_1^{\bar{s}p} = \frac{1}{L^3(\bar{B}_t)} \int_{\bar{B}_t} x' \wedge A_{mic} x' \,\mathrm{dL}^3(x') = \frac{\ell^2}{12} e_3 \wedge A_{mic} e_3 + \frac{r^2}{12}(e_1 \wedge A_{mic} e_1 + e_2 \wedge A_{mic} e_2)\,,$$

where $t \mapsto A_{mic}(t)$ is the antisymmetric objective matrix related to the self-rotation of the bar. Now we approximate the 1-dimensional stick B from the 3-dimensional bar $\bar{B}$ by letting $r \to 0$ and we get

$$\mathscr{S}_{1B}^{sp} = \frac{\ell^2}{12} e_3 \wedge A_{mic} e_3\,,$$

it follows

$$|\mathscr{S}_{1B}^{sp}|^2 = \left|\frac{\ell^2}{12} e_3 \wedge A_{mic} e_3\right|^2 = \frac{2\ell^4}{12^2} |A_{mic} e_3|^2\,.$$

We calculate analogously the kinetic energy of B by

$$|\bar{u}_B|^2 = \frac{1}{L^3(\bar{B}_t)} \int_{\bar{B}_t} |A_{mic} x'|^2 \,\mathrm{dL}^3(x') = \frac{1}{\ell r^2} \sum_{l,k} \int_{\bar{B}_t} x'_l x'_k \,\mathrm{dL}^3(x') \cdot (A_{mic} e_l)\bullet(A_{mic} e_k)$$
$$= \frac{\ell^2}{12} |A_{mic} e_3|^2 + \frac{r^2}{12}(|A_{mic} e_1|^2 + |A_{mic} e_2|^2)$$

and letting $r \to 0$

$$|u_B|^2 = \frac{\ell^2}{12} |A_{mic} e_3|^2\,.$$

□

3 The theorems

First we rewrite the general system (1.7) in terms of the material time derivatives

$$\begin{gathered}
\overset{\circ}{\varrho} + \varrho\,\mathrm{div} v = 0\,,\\
\varrho\overset{\circ}{v} + \mathrm{div}\Pi = \mathbf{f}\,, \quad \Pi^{\mathrm{A}} = \Pi_0^{\mathrm{A}} + \Pi_1^{\mathrm{A}}\,,\\
\varrho(\mathscr{S}_m^{sp})^{\circ} + \mathrm{div}\Sigma_m = 2\Pi_m^{\mathrm{A}} + \Gamma_m =: \mathrm{H}_m\,, \quad m = 0, 1,\\
\overset{\circ}{\varepsilon} + \varepsilon\,\mathrm{div} v + \mathrm{div} q = -(\mathrm{D}v)^{\mathrm{S}} : \Pi^{\mathrm{S}} - \sum_m \tau_m\big(\mathrm{D}\mathscr{S}_m^{sp} : \Sigma_m + \mathscr{S}_m^{sp} : \overline{\mathrm{H}}_m\big)\,.
\end{gathered} \tag{3.1}$$

To prove (3.1), we remark that for every real function w defining $\overline{w} := \varrho w$ and using the mass equation of (1.7)

$$\begin{aligned}
&\overset{\circ}{\overline{w}} + \overline{w}\,\mathrm{div} v = \partial_t \overline{w} + \mathrm{div}(\overline{w} v) = \partial_t(\varrho w) + \mathrm{div}(\varrho w v)\\
&= w(\partial_t \varrho + \mathrm{div}(\varrho v)) + \varrho(\partial_t w + v\bullet\nabla w) = \varrho(\partial_t w + v\bullet\nabla w) = \varrho\overset{\circ}{w}\,.
\end{aligned} \tag{3.2}$$

For the mass equation in (3.1) we take $w = 1$ hence $\overline{w} = \varrho$ and we get

$$\overset{\circ}{\varrho} + \varrho\,\mathrm{div}v = 0\,.$$

For the momentum equation in (1.7) we take $w = v_i$ hence $\overline{w} = \varrho v_i$ and get

$$\varrho\overset{\circ}{v} = \partial_t(\varrho v) + \mathrm{div}(\varrho v\,v^{\mathrm{T}})\,,$$

therefore the second equation in (3.1). Similarly we get for the spin equations in (1.7) by taking $w = \big(\mathscr{S}_m^{sp}\big)_{kl}$ hence $\overline{w} = \big(\mathscr{S}_m\big)_{kl}$

$$\varrho\overset{\circ}{\mathscr{S}_m^{sp}} = \partial_t\mathscr{S}_m + \mathrm{div}(\mathscr{S}_m v^{\mathrm{T}})\,,$$

therefore the spin equations in (3.1). For the energy we have in (1.7) the identity

$$e = \frac{\varrho}{2}|v|^2 + \frac{\varrho}{2}\sum_m \tau_m|\mathscr{S}_m^{sp}|^2 + \varepsilon\,, \tag{3.3}$$

hence from the energy equation in (1.7) we will derive an equation for ε. To get the balance law for the kinetic energy we multiply the momentum equation in (3.1) with v and obtain $v\bullet(\varrho\overset{\circ}{v}) = \frac{\varrho}{2}\big(|v|^2\big)^\circ$ and $v\bullet\,\mathrm{div}\Pi = \mathrm{div}(\Pi^{\mathrm{T}}\,v) - \mathrm{D}v\,\boldsymbol{:}\,\Pi$. Hence we obtain*

$$\frac{\varrho}{2}\big(|v|^2\big)^\circ + \mathrm{div}(\Pi^{\mathrm{T}}\,v) = v\bullet\mathbf{f} + \mathrm{D}v\,\boldsymbol{:}\,\Pi\,.$$

By taking $w = |v|^2$ hence $\overline{w} = \varrho|v|^2$ we get

$$\Big(\frac{\varrho}{2}|v|^2\Big)^\circ + \frac{\varrho}{2}|v|^2\,\mathrm{div}v = \frac{\varrho}{2}\big(|v|^2\big)^\circ$$

and then for the kinetic energy

$$\Big(\frac{\varrho}{2}|v|^2\Big)^\circ + \frac{\varrho}{2}|v|^2\,\mathrm{div}v + \mathrm{div}(\Pi^{\mathrm{T}}\,v) = v\bullet\mathbf{f} + \mathrm{D}v\,\boldsymbol{:}\,\Pi\,. \tag{3.4}$$

Similarly we obtain for the spin equation by multiplying it with $\mathscr{S}_m^{sp}$

$$\mathscr{S}_m^{sp}\,\boldsymbol{:}\,\big(\varrho\overset{\circ}{\mathscr{S}_m^{sp}}\big) = \frac{\varrho}{2}\big(|\mathscr{S}_m^{sp}|^2\big)^\circ\,, \quad \mathscr{S}_m^{sp}\,\boldsymbol{:}\,\mathrm{div}\Sigma_m = \mathrm{div}(\mathscr{S}_m^{sp}\,\boldsymbol{:}\,\Sigma_m) - \mathrm{D}\mathscr{S}_m^{sp}\,\boldsymbol{:}\,\Sigma_m\,,$$

so that for $|\mathscr{S}_m^{sp}|^2$

$$\frac{\varrho}{2}\big(|\mathscr{S}_m^{sp}|^2\big)^\circ + \mathrm{div}(\mathscr{S}_m^{sp}\,\boldsymbol{:}\,\Sigma_m) = \mathrm{D}\mathscr{S}_m^{sp}\,\boldsymbol{:}\,\Sigma_m + \mathscr{S}_m^{sp}\,\boldsymbol{:}\,\mathrm{H}_m\,.$$

By taking $w = |\mathscr{S}_m^{sp}|^2$ hence $\overline{w} = \varrho|\mathscr{S}_m^{sp}|^2$ we get

$$\Big(\frac{\varrho}{2}|\mathscr{S}_m^{sp}|^2\Big)^\circ + \frac{\varrho}{2}|\mathscr{S}_m^{sp}|^2\,\mathrm{div}v = \frac{\varrho}{2}\big(|\mathscr{S}_m^{sp}|^2\big)^\circ$$

*In the paper [2 : between (2.5) and (2.6)] a copy mistake is made, it is twice written ϱv instead of v. We are sorry for this mistake.

and then we obtain for the energy part of the spin

$$\Big(\frac{\varrho}{2}|\mathscr{S}_m^{sp}|^2\Big)^\circ + \frac{\varrho}{2}|\mathscr{S}_m^{sp}|^2\,\mathrm{div}v + \,\mathrm{div}(\mathscr{S}_m^{sp}\mathbin{:}\Sigma_m) = \mathrm{D}\mathscr{S}_m^{sp}\mathbin{:}\Sigma_m + \mathscr{S}_m^{sp}\mathbin{:}\mathrm{H}_m\,. \tag{3.5}$$

Now, we come to the equation for the energy in (1.7). The total energy equation is stated is (3.3). We subtract the equations (3.4) and (3.5) from the energy equation (1.7) and get the equation for the internal energy ε with $\widetilde{q} = \Pi^{\mathrm{T}}\,v + \sum_m \tau_m \mathscr{S}_m^{sp}\mathbin{:}\Sigma_m + q$

$$\begin{aligned}\partial_t\varepsilon + \,\mathrm{div}(\varepsilon v + q) &= -\mathrm{D}v\mathbin{:}\Pi^{\mathrm{S}} - \textstyle\sum_m \tau_m\big(\mathrm{D}\mathscr{S}_m^{sp}\mathbin{:}\Sigma_m + \mathscr{S}_m^{sp}\mathbin{:}\mathrm{H}_m\big)\\ &= -\,(\mathrm{D}v)^{\mathrm{S}}\mathbin{:}\Pi^{\mathrm{S}} - \textstyle\sum_m \tau_m\big(\mathrm{D}\mathscr{S}_m^{sp}\mathbin{:}\Sigma_m + \mathscr{S}_m^{sp}\mathbin{:}\overline{\mathrm{H}}_m\big)\,,\end{aligned} \tag{3.6}$$

since $\mathrm{D}v\mathbin{:}\Pi^{\mathrm{S}} = (\mathrm{D}v)^{\mathrm{S}}\mathbin{:}\Pi^{\mathrm{S}}$, and in [2 : 5.2 Lemma] it was proved that

$$\mathscr{S}_m^{sp}\mathbin{:}\mathrm{H}_m = \mathscr{S}_m^{sp}\mathbin{:}\overline{\mathrm{H}}_m\,. \tag{3.7}$$

Hence all terms of the right side of the ε-equation are objective, and so the entire right side is an objective scalar. Altogether we have shown that the system (3.1) is true.

From system (3.1) we obtain the following pre-version of the main theorem.

3.1 Theorem. Consider a solution of system (1.7) for a liquid crystal with a director d of length $|d| = \ell = \text{const} > 0$. Assume that the spin satisfies (1.6)

$$\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1 \quad \text{with} \quad \mathscr{S}_1 = \varrho\, d \wedge d'\,, \quad \text{where} \quad d' := \overset{\circ}{d} - A_\xi d\,. \tag{3.8}$$

Let the entropy η be of the form

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d)\,, \tag{3.9}$$

then the residual inequality reads

$$\begin{aligned}0 \le \sigma = (\mathrm{D}v)^{\mathrm{S}}\mathbin{:}\big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi^{\mathrm{S}}\big) + \,\mathrm{div}(\psi - \eta v - \eta'_\varepsilon q)\\ +\nabla\eta'_\varepsilon\bullet q + \eta'_\varepsilon\tau_0\sigma_0 + (\sigma_d + \eta'_\varepsilon\tau_1\sigma_1)\,,\end{aligned} \tag{3.10}$$

where

$$\sigma_m := -\big(\mathrm{D}\mathscr{S}_m^{sp}\mathbin{:}\Sigma_m + \mathscr{S}_m^{sp}\mathbin{:}\overline{\mathrm{H}}_m\big)\,, \quad \sigma_d := \eta'_d\bullet\overset{\circ}{d} + \eta'_{\mathrm{D}d}\mathbin{:}(\mathrm{D}d)^\circ\,. \tag{3.11}$$

Proof. See the first part of Section 4 up to (4.4). This gives the result for the entropy production σ in the inequality (3.10), where the definition (3.11) are in (4.3). We only mention that we got from a model

$$\mathscr{S}_1 = \varrho\, d \wedge Ad\,, \tag{3.12}$$

where A is an antisymmetric matrix with $A \circ Y = QA^*Q^{\mathrm{T}}$ as transformation rule, i.e. A is an objective matrix, and not $\mathscr{S}_1 = \varrho\, d \wedge d'$ as in (3.8). But in [2 : (5.15) and (5.16), 5.4 Addendum] we have shown that (3.12) holds. □

From Theorem 3.1 we derive the following main theorem.

3.2 Main theorem. Consider a solution of system (1.7) for a liquid crystal with a director d of length $|d| = \ell = \text{const} > 0$. Assume that the spin satisfies (1.6)

$$\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1 \quad \text{with} \quad \mathscr{S}_1 = \varrho\, d \wedge d', \quad \text{where} \quad d' = \overset{\circ}{d} - A_\xi d\,. \tag{3.13}$$

Let the entropy η and the entropy flux ψ be of the form

$$\begin{aligned} &\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d)\,, \quad \eta'_\varepsilon = \frac{1}{\theta} > 0\,, \quad \theta \text{ the absolute temperature,} \\ &\psi_j = \eta v_j + \eta'_\varepsilon q_j - \sum_k d'_k \big(\eta'_{d_{k,j}} + 2\tau_1 \eta'_\varepsilon \sum_l d_l \Sigma_{1klj}\big)\,, \end{aligned} \tag{3.14}$$

and furthermore let

$$\begin{gathered} P := p\mathrm{Id} - \theta \sum_i \nabla d_i \otimes \eta'_{\nabla d_i}\,, \quad p := \theta(\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\,, \\ u := v - v_\xi\,. \end{gathered} \tag{3.15}$$

Then the entropy principle (1.8) is satisfied, if the entropy production satisfies the residual inequality

$$\begin{aligned} &0 \le \sigma = \eta'_\varepsilon \mathrm{D}(v - v_\xi) \mathbin{:} (P - \Pi^{\mathrm{S}}) \\ &+ \nabla\eta'_\varepsilon \bullet q - \eta'_\varepsilon \tau_0 \big(\mathrm{D}\mathscr{S}_0^{sp} \mathbin{:} \Sigma_0 + \mathscr{S}_0^{sp} \mathbin{:} \overline{\mathrm{H}}_0\big) \\ &+ d' \bullet \Big(\frac{\delta\eta}{\delta d} + 2\tau_1 \big(\eta'_\varepsilon \overline{\mathrm{H}}_1 - \operatorname{div}(\eta'_\varepsilon \Sigma_1)\big) d\Big)\,. \end{aligned} \tag{3.16}$$

The first variation of η with respect to d is

$$\frac{\delta\eta}{\delta d_k} := \eta'_{d_k} - \sum_j \partial_j \eta'_{d_{k,j}}\,,$$

and for a representation of (3.16) in components see the inequality (3.17).

The first term of σ is the standard term for Π^{S} and the second term is the heat flux term. The rest terms are due to the spin equations, where here the terms of $\mathscr{S}_0$ and $\mathscr{S}_1$ are together in one entropy production, while in [2] they appear in separate theorems.

Proof. The proof is in Section 4, where (4.4) is the preliminary result stated in Theorem 3.1. And in (4.6) we make the assumption that the entropy flux ψ is given by (3.14), so that the first term vanishes. The pressure p is given by Gibbs relation, and a pressure matrix P takes care of the term containing the director d, see (3.15), which gives

$$\begin{aligned} &(\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \sum_j \nabla d_j \otimes \eta'_{\nabla d_j} - \eta'_\varepsilon \Pi^{\mathrm{S}} \\ &= \eta'_\varepsilon p\mathrm{Id} - \sum_j \nabla d_j \otimes \eta'_{\nabla d_j} - \eta'_\varepsilon \Pi^{\mathrm{S}} \\ &= \eta'_\varepsilon P - \eta'_\varepsilon \Pi^{\mathrm{S}} = \eta'_\varepsilon (P - \Pi^{\mathrm{S}})\,. \end{aligned}$$

We also write in terms of components

$$\mathrm{D}\mathscr{S}_0^{sp} \mathbin{:} \Sigma_0 + \mathscr{S}_0^{sp} \mathbin{:} \overline{\mathrm{H}}_0 = \sum_{k,l} \big(\mathscr{S}^{sp}_{0kl} \overline{\mathrm{H}}_{0kl} + \sum_j \mathscr{S}^{sp}_{0kl}{}'_j \Sigma_{0klj}\big)\,.$$

With this the entropy production becomes

$$\begin{aligned} 0 &\le \sigma = \eta'_{\varepsilon}\mathrm{D}(v-v_\xi) \mathbin{:} (P-\Pi^{\mathrm{S}}) \\ &+\nabla\eta'_{\varepsilon}\bullet q - \eta'_{\varepsilon}\tau_0 \sum_{k,l}\big(\mathscr{S}^{sp}_{0kl}\overline{\mathrm{H}}_{0kl} + \sum_j \mathscr{S}^{sp}_{0kl}{}'_j \Sigma_{0klj}\big) \\ &+\sum_k d'_k\Big(\frac{\delta\eta}{\delta d_k} + 2\tau_1 \sum_l d_l\big(\eta'_{\varepsilon}\overline{\mathrm{H}}_{1kl} - \sum_j \partial_j(\eta'_{\varepsilon}\Sigma_{1klj})\big)\Big)\,. \end{aligned} \tag{3.17}$$

This inequality is identical with (3.16). □

A more sophisticated version of the main theorem one becomes by a combination of the model of Grad and the theory of Ericksen & Leslie. The latter uses explicite formulas for the spin equation (see (3.20), here denoted by $\mathscr{S}_1$ etc.), where new unknowns π and g are introduced, which are the notations of the original papers.

3.3 Lemma. Assume

$$\mathscr{S}^{sp}_1 = d\wedge d'\,,\quad \Sigma_1 = d\wedge\pi\,,\quad \overline{\mathrm{H}}_1 = d\wedge g + \sum_j \partial_j d\wedge \pi_{\bullet j}\,, \tag{3.18}$$

where the first equation is already contained in (3.8). Then system (3.1) reads

$$\begin{aligned} \mathring{\varrho} + \varrho\,\mathrm{div}v &= 0\,, \\ \varrho\mathring{v} + \mathrm{div}\Pi = \mathbf{f}\,, &\quad \text{where } \Pi^{\mathrm{A}} = \Pi^{\mathrm{A}}_0 + \Pi^{\mathrm{A}}_1\,, \\ \big(\varrho(\mathscr{S}^{sp}_0)^\circ - \mathrm{H}^0_0\big) + \mathrm{div}_x\Sigma_0 &= \overline{\mathrm{H}}_0 = 2\Pi^{\mathrm{A}}_0 + \overline{\Gamma}_0\,, \\ d\wedge(\varrho d'' + \mathrm{div}\pi - g) = 0\,, &\quad 2\Pi^{\mathrm{A}}_1 + \overline{\Gamma}_1 = \overline{\mathrm{H}}_1 = d\wedge g + \textstyle\sum_j \partial_j d\wedge\pi_{\bullet j}\,, \\ \mathring{\varepsilon} + \varepsilon\,\mathrm{div}\,v = -\,\mathrm{div}\,q - (\mathrm{D}v)^{\mathrm{S}} &\mathbin{:} \Pi^{\mathrm{S}} + \tau_0\sigma_0 - 2\ell^2\tau_1\big(\mathrm{D}d' \mathbin{:} \pi + d'\bullet g\big)\,, \end{aligned} \tag{3.19}$$

where π and g are arbitrary functions, which replace Σ_1 and $\overline{\mathrm{H}}_1$, and where $d'' := \mathring{d'} - A_\xi d'$ from [2: (4.5)]).

Proof. The mass and momentum equations and also the spin equation for $\mathscr{S}^{sp}_0$ are the same as in (3.1) but in the version (2.6). We now treat the spin equation for $\mathscr{S}^{sp}_1$, that is

$$\varrho(\mathscr{S}^{sp}_1)^\circ + \mathrm{div}\Sigma_1 = \mathrm{H}_1 \quad \text{with} \quad \mathscr{S}^{sp}_1 = d\wedge d'\,. \tag{3.20}$$

(This is just the reduced spin equation as in [2: (6.5)]). Then, with assumption (3.18), we can exploit [2: 6.2 Lemma], and get

$$\partial_t(\varrho d') + \mathrm{div}(\varrho d' v^{\mathrm{T}} + \pi) = g_\lambda + \varrho G$$

where $g_\lambda = g + \lambda d$ with λ is a real valued function and $G = A_\xi d'$. By [2: 6.1 Lemma] we see that this is equivalent to

$$\varrho d'' + \mathrm{div}\pi = g_\lambda \quad \text{(see [2: (6.4)])},$$

in other words

$$d\wedge(\varrho d'' + \mathrm{div}\pi - g) = 0\,. \tag{3.21}$$

We look at the internal energy equation where only the summand with $m = 1$ has to be treated. We repeat a part of proof of [2: 3.1 Special case]: We calculate out using (3.18)

$$\mathrm{D}\mathscr{S}_1^{sp} \mathbin{:} \Sigma_1 + \mathscr{S}_1^{sp} \mathbin{:} \overline{\mathrm{H}}_1 = \mathrm{D}(d \wedge d') \mathbin{:} (d \wedge \pi) + (d \wedge d') \mathbin{:} (d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j}) .$$

Using the rule $(\vec{a} \wedge \vec{b}) \mathbin{:} (\vec{c} \wedge \vec{d}) = 2(\vec{a} \bullet \vec{c})(\vec{b} \bullet \vec{d}) - 2(\vec{a} \bullet \vec{d})(\vec{b} \bullet \vec{c})$ we get with the help of $d \bullet d' = 0$ and $d \bullet \partial_j d = 0$

$$\begin{aligned}\mathrm{D}(d \wedge d') \mathbin{:} (d \wedge \pi) &= \sum_j (\partial_j d \wedge d' + d \wedge \partial_j d') \mathbin{:} (d \wedge \pi_{\bullet j}) = \sum_j (d \wedge \partial_j d') \mathbin{:} (d \wedge \pi_{\bullet j}) \\ &= 2|d|^2 \mathrm{D}d' \mathbin{:} \pi - 2 \sum_j (d \bullet \pi_{\bullet j})((\partial_j d') \bullet d) , \\ (d \wedge d') \mathbin{:} (d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j}) &= 2|d|^2 d' \bullet g - 2 \sum_j (d \bullet \pi_{\bullet j})(d' \bullet (\partial_j d)) .\end{aligned}$$

From $d \bullet d' = 0$ we get $\partial_j d \bullet d' + d \bullet \partial_j d' = 0$ and thus

$$\sum_j (d \bullet \pi_{\bullet j})((\partial_j d') \bullet d) + \sum_j (d \bullet \pi_{\bullet j})(d' \bullet (\partial_j d)) = 0 .$$

So with $|d|^2 = \ell^2$ we have proven $\mathrm{D}\mathscr{S}_1^{sp} \mathbin{:} \Sigma_1 + \mathscr{S}_1^{sp} \mathbin{:} \overline{\mathrm{H}}_1 = 2\ell^2(\mathrm{D}d' \mathbin{:} \pi + d' \bullet g)$. □

We insert now the functions π and g, defined in (3.18), into the entropy production.

3.4 Lemma. Denote the last summand of (3.16) by

$$\overline{\sigma}_d := d' \bullet \Big(\frac{\delta \eta}{\delta d} + 2\tau_1 \big(\eta_{'\varepsilon} \overline{\mathrm{H}}_1 - \operatorname{div}(\eta_{'\varepsilon} \Sigma_1) \big) d \Big) .$$

Then in terms of π and g it reads

$$\overline{\sigma}_d = \sum_k d_k' \Big(\eta_{'d_k} - \operatorname{div}\big(\eta_{'\nabla d_k} - 2\tau_1 \ell^2 \eta_{'\varepsilon} \pi_{k \bullet} \big) - 2\tau_1 \ell^2 \eta_{'\varepsilon} g_k \Big) .$$

Proof. We compute with the product rule

$$\overline{\sigma}_d = d' \bullet \Big(\frac{\delta \eta}{\delta d} + 2\tau_1 \eta_{'\varepsilon} \big(\overline{\mathrm{H}}_1 - \operatorname{div}\Sigma_1 \big) d - 2\tau_1 \eta_{'\varepsilon} \Sigma_1 \mathbin{:} (d \otimes \nabla \eta_{'\varepsilon}) \Big) ,$$

where the main part of the second summand is by (2.6)

$$d' \bullet \big(\overline{\mathrm{H}}_1 - \operatorname{div}\Sigma_1 \big) d = d' \bullet \big(\varrho (\mathscr{S}_1^{sp})^{\circ} - \mathrm{H}_1^0 \big) d . \tag{3.22}$$

Now it holds

$$\varrho (\mathscr{S}_1^{sp})^{\circ} - \mathrm{H}_1^0 = \varrho \, d \wedge d'' , \tag{3.23}$$

since using the definition of d' and d'' (see also [2: 5.5 Lemma])

$$\begin{aligned}&\Big(d \wedge d' \Big)^{\circ} = \overset{\circ}{d} \wedge d' + d \wedge \overset{\circ}{d'} = (d' + A_\xi d) \wedge d' + d \wedge (d'' + A_\xi d') \\ &= (A_\xi d) \wedge d' + d \wedge (A_\xi d') + d \wedge d'' = A_\xi (d \wedge d') + (d \wedge d)' {A_\xi}^{\mathrm{T}} + d \wedge d'' \\ &= A_\xi \mathscr{S}_1^{sp} + \mathscr{S}_1^{sp} {A_\xi}^{\mathrm{T}} + d \wedge d'' .\end{aligned}$$

After multiplying with ϱ and using (2.5) we get (3.23). Therefore (3.22) becomes

$$\begin{aligned} d'\bullet\big(\overline{\mathrm{H}}_1 - \operatorname{div}\Sigma_1\big)d = d'\bullet\big(\varrho(d\wedge d'')d\big) &= \varrho\, d'\bullet\big((dd''^{\mathrm{T}} - d''d^{\mathrm{T}})d\big) \\ &= \varrho\; d'\bullet d\; d''\bullet d - \varrho\; d'\bullet d''\;\ell^2 = -\ell^2 d'\bullet(\varrho d'') \end{aligned}$$

since $d'\bullet d = 0$. By using the identity (3.21)

$$-\ell^2 d'\bullet(\varrho d'') = \ell^2 d'\bullet(\operatorname{div}\pi - g)\,,$$

hence

$$2\tau_1\eta_{'\varepsilon}\sum_{k,l} d_k' d_l(\overline{\mathrm{H}}_1 - \operatorname{div}\Sigma_1)_{kl} = 2\tau_1\eta_{'\varepsilon}\ell^2 d'\bullet(\operatorname{div}\pi - g)\,.$$

Also by assumption (3.18)

$$\begin{aligned} -2\tau_1\sum_{k,j,l} d_k' d_l(\partial_j\eta_{'\varepsilon})\Sigma_{1\,klj} &= -2\tau_1\sum_{k,l,j} d_k' d_l(\partial_j\eta_{'\varepsilon})(d_k\pi_{jl} - \pi_{kj}d_l) \\ = -2\tau_1 d'\bullet d\sum_{l,j} d_l\partial_j(\eta_{'\varepsilon})\pi_{jl} &+ 2\tau_1\ell^2\sum_{k,j} d_k'\partial_j(\eta_{'\varepsilon})\pi_{kj} = 2\tau_1\ell^2 d'\bullet(\pi\nabla\eta_{'\varepsilon}) \end{aligned}$$

since $d'\bullet d = 0$. Then it holds all in all

$$\begin{aligned} \overline{\sigma}_d &= \sum_k d_k'\Big(\frac{\delta\eta}{\delta d_k} + 2\tau_1\ell^2\eta_{'\varepsilon}(\operatorname{div}(\pi) - g)_k + 2\tau_1\ell^2(\pi\nabla\eta_{'\varepsilon})_k\Big) \\ &= \sum_k d_k'\Big(\eta_{'d_k} - \operatorname{div}(\eta_{'\nabla d_k}) + 2\tau_1\ell^2\operatorname{div}(\eta_{'\varepsilon}\pi)_k - 2\tau_1\ell^2\eta_{'\varepsilon}g_k\Big) \\ &= \sum_k d_k'\Big(\eta_{'d_k} - \operatorname{div}\big(\eta_{'\nabla d_k} - 2\tau_1\ell^2\eta_{'\varepsilon}\pi_{k\bullet}\big) - 2\tau_1\ell^2\eta_{'\varepsilon}g_k\Big)\,. \end{aligned}$$

□

We now consider both effects, and obtain from 3.2 and 3.4 the following theorem.

3.5 Final theorem. Consider a solution of system (1.7) and let be as usual

$$\Pi = P - S\,. \tag{3.24}$$

Assume that (3.13) is satisfied for the spin and further assume (3.18) for Σ_1 and $\overline{\mathrm{H}}_1$. If the entropy η and the entropy flux ψ is given by (3.14), that is

$$\begin{aligned} \eta &= \widehat{\eta}(\varrho,\varepsilon,d,\mathrm{D}d) \\ \psi_j &= \eta v_j + \eta_{'\varepsilon}q_j - \sum_k d_k'\big(\eta_{'d_{k,j}} - 2\tau_1\ell^2\eta_{'\varepsilon}\pi_{kj}\big)\,, \end{aligned} \tag{3.25}$$

we have for the residual inequality

$$0 \le \sigma = \eta_{'\varepsilon}(\mathrm{D}v)^{\mathrm{S}} : S^{\mathrm{S}} + \nabla\eta_{'\varepsilon}\bullet q + \eta_{'\varepsilon}(\mathrm{D}u)^{\mathrm{A}} : P^{\mathrm{A}} + \overline{\sigma}_0 + \overline{\sigma}_d\,, \tag{3.26}$$

where $u := v - v_\xi$ and

$$\begin{aligned} \overline{\sigma}_0 &:= \eta_{'\varepsilon}\tau_0\sigma_0 = -\eta_{'\varepsilon}\tau_0\big(\mathrm{D}\mathscr{S}_0^{sp} : \Sigma_0 + \mathscr{S}_0^{sp} : \overline{\mathrm{H}}_0\big) \\ \overline{\sigma}_d &= d'\bullet\Big(\eta_{'\nabla d} - \operatorname{div}\big(\eta_{'\mathrm{D}d} - 2\tau_1\ell^2\eta_{'\varepsilon}\pi\big) - 2\tau_1\ell^2\eta_{'\varepsilon}g\Big)\,. \end{aligned} \tag{3.27}$$

Definition: We set $\Pi_m^{\mathrm{A}} = P_m^{\mathrm{A}} - S_m^{\mathrm{A}}$ when $S^{\mathrm{A}} = S_0^{\mathrm{A}} + S_1^{\mathrm{A}}$ and $P^{\mathrm{A}} = P_0^{\mathrm{A}} + P_1^{\mathrm{A}}$. We mention that below we assume $P_0^{\mathrm{A}} = 0$ and $P_1^{\mathrm{A}} = 0$.

Proof. We substitute in (3.14) the term for Σ_1, that is $\Sigma_{1klj} = d_k\pi_{lj} - \pi_{kj}d_l$ by (3.18). For the last summand of ψ we compute

$$
\begin{aligned}
&-\sum_k d_k^{'}\big(\eta_{'d_k{'}_j} + \sum_l 2\tau_1\eta_{'\varepsilon}d_l\Sigma_{1klj}\big)\\
&= -\sum_k d_k^{'}\big(\eta_{'d_k{'}_j} + \sum_l 2\tau_1\eta_{'\varepsilon}d_l(d_k\pi_{lj} - \pi_{kj}d_l)\big)\\
&= -\sum_k d_k^{'}\eta_{'d_k{'}_j} - 2\tau_1\eta_{'\varepsilon}\sum_{kl} d_k^{'}d_l(d_k\pi_{lj} - \pi_{kj}d_l)\\
&= -\sum_k d_k^{'}\eta_{'d_k{'}_j} - 2\tau_1\eta_{'\varepsilon}\sum_{kl} d_k^{'}d_ld_k\pi_{lj} + 2\tau_1\eta_{'\varepsilon}\sum_{kl} d_k^{'}d_l\pi_{kj}d_l\\
&= -\sum_k d_k^{'}\eta_{'d_k{'}_j} + 2\tau_1\eta_{'\varepsilon}\ell^2\sum_k d_k^{'}\pi_{kj} = -\sum_k d_k^{'}\big(\eta_{'d_k{'}_j} - 2\tau_1\ell^2\eta_{'\varepsilon}\pi_{kj}\big)\,,
\end{aligned}
$$

since $d^{'}\bullet d = 0$. Hence the representation of ψ in (3.25) follows. In the residual inequality (3.16) we use the definition of $\overline{\sigma}_0$ in (3.27), whereas $\overline{\sigma}_d$ was defined and calculated in 3.4. For the first summand of (3.16) by the setting of Π we get

$$\mathrm{D}(v - v_\xi)\,\mathbf{:}\,(P - \Pi^{\mathrm{S}}) = (\mathrm{D}v)^{\mathrm{S}}\,\mathbf{:}\,S^{\mathrm{S}} + (\mathrm{D}u)^{\mathrm{A}}\,\mathbf{:}\,P^{\mathrm{A}}$$

if we set $u := v - v_\xi$. From this (3.26) follows immediately. □

If we now take special forms of π and g this implies the following theorem. These special forms essentially depend on the entropy η and on g_1 as independent variable, the notation of which is also taken from the original papers.

3.6 Theorem. We now assume that we set $P_0^{\mathrm{A}} = 0$, that is $P^A = P_1^{\mathrm{A}}$, and we define

$$\pi := \frac{1}{2\tau_1\ell^2\eta_{'\varepsilon}}\eta_{'\mathrm{D}d} - d\otimes\beta\,, \qquad g := \frac{1}{2\tau_1\ell^2\eta_{'\varepsilon}}\eta_{'d} - (\gamma d + \mathrm{D}d\,\beta) - g_1\,, \tag{3.28}$$

where g_1 is an arbitrary function, and β and γ are degrees of freedom which don't appear in the residual inequality. Then it follows

$$\Big(2 - \tfrac{1}{\tau_1\ell^2}\Big)\,P^{\mathrm{A}} = 2\,S_1^{\mathrm{A}} - d\wedge g_1 - \overline{\Gamma}_1 \quad \text{if} \quad \tau_1\ell^2 > \tfrac{1}{2}\,, \tag{3.29}$$

$$\overline{\sigma}_d = 2\tau_1\ell^2\eta_{'\varepsilon}d^{'}\bullet g_1\,. \tag{3.30}$$

Proof of P^{A}. Since $2\Pi_1^{\mathrm{A}} + \overline{\Gamma}_1 = \overline{\mathrm{H}}_1$ and because of the representation of $\overline{\mathrm{H}}_1$ in (3.18) we get

$$d\wedge g = 2\,\Pi_1^{\mathrm{A}} - \sum_j(\partial_j d)\wedge\pi_{\bullet j} + \overline{\Gamma}_1\,.$$

With the constitutive equations (3.28) for π and g this is

$$
\begin{aligned}
&d\wedge\big(\tfrac{1}{2\tau_1\ell^2\eta_{'\varepsilon}}\eta_{'d} - g_1 - (\gamma d + \mathrm{D}d\,\beta)\big)\\
&= 2\,\Pi_1^{\mathrm{A}} - \sum_j(\partial_j d)\wedge\big(\frac{1}{2\tau_1\ell^2\eta_{'\varepsilon}}\eta_{'\partial_j d} - \beta_j d\big) + \overline{\Gamma}_1\,.
\end{aligned}
$$

The γ-term vanishes and the β-terms on both sides are equal, since

$$-d \wedge (\mathrm{D}d\,\beta) = \sum_j \beta_j\ d_{\prime j} \wedge d = \sum_j (\partial_j d) \wedge (\beta_j d)\,,$$

therefore

$$d \wedge \big(\frac{1}{2\tau_1 \ell^2 \eta_{\prime\varepsilon}} \eta_{\prime d} - g_1\big) = 2\,\Pi_1^{\mathrm{A}} - \sum_j (\partial_j d) \wedge \big(\frac{1}{2\tau_1 \ell^2 \eta_{\prime\varepsilon}} \eta_{\prime \partial_j d}\big) + \overline{\Gamma}_1\,.$$

Now with $\Pi_1^{\mathrm{A}} = -\,S_1^{\mathrm{A}} + P^{\mathrm{A}}$ by the Definition in 3.5 and the assumption $P_1^{\mathrm{A}} = P^{A}$ it turns out

$$\begin{aligned} d \wedge g_1 + \overline{\Gamma}_1 &= \tfrac{1}{2\tau_1 \ell^2 \eta_{\prime\varepsilon}} \big(d \wedge \eta_{\prime d} + \sum_j (\partial_j d) \wedge \eta_{\prime \partial_j d}\big) - 2\,\Pi_1^{\mathrm{A}} \\ &= \frac{1}{\tau_1 \ell^2 \eta_{\prime\varepsilon}} \big(d \otimes \eta_{\prime d} + \sum_j (\partial_j d) \otimes \eta_{\prime \partial_j d}\big)^{\mathrm{A}} - 2\,\Pi_1^{\mathrm{A}} \\ &= -\frac{1}{\tau_1 \ell^2 \eta_{\prime\varepsilon}} \big(\sum_k (\nabla d_k) \otimes \eta_{\prime \nabla d_k}\big)^{\mathrm{A}} - 2\,\Pi_1^{\mathrm{A}} = \frac{1}{\tau_1 \ell^2}\, P^{\mathrm{A}} - 2\,\Pi_1^{\mathrm{A}} \end{aligned}$$

by Condition 1.1 and the definition (3.15). Since $P^{\mathrm{A}} - 2\,\Pi_1^{\mathrm{A}} = 2\,S_1^{\mathrm{A}} - P^{\mathrm{A}}$ it follows

$$d \wedge g_1 = 2\,S_1^{\mathrm{A}} - \big(2 - \frac{1}{\tau_1 \ell^2}\big)\, P^{\mathrm{A}} - \overline{\Gamma}_1\,,$$

that is the claim. For the condition $\tau_1 \ell^2 > \frac{1}{2}$ we computed in a charakteristic special case in 2.2 that $\tau_1 \ell^2 = 6$. □

Proof of $\overline{\sigma}_d$. Substituting (3.28) into the divergence term of (3.27) yields

$$\begin{aligned} \sum_j \partial_j \big(\eta_{\prime d_{k,j}} - 2\tau_1 \ell^2 \eta_{\prime\varepsilon} \pi_{kj}\big) &= 2\tau_1 \ell^2 \sum_j \partial_j \big(\eta_{\prime\varepsilon} d_k \beta_j\big) \\ &= 2\tau_1 \ell^2 \operatorname{div}(\eta_{\prime\varepsilon} \beta)\, d_k + 2\tau_1 \ell^2 \sum_j \eta_{\prime\varepsilon} d_{k\prime j} \beta_j\,, \end{aligned}$$

and for the g-term it holds

$$\eta_{\prime d_k} - 2\tau_1 \ell^2 \eta_{\prime\varepsilon} g_k = 2\tau_1 \ell^2 \eta_{\prime\varepsilon} \big(\gamma d_k + \sum_j d_{k\prime j} \beta_j + g_{1k}\big)\,.$$

Hence

$$\begin{aligned} \eta_{\prime d_k} - \operatorname{div}\big(\eta_{\prime \nabla d_k} - 2\tau_1 \ell^2 \eta_{\prime\varepsilon} \pi_{k\bullet}\big) - 2\tau_1 \ell^2 \eta_{\prime\varepsilon} g_k \\ = 2\tau_1 \ell^2 \eta_{\prime\varepsilon} g_{1k} + 2\tau_1 \ell^2 \big(\eta_{\prime\varepsilon} \gamma - \operatorname{div}(\eta_{\prime\varepsilon} \beta)\big) d_k\,, \end{aligned}$$

therefore it follows (3.30) since $d \bullet d' = 0$. Thus the d'-term in the formula (3.16) of σ is rewritten as $d' \bullet (2\tau \ell^2 \eta_{\prime\varepsilon} g_1)$. □

Assuming that $\overline{\Gamma}_m = 0$, $m = 0, 1$, and $P^{A} = 0$ we derive the following application. It would be very interesting to have applications in the situation that $\overline{\Gamma}_m$ are given by Maxwell equations.

3.7 Application. Let $\overline{\Gamma}_m = 0$ for $m = 0, 1$ and assume $P^A = 0$ (that is an assumption on η, we assume $P_0^{\mathrm{A}} = 0$ and $P_1^{\mathrm{A}} = 0$). Further assume that $\Sigma_0 = 0$. Then the residual inequality reads

$$0 \leq \sigma = \eta_{'\varepsilon} (\mathrm{D}v)^{\mathrm{S}} \mathbin{:} S^{\mathrm{S}} + \nabla \eta_{'\varepsilon} \bullet q + 2\eta_{'\varepsilon} \tau_0 \mathscr{S}_0^{sp} \mathbin{:} S_0^{\mathrm{A}} + 2\tau_1 \ell^2 \eta_{'\varepsilon} d' \bullet g_1 \,, \tag{3.31}$$

Proof. From $P_0^{\mathrm{A}} = 0$ and $\overline{\Gamma}_0 = 0$ it follows

$$\overline{\mathrm{H}}_0 = 2\Pi_0^{\mathrm{A}} + \overline{\Gamma}_0 = 2\Pi_0^{\mathrm{A}} = -2\,S_0^{\mathrm{A}} \ ,$$

that is, $\overline{\sigma}_0$ in (3.27) reads

$$\overline{\sigma}_0 = -\eta_{'\varepsilon} \tau_0 \mathrm{D}\mathscr{S}_0^{sp} \mathbin{:} \Sigma_0 + 2\eta_{'\varepsilon} \tau_0 \mathscr{S}_0^{sp} \mathbin{:} S_0^{\mathrm{A}}$$

Using $P^{\mathrm{A}} = 0$ and (3.30) for $\overline{\sigma}_d$ the claim follows immediately from (3.26). □

4 Entropy principle

The main part of the proof is presented in this section. For the entropy we have by (3.14)

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) \tag{4.1}$$

and for the spin we have by (3.8)

$$\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_1 \quad \text{with} \quad \mathscr{S}_1 = \varrho\, d \wedge d' \,, \quad \text{where} \quad d' = \overset{\circ}{d} - A_\xi d \tag{4.2}$$

is an objective vector, see [2 : 4.3 Lemma, (4.4)], so that $\mathscr{S}_1$ is an objective matrix. With the entropy in (4.1) we compute for the entropy production

$$\begin{aligned}\sigma &= \partial_t \eta + \operatorname{div}\psi = \overset{\circ}{\eta} + \eta \operatorname{div} v + \operatorname{div}(\psi - \eta v)\\ &= \eta_{'\varrho} \overset{\circ}{\varrho} + \eta_{'\varepsilon} \overset{\circ}{\varepsilon} + \eta_{'d} \bullet \overset{\circ}{d} + \eta_{'\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^{\circ} + \eta \operatorname{div} v + \operatorname{div}(\psi - \eta v) \,.\end{aligned}$$

By the general system (1.7), which is equivalent to (3.1), the entropy production becomes

$$\begin{aligned}\sigma &= (\eta - \varrho \eta_{'\varrho} - \varepsilon \eta_{'\varepsilon}) \operatorname{div} v + \operatorname{div}(\psi - \eta v) - \eta_{'\varepsilon} \operatorname{div} q - \eta_{'\varepsilon} \mathrm{D}v \mathbin{:} \Pi^{\mathrm{S}}\\ &\qquad + \eta_{'d} \bullet \overset{\circ}{d} + \eta_{'\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^{\circ} - \eta_{'\varepsilon} \sum_m \tau_m \big(\mathrm{D}\mathscr{S}_m^{sp} \mathbin{:} \Sigma_m + \mathscr{S}_m^{sp} \mathbin{:} \overline{\mathrm{H}}_m \big)\\ &= \mathrm{D}v \mathbin{:} \big((\eta - \varrho \eta_{'\varrho} - \varepsilon \eta_{'\varepsilon}) \mathrm{Id} - \eta_{'\varepsilon} \Pi^{\mathrm{S}} \big) + \operatorname{div}(\psi - \eta v - \eta_{'\varepsilon} q)\\ &\quad + \nabla \eta_{'\varepsilon} \bullet q + \eta_{'d} \bullet \overset{\circ}{d} + \eta_{'\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^{\circ} - \eta_{'\varepsilon} \sum_m \tau_m \big(\mathrm{D}\mathscr{S}_m^{sp} \mathbin{:} \Sigma_m + \mathscr{S}_m^{sp} \mathbin{:} \overline{\mathrm{H}}_m \big) \,.\end{aligned}$$

Therefore, if we define the abbreviations (3.11)

$$\begin{aligned}\sigma_m &:= -\big(\mathrm{D}\mathscr{S}_m^{sp} \mathbin{:} \Sigma_m + \mathscr{S}_m^{sp} \mathbin{:} \overline{\mathrm{H}}_m \big) \,,\\ \sigma_d &:= \eta_{'d} \bullet \overset{\circ}{d} + \eta_{'\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^{\circ} \,,\end{aligned} \tag{4.3}$$

the whole expression can be written as

$$
\begin{aligned}
0 \le \sigma = \mathrm{D}v \mathbin{:} \big((\eta - \varrho\eta'_{\varrho} - \varepsilon\eta'_{\varepsilon})\mathrm{Id} - \eta'_{\varepsilon}\Pi^{\mathrm{S}}\big) + \operatorname{div}(\psi - \eta v - \eta'_{\varepsilon} q) \\
+\nabla\eta'_{\varepsilon}\bullet q + \eta'_{\varepsilon}\tau_0\sigma_0 + (\sigma_d + \eta'_{\varepsilon}\tau_1\sigma_1)\,.
\end{aligned}
\tag{4.4}
$$

This is the residual inequality in Theorem 3.1.

The most important term in (4.4) is the σ_d term, since $\overset{\circ}{d}$ and $(\mathrm{D}d)^{\circ}$ have to be expressed by the d'-terms, where d' is part of $\mathscr{S}_1^{sp}$. This is done in Section 5 and gives the result

$$
\begin{aligned}
\sigma_d &= \eta'_d \bullet \overset{\circ}{d} + \eta'_{\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^{\circ} \\
&= \sum_i \eta'_{d_i} d_i' + \sum_{i,j} \eta'_{d_{i,j}} (d_i')'_j - \mathrm{D}(v - v_\xi) \mathbin{:} \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big)\,.
\end{aligned}
\tag{4.5}
$$

Also the σ_d-term has only objective quantities. Now we go to the σ_1-term. Plugging the definition (4.2) of $\mathscr{S}_1$ we obtain

$$
\begin{aligned}
\sigma_1 &= -\big(\mathrm{D}\mathscr{S}_1^{sp} \mathbin{:} \Sigma_1 + \mathscr{S}_1^{sp} \mathbin{:} \overline{\mathrm{H}}_1\big) = -\big(\mathrm{D}(d \wedge d') \mathbin{:} \Sigma_1 + d \wedge d' \mathbin{:} \overline{\mathrm{H}}_1\big) \\
&= -\sum_{k,l} \big(\sum_j \partial_j (d_k d_l' - d_k' d_l)\Sigma_{1klj} + (d_k d_l' - d_k' d_l)\overline{\mathrm{H}}_{1kl}\big) \\
&= 2\sum_{k,l} \big(\sum_j (d'_{k}{}'_j d_l + d_k' d_l{}'_j)\Sigma_{1klj} + d_k' d_l \overline{\mathrm{H}}_{1kl}\big) \\
&= \sum_{k,j} d'_{k}{}'_j \sum_l 2 d_l \Sigma_{1klj} + \sum_k d_k' \big(\sum_{l,j} 2 d_l{}'_j \Sigma_{1klj} + \sum_l 2 d_l \overline{\mathrm{H}}_{1kl}\big)\,.
\end{aligned}
$$

Therefore the part of the entropy production which have contributions of the director is

$$
\begin{aligned}
&\sigma_d + \eta'_{\varepsilon}\tau_1\sigma_1 = \\
&= \sum_{k,j} d'_{k}{}'_j \big(\eta'_{d_{k,j}} + \sum_l 2\tau_1 \eta'_{\varepsilon} d_l \Sigma_{1klj}\big) + \sum_k d_k' \big(\eta'_{d_k} + \sum_{l,j} 2\tau_1 \eta'_{\varepsilon} d_l{}'_j \Sigma_{1klj} + \sum_l 2\tau_1 \eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{1kl}\big) \\
&\qquad - \mathrm{D}(v - v_\xi) \mathbin{:} \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big) \\
&= \sum_j \partial_j \Big(\sum_k d_k' \big(\eta'_{d_{k,j}} + \sum_l 2\tau_1 \eta'_{\varepsilon} d_l \Sigma_{1klj}\big)\Big) \\
&\quad + \sum_k d_k' \Big(\eta'_{d_k} - \sum_j \partial_j \eta'_{d_{k,j}} + 2\tau_1 \sum_{l,j} \Big(\eta'_{\varepsilon} d_l{}'_j \Sigma_{1klj} - \partial_j(\eta'_{\varepsilon} d_l \Sigma_{1klj})\Big) + \sum_l 2\tau_1 \eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{1kl}\Big) \\
&\qquad - \mathrm{D}(v - v_\xi) \mathbin{:} \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big)\,.
\end{aligned}
$$

Moreover, in the middle term the coefficient of d_k' becomes

$$
\begin{aligned}
\eta'_{d_k} - \sum_j \partial_j \eta'_{d_{k,j}} + 2\tau_1 \sum_{l,j} \Big(\eta'_{\varepsilon} d_l{}'_j \Sigma_{1klj} - \partial_j(\eta'_{\varepsilon} d_l \Sigma_{1klj})\Big) + \sum_l 2\tau_1 \eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{1kl} \\
= \frac{\delta\eta}{\delta d_k} + 2\tau_1 \sum_l d_l \Big(\eta'_{\varepsilon} \overline{\mathrm{H}}_{1kl} - \sum_j \partial_j(\eta'_{\varepsilon} \Sigma_{1klj})\Big)\,,
\end{aligned}
$$

hence

$$\sigma_d + \eta'_\varepsilon \tau_1 \sigma_1 = \sum_j \partial_j \Big(\sum_k d_k' \big(\eta'_{d_{k,j}} + \sum_l 2\tau_1 \eta'_\varepsilon d_l \Sigma_{1klj} \big) \Big)$$
$$+ \sum_k d_k' \Big(\frac{\delta\eta}{\delta d_k} + 2\tau_1 \sum_l d_l \Big(\eta'_\varepsilon \overline{\mathrm{H}}_{1kl} - \sum_j \partial_j (\eta'_\varepsilon \Sigma_{1klj}) \Big) \Big) - \mathrm{D}(v - v_\xi) \mathbin{:} \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j} \big) .$$

The first term on the right side goes to the div-term in σ and the last term goes together with the Dv-term, since $(\mathrm{D}v)^{\mathrm{S}} = (\mathrm{D}(v - v_\xi))^{\mathrm{S}}$. Therefore we finally obtain for the entropy production σ

$$\begin{aligned}
0 \le \sigma &= (\mathrm{D}v)^{\mathrm{S}} \mathbin{:} \big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon \Pi^{\mathrm{S}} \big) + \operatorname{div}(\psi - \eta v - \eta'_\varepsilon q) \\
&\qquad + \nabla\eta'_\varepsilon \bullet q + \eta'_\varepsilon \tau_0 \sigma_0 + (\sigma_d + \eta'_\varepsilon \tau_1 \sigma_1) \\
&= \sum_j \partial_j \Big(\psi_j - \eta v_j - \eta'_\varepsilon q_j + \sum_k d_k' \big(\eta'_{d_{k,j}} + \sum_l 2\tau_1 \eta'_\varepsilon d_l \Sigma_{1klj} \big) \Big) \\
&\qquad + \mathrm{D}(v - v_\xi) \mathbin{:} \Big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \sum_j \nabla d_j \otimes \eta'_{\nabla d_j} - \eta'_\varepsilon \Pi^{\mathrm{S}} \Big) \\
&\qquad + \nabla\eta'_\varepsilon \bullet q - \eta'_\varepsilon \tau_0 \big(\mathrm{D}\mathscr{S}_0^{sp} \mathbin{:} \Sigma_0 + \mathscr{S}_0^{sp} \mathbin{:} \overline{\mathrm{H}}_0 \big) \\
&\qquad + \sum_k d_k' \Big(\frac{\delta\eta}{\delta d_k} + 2\tau_1 \sum_l d_l \Big(\eta'_\varepsilon \overline{\mathrm{H}}_{1kl} - \sum_j \partial_j (\eta'_\varepsilon \Sigma_{1klj}) \Big) \Big) .
\end{aligned} \tag{4.6}$$

This formula for the entropy production σ is the general form of the entropy production under the assumption that the entropy fulfills (4.1) and the spin is given by (4.2). This formula is further evaluated in the proofs of Section 3.

5 Director part

The purpose of this section is to handle the term σ_d which occurred in the residual inequality in Section 4. This part of the entropy principle is contained in [2 : Sec. 8], although under different assumptions. Let us use the fact that η must be an objective scalar, consequently by Condition 1.1 there holds for any antisymmetric matrix B

$$0 = \eta'_d \bullet Bd + \eta'_{\mathrm{D}d} \mathbin{:} (B\mathrm{D}d + \mathrm{D}d\, B^{\mathrm{T}}) .$$

We take B satisfying the transformation rule $B \circ Y = \dot{Q} Q^{\mathrm{T}} + QB^* Q^{\mathrm{T}}$, and an example is $B := (\mathrm{D}v)^{\mathrm{A}}$. With this we get for the d-terms in the entropy production

$$\begin{aligned}
\sigma_d &:= \eta'_d \bullet \overset{\circ}{d} + \eta'_{\mathrm{D}d} \mathbin{:} (\mathrm{D}d)^\circ \\
&= \eta'_d \bullet (\overset{\circ}{d} - Bd) + \eta'_{\mathrm{D}d} \mathbin{:} \big((\mathrm{D}d)^\circ - (B\mathrm{D}d + \mathrm{D}d\, B^{\mathrm{T}}) \big) \\
&= \eta'_d \bullet d^\eta + \eta'_{\mathrm{D}d} \mathbin{:} D^\eta ,
\end{aligned}$$

where $d^\eta := \overset{\circ}{d} - Bd$ is an objective vector, which we have proved in [2: 4.3(2)]. And $D^\eta := (\mathrm{D}d)^\circ - (B\mathrm{D}d + \mathrm{D}d\,B^{\mathrm{T}})$ satisfies the representation

$$\begin{aligned}(\mathrm{D}d)^\circ{}_{ij} &= (d_{i\,'j})^\circ = d_{i\,'jt} + \sum_k v_k d_{i\,'jk} \\ &= \Big(d_{i\,'t} + \sum_k v_k d_{i\,'k}\Big)_{'j} - \sum_k v_{k\,'j} d_{i\,'k} = \big(\overset{\circ}{d_i}\big)_{'j} - \sum_k v_{k\,'j} d_{i\,'k}\,,\end{aligned}$$

hence, where we use now $B = (\mathrm{D}v)^{\mathrm{A}}$,

$$\begin{aligned}D^\eta_{ij} &= (\mathrm{D}d)^\circ{}_{ij} - (B\mathrm{D}d)_{ij} - \frac{1}{2}\sum_k d_{i\,'k}(v_{j\,'k} - v_{k\,'j}) \\ &= \big(\overset{\circ}{d_i}\big)_{'j} - (B\mathrm{D}d)_{ij} - \frac{1}{2}\sum_k d_{i\,'k}(v_{j\,'k} + v_{k\,'j}) \\ &= \Big(\big(\overset{\circ}{d_i}\big)_{'j} - \sum_k B_{ik} d_{k\,'j}\Big) - \sum_k d_{i\,'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj}\,.\end{aligned}$$

The first term (in bracket) is an objective tensor in (i,j), see [2: 4.3(3)], also, of course, the second term with the symmetric part of the velocity gradient, which is easily to see. Since $d^\eta = \overset{\circ}{d} - Bd$ we get

$$\big(\overset{\circ}{d_i}\big)_{'j} - \sum_k B_{ik} d_{k\,'j} = \Big(d^\eta_i + \sum_k B_{ik} d_k\Big)_{'j} - \sum_k B_{ik} d_{k\,'j} = (d^\eta_i)_{'j} + \sum_k B_{ik\,'j} d_k\,,$$

which finally gives

$$D^\eta_{ij} = (d^\eta_i)_{'j} + \sum_k B_{ik\,'j} d_k - \sum_k d_{i\,'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj}\,,$$

where now all three terms are objective tensors. This is because B_{ik} in (i,k) is a tensor which transforms like the derivative of a velocity, where the inhomogeneous part of this transformation depends only on time. Therefore $B_{ik\,'j}$ has not this inhomogeneous part and is in (i,k,j) an objective 3-tensor.

The matrix A_ξ in 2.1 is an antisymmetric matrix depending only on t and transforms as B like the derivative of a velocity. Therefore we see that $\overline{B} := B - A_\xi$ is an objective tensor, and since A_ξ depends only on time we conclude that $\overline{B}_{ik\,'j} = B_{ik\,'j}$ is, as said, an objective 3-tensor. Therefore using (3.8)

$$d^\eta + \overline{B}d = \overset{\circ}{d} - A_\xi d = d'\,. \tag{5.1}$$

Hence

$$\begin{aligned}(d^\eta_i)_{'j} &+ \sum_k B_{ik\,'j} d_k = (d^\eta_i)_{'j} + \sum_k \overline{B}_{ik\,'j} d_k \\ &= \Big(d^\eta_i + \sum_k \overline{B}_{ik} d_k\Big)_{'j} - \sum_k \overline{B}_{ik} d_{k\,'j} = (d'_i)_{'j} - \sum_k \overline{B}_{ik} d_{k\,'j}\,,\end{aligned}$$

and therefore

$$D^\eta_{ij} = (d'_i)_{'j} - \sum_k \overline{B}_{ik} d_{k\,'j} - \sum_k d_{i\,'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj}\,,$$

$$d^\eta_i = d'_i - \sum_k \overline{B}_{ik} d_k\,.$$

Hence the contribution of σ_d is

$$\begin{aligned}\sigma_d &= \eta_{'d} \bullet d^\eta + \eta_{'\mathrm{D}d} : D^\eta \\ &= \sum_i \eta_{'d_i}\big(d_i' - \sum_k \overline{B}_{ik} d_k\big) + \sum_{i,j} \eta_{'d_{i,j}}\big((d_i')_{'j} - \sum_k \overline{B}_{ik} d_{k'j} - \sum_k d_{i'k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj}\big) \\ &= \sum_i \eta_{'d_i} d_i' + \sum_{i,j} \eta_{'d_{i,j}} (d_i')_{'j} \\ &\qquad - \sum_{i,k} \big(\eta_{'d_i} d_k + \sum_j \eta_{'d_{i,j}} d_{k'j}\big) \overline{B}_{ik} - \sum_{i,j,k} \eta_{'d_{i,j}} d_{i'k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj}\,.\end{aligned}$$

Now we make again usage of the Condition 1.1 and this gives for the $\overline{B}$-term

$$\begin{aligned}&\sum_{i,k} \big(\eta_{'d_i} d_k + \sum_j \eta_{'d_{i,j}} d_{k'j}\big) \overline{B}_{ik} = - \sum_{i,k,j} \eta_{'d_{j,i}} d_{j'k} \overline{B}_{ik} \\ &= -\overline{B} : \big(\sum_j \eta_{'\nabla d_j} \otimes \nabla d_j\big) = \overline{B} : \big(\sum_j \nabla d_j \otimes \eta_{'\nabla d_j}\big) = (\mathrm{D}(v - v_\xi))^{\mathrm{A}} : \big(\sum_j \nabla d_j \otimes \eta_{'\nabla d_j}\big)\,,\end{aligned}$$

since $A_\xi = \mathrm{D}v_\xi$ by 2.1 is antisymmetric and therefore $\overline{B} = (\mathrm{D}v)^{\mathrm{A}} - A_\xi = (\mathrm{D}(v - v_\xi))^{\mathrm{A}}$. Also $(\mathrm{D}v)^{\mathrm{S}} = (\mathrm{D}(v - v_\xi))^{\mathrm{S}}$ and therefore

$$\begin{aligned}\sum_{i,j,k} \eta_{'d_{i,j}} d_{i'k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj} &= \sum_{j,k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj} \sum_i \eta_{'d_{i,j}} d_{i'k} = (\mathrm{D}v)^{\mathrm{S}} : \big(\sum_i \nabla d_i \otimes \eta_{'\nabla d_i}\big) \\ &= (\mathrm{D}(v - v_\xi))^{\mathrm{S}} : \big(\sum_i \nabla d_i \otimes \eta_{'\nabla d_i}\big)\,.\end{aligned}$$

Together we obtain for the contribution σ_d

$$\begin{aligned}\sigma_d &= \eta_{'d} \bullet d^\eta + \eta_{'\mathrm{D}d} : D^\eta \\ &= \sum_i \eta_{'d_i} d_i' + \sum_{i,j} \eta_{'d_{i,j}} (d_i')_{'j} - \mathrm{D}(v - v_\xi) : \big(\sum_j \nabla d_j \otimes \eta_{'\nabla d_j}\big)\,.\end{aligned} \tag{5.2}$$

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