Advances in Mathematical
Sciences and Applications
Vol. 31, No. 2 (2022), pp. 343–382

GAKKOTOSHO
TOKYO JAPAN

A NEW METHOD FOR LIQUID CRYSTALS BASED ON THE SPIN EQUATION

Hans Wilhelm Alt
Technische Universität München
Zentrum Mathematik, Boltzmannstraße 3, 85747 Garching bei München
(E-mail: alt@ma.tum.de)

and

Gabriele Witterstein
Technische Universität München
Zentrum Mathematik, Boltzmannstraße 3, 85747 Garching bei München
(E-mail: gw@ma.tum.de)

Abstract. This paper is concerned with the dynamics of liquid crystals with a non-symmetric form of the pressure tensor. The formulation uses the classical spin equation, where the spin depends on the director d of the local embedded crystals. Thus the mass-momentum equations are completed by the spin equation, and since we consider the temperature dependent case, in addition an energy equation is necessary. The temperature dependent free energy is given as usual by the entropy through $f = \varepsilon - \theta\eta$ and in the stationary case this reduces to well known results. But the main thing is that we apply the entropy principle and use an equivalent formulation of the fact that the entropy is an objective scalar. This way we prove our general theorem by showing that the entropy production is positiv. As application of this theorems we show that important results of H. Grad, Ericksen & Leslie, I. Müller, and Chandrasekhar are correct.

Communicated by Editors; Received May 19, 2022
Mathematics Subject Classification 2020: 35Q35, 35Q79, 76A15
Physics and Astronomy Classification Scheme 2010: 02.30.Jr, 05.70.-a, 47.57.Lj, 65.40.Gr
Keywords: Liquid Crystals, Partial differential equations, Thermodynamics, Entropy principle

1 Introduction

We consider liquid crystals in the nematic state, which are in principle fluids, but they show different behaviour in different temperature intervals. Some of them change the physical conditions abruptly if crossing a particular temperature value. For a general impression of such media see the contribution in [Wikipedia: Liquid crystal (Mar 2022)]. Therefore it is clear that this subject is important for mathematical science, especially since nowadays there is a great interest from material science in liquid crystals. The theoretical literature about nematic liquid crystals has developed mainly during the period between 1950 and 1990. We will give a short overview of the history at the end of this section.

The angular momentum played an important role from the very beginning since the models of liquid crystals have a nonsymmmetric pressure tensor Π as standard feature. Here Π stands for all objective contributions in the momentum equation, that is, at least it contains the pressure and the viscous stress tensor. The nonsymmetric character of Π becomes clear if one looks at the equation of angular momentum (5.5), which is based on the mass-momentum system for (ϱ, v),

$$\partial_t \mathscr{J} + \operatorname{div}\big(\mathscr{J}\, v^{\mathrm{T}} + (x-\xi)\wedge\Pi + \Sigma\big) = (x-\xi)\wedge(\mathbf{f} - \varrho\ddot{\xi}) + \Gamma\,, \tag{1.1}$$

where the tensor $\mathscr{J}$ is antisymmetric and Σ, Γ are explained in (5.7). The moving reference point ξ is described below. This law of angular momntum is stated in the book of DeGroot & Mazur [4 : Chap.XII §1(3)]. Now $\mathscr{J} = \mathscr{L} + \mathscr{S}$ contains the orbital angular momentum $\mathscr{L} = (x-\xi)\wedge\varrho(v-\dot{\xi})$ which satisfies

$$\partial_t \mathscr{L} + \operatorname{div}\big(\mathscr{L}\, v^{\mathrm{T}} + (x-\xi)\wedge\Pi\big) = -2\,\Pi^{\mathrm{A}} + (x-\xi)\wedge\big(\mathbf{f} - \varrho\ddot{\xi}\big)\,, \tag{1.2}$$

and this equation is a direct consequence of the underlying mass-momentum theory with an arbitrary tensor Π, which contains the antisymmetric part Π^{A}. It is a classical formula which you find in the Script of Alt [1 : Sec.II.6] and in all literature with an arbitrary Π. (The formula (1.2) is common in literature which makes the Cauchy assumption that Π is symmetric.) So the difference $\mathscr{S} := \mathscr{J} - \mathscr{L}$, which is called the spin, satisfies the spin equation (5.6)

$$\partial_t \mathscr{S} + \operatorname{div}_x(\mathscr{S}\, v^{\mathrm{T}} + \Sigma) = 2\,\Pi^{\mathrm{A}} + \Gamma\,, \tag{1.3}$$

which is usually written in the form* $\varrho\overset{\circ}{\mathscr{S}}{}^{sp} + \operatorname{div}\Sigma = 2\,\Pi^{\mathrm{A}} + \Gamma$ where $\mathscr{S}^{sp}$ is the specific spin. We are working with the spin equation and we add this to the mass-momentum system, so that

$$\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho v) &= 0\,,\\ \partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) &= \mathbf{f}\,,\\ \partial_t\mathscr{S} + \operatorname{div}(\mathscr{S}\, v^{\mathrm{T}} + \Sigma) &= 2\,\Pi^{\mathrm{A}} + \Gamma\,, \end{aligned} \tag{1.4}$$

is the system, see (2.1), which is the basis of our paper. We want to emphasize that all the equations used here are observer independent. In the literature usually one uses

*The material derivative $\overset{\circ}{h} := \partial_t h + v\bullet\nabla h$ is usually denoted by $\dot{h}$

the special case $\xi = 0$, which means that the angular momentum is meant with respect to the actual observer, and only for him. Therefore we have introduced the trajectory $t \in \mathbb{R} \mapsto \xi(t) \in \mathbb{R}^3$ as a freely movable reference point with respect to which the angular momentum is measured, see Section 5. And the spin should describe a property of the body itself, it should transform itself like an objective quantity. Altogether this is the observer independent, i.e. frame indifferent, method.

Since we consider the temperature dependent case, we have to add the differential equation of the total energy e, which transforms like the kinetic energy,

$$\begin{aligned} \partial_t e + \operatorname{div} \widetilde{q} = \widetilde{g} &:= v \bullet \mathbf{f} + \mathrm{D}v \,\mathbf{:}\, \Pi^{\mathrm{A}} + \mathbf{g}\,, \\ e &= \tfrac{\varrho}{2}|v|^2 + \tfrac{\varrho}{2}\tau|\mathscr{S}^{sp}|^2 + \varepsilon\,, \end{aligned} \tag{1.5}$$

where ε contains the absolute temperature θ. The contributions of $\widetilde{g}$, that is the v dependent terms, have to be there, and then $\mathbf{g}$ is an objective scalar (see e.g. [1 : II.3.14 Mass-momentum-energy theorem]). Since we have no other contribution to the energy, this e is regarded as the total energy, which is assumed to be conserved by the energy principle. It explains why the objective scalar $\mathbf{g}$ can be set to $\mathbf{g} = 0$. Therefore the equation for ε becomes using the system (1.4) and with $\widetilde{q} = \Pi^{\mathrm{T}}\, v + \tau\mathscr{S}^{sp} \,\mathbf{:}\, \Sigma + q$, where q is the heat flux,

$$\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) = -\mathrm{D}v \,\mathbf{:}\, \Pi^{\mathrm{S}} - \tau\big(\mathrm{D}\mathscr{S}^{sp} \,\mathbf{:}\, \Sigma + \mathscr{S}^{sp} \,\mathbf{:}\, \mathrm{H}\big)\,. \tag{1.6}$$

We add this to the system (1.4) and get system (2.7), which is the basis for the entropy principle. We mention that the equation (1.6) is also used by Leslie [17 : (3.13)], where to obtain this he used the director equation (6.4) instead of the spin equation.

At the end of the paper in Section 9 we deal with a classical entropy

$$\eta = \widehat{\eta}(\varrho, \varepsilon) \tag{1.7}$$

and perform the entropy principle. This was also done by DeGroot & Mazur in the book [4 : Non-Equilibrium Thermodynamics], but without stating objectivity and so with a different energy equation. We go to system (2.7) as basis and derive in Theorem 9.1 an entropy production (9.4). Here the entropy principle is true for arbitrary spin. If the spin part gives a positive contribution to the entropy production this essentially says that the spin equation (1.3) has a dissipative behaviour.

In Section 2 we deal with nematic liquid crystals, that is, we use the standard model with a so called director d where $|d| = \ell = \text{const} > 0$ and with the spin

$$\mathscr{S} = \varrho\, d \wedge d'\,, \qquad d' := \overset{\circ}{d} - A_\xi d\,. \tag{1.8}$$

Note that here d' is an objective vector. We remark that we use the variable d with length ℓ which has the advantage that the magnitude of the spin can be adjusted. Then we apply the entropy principle with

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) \tag{1.9}$$

where Dd are the spatial derivatives of the director d which is a vector, see e.g. in the case of incompressible equilibria Hardt & Kinderlehrer [15]. With the constitutive equation (1.9) for η we obtain the central theorem of this paper, with the same system (2.7) as basis. We call this result the Main Theorem 2.2 and it has the following entropy production (2.15)

$$\begin{aligned} 0 \leq \sigma = \eta'_{\varepsilon} \mathrm{D}u \mathbin{:} (P - \Pi^{\mathrm{S}}) + \nabla \eta'_{\varepsilon} \bullet q \\ + d' \bullet \Big(\frac{\delta \eta}{\delta d} + 2\tau \big(\eta'_{\varepsilon} (2\Pi^{\mathrm{A}} + \overline{\Gamma}) - \operatorname{div}(\eta'_{\varepsilon} \Sigma) \big) d \Big) \,. \end{aligned} \tag{1.10}$$

Here the d' term is the contribution which is due to the special form $\mathscr{S} = \varrho\, d \wedge d'$ of the spin (1.8), or one can interprete this as coming from the dependence of η on $(d, \mathrm{D}d)$.

Setting $\Pi = P - S$ with a general $S = S^{\mathrm{S}} + S^{\mathrm{A}}$, see the text following equation (2.18), the inequality (2.15) is equivalent to (2.19)

$$\begin{aligned} 0 \leq \sigma = \eta'_{\varepsilon} \mathrm{D}u \mathbin{:} (P^{\mathrm{A}} + S^{\mathrm{S}}) + \nabla \eta'_{\varepsilon} \bullet q \\ + d' \bullet \Big(\frac{\delta \eta}{\delta d} + 2\tau (\eta'_{\varepsilon} \overline{\mathrm{H}} - \operatorname{div}(\eta'_{\varepsilon} \Sigma)) d \Big) \,, \end{aligned} \tag{1.11}$$

where, see (5.14),

$$\overline{\mathrm{H}} = \overline{\Gamma} + 2 P^{\mathrm{A}} - 2 S^{\mathrm{A}} \,. \tag{1.12}$$

If we assume $\overline{\Gamma} = 0$ the inequality $\sigma \geq 0$ is a condition on S^{S}, S^{A} and q, and the system (1.4) has the form, using (5.13),

$$\begin{gathered} \partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,, \\ \partial_t (\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + P - S) = \mathbf{f} \,, \\ \varrho(\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0) + \operatorname{div}_x \Sigma = \overline{\mathrm{H}} \,, \qquad \mathscr{S}^{sp} = d \wedge d' \,, \qquad \overline{\mathrm{H}} = 2\Pi^{\mathrm{A}} = 2 P^{\mathrm{A}} - 2 S^{\mathrm{A}} \,, \end{gathered} \tag{1.13}$$

and in addition the energy equation (1.6) for the inner energy is fulfilled. This is true in the general case.

The following Section 3 deals with special applications of the Main Theorem 2.2. It are the results of Ericksen & Leslie and Chandrasekhar, where the essential assumption is

$$\begin{aligned} \Sigma = d \wedge \pi \,, \quad & \pi \text{ an objective tensor,} \\ \overline{\mathrm{H}} = d \wedge g + \textstyle\sum_j (\partial_j d) \wedge \pi_{\bullet j} \,, \quad & g \text{ an objective vector.} \end{aligned} \tag{1.14}$$

In Lemma 5.6 it is shown that under these assumptions the reduced spin equation is equivalent to

$$d \wedge (\varrho d'' + \operatorname{div} \pi - g) = 0 \,, \tag{1.15}$$

which we call the “director equation”. And for the antisymmetric part of Π an “additional equation” has to be satisfied

$$2\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}} = d \wedge g + \textstyle\sum_j (\partial_j d) \wedge \pi_{\bullet j} \,. \tag{1.16}$$

In 3.1 both equations (1.15) and (1.16) together are proved to imply the (full) spin equation. In fact, they are equivalent to the (full) spin equation with the condition (1.14).

This is the reason why the system (3.2) implies system (2.7). In other words, we mention that the models of Ericksen & Leslie and of Chandrasekhar turned out to be a special case of our system.

We should point out the fact that the entropy (because of the nonnegativity of the entropy production) has to be an objective scalar. Depending on the special constitutive equation (1.9) on η this has an handy equivalent formulation in Condition 7.1. This condition is vary helpful for the proof of the form of σ in Section 8. And this condition for the free energy is also contained in the literature see e.g. Ericksen [8 : Sec.VII], Leslie [17 : (4.10)] and Chandrasekhar [3 : (3.1.15)].

We now come to the historical development.

Around one century ago in 1933 there is the paper [23] of Oseen, where he introduced in the compressible case a system of differential equations, the equations of mass-momentum and entropy and a director equation, a second order equation for d, see [23 : p.896] ($d \rightsquigarrow \mathrm{L}$). The right-hand side of the director equation contains a multiple of d, as in (6.1). He postulated an entropy equation, but by a little computation shows that the system with entropy is equivalent to the system with inner energy. The momentum equation is coupled with the director equation by an "intrinsic part" given by the director distribution in the neighborhood of the element (we would interpret this term saying that it is depending on ∇d etc.). Hence his system is a full system. However, the stress tensor is symmetric, which is why we do not treat this here.

In 1952 H.Grad [14] considered viscous flows letting the standard assumption that the pressure tensor Π is symmetric behind. In [14 : §4] he sets up the general equation

$$\partial_t(\varrho \mathbf{a}) + \operatorname{div}(\varrho \mathbf{a} v + \mathbf{A}) = \boldsymbol{\alpha}$$

for $(\mathbf{a}, \mathbf{A}, \boldsymbol{\alpha})$ and uses it to derive a system consisting of four equations, that are mass, momentum, energy and angular momentum, see [14 : (4.6), (4.11), (4.15), (4.12)], where the quantities of this system partly deviate from the standard quantities. He motivates this by considerations on statistical mechanics. The conservation of angular momentum is given in [14 : (4.12)] ($\mathscr{J}^{sp} \rightsquigarrow M + \mu$, $\Sigma \rightsquigarrow Q$)†, where the external angular momentum is written down in [14 : (4.13)] ($\mathscr{L}^{sp} \rightsquigarrow M$) and the internal angular momentum in [14 : (4.14)] ($\mathscr{S}^{sp} \rightsquigarrow \mu$) hence $\mathscr{J} = \mathscr{L} + \mathscr{S}$. This general treatment were later in 1962 adopted from DeGroot & Mazur in the book [4 : Non-Equilibrium Thermodynamics]. We chose this approach because it is a general description of angular momentum. In this paper in Section 9, we prove the entropy principle, where the entropy η is taken according to the classical representation $\eta = \widehat{\eta}(\varrho, \varepsilon)$, which means that the Gibbs relation [4 : Chap.XII§1 (20)] ($\eta \rightsquigarrow \varrho s$, $\Pi \rightsquigarrow P$) holds. We carried out this accurately and have been able to verify the representation of Grad's pressure tensor [14 : (4.30)] ($\Pi_{ij} \rightsquigarrow P_{ij} = p\delta_{ij} + p_{ij}$). It should be noted that the authors use the internal energy equation [14 : (4.21)] or [4 : Chap.XII§1 (13)] with the term $\Pi : \mathrm{D}v$ instead of the objective scalar $\Pi : (\mathrm{D}v)^{\mathrm{S}}$, which contradicts the objectivity of this equation.

†We want to show with the remark in brackets, how the quantities in this paper correspond to the quantities in literature.

In 1960, in his paper [7], Ericksen first wrote down the standard equations [7 : (2.1)-(2.4)] for mass, momentum, energy and angular momentum, which are generally known and were derived for symmetric pressure tensors. Then he sets up his system of equations [7 : (2.9)-(2.13)], where equation [7 : (2.11)] is Oseen's director equation, as he says, in a very abbreviated form. So, to make the system complete again, his main concern is equation [7 : (2.13)], which contains the antisymmetric part Π^{A}. For the term Π^{A} he mentions a work by Toupin (1956) and section [14 : §4] of Grad. However, he does not see that the equation [14 : (4.14)], after some thinking, is the one he wishes to have. He only says: "Grad discusses conservation laws more general than those used here. In doing so, he introduces an equation similar to (2.13)." In 1961 Ericksen [8] quoted from Grad general conservation laws [8 : (1)-(4)] and he combines his director equation [8 : (16)] with an equation for the antisymmetric part of the pressure tensor [8 : (42)], which together with the coupling [8 : (35)] imply the equation of angular momentum [8 : (44)] ($x \wedge \dot{x} + n \wedge \dot{n}$, $d \rightsquigarrow \ell n$). So he gave, without saying this, a different reasoning for the model of angular momentum with $\mathscr{S} = \varrho\, d \wedge d'$, which for general $\mathscr{S}$ Grad stated in [14 : (4.12)-(4.14)] ($\mathscr{S} \rightsquigarrow \varrho\mu$) one decade before. In Section 6 we come back to this.

In 1967 I. Müller had with a fundamental paper [21 : On the entropy inequality] a major impact on the theory. Leslie was also inspired by the work of I. Müller and in the paper [17] from 1968 he systemizes all previous approaches. In the entire paper he considered for the first time the objectivity of all quantities and conservation equations. Especially he writes the inner energy equation independent of the observer, that is, there is only the contribution $\mathrm{D}v \boldsymbol{:} \Pi^{\mathrm{S}}$ on the right-hand side of equation [17 : (3.13)], see also Chandrasekhar [3 : (3.1.8)]. In the case of the entropy inequality [17 : (3.14)-(3.16)], according to I. Müller, the entropy flow is assumed arbitrarily and with usage of the existing conservation equations the entropy inequality [17 : (3.16) and (4.8)] is written in a general setting. Then [17 : (4.9)] the representation of of the free energy $f = \widehat{f}(\varrho, \theta, d, \mathrm{D}d)$ is shown. After the entropy flux has been determined, he obtains the residual inequality [17 : (4.14)]. This inequality allows him to write the flux and production terms in standard terms and remainder terms, the latter then must satisfy the final residual inequality [17 : (4.18)]. This procedure is also adopted in the book by Chandrasekhar [3 : 3 Continuum theory of the nematic state].

I.Müller 1985 presented in his book [22 : 10 Thermodynamics of Nematic Liquid Crystals] a chapter about the Ericksen & Leslie theory. As introduction he consideres a rigid body approximated by mass points. He derived from the angular momentum and energy for these collection of points [22 : (10.4) and (10.5)], in a way which is comparable with Grad, the equations of angular momentum and energy for the rigid body [22 : (10.16)]. In this representation additional terms occur, which also are there in the continuum limit [22 : (10.19)]. He also proves in [22 : 10.1.2.2 Balance of spin and director balance] that for the spin $\mathscr{S} = \varrho\, d \wedge d'$ the balance equation is equivalent to the director equation plus the equation for the antisymmetric part of the pressure tensor.

In the book [10 : Theory and Applications of Liquid Crystals] in 1987 by Ericksen & Kinderlehrer were published essays of various authors on the subject of liquid crystals, one articel by Hardt & Kinderlehrer [15 : Mathematical Questions of Liquid Crystal Theory]. This article treats stationary problems and in particular uses free energies [15 : (1.1)]

($f \rightsquigarrow W$, $d = \ell \mathrm{n}$) of the form

$$2\widehat{f}(\mathrm{n}, \nabla \mathrm{n}) = \kappa_1(\operatorname{div}\mathrm{n})^2 + \kappa_2(\mathrm{n}\bullet \operatorname{curl}\mathrm{n} + q)^2 + \kappa_3|\mathrm{n}\wedge \operatorname{curl}\mathrm{n}|^2 \\ +(\kappa_2+\kappa_4)(\operatorname{trace}(\nabla \mathrm{n})^2 - (\operatorname{div}\mathrm{n})^2)\,,$$

see Frank [13]. "Other contributions to the energy may be given by magnetic or electric fields", see [15 : 2 Existence theory for nematics and cholesterics]. It is studied, how stationary solutions look like.

In 1969 Ericksen writes down the equations [9 : (1)-(12)] and says: "We consider the fluids to be incompressible and ignore thermal effects". In 1991 he introduces the first steps of the Q-theory in [12] because "the interest in liquid crystals is associated with efforts to control their orientation". In 1992 Leslie in [19] considers a form of total angular momentum [19 : (2.2)], but he says: "The inertial term associated with local rotation of the material element is omitted because in general it is negligible."

In 1995 Lin & Liu [20] made several simplifications of the model which, among other things, lead to a symmetric pressure tensor. The outcome of this procedure is "the simplest mathematical model one can derive, without destroying the basic nonlinear structure" (from [20]). This model has a symmetric pressure tensor, and several papers about existence theory have used this model. As far as we know, since then only Lasarzig with his Dissertation [16] has written a mathematical paper in which a nonsymmetric stress tensor has been modeled for showing that a weak solution of the problem exists.

2 Main theorem

The problem is based on the following system of differential equations

$$\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho v) &= 0\,, \\ \partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) &= \mathbf{f}\,, \\ \partial_t\mathscr{S} + \operatorname{div}(\mathscr{S} v^{\mathrm{T}} + \Sigma) &= \mathrm{H} := 2\Pi^{\mathrm{A}} + \Gamma\,. \end{aligned} \tag{2.1}$$

Here Π is the pressure tensor in the general setting, that is $\Pi = \Pi^{\mathrm{S}} + \Pi^{\mathrm{A}}$ with an antisymmetric part Π^{A}. This part appears as right-hand side of the spin equation. The spin $\mathscr{S}$ is by (5.8) an objective matrix, and Σ is an objective 3-tensor. In general the spin equation is written as a specific quantity $\mathscr{S}^{sp}$, i.e. $\mathscr{S} = \varrho\mathscr{S}^{sp}$, so it follows using the mass equation $\partial_t\varrho + \operatorname{div}(\varrho v) = 0$

$$\varrho\overset{\circ}{\mathscr{S}}{}^{sp} + \operatorname{div}\Sigma = \mathrm{H}\,, \tag{2.2}$$

where $\overset{\circ}{\mathscr{S}}{}^{sp}$ is the material derivative of $\mathscr{S}^{sp}$. We mention that we call (2.2) the "reduced spin equation", because H has been defined to make the equation

$$2\Pi^{\mathrm{A}} + \Gamma = \mathrm{H} \tag{2.3}$$

an independent relation.

Since we consider the temperature dependent case, we have in addition the differential equation of the total energy e (see e.g. [1 : II.3.14 Mass-momentum-energy theorem])

$$\partial_t e + \operatorname{div} \widetilde{q} = \widetilde{g} := v \bullet \mathbf{f} + \mathrm{D}v \,\mathbf{:}\, \Pi^{\mathrm{A}} + \mathbf{g} \,. \tag{2.4}$$

The objective scalar $\mathbf{g}$ is set to be 0, since e is the total energy which is assumed to be conserved by the energy principle. The other contributions of $\widetilde{g}$ depend on v and therefore are not available for such setting. Of course, e contains the kinetic energy, but also the spin has a contribution to e. Therefore, since we assume that there is no other contribution to the energy equation, we write

$$e = \frac{\varrho}{2}|v|^2 + \frac{\varrho}{2}\tau|\mathscr{S}^{sp}|^2 + \varepsilon \,, \tag{2.5}$$

where the remaining term ε contains the absolute temperature θ, and where $\tau > 0$ is assumed to be a constant. (It can be generalised to $\mathscr{S}^{sp} \mathbf{:} \boldsymbol{\tau} \mathscr{S}^{sp}$ where $\boldsymbol{\tau}$ is a positive definite 4-tensor, see [2].) We mention that the whole spin is assumed to be an energetic term. Now, the kinetic energy satisfies (see e.g. [1 : III.2.2])

$$\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \operatorname{div}\Big(\frac{\varrho}{2}|v|^2 \varrho v + \Pi^{\mathrm{T}} v\Big) = v \bullet \mathbf{f} + \mathrm{D}v \,\mathbf{:}\, \Pi \,,$$

and the spin energy, by multiplying (2.2) by $\mathscr{S}^{sp}\mathbf{:}$ and doing an operation as above,

$$\partial_t\Big(\frac{\varrho}{2}|\mathscr{S}^{sp}|^2\Big) + \operatorname{div}\Big(\frac{\varrho}{2}|\mathscr{S}^{sp}|^2 \varrho v + \mathscr{S}^{sp}\mathbf{:}\Sigma\Big) = \mathrm{D}\mathscr{S}^{sp}\mathbf{:}\Sigma + \mathscr{S}^{sp}\mathbf{:}\mathrm{H} \,,$$

here

$$\mathscr{S}^{sp}\mathbf{:}\Sigma = \Bigg(\sum_{k,l} \mathscr{S}^{sp}{}_{kl}\Sigma_{klj}\Bigg)_{j=1,2,3} \quad \text{and} \quad \mathrm{D}\mathscr{S}^{sp}\mathbf{:}\Sigma = \sum_{k,l,j} \mathscr{S}^{sp}{}_{kl}{}'{}_{j}\Sigma_{klj} \,.$$

Subtracting these equations from (2.4) we obtain with $\widetilde{q} = \Pi^{\mathrm{T}} v + \tau \mathscr{S}^{sp}\mathbf{:}\Sigma + q$

$$\partial_t \varepsilon + \operatorname{div}(\varepsilon v + q) = -\mathrm{D}v \,\mathbf{:}\, \Pi^{\mathrm{S}} - \tau\big(\mathrm{D}\mathscr{S}^{sp}\mathbf{:}\Sigma + \mathscr{S}^{sp}\mathbf{:}\mathrm{H}\big) \,. \tag{2.6}$$

Altogether the equations for the temperature depending system are (2.1) plus (2.4). This system reads, since $\partial_t \varepsilon + \operatorname{div}(\varepsilon v) = \mathring{\varepsilon} + \varepsilon \operatorname{div} v$ and since $\mathscr{S}\mathbf{:}\mathrm{H} = \mathscr{S}\mathbf{:}\overline{\mathrm{H}}$ by 5.2,

$$\begin{gathered}
\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,, \\
\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f} \,, \\
\varrho \mathring{\mathscr{S}}^{sp} + \operatorname{div}\Sigma = \mathrm{H} \,, \quad 2\Pi^{\mathrm{A}} + \Gamma = \mathrm{H} \,, \\
\mathring{\varepsilon} + \varepsilon \operatorname{div} v = -\operatorname{div} q - \mathrm{D}v \,\mathbf{:}\, \Pi^{\mathrm{S}} - \tau\big(\mathrm{D}\mathscr{S}^{sp}\mathbf{:}\Sigma + \mathscr{S}^{sp}\mathbf{:}\overline{\mathrm{H}}\big) \,.
\end{gathered} \tag{2.7}$$

These are the general equations we consider. There is a property which solutions of system (2.7) have to satisfy, it is the entropy principle. It means that the inequality

$$\sigma := \partial_t \eta + \operatorname{div}\psi \geq 0 \tag{2.8}$$

has to hold, where η is the entropy and ψ the entropy flux. We mention that this inequality is the reason why η has to be an objective scalar.

2.1 Assumption. We assume that η is an objective scalar. Therefore, if $\widehat{\eta}$ is defined as in (7.3), the Condition 7.1 is applicable.

If η depends on the variables (ϱ, ε) only, the entropy principle (2.8) is proved in Section 9 and we show in the theorem 9.1 that the pressure tensor Π has to be nonsymmetric. As we point out, this nonsymmetry is strongly related to the spin, which is explained in Section 5. This is one model.

In models of nematic liquid crystals the main quantity is, in accordance with well known theories, the director d of the corresponding fluid particles. The director d of constant length $\ell > 0$ is the characteristic quantity for crystals. We prove the main principle (2.8) if η depends besides on (ϱ, ε) on the values and first derivatives of the director d, that is,

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d)\,, \tag{2.9}$$

for which the Condition 7.1 applies. Also here the pressure tensor Π is antisymmetric. It is clear that this antisymmetry is strongly connected to the spin, which is explained in Section 5. Here the spin depends on the director d and has the following representation

$$\mathscr{S}^{sp} = d \wedge d'\,, \tag{2.10}$$

where d is an objective vector. Now $\mathscr{S}^{sp}$ has to be an objective matrix, and this is true if d' is an objective vector, which is the case when d' equals the relative value, see 5.3,

$$d' := \overset{\circ}{d} - A_\xi d\,. \tag{2.11}$$

The main theorem is the following, which is a theorem on the entropy principle.

2.2 Main theorem. Consider solutions of system (2.7) with the spin satisfying (2.10)

$$\mathscr{S} = \varrho\, d \wedge d' \tag{2.12}$$

with d the director of the fluid. Let the entropy η and the entropy flux ψ be of the form

$$\begin{aligned} &\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d)\,, \quad \eta'_\varepsilon = \frac{1}{\theta} > 0\,, \quad \theta \text{ the absolute temperature,} \\ &\psi_j = \eta v_j + \eta'_\varepsilon q_j - \sum_k d'_k \big(\eta'_{d_{k,j}} + \sum_l 2\tau \eta'_\varepsilon d_l \Sigma_{klj}\big)\,, \end{aligned} \tag{2.13}$$

where η satisfies 2.1, and furthermore let

$$P := p\mathrm{Id} - \theta \sum_i \nabla d_i \otimes \eta'_{\nabla d_i}\,, \quad p := \theta(\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\,, \tag{2.14}$$

and let the relative velocity $u := v - v_\xi$. Then the entropy principle (2.8) is satisfied, if the entropy production satisfies the residual inequality

$$\begin{aligned} &0 \le \sigma = \eta'_\varepsilon \mathrm{D}u \,\boldsymbol{:}\, (P - \Pi^{\mathrm{S}}) + \nabla \eta'_\varepsilon \bullet q \\ &+ d' \bullet \Big(\frac{\delta\eta}{\delta d} + 2\tau\big(\eta'_\varepsilon(2\Pi^{\mathrm{A}} + \overline{\Gamma}) - \operatorname{div}(\eta'_\varepsilon \Sigma)\big)d\Big)\,, \end{aligned} \tag{2.15}$$

where $\overline{\Gamma}$ is defined in (5.14).
Remark: The last line written in components is

$$\sum_k d_k'\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\sum_l d_l\big(\eta'_\varepsilon(2\Pi^{\mathrm{A}}_{kl} + \overline{\Gamma}_{kl}) - \sum_j \partial_j(\eta'_\varepsilon\Sigma_{klj})\big)\Big)\,,$$

and the first variation of η with respect to d is

$$\frac{\delta\eta}{\delta d_k} = \eta'_{d_k} - \sum_j \partial_j\eta'_{d_{k,j}}\,. \tag{2.16}$$

The costitutive equation of the entropy is well known, for example, see in the incompressible case the free energy in Ericksen [9: (8)], Hardt & Kinderlehrer in Ericksen & Kinderlehrer [10: (1.1) on p.152], and Chandrasekhar [3: (3.1.14)], and in the compressible case see the entropy in Leslie [17: (4.9)] ($\eta \rightsquigarrow \varrho S$) and I.Müller [22: (10.39)] ($\eta \rightsquigarrow \varrho\eta$, the additional terms are due to a not correct energy equation, see [22: $(3.45)_1$]). Observe that the entropy flux ψ in (2.13) has compared with Clausius-Duhem an additional term in d' and such term is well known from phase-field theories. The form of the pressure tensor Π, especially the fact that it is nonsymmetric, is also very common, and the representation $\Pi = P - S$ where S is the stress tensor often arises. All fluxes (Π, q, Σ) of the underlying theorem (2.7) are subject to restrictions due to the residual inequality (2.15).

Proof. With an arbitrary entropy function, of course being an objective scalar, and with the dependence in (2.13) the main part of the proof is done in Section 8. The result of Section 8 is the equation for the entropy production σ in (8.5). The last line of (8.5) gives rise to the form of the the entropy flux ψ defined in (2.13). It contains terms which are also in the Clausius-Duhem form and it contains an additional d' term, which is familiar with corresponding terms in phase field models.

The rest consists of three terms, where the second one is due to heat transfer q and the first term gives rise to define the Gibbs relation with the pressure p, see (2.14). The entire pressure part P contains $-\sum_i \nabla d_i \otimes \eta'_{\nabla d_i}$ which may have an antisymmetric part. The antisymmetry is also the case for $\mathrm{D}u$ with $u := v - v_\xi$, which is important for applications. So the whole term reads

$$\mathrm{D}u : (\eta'_\varepsilon P - \eta'_\varepsilon \Pi^{\mathrm{S}})\,,$$

which below in (2.19) is rewritten with $\Pi = P - S$. So finally the d' term remains, and this is the essential one which is discussed in further sections. Here the identity $\overline{\mathrm{H}} = 2\Pi^{\mathrm{A}} + \overline{\Gamma}$ is used. $\square$

The residual inequality says that one can as usual define the stress tensor S by

$$\Pi = P - S\,, \tag{2.17}$$

but where S here and in general has an antisymmetric part S^{A}, and the pressure part P as defined in (2.14) and is not only a scalar pressure p times the Identity. With this definition one can formulate everything in terms of $(S^{\mathrm{S}}, \overline{\mathrm{H}})$ instead of Π. It is

$$\begin{gathered} P - \Pi^{\mathrm{S}} = P^{\mathrm{A}} + S^{\mathrm{S}}\,, \qquad P^{\mathrm{A}} = -\theta\textstyle\sum_i (\nabla d_i \otimes \eta'_{\nabla d_i})^{\mathrm{A}}\,, \\ \Pi^{\mathrm{A}} = P^{\mathrm{A}} - S^{\mathrm{A}}\,, \qquad 2\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}}\,, \end{gathered} \tag{2.18}$$

hence the residual inequality (2.15) becomes

$$
\begin{aligned}
0 \le \sigma = {} & \eta_{'\varepsilon}\mathrm{D}u \mathbin{:} (P^{\mathrm{A}} + S^{\mathrm{S}}) + \nabla\eta_{'\varepsilon} \bullet q \\
& + d' \bullet \Big(\frac{\delta\eta}{\delta d} + 2\tau(\eta_{'\varepsilon}\overline{\mathrm{H}} - \operatorname{div}(\eta_{'\varepsilon}\Sigma))d\Big) \,.
\end{aligned}
\tag{2.19}
$$

Hence this entropy production (2.19) is equivalent to (2.15) and has several realizations. This will be exploited in the next section and we will show that the main results of the existing theory for liquid crystals are correct. The representation of the d' coefficient together with the spin equation (2.2) looks similar to Emmrich et al. [5 : (46a) and (48)] in the Q-tensor theory.

3 Applications

Here we apply the main theorem 2.2 in order to show that results of Ericksen & Leslie and Chandrasekhar in the incompressible case and I. Müller in the compressible case are correct, although they prove this with different assumptions. We treat here the system under the following conditions

$$
\begin{aligned}
\mathscr{S}^{sp} &= d \wedge d' \ \big(\text{in detail: } \mathscr{S}_{kl} = \varrho(d_k d_l' - d_k' d_l)\big) \\
\Sigma &= d \wedge \pi \ \big(\text{in detail: } \Sigma_{klj} = d_k \pi_{lj} - \pi_{kj} d_l\big) \\
\overline{\mathrm{H}} &= d \wedge g + \textstyle\sum_j \partial_j d \wedge \pi_{\bullet j} \ \big(\text{in detail: } \overline{\mathrm{H}}_{kl} = d_k g_l - g_k d_l + \sum_j (d_{k'j}\pi_{lj} - \pi_{kj} d_{l'j})\big)
\end{aligned}
\tag{3.1}
$$

where π is an objective matrix and g an objective vector. We show now that this is a special case of system (2.7). Afterwards we prove that main results in the literature follow from system (2.7).

3.1 Special case. Let an objective matrix π and an objective vector g be given with (3.1), and assume that

$$
\begin{aligned}
& \partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,, \\
& \partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f} \,, \\
& d \wedge (\varrho d'' + \operatorname{div}\pi - g) = 0 \,, \quad 2\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}} = d \wedge g + \textstyle\sum_j \partial_j d \wedge \pi_{\bullet j} \,, \\
& \mathring{\varepsilon} + \varepsilon \operatorname{div} v = -\operatorname{div} q - \mathrm{D}v \mathbin{:} \Pi^{\mathrm{S}} - 2\ell^2\tau\big(\mathrm{D}d' \mathbin{:} \pi + d' \bullet g\big) \,.
\end{aligned}
\tag{3.2}
$$

is true. Then system (2.7) is satisfied.

The third line of (3.2) is a consequence of the model by Leslie [18 : (15) and (14)] and Chandrasekhar [3 : (3.1.9) and (3.1.10)]. Already Ericksen in [8 : (16),(42) with coupling (35)] saw the connection with angular momentum in [8 : (44)].

Proof. The mass und momentum conservations in (3.2) and (2.7) are the same. To show the reduced spin equation in (2.7), we notice that the first part of the third line in (3.2) (equivalent to the director equation (6.4)) is just (5.21). With the help of Lemma 5.6 we see that under assumption (3.1) this equation (5.21) is equivalent to the reduced spin equation (5.13), which can be reformulated, since $\mathrm{H} = \bar{\mathrm{H}} + \varrho\Gamma^0$, into the reduced spin equation in (2.7).

The associated conditions on Π^{A} in both, (2.7) and (3.2), are the same, due to $\Gamma = \bar{\Gamma} + \varrho\Gamma^0$.

For the internal energy equation in (2.7) we calculate out using (3.1)

$$\mathrm{D}\mathscr{S}^{sp}\mathbin{\vdots}\Sigma + \mathscr{S}^{sp}\mathbin{\vdots}\bar{\mathrm{H}} = \mathrm{D}(d \wedge d')\mathbin{\vdots}(d \wedge \pi) + (d \wedge d')\mathbin{\vdots}(d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j})\,.$$

Using the rule $(\vec{a}\wedge\vec{b})\mathbin{\vdots}(\vec{c}\wedge\vec{d}) = 2(\vec{a}\bullet\vec{c})(\vec{b}\bullet\vec{d}) - 2(\vec{a}\bullet\vec{d})(\vec{b}\bullet\vec{c})$ we get with the help of $d\bullet d' = 0$ and $d\bullet\partial_j d = 0$

$$\begin{aligned}\mathrm{D}(d \wedge d')\mathbin{\vdots}(d \wedge \pi) &= \sum_j(\partial_j d \wedge d' + d \wedge \partial_j d')\mathbin{\vdots}(d \wedge \pi_{\bullet j}) = \sum_j(d \wedge \partial_j d')\mathbin{\vdots}(d \wedge \pi_{\bullet j})\\ &= 2|d|^2\mathrm{D}d'\mathbin{\vdots}\pi - 2\sum_j(d\bullet\pi_{\bullet j})((\partial_j d')\bullet d)\,,\end{aligned}$$

$$(d \wedge d')\mathbin{\vdots}(d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j}) = 2|d|^2 d'\bullet g - 2\sum_j(d\bullet\pi_{\bullet j})(d'\bullet(\partial_j d))\,.$$

From $d\bullet d' = 0$ we get $\partial_j d\bullet d' + d\bullet\partial_j d' = 0$ and thus

$$\sum_j(d\bullet\pi_{\bullet j})((\partial_j d')\bullet d) + \sum_j(d\bullet\pi_{\bullet j})(d'\bullet(\partial_j d)) = 0\,.$$

So with $|d|^2 = \ell^2$ we have proven $\mathrm{D}\mathscr{S}^{sp}\mathbin{\vdots}\Sigma + \mathscr{S}^{sp}\mathbin{\vdots}\bar{\mathrm{H}} = 2\ell^2(\mathrm{D}d'\mathbin{\vdots}\pi + d\bullet g)$. □

We do not refer to this system (3.2) because we know from 3.1 that system (2.7) is satisfied, which implies that also all consequences we have made are applicable here with in addition only using (3.1). Now, with the assumptions (3.1) we prove a special representation of the d' term in the residual inequality (2.15).

3.2 Theorem. Let the assumptions of the main theorem 2.2 be true and assume (3.1) for Σ and $\overline{\mathrm{H}}$. Then the residual inequality (2.15) becomes

$$\begin{aligned}0 \le \sigma = {}& \eta'_\varepsilon \mathrm{D}u\mathbin{\vdots}(P^{\mathrm{A}} + S^{\mathrm{S}}) + \nabla\eta'_\varepsilon\bullet q\\ &+ \textstyle\sum_k d'_k\Big(\eta'_{d_k} - \operatorname{div}\big(\eta'_{\nabla d_k} - 2\ell^2\tau\eta'_\varepsilon\pi_{k\bullet}\big) - 2\ell^2\tau\eta'_\varepsilon g_k\Big)\,.\end{aligned} \tag{3.3}$$

Proof. In the main theorem 2.2 the d' term was

$$\begin{aligned}&\sum_k d'_k\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\sum_l d_l\big(\eta'_\varepsilon\overline{\mathrm{H}}_{kl} - \sum_j \partial_j(\eta'_\varepsilon\Sigma_{klj})\big)\Big)\\ &= \sum_k d'_k\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\eta'_\varepsilon\sum_l d_l(\overline{\mathrm{H}} - \operatorname{div}\Sigma)_{kl} - 2\tau\sum_{j,l} d_l(\partial_j\eta'_\varepsilon)\Sigma_{klj}\Big)\,.\end{aligned}$$

Using the spin equation of the system (2.7) in the version (5.13) and lemma 5.5

$$\begin{aligned}\sum_{k,l} d'_k d_l(\overline{\mathrm{H}} - \operatorname{div}\Sigma)_{kl} &= \sum_{k,l} d'_k d_l\varrho(\mathring{\mathscr{S}}^{sp} - \Gamma^0)_{kl} = \varrho\sum_{k,l} d'_k d_l(d \wedge d'')_{kl}\\ &= \varrho\sum_{k,l} d'_k d_l(d_k d''_l - d''_k d_l) = \varrho\big(d'\bullet d\; d\bullet d'' - d'\bullet d''\; d\bullet d\big) = -\ell^2 d'\bullet(\varrho d'')\,.\end{aligned}$$

Using lemma 5.6, exactly (5.22), we see due to $d^{'}\bullet d = 0$ that

$$-\ell^2 d^{'}\bullet(\varrho d^{''}) = \ell^2 d^{'}\bullet(\,\mathrm{div}\pi - g)\,,$$

hence

$$2\tau\eta_{'\varepsilon}\sum_{k,l} d_k^{'} d_l(\overline{\mathrm{H}} -\, \mathrm{div}\Sigma)_{kl} = 2\tau\eta_{'\varepsilon}\ell^2 d^{'}\bullet(\,\mathrm{div}\pi - g)\,.$$

Also by assumption (3.1)

$$\begin{aligned}
-2\tau\sum_{k,j,l} d_k^{'} d_l(\partial_j\eta_{'\varepsilon})\Sigma_{klj} &= -2\tau\sum_{k,l,j} d_k^{'} d_l(\partial_j\eta_{'\varepsilon})(d_k\pi_{jl} - \pi_{kj}d_l) \\
&= -2\tau d^{'}\bullet d\sum_{l,j} d_l\partial_j(\eta_{'\varepsilon})\pi_{jl} + 2\tau\ell^2\sum_{k,j} d_k^{'}\partial_j(\eta_{'\varepsilon})\pi_{kj} = 2\tau\ell^2 d^{'}\bullet(\pi\nabla\eta_{'\varepsilon})
\end{aligned}$$

Then it holds all in all

$$\begin{aligned}
&\sum_k d_k^{'}\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\sum_l d_l\big(\eta_{'\varepsilon}\overline{\mathrm{H}}_{kl} - \sum_j\partial_j(\eta_{'\varepsilon}\Sigma_{klj})\big)\Big) \\
&= \sum_k d_k^{'}\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\ell^2\eta_{'\varepsilon}(\,\mathrm{div}(\pi) - g)_k + 2\tau\ell^2(\pi\nabla\eta_{'\varepsilon})_k\Big) \\
&= \sum_k d_k^{'}\Big(\eta_{'d_k} -\, \mathrm{div}(\eta_{'\nabla d_k}) + 2\tau\ell^2\,\mathrm{div}(\eta_{'\varepsilon}\pi)_k - 2\tau\ell^2\eta_{'\varepsilon}g_k\Big) \\
&= \sum_k d_k^{'}\Big(\eta_{'d_k} -\, \mathrm{div}\big(\eta_{'\nabla d_k} - 2\tau\ell^2\eta_{'\varepsilon}\pi_{k\bullet}\big) - 2\tau\ell^2\eta_{'\varepsilon}g_k\Big)\,.
\end{aligned}$$

□

In the following we do something which is similar of introducing the S term as coefficient of $\mathrm{D}u$ in the entropy production. We introduce an undetermined g_1 so that the coefficient of the $d^{'}$ term in (3.3) is

$$\eta_{'d} -\, \mathrm{div}\big(\eta_{'\mathrm{D}d} - 2\tau\ell^2\eta_{'\varepsilon}\pi\big) - 2\tau\ell^2\eta_{'\varepsilon}g \;\in\; 2\tau\ell^2\eta_{'\varepsilon}g_1 + \{d^{'}\}^{\perp}\,. \tag{3.4}$$

Therefore given g_1 we assume the following for π and g in terms of derivatives of the entropy η

$$\begin{aligned}
\pi &:= \frac{1}{2\tau\ell^2\eta_{'\varepsilon}}\eta_{'\mathrm{D}d} - d\otimes\beta\,, \qquad g := \frac{1}{2\tau\ell^2\eta_{'\varepsilon}}\eta_{'d} - (\gamma d + \mathrm{D}d\,\beta) - g_1\,, \\
\text{in detail:}\quad \pi_{kj} &:= \frac{\theta}{2\tau\ell^2}\eta_{'d_{k,j}} - d_k\beta_j\,,\quad g_k := \frac{\theta}{2\tau\ell^2}\eta_{'d_k} - \big(\gamma d_k + \sum_j d_{k'j}\beta_j\big) - g_{1k}\,,
\end{aligned} \tag{3.5}$$

where the vectors $\beta, \gamma \in \mathbb{R}^3$ are due to the indeterminacy of π and g. In Chandrasekhar in the incompressible case [3: (3.1.16) and (3.1.20)] (with $\pi \rightsquigarrow -\pi$ and $g \rightsquigarrow -g$ and $f \rightsquigarrow F$) these definitions are done for the free energy function $f = \varepsilon - \theta\eta$, in detail

$$f = \widehat{f}(\varrho, \theta, d, \mathrm{D}d) = \varepsilon - \theta\widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) \quad \text{for} \quad \theta\widehat{\eta}_{'\varepsilon}(\varrho, \varepsilon, d, \mathrm{D}d) = 1\,,$$

where then it is assumed that in the used range η is a convex function of ε. Then the definitions above read

$$\pi_{kj} := -\frac{1}{2\tau\ell^2}f_{'d_{k,j}} - d_k\beta_j\,,\quad g_k := -\frac{1}{2\tau\ell^2}f_{'d_k} - \big(\gamma d_k + \sum_j d_{k'j}\beta_j\big) - g_{1k}\,. \tag{3.6}$$

These definitions are also contained in I.Müller [22: $(10.62)_6$ and $(10.62)_7$] in the compressible case (with $\beta \rightsquigarrow \widehat{\beta} - \varrho f_{'\nabla d} d$ and $\pi \rightsquigarrow -\Pi$, $g \rightsquigarrow g$, $f \rightsquigarrow \psi$, $\theta \rightsquigarrow T$). We obtain the following theorem.

3.3 Theorem. Let the assumptions of the theorem 3.2 be true. Then the constitutive equations (3.5) imply that the relation (3.4) is satisfied. Moreover, it follows that the entropy flux is of the Clausius-Duhem form

$$\psi_j = \eta v_j + \eta_{'\varepsilon} q_j \,,$$

and the residual inequality (3.3) becomes

$$0 \le \sigma = \eta_{'\varepsilon} \mathrm{D}u \mathbin{:} (P^{\mathrm{A}} + S^{\mathrm{S}}) + \nabla \eta_{'\varepsilon} \bullet q + 2\tau\ell^2 \eta_{'\varepsilon}\, d' \bullet g_1 \,. \tag{3.7}$$

Proof. From (3.5) it follows

$$\begin{aligned}
\sum_j \partial_j \big(\eta_{'d_{k,j}} - 2\tau\ell^2 \eta_{'\varepsilon} \pi_{kj}\big) &= 2\tau\ell^2 \sum_j \partial_j \big(\eta_{'\varepsilon} d_k \beta_j\big) \\
&= 2\tau\ell^2 \operatorname{div}(\eta_{'\varepsilon}\beta)\, d_k + 2\tau\ell^2 \sum_j \eta_{'\varepsilon} d_{k'j} \beta_j \,, \\
\eta_{'d_k} - 2\tau\ell^2 \eta_{'\varepsilon} g_k &= 2\tau\ell^2 \eta_{'\varepsilon} \big(\gamma d_k + \sum_j d_{k'j} \beta_j + g_{1k}\big) \,,
\end{aligned}$$

hence

$$\eta_{'d_k} - \operatorname{div}\big(\eta_{'\nabla d_k} - 2\tau\ell^2 \eta_{'\varepsilon} \pi_{k\bullet}\big) - 2\tau\ell^2 \eta_{'\varepsilon} g_k = 2\tau\ell^2 \eta_{'\varepsilon} g_{1k} + 2\tau\ell^2 \big(\eta_{'\varepsilon}\gamma - \operatorname{div}(\eta_{'\varepsilon}\beta)\big) d_k \,,$$

therefore (3.4) is satisfied since $d \bullet d' = 0$.

For the entropy flux it is in (2.13) by (3.5)

$$\begin{aligned}
&\sum_l 2\tau \eta_{'\varepsilon} d_l \Sigma_{klj} = \sum_l 2\tau \eta_{'\varepsilon} d_l (d_k \pi_{lj} - d_l \pi_{kj}) \\
&= \sum_l 2\tau \eta_{'\varepsilon} d_l d_k \pi_{lj} - 2\tau\ell^2 \eta_{'\varepsilon} \pi_{kj} = \sum_l 2\tau \eta_{'\varepsilon} d_l d_k \pi_{lj} + 2\tau\ell^2 \eta_{'\varepsilon} d_k \beta_j - \eta_{'d_{k,j}} \,,
\end{aligned}$$

and therefore

$$\sum_k d'_k \big(\eta_{'d_{k,j}} + \sum_l 2\tau \eta_{'\varepsilon} d_l \Sigma_{klj}\big) = 2\tau \eta_{'\varepsilon} \sum_k d'_k d_k \big(\sum_l d_l \pi_{lj} + \ell^2 \beta_j\big) = 0$$

since $d \bullet d' = 0$.

The two constitutive relations (3.5) imply that by (3.4) the d' term in the formula of σ is rewritten as $d' \bullet (2\tau\ell^2 \eta_{'\varepsilon} g_1)$. □

The following lemma uses the fact that P^{A} can be expressed by S^{A} and g_1.

3.4 Lemma. Let the assumptions of the main theorem 2.2 be true and in addition assume that $2\tau\ell^2 > 1$ and take the special cases (3.1) with (3.5) for π and g. Then the residual inequality reads

$$0 \le \sigma = \eta_{'\varepsilon} \mathrm{D}u \mathbin{:} \Big(S^{\mathrm{S}} + \tfrac{\tau\ell^2}{2\tau\ell^2 - 1} \big(2\, S^{\mathrm{A}} - d \wedge g_1 - \overline{\Gamma}\big)\Big) + \nabla \eta_{'\varepsilon} \bullet q + 2\tau\ell^2 \eta_{'\varepsilon} d' \bullet g_1 \tag{3.8}$$

Remark: In applications usually it is assumed that $\overline{\Gamma} = 0$.

Proof. By using theorem 3.3 the residual inequality (3.3) reads

$$0 \le \sigma = \eta'_\varepsilon \mathrm{D}u \mathbin{:} (P^{\mathrm{A}} + S^{\mathrm{S}}) + \nabla\eta'_\varepsilon \bullet q + 2\tau\ell^2\eta'_\varepsilon d' \bullet g_1 \,.$$

We have to show that

$$\big(2 - \tfrac{1}{\tau\ell^2}\big)\, P^{\mathrm{A}} = 2\, S^{\mathrm{A}} - d \wedge g_1 - \overline{\Gamma}\,. \tag{3.9}$$

Since $2\,\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}}$ and because of the representation of $\overline{\mathrm{H}}$ in (3.1) we get

$$d \wedge g = 2\Pi^{\mathrm{A}} - \sum_j (\partial_j d) \wedge \pi_{\bullet j} + \overline{\Gamma}\,.$$

With the constitutive equations for π and g (3.5) this is

$$d \wedge \Big(\frac{1}{2\tau\ell^2\eta'_\varepsilon}\eta'_d - g_1 - (\gamma d + \mathrm{D}d\,\beta)\Big) = 2\Pi^{\mathrm{A}} - \sum_j (\partial_j d) \wedge \Big(\frac{1}{2\tau\ell^2\eta'_\varepsilon}\eta'_{\partial_j d} - \beta_j d\Big) + \overline{\Gamma}\,.$$

The γ term vanishes and the β terms on both sides are equal, since

$$-d \wedge (\mathrm{D}d\,\beta) = \sum_j \beta_j\; d'_j \wedge d = \sum_j (\partial_j d) \wedge (\beta_j d)\,,$$

therefore

$$d \wedge \Big(\frac{1}{2\tau\ell^2\eta'_\varepsilon}\eta'_d - g_1\Big) = 2\Pi^{\mathrm{A}} - \sum_j (\partial_j d) \wedge \Big(\frac{1}{2\tau\ell^2\eta'_\varepsilon}\eta'_{\partial_j d}\Big) + \overline{\Gamma}\,.$$

Now with $\Pi^{\mathrm{A}} = -\,S^{\mathrm{A}} + P^{\mathrm{A}}$ it results into

$$\begin{aligned} d \wedge g_1 + \overline{\Gamma} &= \frac{1}{2\tau\ell^2\eta'_\varepsilon}\Big(d \wedge \eta'_d + \sum_j (\partial_j d) \wedge \eta'_{\partial_j d}\Big) - 2\Pi^{\mathrm{A}} \\ &= \frac{1}{\tau\ell^2\eta'_\varepsilon}\,\Big(d \otimes \eta'_d + \sum_j (\partial_j d) \otimes \eta'_{\partial_j d}\Big)^{\mathrm{A}} - 2\,\Pi^{\mathrm{A}} \\ &= -\frac{1}{\tau\ell^2\eta'_\varepsilon}\,\Big(\sum_k (\nabla d_k) \otimes \eta'_{\nabla d_k}\Big)^{\mathrm{A}} - 2\,\Pi^{\mathrm{A}} = \frac{1}{\tau\ell^2}\,P^{\mathrm{A}} - 2\,\Pi^{\mathrm{A}} \end{aligned}$$

by Condition 7.1. Since $P^{\mathrm{A}} - 2\,\Pi^{\mathrm{A}} = 2\,S^{\mathrm{A}} - P^{\mathrm{A}}$ it follows

$$d \wedge g_1 = 2\,S^{\mathrm{A}} - \Big(2 - \frac{1}{\tau\ell^2}\Big)\,P^{\mathrm{A}} - \overline{\Gamma}\,,$$

that is (3.9). $\square$

In the following we give applications to the residual inequality (3.8) where first we use a connection between S^{A} and g_1. This connection leads to the belief that the Du term and the d' term in the residual inequality have to be considered together as one term, and here it is given as quadratic form. Suppose S and g_1 are given in the form

$$\begin{aligned} &S = \widehat{S}(\varrho, \varepsilon, d, \mathrm{D}d, d', \mathcal{D}) \\ &:= \mu_1\big(d \bullet \mathcal{D}^0 d\big)\, d \otimes d + \big(\mu_2 d \otimes d' + \mu_3 d' \otimes d\big) + \mu_4 \mathcal{D}^0 + \nu(\mathrm{div}v)\,\mathrm{Id} \\ &\qquad\qquad + \big(\mu_5 d \otimes (\mathcal{D}^0 d) + \mu_6 (\mathcal{D}^0 d) \otimes d\big)\,, \\ &g_1 = \widehat{g}_1(\varrho, \varepsilon, d, \mathrm{D}d, d', \mathcal{D}^0) := \lambda_1 d' + \lambda_2 \mathcal{D}^0 d\,, \\ &\text{where}\quad \mathcal{D} := (\mathrm{D}v)^{\mathrm{S}} \quad\text{and}\quad \mathcal{D}^0 = \mathcal{D} - \tfrac{1}{3}(\,\mathrm{div}v)\,\mathrm{Id}. \end{aligned} \tag{3.10}$$

Here the coefficients μ_k, $k=1,\dots,6$, and λ_k, $k=1,2$, and ν are objective scalars and can be functions of $(\varrho,\varepsilon,d,\mathrm{D}d,d',\mathcal{D})$. This defines the stress tensor S as objective tensor and these definitions are the compressible generalisations of the definitions in Ericksen [9 : (13)] and later in Chandrasekhar [3 : (3.1.36) and (3.1.37)]. The assumptions (3.11) on the coefficients one can find in Ericksen [9 : (9) and (10)] and Chandrasekhar [3 : (3.1.34)] and in I.Müller [22 : (10.65)].

3.5 Lemma. Suppose S and g_1 are given as in (3.10) with general coefficients λ_k, μ_k and ν. Then under the condition

$$\lambda_1=\mu_2-\mu_3\,,\qquad \lambda_2=\mu_5-\mu_6\,, \tag{3.11}$$

it holds

$$2\,S^{\mathrm{A}}-d\wedge g_1=0\,.$$

Proof. Due to $d\wedge d'=2\,(d\otimes d')^{\mathrm{A}}$ one gets

$$\begin{aligned}2\,S^{\mathrm{A}}&=2(\mu_2-\mu_3)\,(d\otimes d')^{\mathrm{A}}+2(\mu_5-\mu_6)\,(d\otimes\mathcal{D}^0d)^{\mathrm{A}}\\&=(\mu_2-\mu_3)\,d\wedge d'+(\mu_5-\mu_6)\,d\wedge\mathcal{D}^0d=d\wedge g_1\,.\end{aligned}$$

□

We now compute the quadratic form in the entropy production, where we mention that the assumptions (3.10) have been made so that the term $\sigma_g:=(\mathrm{D}v)^{\mathrm{S}}:S^{\mathrm{S}}+2\tau\ell^2d'\bullet g_1$ of the entropy production has this quadratic form.

3.6 Theorem. Assume S and g_1 are given as in (3.10) with the constraint (3.11) and assume $\overline{\Gamma}=0$. Then the residual inequality reads

$$\begin{aligned}0\le\sigma&=\eta'_\varepsilon\big((\mathrm{D}u)^{\mathrm{S}}:S^{\mathrm{S}}+2\tau\ell^2d'\bullet g_1\big)+\nabla\eta'_\varepsilon\bullet q\\&=\eta'_\varepsilon\Big(\nu\mathrm{V}^2+\ell^2(\mu_1\ell^2+\mu_5+\mu_6)\mathrm{V}\Lambda_{11}+\big(\mu_4+\ell^2(\mu_1\ell^2+\mu_5+\mu_6)\big)\Lambda_{11}^2\\&\qquad+\mu_4(\Lambda_{22}^2+\Lambda_{33}^2+2\Lambda_{32}^2)+(2\mu_4+\ell^2(\mu_5+\mu_6))\Lambda_{31}^2\\&\quad+2\tau\ell^2\lambda_1\mathrm{T}^2+\ell(\mu_2+\mu_3+2\tau\ell^2\lambda_2)\mathrm{T}\Lambda_{21}+(2\mu_4+\ell^2(\mu_5+\mu_6))\Lambda_{21}^2\Big)+\nabla\eta'_\varepsilon\bullet q\end{aligned} \tag{3.12}$$

with $\mathrm{V}:=\operatorname{div}v$ and $\mathrm{T}:=|d'|$ and $\Lambda_{kl}:=e_k\bullet\mathcal{D}^0e_l$, where $\{e_1,e_2,e_3\}$ is an orthonormal moving frame of $\mathbb{R}^3$ with $d=\ell e_1$ and $d'=|d'|e_2=\mathrm{T}e_2$ if $\mathrm{T}\neq0$ and $e_3=e_1\times e_2$.

Proof. Of course $\mathcal{D}^0=\mathcal{D}-\mathrm{V}\,\mathrm{Id}$ with $\mathcal{D}^0:\mathrm{Id}=0$. This said, the residual inequality (3.8) becomes, because of lemma 3.5 and $\overline{\Gamma}=0$,

$$0\le\sigma=\eta'_\varepsilon\,(\mathrm{D}v)^{\mathrm{S}}:S^{\mathrm{S}}+\nabla\eta'_\varepsilon\bullet q+2\tau\ell^2\eta'_\varepsilon d'\bullet g_1\,. \tag{3.13}$$

We are interested into the first and last term. Since

$$\begin{aligned}&\mathcal{D}^0:d\otimes d=d\bullet\mathcal{D}^0d\,,\\&\mathcal{D}^0:(d\otimes d')^{\mathrm{S}}=d'\bullet\mathcal{D}^0d\,,\\&\mathcal{D}^0:\big(d\otimes\mathcal{D}^0d\big)^{\mathrm{S}}=\big|\mathcal{D}^0d\big|^2\,,\\&\mathrm{VId}:S^{\mathrm{S}}=\mathrm{V}\cdot\operatorname{trace}(S^{\mathrm{S}})=\nu\mathrm{V}^2+\mathrm{V}(\mu_1\ell^2+\mu_5+\mu_6)\,d\bullet\mathcal{D}^0d\,,\\&|\mathcal{D}^0|^2=\sum_{k,l}\Lambda_{kl}^2=\Lambda_{11}^2+\Lambda_{22}^2+\Lambda_{33}^2+2(\Lambda_{21}^2+\Lambda_{32}^2+\Lambda_{31}^2)\,,\end{aligned}$$

we obtain, without the multiplication by η'_ε,

$$
\begin{aligned}
\sigma_g &:= (\mathrm{D}v)^{\mathrm{S}} \,\boldsymbol{:}\, S^{\mathrm{S}} + 2\tau\ell^2 d' \bullet g_1 = (\mathcal{D}^0 + \mathrm{V}\mathrm{Id}) \,\boldsymbol{:}\, S^{\mathrm{S}} + 2\tau\ell^2 d' \bullet g_1 \\
&= \mu_1 |d \bullet \mathcal{D}^0 d|^2 + (\mu_2 + \mu_3)\, d' \bullet \mathcal{D}^0 d + \mu_4 |\mathcal{D}^0|^2 \\
&\quad + \nu \mathrm{V}^2 + (\mu_1 \ell^2 + \mu_5 + \mu_6) \mathrm{V}\, d \bullet \mathcal{D}^0 d + (\mu_5 + \mu_6) |\mathcal{D}^0 d|^2 \\
&\quad + 2\tau\ell^2 \big(\lambda_1 |d'|^2 + \lambda_2\, d' \bullet (\mathcal{D}^0 d)\big) \\
&= \ell^4 \mu_1 |e_1 \bullet (\mathcal{D}^0 e_1)|^2 + \mu_4 |\mathcal{D}^0|^2 \\
&\quad + \nu \mathrm{V}^2 + (\mu_1 \ell^2 + \mu_5 + \mu_6) \ell^2 \mathrm{V}\, e_1 \bullet \mathcal{D}^0 e_1 + \ell^2 (\mu_5 + \mu_6) |\mathcal{D}^0 e_1|^2 \\
&\quad + 2\tau\ell^2 \lambda_1 \mathrm{T}^2 + \ell(\mu_2 + \mu_3 + 2\tau\ell^2\lambda_2) \mathrm{T}\, e_2 \bullet (\mathcal{D}^0 e_1) \\
&= \ell^4 \mu_1 \Lambda_{11}^2 + \mu_4 \big(\Lambda_{11}^2 + \Lambda_{22}^2 + \Lambda_{33}^2 + 2(\Lambda_{21}^2 + \Lambda_{32}^2 + \Lambda_{31}^2)\big) \\
&\quad + \nu \mathrm{V}^2 + \ell^2 (\mu_1 \ell^2 + \mu_5 + \mu_6) \mathrm{V} \Lambda_{11} + \ell^2 (\mu_5 + \mu_6)(\Lambda_{11}^2 + \Lambda_{21}^2 + \Lambda_{31}^2) \\
&\quad + 2\tau\ell^2 \lambda_1 \mathrm{T}^2 + \ell(\mu_2 + \mu_3 + 2\tau\ell^2\lambda_2) \mathrm{T} \Lambda_{21}\,.
\end{aligned}
$$

The entropy principle says $0 \le \sigma = \eta'_\varepsilon \sigma_g + \nabla\eta'_\varepsilon \bullet q$ where σ_g has the asserted form in the variables $(\mathrm{T}, \Lambda, \mathrm{V})$. □

For the coefficients in definition (3.10) the following is true under the assumption that the part $\sigma_g \ge 0$.

3.7 Lemma. Assume S and g_1 are given as in (3.10) with constraint (3.11). Assume further that (like the entropy) the coefficients μ_k, $k = 1, \ldots, 6$, and λ_k, $k = 1, 2$, and ν are functions of $(\varrho, \varepsilon, d, \mathrm{D}d)$. Then if $\sigma_g \ge 0$ for all solutions this implies the inequalities

$$
\begin{gathered}
\mu_4 \ge 0\,, \quad \nu \ge 0\,, \quad \lambda_1 \ge 0\,, \\
\big(\; \mu_4 + \ell^2(\mu_1 \ell^2 + \mu_5 + \mu_6) \ge 0\,, \quad 2\mu_4 + \ell^2(\mu_5 + \mu_6) \ge 0\,, \;\big) \\
4\nu(\mu_4 + \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6)) \ge \ell^4 (\mu_1 \ell^2 + \mu_5 + \mu_6)^2\,, \\
8\lambda_1 (2\mu_4 + \ell^2(\mu_5 + \mu_6)) \ge (\mu_2 + \mu_3 + 2\ell^2\lambda_2)^2\,,
\end{gathered}
$$

have to be satisfied. The second line is redundant if $\nu > 0$ and $\lambda_1 > 0$.
Addition: If in addition $q = \widehat{q}(\varrho, \varepsilon, d, \mathrm{D}d, \nabla\varrho, \nabla\varepsilon)$ then the entropy principle $\sigma \ge 0$ is always satisfied if and only if always $\sigma_g \ge 0$ and $\nabla\eta'_\varepsilon \bullet q \ge 0$.

Proof. The coefficients λ, μ, and ν are independent of $(\mathrm{T}, \Lambda, \mathrm{V})$. First of all the terms with T^2, Λ_{kl}^2, and V^2 must have nonnegative coefficients. Second and last there are two quadratic forms in $(\mathrm{V}, \Lambda_{11})$ and $(\mathrm{T}, \Lambda_{21})$ which have to be nonnegative. Therefore the corresponding symmetric matrices must have a nonnegative determinant. These are the inequalities. This is true, if $\overline{\Gamma} = 0$. With $\sigma_q := \nabla\eta'_\varepsilon \bullet q$ we then get that $\sigma = \eta'_\varepsilon \sigma_g + \sigma_q$ is always greater or equal to 0 if (and only if) always $\sigma_g \ge 0$ and $\sigma_q \ge 0$. □

Now we treat the case, that the heat is coupled to the spin. In the following application this coupling lets all the three terms in the entropy production (3.8) depend on each other, that is, S^{A} and q and g_1 are related one to another. This application requires that we are in a neighbourhood where θ instead of ε is the independent variable, that is

$$
\theta \cdot \eta'_\varepsilon(\varrho, \varepsilon, d, \mathrm{D}d) = 1 \quad \text{and } \eta \text{ is a concave function of } \varepsilon.
$$

Then we assume

$$
\begin{aligned}
&S = \widehat{S}(\varrho,\theta,\nabla\theta,d,\mathrm{D}d,d',\mathcal{D})\\
&:= \mu_1(d\bullet\mathcal{D}^0 d)d\otimes d + \big(\mu_2 d\otimes d' + \mu_3 d'\otimes d\big) + \mu_4\mathcal{D}^0 + \nu(\mathrm{div}v)\,\mathrm{Id}\\
&\quad +\big(\mu_5 d\otimes(\mathcal{D}^0 d) + \mu_6(\mathcal{D}^0 d)\otimes d\big) + \big(\mu_7 d\otimes(d\times\nabla\theta^{-1}) + \mu_8(d\times\nabla\theta^{-1})\otimes d\big)\,,\\
&q = \widetilde{q}(\varrho,\theta,\nabla\theta,d,\mathrm{D}d,d',\mathcal{D})\\
&:= \theta^{-1}\left(\kappa_1\nabla\theta^{-1} + \kappa_2(d\bullet\nabla\theta^{-1})\,d + \kappa_3 d\times d' + \kappa_4 d\times(\mathcal{D}^0 d)\right),\\
&g_1 = \widehat{g}_1(\varrho,\theta,\nabla\theta,d,\mathrm{D}d,d',\mathcal{D}) := \lambda_1 d' + \lambda_2\mathcal{D}^0 d + \lambda_3 d\times\nabla\theta^{-1}\,.
\end{aligned}
\tag{3.14}
$$

Here the coefficients μ_k, $k = 1,\dots,8$, and κ_k, $k = 1,\dots,4$, and λ_k, $k = 1,2,3$, and ν are objective scalars and can be functions of $(\varrho,\theta,\nabla\theta,d,\mathrm{D}d,d',\mathcal{D})$. This defines S as objective tensor and q as objective vector, and these definitions are the compressible generalisations of the definitions in Leslie [18 : (41)], where the κ_k terms are normalised differently. In [22 : $(10.62)_4$] I.Müller has defined the heat flux (in our notation)

$$
q = -\widetilde{\kappa}_1\nabla\theta - \widetilde{\kappa}_2(d\bullet\nabla\theta)\,d \quad (\widetilde{\kappa}_l = \theta^{-3}\,\kappa_l)\,,
$$

hence the rest terms are both zero. The assumptions (3.15) on the coefficients one can find in Leslie [18 : (42)].

3.8 Lemma. Suppose S, q and g_1 are given as in (3.14) with general coefficients λ_k, κ_k, μ_k, and ν. If

$$
\lambda_1 = \mu_2 - \mu_3\,, \qquad \lambda_2 = \mu_5 - \mu_6\,, \qquad \lambda_3 = \mu_7 - \mu_8\,,
\tag{3.15}
$$

it holds

$$
2\,S^{\mathrm{A}} - d\wedge g_1 = 0\,.
$$

Proof. Due to the rule $\vec{a}\wedge\vec{b} = 2\left(\vec{a}\otimes\vec{b}\right)^{\mathrm{A}}$ for $\vec{a},\vec{b}\in\mathbb{R}^3$ we have that

$$
\begin{aligned}
&2\,S^{\mathrm{A}} = 2(\mu_2-\mu_3)\,(d\otimes d')^{\mathrm{A}} + 2(\mu_5-\mu_6)\,(d\otimes(\mathcal{D}^0 d))^{\mathrm{A}} + 2(\mu_7-\mu_8)\,(d\otimes(d\times\nabla\theta^{-1}))^{\mathrm{A}}\\
&= (\mu_2-\mu_3)d\wedge d' + (\mu_5-\mu_6)d\wedge(\mathcal{D}^0 d) + (\mu_7-\mu_8)d\wedge(d\times\nabla\theta^{-1})\\
&= d\wedge\left(\lambda_1 d' + \lambda_2\mathcal{D}^0 d + \lambda_3 d\times\nabla\theta^{-1}\right) = d\wedge g_1\,.
\end{aligned}
$$

□

The entropy inequality becomes in this case totally a quadratic form.

3.9 Theorem. Assume (3.14) is given with the constraint (3.15) and assume $\overline{\Gamma} = 0$. Then the residual inequality is a quadratic form in $(\mathrm{T},\Lambda,\mathrm{V},\Theta)$ and reads

$$
\begin{aligned}
0 \le \theta\sigma &= (\mathrm{D}u)^{\mathrm{S}} : S^{\mathrm{S}} + 2\tau\ell^2 d'\bullet g_1 + \theta\nabla\theta^{-1}\bullet q\\
&= \Big(\nu\mathrm{V}^2 + \ell^2(\mu_1\ell^2+\mu_5+\mu_6)\mathrm{V}\Lambda_{11} + (\mu_4 + \ell^2(\mu_1\ell^2+\mu_5+\mu_6))\Lambda_{11}^2\\
&\qquad +\mu_4\big(\Lambda_{22}^2+\Lambda_{33}^2+2\Lambda_{32}^2\big) + (2\mu_4+\ell^2(\mu_5+\mu_6))\Lambda_{31}^2\\
&\qquad +2\tau\ell^2\lambda_1\mathrm{T}^2 + \ell(\mu_3+\mu_4+2\tau\ell^2\lambda_2)\mathrm{T}\Lambda_{21} + (2\mu_4+\ell^2(\mu_5+\mu_6))\Lambda_{21}^2\Big)\\
&\quad +\Big((\kappa_1+\ell^2\kappa_2)|\Theta_1|^2 + \kappa_1\big(|\Theta_2|^2+|\Theta_3|^2\big) + (\kappa_3 - 2\tau\ell^2\lambda_3)\ell\mathrm{T}\Theta_3\\
&\qquad +(\kappa_4 - (\mu_7+\mu_8))\ell^2(\Theta_3\Lambda_{21} - \Theta_2\Lambda_{31})\Big)
\end{aligned}
$$

where $\Theta_k := e_k \bullet \nabla\theta^{-1}$ and T, V and Λ_{ij} as in Theorem 3.6.

The terms in the first big paranthesis are those which were there already in Theorem 3.6, the additional terms are the terms in the last big paranthesis.

Proof. We only treat the terms containing the Θ_k variables, since the other terms are already contained in the proof of Theorem 3.6. Let us first consider the g_1 term

$$2\tau\ell^2 d^{'}\bullet g_1 = 2\tau\ell^2\lambda_1|d^{'}|^2 + 2\tau\ell^2\lambda_2 d^{'}\bullet(\mathcal{D}^0 d) + 2\tau\ell^2\lambda_3 d^{'}\bullet(d\times\nabla\theta^{-1})\,.$$

For the last summand we get

$$\begin{aligned}
&d\times d^{'} = \ell\mathrm{T}e_1\times e_2 = \ell\mathrm{T}e_3 \quad \text{hence} \quad (d\times d^{'})\bullet\nabla\theta^{-1} = \ell\mathrm{T}\Theta_3\,,\\
&\text{thus} \quad 2\tau\ell^2\lambda_3 d^{'}\bullet(d\times\nabla\theta^{-1}) = -2\tau\ell^2\lambda_3\nabla\theta^{-1}\bullet(d\times d^{'}) = -2\tau\ell^3\lambda_3\mathrm{T}\Theta_3\,.
\end{aligned}$$

Now we consider the q term

$$\begin{aligned}
\theta\nabla\theta^{-1}\bullet q = {}& \kappa_1|\nabla\theta^{-1}|^2 + \kappa_2|\nabla\theta^{-1}\bullet d|^2\\
&+\kappa_3\nabla\theta^{-1}\bullet(d\times d^{'}) + \kappa_4\nabla\theta^{-1}\bullet(d\times(\mathcal{D}^0 d))\,.
\end{aligned}$$

It is easily to see that

$$\kappa_1|\nabla\theta^{-1}|^2 + \kappa_2|\nabla\theta^{-1}\bullet d|^2 = (\kappa_1 + \ell^2\kappa_2)|\Theta_1|^2 + \kappa_1(|\Theta_2|^2 + |\Theta_3|^2)\,,$$

and further

$$\begin{aligned}
&d\times d^{'} = \ell\mathrm{T}e_1\times e_2 = \ell\mathrm{T}e_3 \quad \text{thus} \quad \kappa_3(d\times d^{'})\bullet\nabla\theta^{-1} = \kappa_3\ell\mathrm{T}\Theta_3\,,\\
&d\times(\mathcal{D}^0 d) = \ell^2 e_1\times(\mathcal{D}^0 e_1) = \ell^2 e_1\times(\Lambda_{21}e_2 + \Lambda_{31}e_3) = \ell^2\big(\Lambda_{21}e_3 - \Lambda_{31}e_2\big)\,,\\
&\text{thus} \quad \kappa_4(d\times(\mathcal{D}^0 d))\bullet\nabla\theta^{-1} = \kappa_4\ell^2(\Theta_3\Lambda_{21} - \Theta_2\Lambda_{31})\,.
\end{aligned}$$

So together we obtain

$$\begin{aligned}
\theta\nabla\theta^{-1}\bullet q = {}& (\kappa_1 + \ell^2\kappa_2)|\Theta_1|^2 + \kappa_1(|\Theta_2|^2 + |\Theta_3|^2)\\
&+\kappa_3\ell\mathrm{T}\Theta_3 + \kappa_4\ell^2(\Theta_3\Lambda_{21} - \Theta_2\Lambda_{31})\,.
\end{aligned}$$

Finally we consider the S term. The symmetric part of the stress tensor reads

$$\begin{aligned}
S^{\mathrm{S}} = {}& \mu_1(d\bullet\mathcal{D}^0 d)d\otimes d + (\mu_2 + \mu_3)\big(d\otimes d^{'}\big)^{\mathrm{S}} + \mu_4\mathcal{D}^0 + \nu(\mathrm{div}v)\mathrm{Id}\\
&+(\mu_5 + \mu_6)\big(d\otimes(\mathcal{D}^0 d)\big)^{\mathrm{S}} + (\mu_7 + \mu_8)\big(d\otimes(d\times\nabla\theta^{-1})\big)^{\mathrm{S}}\,.
\end{aligned}$$

For all terms except the last summand on the right-hand side, we have calculated the scalar product with $(\mathrm{D}u)^{\mathrm{S}}$ in the proof of Theorem 3.6. Now we treat the last summand. Using $(\mathrm{D}u)^{\mathrm{S}} = (\mathrm{D}v)^{\mathrm{S}} = \mathcal{D}^0 + (\mathrm{div}v)\mathrm{Id}$ we compute defining $\mu := \mu_7 + \mu_8$

$$\begin{aligned}
&\mu\,(\mathrm{D}u)^{\mathrm{S}} : \big((d\times\nabla\theta^{-1})\otimes d\big)^{\mathrm{S}} = \mu\ell^2\,(\mathrm{D}v)^{\mathrm{S}} : ((e_1\times\nabla\theta^{-1})\otimes e_1)\\
&= \mu\ell^2((\mathcal{D}^0 + (\mathrm{div}v)\mathrm{Id})e_1)\bullet(e_1\times(\Theta_2 e_2 + \Theta_3 e_3))\\
&= \mu\ell^2(\mathcal{D}^0 e_1)\bullet(e_1\times(\Theta_2 e_3 - \Theta_3 e_2)) = \mu\ell^2(\Theta_2\Lambda_{31} - \Theta_3\Lambda_{21})\,.
\end{aligned}$$

Combining these terms with the proof of Theorem 3.6 we get the assertion. $\square$

Reordering the terms of $\theta\sigma$ gives one 3x3 matrix, two 2x2 matrices, and four single values:

$$
\begin{aligned}
0 \le \theta\sigma &= (\mathrm{D}u)^{\mathrm{S}} \mathbin{\vdots} S^{\mathrm{S}} + 2\tau\ell^2 d' \bullet g_1 + \theta\nabla\theta^{-1} \bullet q \\
&= \Big(\nu\mathrm{V}^2 + \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6)\mathrm{V}\Lambda_{11} + (\mu_4 + \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6))\Lambda_{11}^2\Big) \\
&\qquad +\mu_4\big(\Lambda_{22}^2 + \Lambda_{33}^2 + 2\Lambda_{32}^2\big) \\
&\quad +\Big(\kappa_1|\Theta_2|^2 - \ell^2(\kappa_4 - (\mu_7 + \mu_8))\Theta_2\Lambda_{31} + (2\mu_4 + \ell^2(\mu_5 + \mu_6))\Lambda_{31}^2\Big) \\
&\qquad +(\kappa_1 + \ell^2\kappa_2)|\Theta_1|^2 \\
&\quad +\Big(2\tau\ell^2\lambda_1\mathrm{T}^2 + \ell(\mu_3 + \mu_4 + 2\tau\ell^2\lambda_2)\mathrm{T}\Lambda_{21} + (2\mu_4 + \ell^2(\mu_5 + \mu_6))\Lambda_{21}^2 \\
&\qquad +\kappa_1|\Theta_3|^2 + \ell(\kappa_3 - 2\tau\ell^2\lambda_3)\mathrm{T}\Theta_3 + \ell^2(\kappa_4 - (\mu_7 + \mu_8))\Theta_3\Lambda_{21}\Big)
\end{aligned}
\tag{3.16}
$$

If the indicated matrices in (3.16) are positive semidefinite, then the entropy production σ is nonnegative. We write this fact as lemma.

3.10 Lemma. Let S, g_1 and q be given by (3.14) as in the Theorem 3.9 and assume $\overline{\Gamma} = 0$. Further assume that μ_k for $k = 1, \dots, 8$, λ_k for $k = 1, \dots, 3$, κ_k for $k = 1, \dots, 4$ and ν are functions of $(\varrho, \theta, d, \mathrm{D}d)$. Then $\sigma \ge 0$ for all solutions is equivalent to the inequalities

$$
\mu_4 \ge 0\,, \quad \kappa_1 + \ell^2\kappa_2 \ge 0\,,
$$

$$
\begin{bmatrix} 2(\mu_4 + \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6)) & \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6) \\ \ell^2(\mu_1\ell^2 + \mu_5 + \mu_6) & 2\nu \end{bmatrix} \ge 0\,,
$$

$$
\begin{bmatrix} 2(2\mu_4 + \ell^2(\mu_5 + \mu_6)) & -\ell^2(\kappa_4 - (\mu_7 + \mu_8)) \\ -\ell^2(\kappa_4 - (\mu_7 + \mu_8)) & 2\kappa_1 \end{bmatrix} \ge 0\,,
$$

$$
\begin{bmatrix} 4\tau\ell^2\lambda_1 & \ell(\mu_3 + \mu_4 + 2\tau\ell^2\lambda_2) & \ell(\kappa_3 - 2\tau\ell^2\lambda_3) \\ \ell(\mu_3 + \mu_4 + 2\tau\ell^2\lambda_2) & 2(2\mu_4 + \ell^2(\mu_5 + \mu_6)) & \ell^2(\kappa_4 - (\mu_7 + \mu_8)) \\ \ell(\kappa_3 - 2\tau\ell^2\lambda_3) & \ell^2(\kappa_4 - (\mu_7 + \mu_8)) & 2\kappa_1 \end{bmatrix} \ge 0\,.
$$

Proof. This is because all coefficients do not depend on $(\mathrm{T}, \Lambda, \mathrm{V}, \Theta)$. □

This is the same as one can find in Leslie [18: (35) and (44) or pp.13-15]. His equation (35) is based on the heat flux in equation (33), which depends only on κ_1 and κ_2 (as I.Müller in [22: $(10.62)_4$]), and his equation (44) is the statement on a 3x3 matrix, which is exactly our 3x3 matrix.

4 Auxiliary lemmata

We take a vector b with the following transformation rule

$$
b\circ Y = Qb^*\,, \tag{4.1}
$$

where Y is the transformation between the observer with coordinates (t, x) and another observer with coordinates (t^*, x^*). The transformation has the form

$$
\begin{bmatrix} t \\ x \end{bmatrix} = Y\Big(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\Big) = \begin{bmatrix} t^* + \mathrm{a} \\ X(t^*, x^*) \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Q(t^*)x^* + \mathrm{b}(t^*) \end{bmatrix} \tag{4.2}
$$

with an orthogonal matrix Q having determinant 1 and an objective vector b. We denote by $\dot X$ etc. the time derivative w.r.t. t^*. It follows by differentiating the formula (4.1) the following well known lemma.

4.1 Lemma. It is $(\mathrm{D}b)\circ Y = Q\,\mathrm{D}b^* Q^{\mathrm{T}}$, that is, $\mathrm{D}b$ is an objective matrix. This result applies to the director d.

Proof. From (4.1) it follows $\partial_{x_j^*}(b\circ Y) = \partial_{x_j^*}(Qb^*) = Q\partial_{x_j^*} b^*$ and from the chain rule

$$\partial_{x_j^*}(b\circ Y) = \sum_{i\geq 1}(\partial_{x_i} b)\circ Y\cdot Q_{ij}\,.$$

Therefore

$$\sum_{i\geq 1}(\partial_{x_i} b)\circ Y\cdot Q_{ij} = Q\partial_{x_j^*} b^* \tag{4.3}$$

which gives the result. □

4.2 Lemma. We get the following transformation rules for $\overset{\circ}{b}$.

(1) $\overset{\circ}{b}\circ Y = \dot Q b^* + Q\overset{\circ}{b^*}$.

(2) $\mathrm{D}\big(\overset{\circ}{b}\big)\circ Y = \dot Q\,\mathrm{D}b^* Q^{\mathrm{T}} + Q\mathrm{D}\big(\overset{\circ}{b^*}\big)\,Q^{\mathrm{T}}$.

These results apply to the director d.

Proof (1). From (4.1) we obtain by differentiating w.r.t. t^* and x_j^*

$$\begin{aligned}
&\dot Q b^* + Q\partial_{t^*} b^* = \partial_{t^*}(Qb^*) = \partial_{t^*}(b\circ Y) = (\partial_t b)\circ Y + \sum_{i\geq 1}(\partial_{x_i} b)\circ Y\cdot \dot X_i\,,\\
&Q\partial_{x_j^*} b^* = \sum_{i\geq 1}(\partial_{x_i} b)\circ Y\cdot Q_{ij}\quad \text{for } j\geq 1 \text{ by (4.3)},\\
&v_i\circ Y = \dot X_i + \sum_{j\geq 1} Q_{ij} v_j^*\quad \text{for } i\geq 1,
\end{aligned}$$

so by taking the sum of the first line and the second line multiplied by v_j^* we obtain

$$\dot Q b^* + Q\big(\partial_{t^*} b^* + \sum_{j\geq 1} v_j^* \partial_{x_j^*} b^*\big) = (\partial_t b)\circ Y + \sum_{i\geq 1}(\partial_{x_i} b)\circ Y\cdot v_i\circ Y\,,$$

hence $\dot Q b^* + Q\overset{\circ}{b^*} = \overset{\circ}{b}\circ Y$. □

Proof (2). We compute using (1)

$$\begin{aligned}
\sum_j (\overset{\circ}{b_i})_{,j}\circ Y\cdot Q_{j\bar j} &= (\overset{\circ}{b_i}\circ Y)_{,\bar j}\\
&= \sum_{\bar i}\left(\dot Q_{i\bar i} b_{\bar i}^* + Q_{i\bar i}\overset{\circ}{b^*}_{\bar i}\right)_{,\bar j} = \sum_{\bar i}\left(\dot Q_{i\bar i} b^*_{\bar i,\bar j} + Q_{i\bar i}\overset{\circ}{b^*}_{\bar i,\bar j}\right),
\end{aligned}$$

hence for all i,j

$$(\overset{\circ}{b_i})_{,j}\circ Y = \sum_{\bar i,\bar j}\dot Q_{i\bar i} Q_{j\bar j} b^*_{\bar i,\bar j} + \sum_{\bar i,\bar j} Q_{i\bar i} Q_{j\bar j}\overset{\circ}{b^*}_{\bar i,\bar j}$$

or in matrix notation

$$(\mathrm{D}(\overset{\circ}{b}))\circ Y = \dot{Q}\,\mathrm{D}b^*\,Q^{\mathrm{T}} + Q\mathrm{D}\big(\overset{\circ}{b^*}\big)\,Q^{\mathrm{T}}\ .$$

□

4.3 Lemma. Let G_b for any b with (4.1) be a vector satisfying $G_b\circ Y = \dot{Q}b^* + QG^*_{b^*}$ and let B be any antisymmetric matrix satisfying $B\circ Y = \dot{Q}\,Q^{\mathrm{T}} + QB^*\,Q^{\mathrm{T}}$. Then

(1) $\overset{\circ}{b} - G_b$ is an objective vector.

(2) $\overset{\circ}{b} - Bb$ is an objective vector.

(3) $\mathrm{D}\big(\overset{\circ}{b}\big) - B\mathrm{D}d$ is an objective matrix.

These results apply to the director d. In (2) one chooses $B = A_\xi$, that is

$$d' := \overset{\circ}{d} - A_\xi d \quad \text{is an objective vector.} \tag{4.4}$$

Therefore one can apply (1) to d' and $G_{d'} = A_\xi d'$ which gives

$$d'' := \overset{\circ}{d'} - A_\xi d' \quad \text{is an objective vector.} \tag{4.5}$$

Proof (1). To prove (1) we compute using 4.2(1)

$$(\overset{\circ}{b} - G_b)\circ Y = \overset{\circ}{b}\circ Y - G_b\circ Y = \dot{Q}b^* + Q\overset{\circ}{b^*} - \dot{Q}b^* - QG^*_{b^*} = Q(\overset{\circ}{b^*} - G^*_{b^*})\,,$$

hence $\overset{\circ}{b} - G_b$ is an objective scalar. □

Proof (2). Set $G_b := Bb$. Then it follows that

$$(Bb)\circ Y = (\dot{Q}\,Q^{\mathrm{T}} + QB^*\,Q^{\mathrm{T}})Qb^* = \dot{Q}b^* + Q(B^*b^*)\,,$$

the rule for G_b. Thus (1) can be applied. □

Proof (3). For the proof of (3) we use 4.2(2)

$$\mathrm{D}\big(\overset{\circ}{d}\big)\circ Y = \dot{Q}\,\mathrm{D}b^*\,Q^{\mathrm{T}} + Q\mathrm{D}\big(\overset{\circ}{b^*}\big)\,Q^{\mathrm{T}}\,.$$

Using 4.1 we combine this with

$$(B\,\mathrm{D}b)\circ Y = (\dot{Q}\,Q^{\mathrm{T}} + QB^*\,Q^{\mathrm{T}})Q\,\mathrm{D}b^*\,Q^{\mathrm{T}} = \dot{Q}\,\mathrm{D}b^*\,Q^{\mathrm{T}} + QB^*\mathrm{D}b^*\,Q^{\mathrm{T}}\,,$$

and we obtain that the difference $\mathrm{D}\big(\overset{\circ}{b}\big) - B\mathrm{D}d$ is objective. □

Proof of rest. We take for A_ξ the transformation formula (5.3), which is the transformation fomula of B, and compute with (4.1)

$$(A_\xi b)\circ Y = (\dot{Q}\,Q^{\mathrm{T}} + QA^*_\xi\,Q^{\mathrm{T}})Qb^* = \dot{Q}b^* + Q(A^*_\xi b^*)$$

which is the transformation formula of G_b. □

5 Spin equation

We start with the usual conservation for mass and momentum

$$\begin{aligned} &\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,,\\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f}\,, \end{aligned} \tag{5.1}$$

where Π is a general matrix including nonsymmetric parts. We define the orbital angular momentum $\mathscr{L} = r \wedge p$ by a matrix consisting of the relative position $r = x - \xi$ and the relative momentum $p = \varrho(v - \dot{\xi})$, where $t \mapsto \xi(t)$ is the reference orbit, that is

$$\mathscr{L} = (x - \xi) \wedge \varrho(v - \dot{\xi})\,.$$

For the reference orbit not only the position $\xi(t)$ is required, but also how it (by which we mean the virtual observer) turns around its body (see [1 : II.6.2]), or equivalently, he as observer is turning with his body.

5.1 The virtual body. The antisymmetric matrix $t \mapsto A_\xi(t)$ describes the velocity of the virtual body, including its rotation, and is given by

$$v_\xi(t,x) := \dot{\xi}(t) + A_\xi(t)(x - \xi(t))\,. \tag{5.2}$$

This means $\mathrm{D}v_\xi = A_\xi$ and that A_ξ satisfies the transformation formula

$$A_\xi \circ Y = \dot{Q}Q^{\mathrm{T}} + QA_\xi^* Q^{\mathrm{T}} \tag{5.3}$$

as derivative of a velocity. The speed of this body in space is $t \mapsto \dot{\xi}(t) = v_\xi(t, \xi(t))$.

For the orbital angular momentum there holds the following equation, which follows directly from the mass-momentum system (5.1) (one can find this in all physics books where Π is not assumed to be symmetric)

$$\partial_t\mathscr{L} + \operatorname{div}\big(\mathscr{L} v^{\mathrm{T}} + (x - \xi) \wedge \Pi\big) = -2\,\Pi^{\mathrm{A}} + (x - \xi) \wedge \big(\mathbf{f} - \varrho\ddot{\xi}\big)\,. \tag{5.4}$$

This is the reason for the following definition of the angular momentum $\mathscr{J}$ (see Grad [14 : (4.13)] ($\mathscr{L} \rightsquigarrow M$, $\Pi \rightsquigarrow P$) and DeGroot & Mazur [4 : Chap.XII §1(3)] and Alt [1 : (II 6.12)])

$$\begin{aligned} &\partial_t\mathscr{J} + \operatorname{div}(\mathscr{J} v^{\mathrm{T}} + \widetilde{\Sigma}) = \widetilde{\Gamma}\,,\\ &\widetilde{\Sigma} = \Big(\widetilde{\Sigma}_{klj}\Big)_{k,l,j=1,2,3}\,,\quad \widetilde{\Gamma} = \Big(\widetilde{\Gamma}_{kl}\Big)_{k,l=1,2,3}\,, \end{aligned} \tag{5.5}$$

and $\mathscr{J}$ satisfies the transformation rule

$$\mathscr{J} \circ Y = \varrho^*(Q(x^* - \xi^*)) \wedge (\dot{Q}(x^* - \xi^*)) + Q\,\mathscr{J}^* Q^{\mathrm{T}}\,,$$

which is the rule that holds for $\mathscr{L}$. The spin is defined by $\mathscr{S} = \mathscr{J} - \mathscr{L}$, i.e. it satisfies the difference of (5.5) and (5.4), which is the spin balance equation (see Grad [14 : (4.14)] ($\mathscr{S} \rightsquigarrow \mu$, $\Sigma \rightsquigarrow Q$, $\Pi \rightsquigarrow P$) and DeGroot & Mazur [4 : Chap.XII §1(8)] and Alt [1 : (II 6.15)])

$$\partial_t\mathscr{S} + \operatorname{div}_x(\mathscr{S} v^{\mathrm{T}} + \Sigma) = 2\,\Pi^{\mathrm{A}} + \Gamma\,, \tag{5.6}$$

where

$$\begin{aligned}\widetilde{\Sigma} &= (x-\xi)\wedge\Pi+\Sigma \quad (\Sigma \text{ couple stress density}),\\ \widetilde{\Gamma} &= (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi})+\Gamma \quad (\Gamma \text{ intrinsic body couple density}).\end{aligned} \tag{5.7}$$

The transformation rule for the spin $\mathscr{S}$ is therefore the difference of the transformation rules for $\mathscr{J}$ and $\mathscr{L}$, which is

$$\mathscr{S}\circ Y = Q\mathscr{S}^* Q^{\mathrm{T}}\ . \tag{5.8}$$

that is, the spin $\mathscr{S}$ is an objective tensor satisfying the spin balance equation (5.6). This equation we write

$$\partial_t\mathscr{S} + \operatorname{div}_x(\mathscr{S}v^{\mathrm{T}}+\Sigma) = \mathrm{H} \quad \text{with} \quad \mathrm{H} := 2\Pi^{\mathrm{A}}+\Gamma\,. \tag{5.9}$$

We mention that this equation satisfies the invariance principle for different observers, see [1 : (I 5.13)], which means, that $\mathscr{S}$ satisfies the transformation rule (5.8) and (after some computation) the following rules

$$\begin{aligned}\Sigma_{kli}\circ Y &= \sum_{\bar k,\bar l,j} Q_{ij}Q_{k\bar k}Q_{l\bar l}\,\Sigma^*_{\bar k\bar l j}\,,\\ \mathrm{H}_{kl}\circ Y &= \sum_{\bar k,\bar l}(Q_{k\bar k}Q_{l\bar l})^{\boldsymbol{\cdot}}\ \mathscr{S}^*_{\bar k\bar l} + \sum_{\bar k,\bar l} Q_{k\bar k}Q_{l\bar l}\,\mathrm{H}^*_{\bar k\bar l}\,.\end{aligned}$$

We mention that then the equation holds for all observers. As consequence of the transformation rule the right side of H can be split into $\mathrm{H} = \mathrm{H}^0 + \overline{\mathrm{H}}$, where H^0 satisfies the transformation rule of H and $\overline{\mathrm{H}}$ is an objective tensor, that is

$$\mathrm{H} = \overline{\mathrm{H}} + \mathrm{H}^0 \quad \text{where} \quad \overline{\mathrm{H}}\circ Y = Q\overline{\mathrm{H}}^* Q^{\mathrm{T}}\,. \tag{5.10}$$

A good choice of H^0 is given by

$$\mathrm{H}^0 := B\mathscr{S} + \mathscr{S}B^{\mathrm{T}}\ , \quad B = A_\xi\,. \tag{5.11}$$

5.2 Lemma. H^0 has the transformation rule of H and $\mathscr{S}\,\mathbf{:}\,\mathrm{H}^0 = 0$. Therefore

$$\mathscr{S}\,\mathbf{:}\,\mathrm{H} = \mathscr{S}\,\mathbf{:}\,\overline{\mathrm{H}}\,.$$

Proof. Here B can be any tensor with transformation rule as A_ξ, that is,

$$B\circ Y = \dot{Q}Q^{\mathrm{T}} + QB^*Q^{\mathrm{T}}\ .$$

Then from (5.11) and (5.8)

$$\begin{aligned}\mathrm{H}^0\circ Y &= B\circ Y\,Q\mathscr{S}^*Q^{\mathrm{T}} + Q\mathscr{S}^*Q^{\mathrm{T}}(B\circ Y)^{\mathrm{T}}\\ &= (B\circ Y\,Q)\mathscr{S}^*Q^{\mathrm{T}} + Q\mathscr{S}^*(B\circ Y Q)^{\mathrm{T}}\\ &= (\dot{Q}+QB^*)\mathscr{S}^*Q^{\mathrm{T}} + Q\mathscr{S}^*(\dot{Q}+QB^*)^{\mathrm{T}}\\ &= \dot{Q}\mathscr{S}^*Q^{\mathrm{T}} + Q\mathscr{S}^*\dot{Q}^{\mathrm{T}} + Q\mathrm{H}^{0*}Q^{\mathrm{T}}\ ,\end{aligned}$$

the assertion on transformation rule for H^0. And

$$\mathscr{S}\,\mathbf{:}\,\mathrm{H}^0 = \mathscr{S}\,\mathbf{:}\,(B\mathscr{S}) + \mathscr{S}\,\mathbf{:}\,(\mathscr{S}B^{\mathrm{T}}) = B\,\mathbf{:}\,(\mathscr{S}\mathscr{S}^{\mathrm{T}} + \mathscr{S}^{\mathrm{T}}\mathscr{S}) = 0$$

since B is antisymmetric and $\mathscr{S}\mathscr{S}^{\mathrm{T}} + \mathscr{S}^{\mathrm{T}}\mathscr{S}$ is symmetric. $\square$

In many cases the spin is given so that one can define the specific spin $\mathscr{S}^{sp}$ by $\mathscr{S} = \varrho\mathscr{S}^{sp}$ and therefore $\mathrm{H}^0 = \varrho\Gamma^0$ with

$$\Gamma^0 := B\mathscr{S}^{sp} + \mathscr{S}^{sp}B^{\mathrm{T}}\ , \quad B = A_\xi\,. \tag{5.12}$$

Then, because of the particular mass equation, the reduced spin equation reads

$$\varrho(\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0) + \operatorname{div}_x\Sigma = \overline{\mathrm{H}}\,, \tag{5.13}$$

where $\mathrm{H} = \overline{\mathrm{H}} + \varrho\Gamma^0$ and

$$2\,\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}} \quad \text{with} \quad \Gamma = \varrho\Gamma^0 + \overline{\Gamma}\,. \tag{5.14}$$

Equation (5.13) is the mostly used form of the reduced spin equation. The equation for the antisymmetric part Π^{A} is then (5.14).

Specific choice of the spin

We now come to a specific choice for the spin. As often explained in the literature, see for example [22 : 10.1.1 Equation of motion of rigid rods], the spin contains a director $d \in \mathbb{R}^3$ with $|d| = \ell = \text{const} > 0$, which is an objective vector $d\circ Y = Qd^*$, and has the form

$$\mathscr{S}^{sp} := d \wedge Ad\,, \quad \text{where } A \text{ is antisymmetric} \tag{5.15}$$

and an objective matrix, i.e. $A\circ Y = QA^*Q^{\mathrm{T}}$ (see [1 : (IV 17.11)]). Hence $\mathscr{S}$, as it should be, satisfies (5.8), that is, is an objective tensor. This form of the representation of $\mathscr{S}$ is known in literature and it is a first approximation of a real material. In literature the specific spin $\mathscr{S}^{sp}$ in this case is usually denoted by $d \wedge \overset{\circ}{d}$, but this is not correct for an observer independent description as explained above, where $\mathscr{S}^{sp}$ has to be an objective tensor. Now $\overset{\circ}{d}$ transforms with $\overset{\circ}{d}\circ Y = \dot{Q}d^* + Q\overset{\circ}{d}{}^*$ by 4.2(1), therefore $\overset{\circ}{d}$ is not an objective vector, and our final claim (see Addendum 5.4) is that

$$\mathscr{S}^{sp} = d \wedge d'\ , \quad d' := \overset{\circ}{d} - A_\xi d\,, \tag{5.16}$$

where A_ξ is the antisymmetric matrix from 5.1. Then the following is true.

5.3 Lemma. d' is an objective vector and the spin $\mathscr{S}$ defined by (5.16) therefore is an objective matrix.

Proof. d' is an objective vector by (4.4). Hence $d \wedge d'$ satisfies

$$(d \wedge d')\circ Y = (Qd^*) \wedge (Qd^{*\prime}) = Q(d^* \wedge d^{*\prime})Q^{\mathrm{T}}\,,$$

this gives that $\mathscr{S}$ defined by (5.16) is an objective matrix. □

We have still to clarify why the formula (5.16) does not contradict the original representation of the spin in (5.15).

5.4 Addendum. In addition to 5.3 there is an antisymmetric and objective matrix A such that $d' = Ad$. Hence the spin $\mathscr{S}^{sp} = d \wedge d'$ in (5.16) is of the form (5.15).

Proof. Because $|d| = \ell = \text{const}$ we have $0 = (|d|^2)^{\circ} = 2d \bullet \overset{\circ}{d}$ hence $\overset{\circ}{d}$ is perpendicular to d. By an algebraic computation $(d \times \overset{\circ}{d}) \times d = d \bullet d \, \overset{\circ}{d} - d \bullet \overset{\circ}{d} \, d = \ell^2 \overset{\circ}{d}$, therefore

$$\overset{\circ}{d} = \boldsymbol{\omega} \times d = \mathcal{R}(-\boldsymbol{\omega}) d$$

$$\text{with} \quad \boldsymbol{\omega} := \frac{1}{\ell^2} d \times \overset{\circ}{d} \quad \text{and} \quad \mathcal{R}\Big(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \Big) := \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} .$$

Since d is an objective vector we have proved the transformation rule $\overset{\circ}{d} \circ Y = \dot{Q} d^* + Q \overset{\circ}{d}{}^*$, see 4.2(1). This implies that

$$\boldsymbol{\omega} \circ Y = \frac{1}{\ell^2} (Q d^*) \times (\dot{Q} d^* + Q \overset{\circ}{d}{}^*) = \frac{1}{\ell^2} (Q d^*) \times (\dot{Q} d^*) + Q \boldsymbol{\omega}^* .$$

Now with $\widetilde{d} := Q d^*$ and $\widetilde{B} := \dot{Q} Q^{\mathrm{T}}$ and $\widetilde{a} := \mathcal{R}^{-1}(-\widetilde{B})$ by an algebraic computation

$$\begin{aligned} (Q d^*) \times (\dot{Q} d^*) &= \widetilde{d} \times (\dot{Q} Q^{\mathrm{T}} \widetilde{d}) = \widetilde{d} \times (\widetilde{B} \widetilde{d}) = \widetilde{d} \times (\widetilde{a} \times \widetilde{d}) \\ &= \widetilde{d} \bullet \widetilde{d} \, \widetilde{a} - \widetilde{a} \bullet \widetilde{d} \, \widetilde{d} = \ell^2 \mathcal{R}^{-1}(-\widetilde{B}) - \mathcal{R}^{-1}(-\widetilde{B}) \bullet \widetilde{d} \, \widetilde{d} , \end{aligned}$$

which gives

$$\boldsymbol{\omega} \circ Y = \mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}}) - \frac{1}{\ell^2} \mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}}) \bullet \widetilde{d} \, \widetilde{d} + Q \boldsymbol{\omega}^* . \tag{5.17}$$

In order to get the correct transformation formula we have to add a multiple λ of d to $\boldsymbol{\omega}$

$$\widetilde{\omega} := \boldsymbol{\omega} + \lambda d ,$$

where $\boldsymbol{\omega}$ by definition is orthogonal to d, therefore the inhomogeneous term in (5.17) is the projection of $\mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}})$ onto the space orthogonal to $\widetilde{d} = Q d^* = d \circ Y$. Since

$$(\lambda d) \circ Y = (\lambda \circ Y - \lambda^*) \widetilde{d} + Q(\lambda^* d^*)$$

we can choose the λ for the observers so that[‡]

$$\lambda \circ Y = \frac{1}{\ell^2} \big(\mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}}) \big) \bullet (Q d^*) + \lambda^* ,$$

that is, λ is chosen equal to 0 for a specific observer and then by this formula for all other observers. Therefore

$$(\lambda d) \circ Y = \frac{1}{\ell^2} \big(\mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}}) \big) \bullet \widetilde{d} \, \widetilde{d} + Q(\lambda^* d^*) . \tag{5.18}$$

Adding the two equations (5.17) and (5.18) we get

$$\widetilde{\omega} \circ Y = \mathcal{R}^{-1}(-\dot{Q} Q^{\mathrm{T}}) + Q \widetilde{\omega}^*$$

[‡]This transformation formula satisfies the associative law

and applying the linear map $\mathcal{R}$ to this equality leads to

$$\mathcal{R}(-\widetilde{\omega})\circ Y = \dot{Q}Q^{\mathrm{T}} + \mathcal{R}(-Q\widetilde{\omega}^*) = \dot{Q}Q^{\mathrm{T}} + Q\mathcal{R}(-\widetilde{\omega}^*)\,Q^{\mathrm{T}} \ .$$

So defining $A_d = \mathcal{R}(-\widetilde{\omega})$ gives

$$A_d\circ Y = \dot{Q}Q^{\mathrm{T}} + QA^*_{d^*}\,Q^{\mathrm{T}} \ , \tag{5.19}$$

the wanted transformation formula for A_d, and we have

$$\overset{\circ}{d} = \boldsymbol{\omega}\times d = (\widetilde{\omega} - \lambda d)\times d = \widetilde{\omega}\times d = \mathcal{R}(-\widetilde{\omega})d = A_d d\,.$$

With this it is easy to define $A := A_d - A_\xi$ which then satisfies $d' = \overset{\circ}{d} - A_\xi d = Ad$. Now A_ξ satisfies (5.3) which is the same as (5.19), therefore A is an objective matrix. □

We now let $\mathscr{S}^{sp} = d\wedge d'$ as in (5.16) and obtain

5.5 Lemma. Using (5.16) we get for the time term

$$\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0 = d\wedge d''\ ,$$

where $d'' = \overset{\circ}{d'} - A_\xi d'$ is an objective vector by (4.5).

Proof. It is

$$\begin{aligned}
\Big(d\wedge d'\Big)^\circ &= \overset{\circ}{d}\wedge d' + d\wedge \overset{\circ}{d'} = (d' + A_\xi d)\wedge d' + d\wedge(d'' + A_\xi d')\\
&= (A_\xi d)\wedge d' + d\wedge(A_\xi d') + d\wedge d'' = A_\xi(d\wedge d') + (d\wedge d)'\,A_\xi{}^{\mathrm{T}} + d\wedge d''\\
&= A_\xi\mathscr{S}^{sp} + \mathscr{S}^{sp}A_\xi{}^{\mathrm{T}} + d\wedge d'' = \Gamma^0 + d\wedge d''\ ,
\end{aligned}$$

hence $\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0 = d\wedge d''$. □

We now use special terms for Σ and $\overline{\mathrm{H}}$ as they occured in the theory of Ericksen & Leslie.

5.6 Lemma. The reduced spin equation (5.13) with $\mathscr{S}^{sp} = d\wedge d'$ and the identities

$$\Sigma = d\wedge\pi\,, \qquad \overline{\mathrm{H}} = d\wedge g + \textstyle\sum_j \partial_j d\wedge \pi_{\bullet j}\,, \tag{5.20}$$

where π is an objective matrix and g an objective vector, is equivalent to

$$d\wedge(\varrho d'' + \mathrm{div}\pi - g) = 0\,, \tag{5.21}$$

in other words

$$\varrho d'' + \mathrm{div}\pi - g \in \mathrm{span}\,\{d\}\,. \tag{5.22}$$

Remark: Hence the director equation (6.1) is true for some real valued function λ.

Proof. We obtain using the identity (5.13) and the previous lemma 5.5

$$\begin{aligned}
&\varrho(\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0) + \operatorname{div}\Sigma = \varrho\, d \wedge d'' + \operatorname{div}\Sigma \\
&= d \wedge (\varrho d'') + \operatorname{div}(d \wedge \pi) = d \wedge (\varrho d'') + \sum_j \partial_j (d \wedge \pi_{\bullet j}) \\
&= d \wedge (\varrho d'' + \operatorname{div}\pi) + \sum_j (\partial_j d) \wedge \pi_{\bullet j}
\end{aligned}$$

and

$$\overline{\mathrm{H}} = d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j}\,,$$

Therefore the spin equation, i.e. the equality of the left sides

$$\varrho(\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0) + \operatorname{div}\Sigma = \overline{\mathrm{H}}\,,$$

is equivalent to the equality of the right sides

$$d \wedge (\varrho d'' + \operatorname{div}\pi) = d \wedge g\,.$$

This is equivalent to

$$d \wedge (\varrho d'' + \operatorname{div}\pi - g) = 0\,,$$

which is equivalent to the assertion. □

6 Director equation

The purpose of this chapter is to give an independent definition of the director equation and show its connection with the spin equation. The director equation is of the form

$$\begin{aligned}
\partial_t(\varrho d') + \operatorname{div}(\varrho d' v^{\mathrm{T}} + \pi) &= g_\lambda + \varrho G \\
g_\lambda := g + \lambda d \quad &\text{with a real valued function } \lambda,
\end{aligned} \tag{6.1}$$

where $d' := \overset{\circ}{d} - A_\xi d$ is an objective vector by (4.4), i.e.

$$d' \circ Y = Q d^{*\prime}\,. \tag{6.2}$$

Moreover the definition says, see the invariance principle [1 : (I 5.13)], that the test functions of this equation transform with $\zeta^* = Q^{\mathrm{T}}\,\zeta\circ Y$. This is true if π is an objective tensor and

$$G \circ Y = \dot{Q} d' + Q G^* \tag{6.3}$$

and the rest g_λ is an objective vector.

6.1 Lemma. The equation (6.1) is equivalent to

$$\varrho d'' + \operatorname{div}\pi = g_\lambda \tag{6.4}$$

where $d'' := \overset{\circ}{d'} - G$ is an objektive vector by 4.3(1) (with $b = d'$ and $G_b = G$) if (6.3) holds. This is a more general definition of d'' than (4.5).

Reference: The equation (6.4) is important in the Ericksen-Leslie theory, see Ericksen [7 : (2.11)] without π and Leslie [17 : (3.7)] ($\pi \rightsquigarrow -\pi$), and for the incompressible case in Chandrasekhar [3 : (3.1.9)]. The term g is called the "intrinsic director body force".

Proof. We have that with the mass equation of (5.1)

$$\partial_t(\varrho d') + \operatorname{div}(\varrho d' v) = \varrho \overset{\circ}{d'} = \varrho(d'' + G)$$

hence (6.1) becomes

$$\begin{aligned} 0 &= \partial_t(\varrho d') + \operatorname{div}(\varrho d' v + \pi) - (g_\lambda + \varrho G) = \\ &= \varrho(d'' + G) + \operatorname{div}\pi - (g_\lambda + \varrho G) = \varrho d'' + \operatorname{div}\pi - g_\lambda\,. \end{aligned}$$

□

In this paper we have assumed a reduced spin equation (2.2)

$$\varrho \overset{\circ}{\mathscr{S}}{}^{sp} + \operatorname{div}\Sigma = \mathrm{H} \quad \text{with} \quad \mathscr{S}^{sp} = d \wedge d'\,, \tag{6.5}$$

where by (5.10) the right side H is equal to $\mathrm{H} = \mathrm{H}^0 + \overline{\mathrm{H}}$ with (5.11)

$$\begin{aligned} \mathrm{H}^0 = \varrho(A_\xi \mathscr{S}^{sp} + \mathscr{S}^{sp} A_\xi{}^{\mathrm{T}}) &= \varrho\big((A_\xi(d \wedge d') + (d \wedge d') A_\xi{}^{\mathrm{T}}\big) \\ &= \varrho\big((A_\xi d) \wedge d' + d \wedge (A_\xi d')\big)\,. \end{aligned} \tag{6.6}$$

We show now that (6.1) implies (6.5) for certain Σ and H.

6.2 Lemma. The following is equivalent:

(1) The director equation (6.1) with $G = A_\xi d'$ for a function λ.

(2) The reduced spin equation (6.5) with

$$\mathscr{S} = \varrho\, d \wedge d' \quad \text{and} \quad \Sigma = d \wedge \pi \quad \text{and} \quad \overline{\mathrm{H}} = d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j}\,. \tag{6.7}$$

And the following holds as consequence: If to the reduced spin equation one adds

$$2\Pi^{\mathrm{A}} + \overline{\Gamma} = \overline{\mathrm{H}} = d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j} \tag{6.8}$$

one gets the spin equation, see (5.9). Equivalently, one takes the director equation for some function λ and adds (6.8) as additional equation.

We see that the (full) spin equation, given as (6.7), contains all information needed for this model of liquid crystals.

Reference: The equation (6.8) one finds in Leslie [17 : (3.10)] ($-\pi^{\mathrm{T}} \rightsquigarrow \pi$, $\overline{\Gamma} = 0$), what he gets from objectivity considerations about the internal energy equation, and for the incompressible case in Chandrasekhar [3 : (3.1.10)] ($-\pi^{\mathrm{T}} \rightsquigarrow \pi$, $\overline{\Gamma} = 0$), what he gets from the conservation of angular momentum together with the director equation.

Proof of the equivalence. The definition $G = A_\xi d'$ implies the transfromation rule (6.3) which has to be satisfied. The spin equation reads by (5.13) with definition (5.12)

$$\varrho \overset{\circ}{\mathscr{S}}{}^{sp} + \operatorname{div}\Sigma = \mathrm{H} = \varrho\Gamma^0 + \overline{\mathrm{H}}\,,$$

where by 5.5

$$\overset{\circ}{\mathscr{S}}{}^{sp} - \Gamma^0 = d \wedge d''\,.$$

Assume (1). To show (2) we compute by the director equation (6.4)

$$\begin{aligned}
&\varrho \overset{\circ}{\mathscr{S}}{}^{sp} - \varrho\Gamma^0 + \operatorname{div}\Sigma = d \wedge (\varrho d'') + \operatorname{div}\Sigma \\
&= d \wedge g_\lambda - d \wedge \operatorname{div}\pi + \operatorname{div}\Sigma \\
&= d \wedge g + \sum_j \partial_j d \wedge \pi_{\bullet j} + \operatorname{div}(\Sigma - d \wedge \pi) = \overline{\mathrm{H}}\,,
\end{aligned}$$

since we assume the identities for Σ and $\overline{\mathrm{H}}$ in (2). Hence the spin equation is fulfilled. Now assume (2). To show (1) we obtain using the identity for $\overset{\circ}{\mathscr{S}}{}^{sp}$ from above and the formula of Σ

$$\begin{aligned}
&\mathrm{H} = \varrho \overset{\circ}{\mathscr{S}}{}^{sp} + \operatorname{div}\Sigma = \varrho(d \wedge d'' + \Gamma^0) + \operatorname{div}\Sigma \\
&= d \wedge (\varrho d'') + \operatorname{div}(d \wedge \pi) + \varrho\Gamma^0 \\
&= d \wedge (\varrho d'' + \operatorname{div}\pi) + \sum_j \partial_j d \wedge \pi_{\bullet j} + \varrho\Gamma^0\,,
\end{aligned}$$

hence by the formula of $\overline{\mathrm{H}}$

$$d \wedge (\varrho d'' + \operatorname{div}\pi) = \mathrm{H} - \varrho\Gamma^0 - \sum_j \partial_j d \wedge \pi_{\bullet j} = d \wedge g\,.$$

This is equivalent to

$$d \wedge (\varrho d'' + \operatorname{div}\pi - g) = 0\,.$$

Hence $\varrho d'' + \operatorname{div}\pi - g = \lambda d$ for some λ. □

Reference: In literature for $\mathscr{S}^{sp} = d \wedge d'$ the connection between angular momentum and the director equation has been treated in Leslie [18 : (8),(14)-(16)] in 1979, where he writes: "It is clear that the conservation law for angular momentum (8) is equivalent to the integral balance (16) provided that the intrinsic director body force g satisfies the relationship (14), and furthermore is indeterminate up to an arbitrary, scalar multiple of the director." Later in 1985 I. Müller treated his view in [22 : 10.1], where he writes down the balance of spin in equation [22 : (10.21)] as a part of the angular momentum, as DeGroot & Mazur [4 : Chap.XII §1] did, and shows that this is equivalent to the equations in [22 : (10.23)], which is exactly our Lemma 6.2. The undeterminacy of g with a multiple of d becomes clear later, it is contained in the constitutive part [22 : $(10.62)_7$].

7 Objective scalar entropy

Since the entropy principle says

$$\sigma := \partial_t \eta + \operatorname{div}\psi \geq 0 \tag{7.1}$$

the entropy η has to be an objectice scalar, that is

$$\eta \circ Y = \eta^* \,, \tag{7.2}$$

where $Y : \mathbb{R}^4 \to \mathbb{R}^4$ is the observer transformation, see (4.2), and η and η^* are the entropy for the two observers. In (2.9) we make the following constitutive assumption on η

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) \,, \tag{7.3}$$

where $\eta^* = \widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*)$ has the same function $\widehat{\eta}$ like η. This is the well known objectivity of constitutive functions. Hence (7.2) becomes

$$\begin{aligned}
&\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \eta^* = \eta \circ Y \\
&= \widehat{\eta}(\varrho \circ Y, \varepsilon \circ Y, d \circ Y, \mathrm{D}d \circ Y) = \widehat{\eta}(\varrho^*, \varepsilon^*, Qd^*, Q\mathrm{D}d^* Q^{\mathrm{T}})
\end{aligned}$$

since ϱ and ε are objective scalars, d is an objective vector, and from there $\mathrm{D}d$ an objective matrix, see 4.1. Thus it follows

$$\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \widehat{\eta}(\varrho^*, \varepsilon^*, Qd^*, Q\mathrm{D}d^* Q^{\mathrm{T}}) \tag{7.4}$$

for every value ϱ^*, ε^*, d^*, $\mathrm{D}d^*$. And this holds for all orthogonal matrices Q with determinant 1. For this matrix take $s \mapsto Q_s$, where s is a real variable and

$$\frac{\mathrm{d}}{\mathrm{d}s} Q_s = A_s Q_s \text{ for } s \geq 0 \text{ and } Q_0 = \mathrm{Id}$$

with a given antisymmetric matrix A_s. Then by (7.4)

$$\widehat{\eta}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) = \widehat{\eta}(\varrho^*, \varepsilon^*, Q_s d^*, Q_s \mathrm{D}d^* {Q_s}^{\mathrm{T}})$$

that is [§]

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s} \widehat{\eta}(\varrho^*, \varepsilon^*, Q_s d^*, Q_s \mathrm{D}d^* {Q_s}^{\mathrm{T}}) \\
&= \widehat{\eta}'_d(\ldots) \bullet (A_s Q_s d^*) + \widehat{\eta}'_{\mathrm{D}d}(\ldots) \mathbin{\vdots} \big(A_s Q_s \mathrm{D}d^* {Q_s}^{\mathrm{T}} + Q_s \mathrm{D}d^* (A_s Q_s)^{\mathrm{T}} \big) \,.
\end{aligned}$$

In particular, for $s = 0$ and $\overline{A} := A_0$

$$0 = \widehat{\eta}'_d(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) \bullet (\overline{A} d^*) + \widehat{\eta}'_{\mathrm{D}d}(\varrho^*, \varepsilon^*, d^*, \mathrm{D}d^*) \mathbin{\vdots} \big(\overline{A} \mathrm{D}d^* + \mathrm{D}d^* \overline{A}^{\mathrm{T}} \big) \,.$$

Or we write this as

[§] we write $\eta'_{\mathrm{D}d}$ where 'Dd' is a place holder for the last variables

7.1 Condition. If η as in (7.3) is an objective scalar, then for all arguments $(\varrho, \varepsilon, d, \mathrm{D}d)$ and for every antisymmetric matrix $\overline{A}$

$$0 = \widehat{\eta}'_d(\varrho, \varepsilon, d, \mathrm{D}d) \bullet (\overline{A}d) + \widehat{\eta}'_{\mathrm{D}d}(\varrho, \varepsilon, d, \mathrm{D}d) : \big(\overline{A}\mathrm{D}d + \mathrm{D}d\overline{A}^{\mathrm{T}}\big) .$$

Factoring out $\overline{A}$ and omitting the argument $(\varrho, \varepsilon, d, \mathrm{D}d)$ we get

$$0 = \overline{A} : \big(\eta'_d \otimes d + \eta'_{\mathrm{D}d} (\mathrm{D}d)^{\mathrm{T}} + (\eta'_{\mathrm{D}d})^{\mathrm{T}} \mathrm{D}d\big) , \tag{7.5}$$

which written in coordinates is

$$0 = \sum_{ik} \overline{A}_{ik} \Big(\widehat{\eta}'_{d_i} d_k + \sum_m \big(\widehat{\eta}'_{d_i{}'_m} d_{k}{}'_m + \widehat{\eta}'_{d_m{}'_i} d_m{}'_k \big) \Big) .$$

Hence the matrix in brackets of (7.5) is symmetric.

One can derive a corresponding formula for any other dependence of η (for d replaced by an objective scalar see [1 : IV.13.5]).

8 Entropy production

The main part of the proof is presented in this section. For the entropy we assume

$$\eta = \widehat{\eta}(\varrho, \varepsilon, d, \mathrm{D}d) \tag{8.1}$$

and for the specific spin we have by (2.10)

$$\mathscr{S}^{sp} = d \wedge d' . \tag{8.2}$$

The entropy principle starts with system (2.7) which gives

$$\begin{aligned}
&\overset{\circ}{\varrho} + \varrho \,\mathrm{div} v = 0 , \\
&\overset{\circ}{\varepsilon} + \varepsilon \,\mathrm{div} v = -\,\mathrm{div} q - \mathrm{D}v : \Pi^{\mathrm{S}} - \tau\big(\mathrm{D}\mathscr{S}^{sp} : \Sigma + \mathscr{S}^{sp} : \mathrm{H}\big) .
\end{aligned}$$

Then we compute for the entropy production

$$\begin{aligned}
\sigma &= \partial_t \eta + \,\mathrm{div}\psi = \overset{\circ}{\eta} + \eta \,\mathrm{div} v + \,\mathrm{div}(\psi - \eta v) \\
&= \eta'_\varrho \overset{\circ}{\varrho} + \eta'_\varepsilon \overset{\circ}{\varepsilon} + \eta'_d \bullet \overset{\circ}{d} + \eta'_{\mathrm{D}d} : (\mathrm{D}d)^\circ + \eta \,\mathrm{div} v + \,\mathrm{div}(\psi - \eta v) ,
\end{aligned}$$

and since $\mathscr{S}^{sp} : \mathrm{H} = \mathscr{S}^{sp} : \overline{\mathrm{H}}$ by 5.2, where $\overline{\mathrm{H}}$ is an objective matrix, this expression becomes

$$\begin{aligned}
\sigma &= (\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon) \,\mathrm{div} v + \,\mathrm{div}(\psi - \eta v) - \eta'_\varepsilon \,\mathrm{div} q - \eta'_\varepsilon \mathrm{D}v : \Pi^{\mathrm{S}} \\
&\qquad + \eta'_d \bullet \overset{\circ}{d} + \eta'_{\mathrm{D}d} : (\mathrm{D}d)^\circ - \eta'_\varepsilon \tau \big(\mathrm{D}\mathscr{S}^{sp} : \Sigma + \mathscr{S}^{sp} : \overline{\mathrm{H}}\big) \\
&= \mathrm{D}v : \big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon \Pi^{\mathrm{S}} \big) + \,\mathrm{div}(\psi - \eta v - \eta'_\varepsilon q) \\
&\qquad + \nabla\eta'_\varepsilon \bullet q + \eta'_d \bullet \overset{\circ}{d} + \eta'_{\mathrm{D}d} : (\mathrm{D}d)^\circ - \eta'_\varepsilon \tau \big(\mathrm{D}\mathscr{S}^{sp} : \Sigma + \mathscr{S}^{sp} : \overline{\mathrm{H}}\big) .
\end{aligned}$$

Here the classical things for the final version of the entropy production are done. So we have essentially to deal with the d terms where the spin $\mathscr{S}$ depends on d and d'. We mention that d' is related to $\overset{\circ}{d}$, see (8.3). But first let us use the fact that η must be an objective scalar, consequently by Condition 7.1 there holds for any antisymmetric matrix B satisfying the transformtion rule $B \circ Y = \dot{Q}Q^{\mathrm{T}} + QB^* Q^{\mathrm{T}}$

$$0 = \eta_{'d} \bullet Bd + \eta_{'\mathrm{D}d} : (B\mathrm{D}d + \mathrm{D}dB^{\mathrm{T}}) ,$$

for example $B := (\mathrm{D}v)^{\mathrm{A}}$. With this we get for the d-terms in the entropy production

$$\begin{aligned}\sigma_d &:= \eta_{'d} \bullet \overset{\circ}{d} + \eta_{'\mathrm{D}d} : (\mathrm{D}d)^{\circ} \\ &= \eta_{'d} \bullet (\overset{\circ}{d} - Bd) + \eta_{'\mathrm{D}d} : \big((\mathrm{D}d)^{\circ} - (B\mathrm{D}d + \mathrm{D}dB^{\mathrm{T}})\big) .\end{aligned}$$

Now $d^\eta := \overset{\circ}{d} - Bd$ is an objective vector by 4.3(2) and $D^\eta := (\mathrm{D}d)^{\circ} - (B\mathrm{D}d + \mathrm{D}dB^{\mathrm{T}})$ satisfies the representation

$$\begin{aligned}(\mathrm{D}d)^{\circ}{}_{ij} &= (d_{i'j})^{\circ} = d_{i'jt} + \sum_k v_k d_{i'jk} \\ &= \big(d_{i't} + \sum_k v_k d_{i'k}\big)_{'j} - \sum_k v_{k'j} d_{i'k} \\ &= \big(\overset{\circ}{d_i}\big)_{'j} - \sum_k v_{k'j} d_{i'k} ,\end{aligned}$$

hence, where we use now $B = (\mathrm{D}v)^{\mathrm{A}}$,

$$\begin{aligned}D^\eta_{ij} &= (\mathrm{D}d)^{\circ}{}_{ij} - (B\mathrm{D}d)_{ij} - \frac{1}{2}\sum_k d_{i'k}(v_{j'k} - v_{k'j}) \\ &= \big(\overset{\circ}{d_i}\big)_{'j} - (B\mathrm{D}d)_{ij} - \frac{1}{2}\sum_k d_{i'k}(v_{j'k} + v_{k'j}) \\ &= \Big(\big(\overset{\circ}{d_i}\big)_{'j} - \sum_k B_{ik} d_{k'j}\Big) - \sum_k d_{i'k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj} .\end{aligned}$$

The first term (in bracket) is by 4.3(3) an objective tensor in (i,j), and also, of course, the second term with the symmetric part of the velocity gradient. Since $d^\eta = \overset{\circ}{d} - Bd$ we get

$$\big(\overset{\circ}{d_i}\big)_{'j} - \sum_k B_{ik} d_{k'j} = \big(d^\eta_i + \sum_k B_{ik} d_k\big)_{'j} - \sum_k B_{ik} d_{k'j} = (d^\eta_i)_{'j} + \sum_k B_{ik'j} d_k ,$$

which finally gives

$$D^\eta_{ij} = (d^\eta_i)_{'j} + \sum_k B_{ik'j} d_k - \sum_k d_{i'k} (\mathrm{D}v)^{\mathrm{S}}{}_{kj} ,$$

where now all three terms are objective tensors. This is because B_{ik} in (i,k) is a tensor which transforms like the derivative of a velocity, where the inhomogeneous part of this transformation depends only on time. Therefore $B_{ik'j}$ has not this part and is in (i,k,j) an objective 3-tensor. Now, tthe matrix A_ξ in 5.1 is an antisymmetric matrix depending only on t and transforms as B like the derivative of a velocity. Then we see that $\overline{B} := B - A_\xi$ is an objective tensor, and it is $B_{ik'j} = \overline{B}_{ik'j}$ as said an objective 3-tensor. Consequently we get using (5.16)

$$d^\eta + \overline{B}d = \overset{\circ}{d} - A_\xi d = d' \tag{8.3}$$

which is an objective vector. Hence

$$\begin{aligned}
&(d_i^\eta)_{'j} + \sum_k B_{ik'j}d_k = (d_i^\eta)_{'j} + \sum_k \overline{B}_{ik'j}d_k \\
&= \big(d_i^\eta + \sum_k \overline{B}_{ik}d_k\big)_{'j} - \sum_k \overline{B}_{ik}d_{k'j} = (d_i^{'})_{'j} - \sum_k \overline{B}_{ik}d_{k'j} \,,
\end{aligned}$$

and therefore

$$\begin{aligned}
D_{ij}^\eta &= (d_i^{'})_{'j} - \sum_k \overline{B}_{ik}d_{k'j} - \sum_k d_{i'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj} \,, \\
d_i^\eta &= d_i^{'} - \sum_k \overline{B}_{ik}d_k \,.
\end{aligned}$$

Hence the contribution of σ_d is

$$\begin{aligned}
\sigma_d &:= \eta_{'d} \bullet d^\eta + \eta_{'\mathrm{D}d} \mathbin{:} D^\eta \\
&= \sum_i \eta_{'d_i}\big(d_i^{'} - \sum_k \overline{B}_{ik}d_k\big) + \sum_{i,j} \eta_{'d_{i,j}}\big((d_i^{'})_{'j} - \sum_k \overline{B}_{ik}d_{k'j} - \sum_k d_{i'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj}\big) \\
&= \sum_i \eta_{'d_i}d_i^{'} + \sum_{i,j} \eta_{'d_{i,j}}(d_i^{'})_{'j} - \sum_{i,k} \eta_{'d_i}d_k\overline{B}_{ik} - \sum_{i,j,k} \eta_{'d_{i,j}}d_{k'j}\overline{B}_{ik} \\
&\qquad - \sum_{i,j,k} \eta_{'d_{i,j}}d_{i'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj} \,.
\end{aligned}$$

In the entropy production σ the terms of σ_d have a counterpart, the spin terms σ_s where $(k,l) \mapsto \Sigma_{klj}, \overline{\mathrm{H}}_{kl}$ are antisymmetric,

$$\begin{aligned}
\sigma_s &:= -\big(\mathrm{D}\mathscr{S}^{sp} \mathbin{:} \Sigma + \mathscr{S}^{sp} \mathbin{:} \overline{\mathrm{H}}\big) = -\big(\mathrm{D}(d \wedge d') \mathbin{:} \Sigma + d \wedge d' \mathbin{:} \overline{\mathrm{H}}\big) \\
&= -\sum_{k,l} \big(\sum_j \partial_j(d_k d_l^{'} - d_k^{'}d_l)\Sigma_{klj} + (d_k d_l^{'} - d_k^{'}d_l)\overline{\mathrm{H}}_{kl}\big) \\
&= 2\sum_{k,l} \big(\sum_j (d_{k'j}^{'}d_l + d_k^{'}d_{l'j})\Sigma_{klj} + d_k^{'}d_l\overline{\mathrm{H}}_{kl}\big) \\
&= \sum_{k,j} d_{k'j}^{'} \sum_l 2d_l\Sigma_{klj} + \sum_k d_k^{'}\big(\sum_{l,j} 2d_{l'j}\Sigma_{klj} + \sum_l 2d_l\overline{\mathrm{H}}_{kl}\big) \,.
\end{aligned}$$

Hence together with σ_d we obtain

$$
\begin{aligned}
&\sigma_d + \eta'_{\varepsilon}\tau\sigma_s = \\
&= \sum_{k,j} d'_{k'j}\big(\eta'_{d_{k,j}} + \sum_l 2\tau\eta'_{\varepsilon} d_l \Sigma_{klj}\big) + \sum_k d'_k\big(\eta'_{d_k} + \sum_{l,j} 2\tau\eta'_{\varepsilon} d_{l'j}\Sigma_{klj} + \sum_l 2\tau\eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{kl}\big) \\
&\qquad\qquad - \sum_{i,k} \eta'_{d_i} d_k \overline{B}_{ik} - \sum_{i,j,k} \eta'_{d_{i,j}} d_{k'j}\overline{B}_{ik} - \sum_{i,j,k} \eta'_{d_{i,j}} d_{i'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj} \\
&= \sum_j \partial_j\Big(\sum_k d'_k\big(\eta'_{d_{k,j}} + \sum_l 2\tau\eta'_{\varepsilon} d_l \Sigma_{klj})\Big) \\
&\quad + \sum_k d'_k\Big(\eta'_{d_k} - \sum_j \partial_j \eta'_{d_{k,j}} + 2\tau\sum_{l,j}\Big(\eta'_{\varepsilon} d_{l'j}\Sigma_{klj} - \partial_j(\eta'_{\varepsilon} d_l \Sigma_{klj})\Big) + \sum_l 2\tau\eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{kl}\Big) \\
&\qquad\qquad - \sum_{i,k} \eta'_{d_i} d_k \overline{B}_{ik} - \sum_{i,j,k} \eta'_{d_{i,j}} d_{k'j}\overline{B}_{ik} - \sum_{i,j,k} \eta'_{d_{i,j}} d_{i'k}\,(\mathrm{D}v)^{\mathrm{S}}{}_{kj}\,.
\end{aligned}
$$

The first term on the right side goes to the div-term in σ and the last term goes to the Dv-term. Moreover, in the middle term the coefficient of d'_k becomes

$$
\begin{aligned}
\eta'_{d_k} - \sum_j \partial_j \eta'_{d_{k,j}} + 2\tau\sum_{l,j}\Big(\eta'_{\varepsilon} d_{l'j}\Sigma_{klj} - \partial_j(\eta'_{\varepsilon} d_l \Sigma_{klj})\Big) + \sum_l 2\tau\eta'_{\varepsilon} d_l \overline{\mathrm{H}}_{kl} \\
= \frac{\delta\eta}{\delta d_k} + 2\tau\sum_l d_l\Big(\eta'_{\varepsilon}\overline{\mathrm{H}}_{kl} - \sum_j \partial_j(\eta'_{\varepsilon}\Sigma_{klj})\Big)\,.
\end{aligned}
$$

With these computations we finally obtain for the entropy production σ

$$
\begin{aligned}
\sigma &= \mathrm{D}v : \big((\eta - \varrho\eta'_{\varrho} - \varepsilon\eta'_{\varepsilon})\mathrm{Id} - \eta'_{\varepsilon}\Pi^{\mathrm{S}}\big) \\
&\qquad\qquad + \nabla\eta'_{\varepsilon}\cdot q + \sigma_d + \eta'_{\varepsilon}\tau\sigma_s + \operatorname{div}(\psi - \eta v - \eta'_{\varepsilon} q) \\
&= \mathrm{D}v : \Big((\eta - \varrho\eta'_{\varrho} - \varepsilon\eta'_{\varepsilon})\mathrm{Id} - \eta'_{\varepsilon}\Pi^{\mathrm{S}} - \sum_i (\nabla d_i \otimes \eta'_{\nabla d_i})^{\mathrm{S}}\Big) + \nabla\eta'_{\varepsilon}\cdot q \\
&\qquad\qquad - \sum_{i,k}\big(\eta'_{d_i} d_k + \sum_j \eta'_{d_{i,j}} d_{k'j}\big)\overline{B}_{ik} \\
&\quad + \sum_k d'_k\Big(\frac{\delta\eta}{\delta d_k} + 2\tau\sum_l d_l\big(\eta'_{\varepsilon}\overline{\mathrm{H}}_{kl} - \sum_j \partial_j(\eta'_{\varepsilon}\Sigma_{klj})\big)\Big) \\
&\quad + \sum_j \partial_j\Big(\psi_j - \eta v_j - \eta'_{\varepsilon} q_j + \sum_k d'_k\big(\eta'_{d_{k,j}} + \sum_l 2\tau\eta'_{\varepsilon} d_l \Sigma_{klj}\big)\Big)\,.
\end{aligned}
\tag{8.4}
$$

Now we make again usage of the Condition 7.1 and this gives for the $\overline{B}$-term

$$
\begin{aligned}
&\sum_{i,k}\big(\eta'_{d_i} d_k + \sum_j \eta'_{d_{i,j}} d_{k'j}\big)\overline{B}_{ik} = -\sum_{i,k,j} \eta'_{d_{j,i}} d_{j'k}\overline{B}_{ik} \\
&= -\overline{B} : \big(\sum_j \eta'_{\nabla d_j} \otimes \nabla d_j\big) = \overline{B} : \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big) \\
&= \mathrm{D}(v - v_\xi)^{\mathrm{A}} : \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big) = \mathrm{D}(v - v_\xi) : \big(\sum_j \nabla d_j \otimes \eta'_{\nabla d_j}\big)^{\mathrm{A}}\,,
\end{aligned}
$$

since $A_\xi = \mathrm{D}v_\xi$ by 5.1 is antisymmetric and therefore $\overline{B} = (\mathrm{D}v)^{\mathrm{A}} - A_\xi = (\mathrm{D}(v - v_\xi))^{\mathrm{A}}$. It also holds $(\mathrm{D}(v - v_\xi))^{\mathrm{S}} = (\mathrm{D}v)^{\mathrm{S}}$ and with this (8.4) becomes

$$\begin{aligned}\sigma = {} & \mathrm{D}(v - v_\xi) \mathbin{:} \Big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \sum_i \nabla d_i \otimes \eta'_{\nabla d_i} - \eta'_\varepsilon \Pi^{\mathrm{S}} \Big) \\ & + \nabla\eta'_\varepsilon \bullet q + \sum_k d_k' \Big(\frac{\delta\eta}{\delta d_k} + 2\tau \sum_l d_l \big(\eta'_\varepsilon \overline{\mathrm{H}}_{kl} - \sum_j \partial_j (\eta'_\varepsilon \Sigma_{klj}) \big) \Big) \\ & + \sum_j \partial_j \Big(\psi_j - \eta v_j - \eta'_\varepsilon q_j + \sum_k d_k' \big(\eta'_{d_{k,j}} + \sum_l 2\tau\eta'_\varepsilon d_l \Sigma_{klj} \big) \Big) \,. \end{aligned} \tag{8.5}$$

This is the general form of the entropy production σ under the assumption that the entropy fulfilles (8.1) and the spin is given by (8.2). This we use for the proof of the main theorem 2.2 and the applications in Section 3.

9 Classical entropy

In this section we will present a version of the entropy principle which is indicated by the approach in DeGroot & Mazur [4: Chap.XII §1], which was adopted from Grad [14: §4]. In [4: Chap.XII §1 (20)] this principle is based on the classical Gibbs relation, which means that η depends only on (ϱ, ε), that is

$$\eta = \widehat{\eta}(\varrho, \varepsilon) \,. \tag{9.1}$$

(It is $\eta = \varrho s$ with the specific entropy s in DeGroot & Mazur, and the Gibbs relation [4: Chap.XII §1 (20)] is identical to [1: Chap.III.1.4(3)].)

Therefore in our framework the entropy principle starts with system (2.7) which gives for ϱ and ε

$$\begin{aligned} & \mathring{\varrho} + \varrho \,\mathrm{div} v = 0 \,, \\ & \mathring{\varepsilon} + \varepsilon \,\mathrm{div} v = -\,\mathrm{div} q - \mathrm{D}v \mathbin{:} \Pi^{\mathrm{S}} - \tau \big(\mathrm{D}\mathscr{S}^{sp} \mathbin{:} \Sigma + \mathscr{S}^{sp} \mathbin{:} \mathrm{H} \big) \,, \end{aligned}$$

where $\mathscr{S}^{sp} \mathbin{:} \mathrm{H} = \mathscr{S}^{sp} \mathbin{:} \overline{\mathrm{H}}$ by 5.2 with an objective matrix $\overline{\mathrm{H}}$. Therefore we conclude for η by (9.1)

$$\begin{aligned} \mathring{\eta} = \eta'_\varrho \mathring{\varrho} + \eta'_\varepsilon \mathring{\varepsilon} = & -(\varrho\eta'_\varrho + \varepsilon\eta'_\varepsilon)\,\mathrm{div} v - \eta'_\varepsilon \,\mathrm{div} q \\ & - \eta'_\varepsilon \mathrm{D}v \mathbin{:} \Pi^{\mathrm{S}} - \eta'_\varepsilon \tau \big(\mathrm{D}\mathscr{S}^{sp} \mathbin{:} \Sigma + \mathscr{S}^{sp} \mathbin{:} \overline{\mathrm{H}} \big) \,. \end{aligned}$$

Now the entropy inequality (7.1) gives, with the Clausius-Duhem term $\psi = \eta v + \eta'_\varepsilon q$,

$$\begin{aligned} 0 \le \sigma & = \partial_t \eta + \mathrm{div}\psi = \partial_t \eta + \mathrm{div}(\eta v + \eta'_\varepsilon q) \\ & = \mathring{\eta} + \eta \,\mathrm{div} v + \mathrm{div}(\eta'_\varepsilon q) \\ & = \mathrm{D}v \mathbin{:} \big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon \Pi^{\mathrm{S}} \big) + \nabla\eta'_\varepsilon \bullet q \\ & \quad - \eta'_\varepsilon \tau \big(\mathrm{D}\mathscr{S}^{sp} \mathbin{:} \Sigma + \mathscr{S}^{sp} \mathbin{:} \overline{\mathrm{H}} \big) \,. \end{aligned} \tag{9.2}$$

This residual inequality gives rise to the following theorem, where $\theta\eta'_\varepsilon = 1$, where we remark that ψ is the classical Clausius-Duhem flux.

9.1 Theorem. Let us consider solutions of system (2.7) and let us consider the classical entropy and Clausius-Duhem flux

$$\eta = \widehat{\eta}(\varrho, \varepsilon)\,, \qquad \psi = \eta v + \eta'_{\varepsilon} q\,.$$

Then the entropy principle is satisfied, where in the system (2.7)

$$\Pi = p\mathrm{Id} - S\,, \quad p := \theta(\eta - \varrho\eta'_{\varrho} - \varepsilon\eta'_{\varepsilon})\,, \quad -2S^{\mathrm{A}} = \overline{\mathrm{H}} - \overline{\Gamma}\,, \tag{9.3}$$

holds, and the residual inequality reads

$$0 \le \sigma = \nabla\eta'_{\varepsilon}\bullet q + \eta'_{\varepsilon}\big(\mathrm{D}v\,\boldsymbol{:}\,S^{\mathrm{S}} - \tau(\mathrm{D}\mathscr{S}^{sp}\boldsymbol{:}\Sigma + \mathscr{S}^{sp}\boldsymbol{:}\overline{\mathrm{H}})\big)\,. \tag{9.4}$$

Remark: In standard settings it is $\overline{\Gamma} = 0$.

Proof. In the entropy estimate in (9.2) we set $\Pi^{\mathrm{S}} = p\mathrm{Id} - S^{\mathrm{S}}$ and obtain the residual inequality (9.4). The antisymmetric part Π^{A} satisfies $-2S^{\mathrm{A}} = 2\Pi^{\mathrm{A}} = \overline{\mathrm{H}} - \overline{\Gamma}$, see the equation (5.14). □

In this theorem the dynamical system (2.7) becomes

$$\begin{aligned}
\partial_t\varrho + \mathrm{div}(\varrho v) &= 0\,,\\
\partial_t(\varrho v) + \mathrm{div}(\varrho v v^{\mathrm{T}} + p\mathrm{Id} - S) &= \mathbf{f}\,,\\
\varrho\big(\overset{\circ}{\mathscr{S}}{}^{sp} - (A_{\xi}\mathscr{S}^{sp})^{\mathrm{A}}\big) + \mathrm{div}\,\Sigma &= \overline{\mathrm{H}} = \overline{\Gamma} - 2S^{\mathrm{A}}\,,
\end{aligned} \tag{9.5}$$

and in addition the energy equation (9.7) has to be satisfied and the residual inequality (9.4) has to hold. We assume that this inequality is satisfied by

$$0 \le \sigma = \underbrace{\nabla\eta'_{\varepsilon}\bullet q + \eta'_{\varepsilon}\mathrm{D}v\,\boldsymbol{:}\,S^{\mathrm{S}}}_{\ge 0} + \eta'_{\varepsilon}\tau\underbrace{\big(-(\mathrm{D}\mathscr{S}^{sp}\boldsymbol{:}\Sigma + \mathscr{S}^{sp}\boldsymbol{:}\overline{\mathrm{H}})\big)}_{=:\,\sigma_s \ge 0}\,. \tag{9.6}$$

The second inequality has as consequence of the spin equation that the estimate

$$\frac{\varrho}{2}\big(|\mathscr{S}^{sp}|^2\big)^{\circ} + \mathrm{div}(\mathscr{S}^{sp}\boldsymbol{:}\Sigma) = \mathrm{D}\mathscr{S}^{sp}\boldsymbol{:}\Sigma + \mathscr{S}^{sp}\boldsymbol{:}\overline{\mathrm{H}} = -\sigma_s \le 0$$

has to hold, which follows since $\mathscr{S}^{sp}\boldsymbol{:}(A_{\xi}\mathscr{S}^{sp})^{\mathrm{A}} = 0$. This is part of the equation for the total energy, see (2.5). Therefore, if $\Sigma = 0$, the L^2 integral of the spin has to stay bounded. We mention that the equation for the inner energy ε becomes

$$\overset{\circ}{\varepsilon} + (\varepsilon + p)\,\mathrm{div}v + \mathrm{div}q = (\mathrm{D}v)^{\mathrm{S}}\boldsymbol{:}S + \tau\sigma_s\,. \tag{9.7}$$

We mention that the pressure tensor Π, as example, has the form of Grad [14 : (4.30)]:

9.2 Example. We take in 9.1 for the spin equation

$$\Sigma := -\mu_1\mathrm{D}\mathscr{S}^{sp}\,, \quad \overline{\mathrm{H}} := -2\mu_2\mathscr{S}^{sp} \quad \text{with } \mu_1, \mu_2 \ge 0\,,$$

and we choose for the symmetric part of the stress tensor S as for Navier-Stokes

$$S^{\mathrm{S}} = \lambda^1\Big(2\,(\mathrm{D}v)^{\mathrm{S}} - \frac{2}{3}(\,\mathrm{div}\,v)\mathrm{Id}\Big) + \lambda^2(\,\mathrm{div}\,v)\,\mathrm{Id} \quad \text{ with } \lambda^k \geq 0 \text{ for } k = 1, 2,$$

and for the antisymmetric part $2\,S^{\mathrm{A}} = -\overline{\mathrm{H}}$ provided $\overline{\Gamma} = 0$, so that with the Fourier law for q by (9.6) the entropy production $\sigma \geq 0$. So together we get for the stress tensor

$$S = \lambda^1\Big(2\,(\mathrm{D}v)^{\mathrm{S}} - \frac{2}{3}(\,\mathrm{div}\,v)\mathrm{Id}\Big) + \lambda^2(\,\mathrm{div}\,v)\,\mathrm{Id} + \mu_2\mathscr{S}^{sp} \tag{9.8}$$

and the spin equation becomes with $\overline{\Gamma} = 0$

$$\varrho\big(\overset{\circ}{\mathscr{S}}{}^{sp} - (A_\xi\mathscr{S}^{sp})^{\mathrm{A}}\,\big) + \,\mathrm{div}(-\mu_1\mathrm{D}\mathscr{S}^{sp}) = -2\mu_2\mathscr{S}^{sp}\,. \tag{9.9}$$

The spin one can imagine as expression of an unknown antisymmetric matrix Ω as

$$-\mathscr{S}^{sp} := \mu_3(\Omega - (\mathrm{D}v)^{\mathrm{A}}) \quad \text{ with } \mu_3 \geq 0\,,$$

where the "internal angular momentum" Ω transforms like $\Omega{\circ}Y = \dot{Q}\,Q^{\mathrm{T}} + Q\Omega^*\,Q^{\mathrm{T}}$, which is the same transformation rule as the "external angular momentum" $(\mathrm{D}v)^{\mathrm{A}}$ has, so that the difference $\mathscr{S}^{sp} = \mu_3((\mathrm{D}v)^{\mathrm{A}} - \Omega)$ is an objective tensor as it should be. With this choice the stress tensor becomes setting $2\lambda^3 := \mu_2\mu_3 \geq 0$

$$S = \lambda^1\Big(2\,(\mathrm{D}v)^{\mathrm{S}} - \frac{2}{3}(\,\mathrm{div}\,v)\mathrm{Id}\Big) + \lambda^2(\,\mathrm{div}\,v)\,\mathrm{Id} + \lambda^3(2\,(\mathrm{D}v)^{\mathrm{A}} - 2\Omega)$$

where "λ^1 and λ^2 are the usual coefficients of shear and bulk viscosity, while λ^3 is a new viscosity coefficient", this is identical to [14 : (4.30)]. It is Ω the internal angular momentum which one sees from outside, the spin $\mathscr{S}^{sp} := \mu_3(\Omega - (\mathrm{D}v)^{\mathrm{A}})$ is the relative eigen-rotation of the particle.

This is the result if the entropy η has no contribution from the spin $\mathscr{S}$. The spin is only present in the total energy e in (2.5), and of course in the underlying system (2.1), which is part of (2.7). This situation is considered in H. Grad (1952) [14] and in DeGroot & Mazur (1962) [4]. They showed this without considering the objectivity at all, and the derivation therefore is generally doubted, because this is due to a not correct energy equation, see [4 : Chap.II§4 (36)]. We have shown in this section by a rigorous application of the energy equation that, despite of what is said previously, the result of Grad is correct.

References

[1] H.W. Alt: *Mathematical Continuum Mechanics*. Script of the Lecture 2011-2022. Technische Universität München (TUM) 2022

[2] H.W. Alt, G. Witterstein: *Nonsymmetric pressure tensors and the spin equation*. In preparation.

[3] S. Chandrasekhar: *Liquid Crystals*. Second Edition. Cambridge University Press 1992

[4] S.R. de Groot, P. Mazur: *Non-Equilibrium Thermodynamics*. North-Holland 1962

[5] Etienne Emmrich, Sabine H.L. Klapp, Robert Lasarzik: *Nonstationary models for liquid crystals: A fresh mathematical perspective*. Journal of Non-Newtonian Fluid Mechanics 259, pp.32-47. 2018

[6] J.L. Ericksen: *Transversely Isotropic Fluids*. Kolloid-Zeitschrift 173 (Heft 2), pp.117-122. 1960

[7] J.L. Ericksen: *Anisotropic Fluids*. Arch. Rational Mech. Anal. 4, pp.231-237. 1960

[8] J.L. Ericksen: *Conservation laws for liquid crystals*. Trans. Soc. Rheol. 5, pp.23-34. 1961

[9] J.L. Ericksen: *Continuum Theory of Liquid Crystals of Nematic Type*. Molecular Crystals 7:1, pp.153-164. 1969

[10] J.L. Ericksen, D. Kinderlehrer (Eds.): *Theory and Applications of Liquid Crystals*. The IMA Volumes in Mathematics and Its Applications Vol.5. Springer-Verlag 1987

[11] J. Ericksen: *Continuum theory of nematic liquid crystals*. Res. Mechanica 21, pp.381-392 (not available). 1987

[12] J. Ericksen: *Liquid crystals with variable degree of orientation*. Arch. Rational Mech. Anal. 113, pp.97-120. 1991

[13] F.C. Frank: *On the theory of liquid crystals*. Discuss. Faraday Soc. 28, pp.19-28. 1958

[14] Harold Grad: *Statistical Mechanics, Thermodynamics, and Fluid Dynamics of Systems with an Arbitrary Number of Integrals*. Comm. Pure and Applied Math. 5, pp.455-494. 1952

[15] Robert Hardt, David Kinderlehrer: *Mathematical Questions of Liquid Crystal Theory. in:* Ericksen, D. Kinderlehrer (Eds.), pp.151-184, The IMA Volumes in Mathematics and Its Applications Vol.5. Springer-Verlag 1987

[16] Robert Lasarzik: *Verallgemeinerte Lösungen der Ericksen-Leslie-Gleichungen zur Beschreibung von Flüssigkristallen*. Von der Fakultät II Mathematik und Naturwissenschaften der Technischen Universität Berlin zur Erlangung des akademischen Grades Doktor der Naturwissenschaften. Technische Universität Berlin 2017

[17] F. M. Leslie: *Some Constitutive Equations for Liquid Crystals*. Arch. Rational Mech. Anal. 28, pp.265-283. 1968

[18] F. M. Leslie: *Theory of Flow Phenomena in Liquid Crystals*. Advances in Liquid Crystals Vol.4. Academic Press 1979

[19] F. M. Leslie: *Continuum theory for nematic liquid crystals*. Continuum Mech. Thermodyn. 4, pp.167-175. Springer-Verlag 1992

[20] Fang-Hua Lin, Chun Liu: *Nonparabolic Dissipative Systems Modeling the Flow of Liquid Crystals*. Communications on Pure and Applied Mathematics XLVIII, pp.501-537. John Wiley & Sons 1995

[21] Ingo Müller: *On the entropy inequality*. Arch. Rational Mech. Anal. 26, pp.118-141. 1967

[22] Ingo Müller: *Thermodynamics*. Interaction of mechanics and mathematics series. Pitman 1985

[23] C.W. Oseen: *The theory of liquid crystals*. Trans. Faraday Soc. 29, pp.883-889. 1933