**Advances in Mathematical Sciences and Applications**
**Vol. 29, No. 1 (2020), pp. 1–33**

GAKKOTOSHO
TOKYO JAPAN

# MAGNETISM AND GRAVITY. A UNIFIED TREATMENT

Hans Wilhelm Alt
Technische Universität München, Fakultät für Mathematik
Boltzmannstr. 3, DE-85747 Garching bei München, Germany
(E-mail: alt@ma.tum.de)

**Abstract.** In this paper we present a unified theory for gravitation and magnetization including electrodynamics. It is based on Maxwell's equations which in the form of Ampère's circuital law is the antisymmetric part of this theory, and the symmetric part is the gravity, which contains Newton's gravitation in the time-dependent relativistic version. Since magnetism and gravity are formulated in one system of differential equations, this new theory combines these two parts, and therefore this combination will probably bring some new insight to related problems which are discussed these days.

We further study the related force in the mass-momentum system. This force consists of the well-known Newton force and the well-known Lorentz force. We show that they are exactly those forces that are predicted by our theory. We prove that these forces are equal to the divergence of a 4-flux in the mass-momentum equation. In this way these forces can be considered as internal expressions. In the proof all 4-fields of the new theory have to be 4-gradients of vector potentials, which is a well-known assumption.

Communicated by Editors; Received December 31, 2019
Mathematics Subject Classification (MSC 2010): 35Q75, 35Q35, 35D05, 35Q61.
Physics and Astronomy Classification Scheme (PACS): 95.30.Sf, 04.50.Kd, 12.10.-g, 03.50.De, 96.12.Fe, 96.15.Ef.
Keywords: Relativity, Unified field theory, Gravity, Maxwell equation, Partial differential equations.

# 1 Introduction

The purpose of this paper is to present a theory that covers electromagnetism as well as the effects of gravity. The picture is that both effects are based on the atomic structure of materials, where magnetism is the bridge between gravity and electrical effects. A natural way to do this is to assume a differential equation in $\mathbb{R}^4$ of the form (3.1)

$$\underline{\operatorname{div}}\,\mathfrak{M} = \widetilde{\mathfrak{j}}\,, \tag{1.1}$$

where $\underline{\operatorname{div}}$ is defined below. The tensor $\mathfrak{M}$ contains terms for gravitation as well as for magnetization, and the source term $\widetilde{\mathfrak{j}}$ contains the masses for gravitation and the currents for electricity. We call the contravariant tensor $\mathfrak{M}$ the "general magnetic tensor" and the 4-vector $\widetilde{\mathfrak{j}}$ is split into $\widetilde{\mathfrak{j}} = \mathfrak{j} + \mathfrak{j}_{\mathfrak{M}}$, where $\mathfrak{j}$ is a contravariant vector and $\mathfrak{j}_{\mathfrak{M}}$ is due to Coriolis effects as described in Lemma 3.1. In a Lorentz frame $\mathfrak{M}$ and $\mathfrak{j}$ have the representation (4.6)

$$\mathfrak{M} = \begin{bmatrix} \dfrac{F_0}{\mathrm{c}^2} & -(F-D)^{\mathrm{T}} \\ -(F+D) & M^s + \mathcal{R}(H) \end{bmatrix}, \quad \mathfrak{j} = \begin{bmatrix} \varrho \\ \mathrm{j} \end{bmatrix}. \tag{1.2}$$

Here the quantities are uniquely determined by the tensor $\mathfrak{M}$, if $M^s$ is symmetric. And $\mathcal{R}(H)$ is defined in (4.2)-(4.4). The 4-vector $\underline{F} = (F_0, F)$ defines "Newton's gravitational law", see 4.2(1), and $D$ and $H$ describe "Ampère's circuital law" in the $kg$-based version. This version is necessary because the gravitational field $F$ is $kg$-based, whereas $D$ and $H$ originally are based on $As$. In detail, $D = \mathrm{k}_0\bar{D}$ and $H = \mathrm{k}_0\bar{H}$ where $\boldsymbol{D} := \bar{D}$ and $\boldsymbol{H} := \bar{H}$ are the well-known quantities in Section 2 with $\mathrm{k}_0$ being the conversion parameter, see Section 7. In Section 2 we give a short overview of the derivation of the classical Maxwell equations. As a consequence our equation (1.1) in a Lorentz frame is equivalent to

$$\begin{aligned} \tfrac{1}{\mathrm{c}^2}\partial_t F_0 - \operatorname{div}_x(F-D) &= \varrho\,, \\ -\partial_t(F+D) + \operatorname{div}_x(M^s + \mathcal{R}(H)) &= \mathrm{j}\,. \end{aligned} \tag{1.3}$$

This system is a coupling of the laws of Newton and of Ampére, and in a Lorentz frame the first scalar equation is the time version of Newton's gravitational law combined with Gauss's law and the second vector equation is Ampére's circuital law combined with a law, which determines the portion $\mathrm{j}^s$ of $\mathrm{j} = \mathrm{j}^s + \mathrm{j}^a$ which corresponds to the symmetric part of the equation. The splitting in symmetric and antisymmetric part is done in Section 3, the symmetric part covers gravity, whereas the antisymmetric part covers in particular electricity. The model will be presented in Section 3 and Section 4. The latter contains the differential equations in a Lorentz frame.

In Section 6 we deal with the existence of the scalar potential and the vector potential, and we derive as an equivalent version of (1.3) the equation

$$\begin{aligned} \frac{1}{\mathrm{c}^2}\partial_t^2\boldsymbol{\phi} + \operatorname{div}_x\big(-\nabla_x\boldsymbol{\phi} + P\big) &= \varrho\,, \\ \frac{1}{\bar{\mu}_0\mathrm{c}^2}\partial_t^2 A - \partial_t P + \operatorname{div}_x\big(M_0 - \frac{1}{\bar{\mu}_0}\mathrm{D}_x A\big) &= \mathrm{j}\,. \end{aligned} \tag{1.4}$$

Here $\boldsymbol{\phi} = \phi^g + \bar{\varepsilon}_0\phi^e$ is the total scalar potential and $A$ is the classical vector potential. The classical gravitational potential is $\phi^g$ and $\phi^e$ is the classical electrodynamical potential.

The polarization $P$ is $kg$-based and $M_0$, see (6.3), is a matrix which is partly determined by magnetization and gravitation. We mention that during the derivation of this equivalent system we had to satisfy the Lorenz gauge condition.

In Section 5 we deal with the so-called Newton-Lorentz force, that is the force density $\underline{\mathbf{f}}_{NL}$ by which $\mathfrak{M}$ acts on the mass-momentum equation. Without polarization and magnetization it is

$$\begin{aligned}\underline{\mathbf{f}}_{NL} &= \mathfrak{g}\mathrm{G}\big(\mathfrak{F}\,\mathrm{j}^s + \widetilde{\mathfrak{H}}\,\mathrm{j}^a\big) \\ &= \mathfrak{g}\begin{bmatrix} 0 \\ \varrho^s F + \bar{\varepsilon}_0(\varrho^a E + \mathrm{j}^a \times B)\end{bmatrix} + \frac{\mathfrak{g}}{\mathrm{c}^2}\begin{bmatrix} -\varrho^s F_0 - F\bullet\mathrm{j}^s + \bar{\varepsilon}_0 E\bullet\mathrm{j}^a \\ M^s\mathrm{j}^s\end{bmatrix}.\end{aligned} \tag{1.5}$$

The first term on the right side contains the well known classical terms as they are used e.g. in MHD. The formula (1.5) is shown via the identity

$$\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\big(\mathfrak{M}\mathrm{G}^{-1}\mathfrak{M}\big)^{\mathrm{S}} - \lambda_{\mathfrak{M}}\mathrm{G}\Big) = \mathrm{G}\big(\mathfrak{F}\,\widetilde{\mathrm{j}}^s + \widetilde{\mathfrak{H}}\,\widetilde{\mathrm{j}}^a\big) \tag{1.6}$$

where $\mathfrak{M}$ is the original matrix in (1.1).

*References:* There are no direct references, but historically there has been a lot of effort to combine the gravitational effects with the electrodynamical effects, for example, the work of Cartan and Einstein circa 1924-1934. We mention the contributions of Vargas & Torr in [12] and of Goenner in [6] and [7], where a detailed historical review is given. Another approach was created by Heaviside in [8] 1893, where the gravitational quantities are defined in analogy with the quantities in Maxwell's theory. This method is called Gravitoelectromagnetism (GEM), see [11].

*Notation:* We denote terms in spacetime $\mathbb{R}^4$ with an underscore which are usually used in space $\mathbb{R}^3$ only. For example in Lorentz frames we write $\underline{q} = (q_0, q)$, and also

$$\underline{\nabla} u = (\partial_t u, \nabla_x u)\,, \quad \underline{\mathrm{div}}\,\underline{q} = (\partial_t q_0,\, \mathrm{div}_x q)\,.$$

For a tensor $T = (T_{ij})_{i,j\ge 0}$ we define

$$\underline{\mathrm{div}}\, T = \Big(\sum_{j\ge 0}\partial_j T_{ij}\Big)_{i\ge 0}.$$

*Definition:* The definition of a contravariant $m$-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m\ge 0}$ is

$$T_{k_1\cdots k_m}\circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_m\ge 0} Y_{k_1}{}'_{\bar{k}_1}\cdots Y_{k_m}{}'_{\bar{k}_m} T^*_{\bar{k}_1\cdots\bar{k}_m}\,, \tag{1.7}$$

and the definition of a covariant $m$-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m\ge 0}$

$$T^*_{\bar{k}_1\cdots\bar{k}_m} = \sum_{k_1,\ldots,k_m\ge 0} Y_{k_1}{}'_{\bar{k}_1}\cdots Y_{k_m}{}'_{\bar{k}_m} T_{k_1\cdots k_m}\circ Y\,. \tag{1.8}$$

Here $y = Y(y^*)$ is the observer transformation, where the coordinates of the different observers are $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ and $y^* = (y^*_0, y^*_1, y^*_2, y^*_3) \in \mathbb{R}^4$. A 0-tensor is a function $f$ satisfying $f^* = f\circ Y$.

## 2 Maxwell equations

In this section we give a short description of the derivation of Maxwell equations in order to show the reader some of the methods which we used in other sections.

Maxwell's equations are given by two things, by "Ampère's circuital law", which we present as a conservation law (2.5), and by "Faraday's law of induction", which we formulate as a constitutive equation (2.7). The result is that Maxwell's equations are equivalent to four differential equations. In a Lorentz frame they are given by (2.1) and (2.3)

$$\begin{aligned} \mathrm{div}_x \boldsymbol{D} &= \boldsymbol{\varrho}\,, \\ -\partial_t \boldsymbol{D} + \mathrm{rot}\,_x \boldsymbol{H} &= \boldsymbol{j}\,, \end{aligned} \tag{2.1}$$

$$\boldsymbol{D} = \varepsilon_0 E + \boldsymbol{P}\,, \quad \boldsymbol{H} = \frac{1}{\mu_0} B - \boldsymbol{M}\,, \quad \varepsilon_0 \mu_0 = \frac{1}{\mathrm{c}^2}\,, \tag{2.2}$$

$$\begin{aligned} \mathrm{div}_x B &= 0\,, \\ \partial_t B + \mathrm{rot}\,_x E &= 0\,. \end{aligned} \tag{2.3}$$

Here $\boldsymbol{\varrho}$ is the charge density and $\boldsymbol{j}$ the current density, and $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{P}$, $\boldsymbol{M}$, $B$, and $E$ are the well-known fields in electrodynamics. (We mention that the boldface quantities are $\boldsymbol{D} = \bar{D}$ and $\boldsymbol{\varrho} = \bar{\varrho}^a$ etc. where the overlined quantities are $As$-based in the general theory. The other quantities are the same in the general theory.) Without polarization $\boldsymbol{P}$ and without magnetization $\boldsymbol{M}$ this is how Maxwell equations usually are presented

$$\begin{aligned} \mathrm{div}_x E &= \frac{\boldsymbol{\varrho}}{\varepsilon_0}\,, \\ -\frac{1}{\mathrm{c}^2}\partial_t E + \mathrm{rot}\,_x B &= \mu_0 \boldsymbol{j}\,, \\ \mathrm{div}_x B &= 0\,, \\ \partial_t B + \mathrm{rot}\,_x E &= 0\,. \end{aligned} \tag{2.4}$$

We have introduced Maxwell equations in a different but equivalent way (see [1 : VI 2]). In a general frame we start (see e.g. Duvaut & Lions [3 : VII 2]) with a conservation law in $\mathbb{R}^4$

$$\begin{aligned} \underline{\mathrm{div}}\,\boldsymbol{\mathfrak{H}} &= \mathbf{j}\,, \\ \boldsymbol{\mathfrak{H}} : \mathbb{R}^4 &\to \mathbb{R}^{4\times4} \text{ antisymmetric.} \end{aligned} \tag{2.5}$$

Here $\boldsymbol{\mathfrak{H}}$ is an antisymmetric contravariant tensor and $\mathbf{j}$ is a contravariant vector due to the fact that $\boldsymbol{\mathfrak{H}}$ is antisymmetric. Also due to the antisymmetry of $\boldsymbol{\mathfrak{H}}$ the vector $\mathbf{j}$ satisfies the conservation of charge (see 3.3)

$$\underline{\mathrm{div}}\,\mathbf{j} = 0\,.$$

In a Lorentz frame we have the representation

$$\boldsymbol{\mathfrak{H}} = \begin{bmatrix} 0 & \boldsymbol{D}_1 & \boldsymbol{D}_2 & \boldsymbol{D}_3 \\ -\boldsymbol{D}_1 & 0 & \boldsymbol{H}_3 & -\boldsymbol{H}_2 \\ -\boldsymbol{D}_2 & -\boldsymbol{H}_3 & 0 & \boldsymbol{H}_1 \\ -\boldsymbol{D}_3 & \boldsymbol{H}_2 & -\boldsymbol{H}_1 & 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} \boldsymbol{\varrho} \\ \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \\ \boldsymbol{j}_3 \end{bmatrix},$$

so that (2.5) means, since $\underline{\mathrm{div}} = (\partial_t, \mathrm{div}_x)$, that the first two differential equations in (2.1) are satisfied. The conservation of charge becomes $\partial_t \boldsymbol{\varrho} + \mathrm{div}_x \boldsymbol{j} = 0$. Now, in a general frame, we write down the "Maxwell-Lorentz aether relations"

$$\boldsymbol{\mathfrak{H}} = \frac{1}{\mu_0} \mathrm{G} \mathfrak{E} \mathrm{G}^{\mathrm{T}} - \boldsymbol{\mathfrak{P}}\,, \tag{2.6}$$

where $\mathfrak{E}$ and $\boldsymbol{\mathfrak{P}}$ again are antisymmetric, and together with $\boldsymbol{\mathfrak{P}}$ is a contravariant tensor. Consequently $\mathfrak{E}$ is a covariant tensor. In a Lorentz frame we have the representation

$$\mathfrak{E} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}, \quad \boldsymbol{\mathfrak{P}} = \begin{bmatrix} 0 & -\boldsymbol{P}_1 & -\boldsymbol{P}_2 & -\boldsymbol{P}_3 \\ \boldsymbol{P}_1 & 0 & \boldsymbol{M}_3 & -\boldsymbol{M}_2 \\ \boldsymbol{P}_2 & -\boldsymbol{M}_3 & 0 & \boldsymbol{M}_1 \\ \boldsymbol{P}_3 & \boldsymbol{M}_2 & -\boldsymbol{M}_1 & 0 \end{bmatrix}.$$

This implies the equivalence of (2.6) and (2.2). To get the second two differential equations of (2.3) we assume (see Landau & Lifschitz [10 : §26 (26.5)]) Faraday's law of induction

$$\partial_l \mathfrak{E}_{jk} + \partial_j \mathfrak{E}_{kl} + \partial_k \mathfrak{E}_{lj} = 0 \quad \text{for } j,k,l = 0,\dots,3. \tag{2.7}$$

Since $\mathfrak{E}$ is antisymmetric, this is relevant only for $\{j,k,l\}$ equal to $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$, and $\{1,2,3\}$. Therefore this is equivalent to four real differential equations. In a Lorentz frame this means, see the proof of 4.3, the equations in (2.3) hold, that is

$$\mathrm{div}_x B = 0\,, \quad \partial_t B + \mathrm{rot}\,_x E = 0\,.$$

Altogether we have shown that (2.5), (2.6) and (2.7) in a Lorentz frame are equivalent to Maxwell equations (2.1), (2.2), (2.3).

The potential $\underline{A}$ is a consequence of these equations. In fact, it follows that Faraday's law (2.7) by the Poincaré lemma is equivalent to the local existence of a vector potential $\underline{A} = (-\phi^e, A_1, A_2, A_3)$, where $\phi^e$ is the electrical potential. The connection to $E$ and $B$, see Section 6, is given by

$$E = -\nabla_x \phi^e - \partial_t A\,, \quad B = \mathrm{rot}\,_x A\,.$$

**Remark:** To compare this with the remaining paper, you have to write boldface quantities as overlined quantities, like $\boldsymbol{\mathfrak{H}} = \bar{\mathfrak{H}}$ and $\boldsymbol{D} = \bar{D}$ etc. as mentioned above. And on the right sides you have to write $\boldsymbol{\varrho} = \bar{\varrho}^a$ and $\boldsymbol{j} = \bar{\mathrm{j}}^a$ as the overlined version of the antisymmetric part. The other quantities $B$, $E$, $\varepsilon_0$, $\mu_0$ stay the same throughout this paper.

A detailed presentation of this derivation you will find in [1 : VI 2 Maxwell equations]. For more details about the vector potential see Section 6 of this paper.

# 3 General Magnetodynamics

We want to describe the gravitational and the electrical fields in a unified way including magnetic materials. So we write down a law for a general $4 \times 4$-matrix, which is a system of four differential equations usually valid in the entire spacetime $\mathbb{R}^4$:

**General law of Magnetodynamics**:

$$\underline{\operatorname{div}}\, \mathfrak{M} = \widetilde{\mathfrak{j}} \tag{3.1}$$

where the test functions are covariant vectors.

We call $\mathfrak{M} = (\mathfrak{M}_{ij})_{i,j=0,\dots,3}$ the ***general magnetic tensor***. For the special case of an antisymmetric matrix this is Ampère's circuital law as presented in Section 2. Written in components the general system reads

$$\sum_{j\geq 0} \partial_{y_j} \mathfrak{M}_{ij} = \widetilde{\mathfrak{j}}_i \quad \text{for } i = 0, \dots, 3\,,$$

where $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ are the coordinates. Here $\mathbb{R}^4$ is called the spacetime, but at the moment it is not said what is the time and what is the space. For this the matrix G is introduced, that is, G describes the geometry and depends on the observer, see for example [2 : 3 Time and space]. The weak formulation of equation (3.1) is

$$\int_{\mathbb{R}^4} \Big( \sum_{i,j\geq 0} \partial_{y_j} \zeta_i \cdot \mathfrak{M}_{ij} + \sum_{i\geq 0} \zeta_i \cdot \widetilde{\mathfrak{j}}_i \Big)\, \mathrm{dL}^4 = 0 \tag{3.2}$$

for test functions $\zeta \in C_0^\infty(\mathbb{R}^4; \mathbb{R}^4)$, where $\zeta$ is a covariant vector. Now this is true if the quantities in the system satisfy the transformation rules (see [1 : Equ.(5.8)] with $Z = \underline{D}Y$)

$$\mathfrak{M}_{ij} \circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}} Y_{j\,'\bar{j}}\, \mathfrak{M}^*_{\bar{i}\,\bar{j}}\,, \tag{3.3}$$

$$\widetilde{\mathfrak{j}}_i \circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}\,\bar{j}}\, \mathfrak{M}^*_{\bar{i}\,\bar{j}} + \sum_{\bar{i}\geq 0} Y_{i\,'\bar{i}}\, \widetilde{\mathfrak{j}}^*_{\bar{i}}\,, \tag{3.4}$$

where $Y : \mathbb{R}^4 \to \mathbb{R}^4$ is the observer transformation. In particular $\mathfrak{M}$ is a contravariant tensor. By a general method applied to the system here, the transformation rule (3.4) implies the following lemma.

**3.1 Lemma.** The equation (3.4) gives rise to the following representation

$$\widetilde{\mathfrak{j}} = \mathfrak{j} + \mathfrak{j}_{\mathfrak{M}}\,, \quad \mathfrak{j}_{\mathfrak{M}} := \sum_{p,q\geq 0} \mathfrak{M}_{pq} \mathfrak{C}^{pq} \tag{3.5}$$

where the vectors $\mathfrak{C}^{pq}$ satisfy the transformation rule

$$\sum_{p,q} Y_{p\,'\bar{p}} Y_{q\,'\bar{q}} \mathfrak{C}^{pq}_i \circ Y = \sum_{\bar{i}} Y_{i\,'\bar{i}} \mathfrak{C}^{*\bar{p}\bar{q}}_{\bar{i}} + Y_{i\,'\bar{p}\bar{q}} \quad \text{for all } i \text{ and } (\bar{p}, \bar{q}). \tag{3.6}$$

Thus (3.4) and (3.6) leads to the fact that $\mathfrak{j}$ is a contravariant vector. Besides this $\mathfrak{M}$ is a contravariant tensor and system (3.1) becomes

$$\underline{\operatorname{div}}\,\mathfrak{M} - \sum_{p,q\geq 0} \mathfrak{M}_{pq}\mathfrak{C}^{pq} = \mathfrak{j}\,. \tag{3.7}$$

Here $\mathfrak{C}^{pq}$ we call ***Coriolis coefficients***. *Remark:* These coefficients are defined for all observers in (3.6). It is a physical statement that for certain observers all coefficients $\mathfrak{C}^{pq}_k$ are 0. These special observers are supposed to be in an "inertial frame".

We note that in differential geometry the same rule (3.6) is satisfied for the negative Christoffel symbols. See also [2 : 6 Momentum equation].

**Symmetric and antisymmetric part.**
The tensor $\mathfrak{M}$ is split into the symmetric part $\mathfrak{M}^{\mathrm{S}}$ and the antisymmetric one $\mathfrak{M}^{\mathrm{A}}$:

$$\mathfrak{M} = \mathfrak{M}^{\mathrm{S}} + \mathfrak{M}^{\mathrm{A}}\,,$$
$$\mathfrak{M}^{\mathrm{S}} = \frac{1}{2}\big(\mathfrak{M} + \mathfrak{M}^{\mathrm{T}}\big)\,, \quad \mathfrak{M}^{\mathrm{A}} = \frac{1}{2}\big(\mathfrak{M} - \mathfrak{M}^{\mathrm{T}}\big)\,.$$

It follows from (3.3) that ${\mathfrak{M}^{\mathrm{T}}}_{ij}\circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}}Y_{j\,'\bar{j}}\,{\mathfrak{M}^{*\mathrm{T}}}_{\bar{i}\bar{j}}$, and therefore besides $\mathfrak{M}$ both $\mathfrak{M}^{\mathrm{S}}$ and $\mathfrak{M}^{\mathrm{A}}$ satisfy the transformation rule (3.3). Let us define (the indices 's' and 'a' belong to the name, whereas the indices 'T','S','A' define a mathematical operation)

$$\widetilde{\mathfrak{j}}^{s} := \underline{\operatorname{div}}\,\mathfrak{M}^{\mathrm{S}}\,, \quad \widetilde{\mathfrak{j}}^{a} := \underline{\operatorname{div}}\,\mathfrak{M}^{\mathrm{A}} \quad \Longrightarrow \quad \widetilde{\mathfrak{j}} = \widetilde{\mathfrak{j}}^{s} + \widetilde{\mathfrak{j}}^{a}\,. \tag{3.8}$$

Later, the symmetric part will correspond to gravitation and the antisymmetric part will cover electrodynamics.

**3.2 Lemma.** The definitions in (3.8) imply the transformation formulas

$$\widetilde{\mathfrak{j}}^{s}_{i}\circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}\bar{j}}\,{\mathfrak{M}^{*\mathrm{S}}}_{\bar{i}\bar{j}} + \sum_{\bar{i}\geq 0} Y_{i\,'\bar{i}}\,\widetilde{\mathfrak{j}}^{s*}_{\bar{i}}\,, \tag{3.9}$$

$$\widetilde{\mathfrak{j}}^{a}_{i}\circ Y = \sum_{\bar{i}\geq 0} Y_{i\,'\bar{i}}\,\widetilde{\mathfrak{j}}^{a*}_{\bar{i}}\,. \tag{3.10}$$

They lead to the fact that

$$\mathfrak{j}^{s} := \widetilde{\mathfrak{j}}^{s} - \sum_{p,q\geq 0} {\mathfrak{M}^{\mathrm{S}}}_{pq}\mathfrak{C}^{pq} \quad \text{and} \quad \mathfrak{j}^{a} := \widetilde{\mathfrak{j}}^{a}$$

are contravariant vectors.

*Proof.* Since $\mathfrak{M}^{\mathrm{S}}$ satisfies (3.3), i.e. ${\mathfrak{M}^{\mathrm{S}}}_{ij}\circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}}Y_{j\,'\bar{j}}{\mathfrak{M}^{*\mathrm{S}}}_{\bar{i}\bar{j}}$, the definition of $\widetilde{\mathfrak{j}}^{s} := \underline{\operatorname{div}}\,\mathfrak{M}^{\mathrm{S}}$ implies (we refer to [1 : Section I.5], see [1 : I.5.3]) that it satisfies the transformation rule

$$\widetilde{\mathfrak{j}}^{s}_{i}\circ Y = \sum_{\bar{i},\bar{j}\geq 0} Y_{i\,'\bar{i}\bar{j}}{\mathfrak{M}^{*\mathrm{S}}}_{\bar{i}\bar{j}} + \sum_{\bar{i}\geq 0} Y_{i\,'\bar{i}}\,\widetilde{\mathfrak{j}}^{s*}_{\bar{i}}\,.$$

Similar $\mathfrak{M}^{\mathrm{A}}$ also satisfies (3.3), i.e. $\mathfrak{M}^{\mathrm{A}}{}_{ij} \circ Y = \sum_{\bar{i},\bar{j}\ge 0} Y_{i'\bar{i}} Y_{j'\bar{j}} \mathfrak{M}^{*\mathrm{A}}{}_{\bar{i}\bar{j}}$, and the definition $\widetilde{\mathrm{j}}^a := \underline{\mathrm{div}}\,\mathfrak{M}^{\mathrm{A}}$ implies

$$\widetilde{\mathrm{j}}^a_i \circ Y = \sum_{\bar{i},\bar{j}\ge 0} Y_{i'\bar{i}\bar{j}} \mathfrak{M}^{*\mathrm{A}}{}_{\bar{i}\bar{j}} + \sum_{\bar{i}\ge 0} Y_{i'\bar{i}}\, \widetilde{\mathrm{j}}^{a*}_{\bar{i}} = \sum_{\bar{i}\ge 0} Y_{i'\bar{i}}\, \widetilde{\mathrm{j}}^{a*}_{\bar{i}} \,,$$

we see that $\mathrm{j}^s$ and $\mathrm{j}^a$ are contravariant vectors. □

It implies that $\mathrm{j}^a$ satisfies the “charge conservation”.

**3.3 Charge conservation.** For the charge density $\mathrm{j}^a$ the conservation law

$$\boxed{\underline{\mathrm{div}}\,\mathrm{j}^a = 0}$$

applies. The test functions of this equation are objective scalars.

*Proof.* The definition $\mathrm{j}^a := \underline{\mathrm{div}}\,\mathfrak{M}^{\mathrm{A}}$ says that in its weak version

$$\int_{\mathbb{R}^4} \Big( \sum_{kl} \partial_l \zeta_k \, \mathfrak{M}^{\mathrm{A}}{}_{kl} + \sum_k \zeta_k \, \mathrm{j}^a_k \Big) \, \mathrm{dL}^4 = 0 \,.$$

The transformation formulas in 3.2, that is (3.3) for $\mathfrak{M}^{\mathrm{A}}$ and (3.10) for $\mathrm{j}^a$, imply that this equation holds for test functions $\zeta$ that are covariant vectors (that is the statement of [1 : Property I.5.2]). We set $\zeta_k = \partial_k \eta$ with a scalar function $\eta$ (the required transformation rule for $\zeta$ follows from $\eta^* = \eta \circ Y$). Then it is

$$\int_{\mathbb{R}^4} \Big( \sum_{kl} \partial_{lk}\eta \, \mathfrak{M}^{\mathrm{A}}{}_{kl} + \sum_k \partial_k \eta \, \mathrm{j}^a_k \Big) \, \mathrm{dL}^4 = 0 \,.$$

Since $\mathfrak{M}^{\mathrm{A}}$ is antisymmetric and $\mathrm{D}^2\eta$ symmetric, the first summand vanishes. Hence

$$\int_{\mathbb{R}^4} \Big( \sum_k \partial_k \eta \, \mathrm{j}^a_k \Big) \, \mathrm{dL}^4 = 0 \,,$$

the assertion. □

Therefore we obtain mainly as symmetric part “Newton's gravitation law” and as antisymmetric part “Ampère's circuital law”:

$$\boxed{\begin{array}{c} \mathfrak{M} = \mathfrak{M}^{\mathrm{S}} + \mathfrak{M}^{\mathrm{A}} = \widetilde{\mathfrak{F}} + \mathfrak{H} \\ \hline \text{Ampère's circuital law:} \quad \underline{\mathrm{div}}\mathfrak{H} = \mathrm{j}^a \\ \mathfrak{H} \text{ is always antisymmetric, and here } \mathfrak{H} = \mathfrak{M}^{\mathrm{A}} \\ \text{Newton's gravitation law:} \quad \underline{\mathrm{div}}\widetilde{\mathfrak{F}} = \widetilde{\mathrm{j}}^s \\ \widetilde{\mathfrak{F}} \text{ contains the symmetric part, and here } \widetilde{\mathfrak{F}} = \mathfrak{M}^{\mathrm{S}} \end{array}} \qquad (3.11)$$

The gravitational part may also contain some antisymmetric contribution, at least it seems so. In any case, the combination of the electrical part with gravitation makes it necessary to combine the $kg$-based gravitation field with the $As$-based electrodynamics. This in detail is carried out in Section 7. For the form of $\mathfrak{H}$ and $\widetilde{\mathfrak{F}}$ in a Lorentz frame see Section 4.

**Representation as a covariant tensor.**

To transform the contravariant tensor $\mathfrak{M}$ into a covariant tensor $\mathfrak{N}$ is done by the identity

$$\boxed{\mathfrak{M} = \mathrm{c}^2 \mathrm{G} \mathfrak{N} \mathrm{G}^{\mathrm{T}} .} \tag{3.12}$$

In fact it holds

**3.4 Lemma.** The matrix $\mathfrak{N}$ in (3.12) is a covariant tensor, that is

$$\mathfrak{N}^* = \mathrm{D}Y^{\mathrm{T}} \mathfrak{N} \circ Y \, \mathrm{D}Y .$$

*Proof.* It holds $\mathrm{G}^{*-1} = \mathrm{D}Y^{\mathrm{T}} \mathrm{G}^{-1} \circ Y \, \mathrm{D}Y$ and $\mathfrak{N} = \frac{1}{\mathrm{c}^2} \mathrm{G}^{-1} \mathfrak{M} \mathrm{G}^{-\mathrm{T}}$. From this it follows

$$\begin{aligned} \mathrm{c}^2 \mathfrak{N}^* &= (\mathrm{G}^*)^{-1} \mathfrak{M}^* (\mathrm{G}^*)^{-\mathrm{T}} = \mathrm{D}Y^{\mathrm{T}} \mathrm{G}^{-1} \circ Y \, \mathrm{D}Y \mathfrak{M}^* \mathrm{D}Y^{\mathrm{T}} \mathrm{G}^{-\mathrm{T}} \circ Y \, \mathrm{D}Y \\ &= \mathrm{D}Y^{\mathrm{T}} \mathrm{G}^{-1} \circ Y \mathfrak{M} \circ Y \mathrm{G}^{-\mathrm{T}} \circ Y \, \mathrm{D}Y = \mathrm{c}^2 \mathrm{D}Y^{\mathrm{T}} \mathfrak{N} \circ Y \, \mathrm{D}Y . \end{aligned}$$

Consequently $\mathfrak{N}$ is a covariant tensor. □

The representation $\mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H}$ from (3.11) yields, because of (3.12),

$$\boxed{\begin{gathered} \mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H} \quad \Longleftrightarrow \quad \mathfrak{N} = \mathfrak{F} + \widetilde{\mathfrak{H}} , \\ \widetilde{\mathfrak{F}} = \mathrm{c}^2 \mathrm{G} \mathfrak{F} \mathrm{G}^{\mathrm{T}} , \quad \mathfrak{H} = \mathrm{c}^2 \mathrm{G} \widetilde{\mathfrak{H}} \mathrm{G}^{\mathrm{T}} , \end{gathered}} \tag{3.13}$$

which defines the quantities $\mathfrak{F}$ and $\widetilde{\mathfrak{H}}$. The electrical quantities $\mathfrak{E}$ are defined by (cf. 2)

$$\boxed{\begin{gathered} \textit{Maxwell-Lorentz aether relation:} \\ \mathfrak{H} = \tfrac{1}{\bar{\mu}_0} \mathrm{G} \mathfrak{E} \mathrm{G}^{\mathrm{T}} - \mathfrak{P} \quad \text{or equivalent} \quad \widetilde{\mathfrak{H}} = \bar{\varepsilon}_o \mathfrak{E} - \widetilde{\mathfrak{P}} \\ \big( \bar{\varepsilon}_0 \bar{\mu}_0 \mathrm{c}^2 = 1 \big) \\ \hline \mathfrak{E} \text{ contains electric field and magnetic flux,} \\ \mathfrak{P} = \mathrm{c}^2 \mathrm{G} \widetilde{\mathfrak{P}} \mathrm{G}^{\mathrm{T}} \text{ contains polarization and magnetization.} \end{gathered}} \tag{3.14}$$

For the form of $\mathfrak{H}$ and $\widetilde{\mathfrak{F}}$ in a Lorentz frame see Section 4, also for $\mathfrak{E}$ and $\mathfrak{P}$.

# 4 The system in Lorentz frames

We describe an observer who has the standard matrix (with $G$ in italic style we denote the gravitational constant)

$$\mathrm{G} = \mathrm{G_c} := \begin{bmatrix} -\frac{1}{\mathrm{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$$

and corresponding coordinates $(t, x) = y \in \mathbb{R}^4$. Such a situation we call Lorentz frame. We mention that working only in Lorentz frames one can not deal with arbitrary observer transformations, and we focus on the fact that we assume that the Coriolis coefficients in (3.7) are zero. In a forthcoming paper we will treat the case of general frames.

In a Lorentz frame let us write

$$\mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H}\,, \quad \widetilde{\mathfrak{F}} =: \begin{bmatrix} \frac{F_0}{\mathrm{c}^2} & -F^{\mathrm{T}} \\ -F & M^s \end{bmatrix}, \quad \mathfrak{H} =: \begin{bmatrix} 0 & D^{\mathrm{T}} \\ -D & \mathcal{R}(H) \end{bmatrix}, \tag{4.1}$$

where for a 3-vector $q$

$$\mathcal{R}(q) := \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix}, \tag{4.2}$$

hence $\mathcal{R}(q)$ is the matrix which satisfies

$$\mathcal{R}(q)z = z \times q \text{ for } z \in \mathbb{R}^3\,. \tag{4.3}$$

Therefore for a vector field $q : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathrm{div}_x \mathcal{R}(q) = \mathrm{rot}\,_x q\,. \tag{4.4}$$

holds. Further we define

$$\begin{bmatrix} \varrho \\ \mathrm{j} \end{bmatrix} := \mathrm{j} = \mathrm{j}^s + \mathrm{j}^a, \quad \mathrm{j}^s =: \begin{bmatrix} \varrho^s \\ \mathrm{j}^s \end{bmatrix}, \quad \mathrm{j}^a =: \begin{bmatrix} \varrho^a \\ \mathrm{j}^a \end{bmatrix}, \tag{4.5}$$

where usually $\varrho = \varrho^s + \varrho^a > 0$ and $\varrho^s > 0$. The general system (3.1) then becomes

***General equations in Lorentz frame:***

$$\mathfrak{M} = \begin{bmatrix} \frac{F_0}{\mathrm{c}^2} & -(F - D)^{\mathrm{T}} \\ -(F + D) & M^s + \mathcal{R}(H) \end{bmatrix},$$
$$\frac{1}{\mathrm{c}^2}\partial_t F_0 - \mathrm{div}_x(F - D) = \varrho\,,$$
$$-\partial_t(F + D) + \mathrm{div}_x(M^s + \mathcal{R}(H)) = \mathrm{j}\,. \tag{4.6}$$

All quantities $\underline{F} := (F_0, F)$, $F := (F_1, F_2, F_3)$, and $M^s$ (if symmetric), $D$, $H$ are uniquely determined by $\mathfrak{M}$.

It is $\mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H}$, where $\widetilde{\mathfrak{F}}$ and $\mathfrak{H}$ are given in (4.1), and by (3.12) this leads to $\mathfrak{N} = \mathfrak{F} + \widetilde{\mathfrak{H}}$, see (3.13), with

$$\mathfrak{F} = \begin{bmatrix} F_0 & F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M^s \end{bmatrix}, \qquad \widetilde{\mathfrak{H}} = \begin{bmatrix} 0 & -D^{\mathrm{T}} \\ D & \frac{1}{\mathrm{c}^2}\mathcal{R}(H) \end{bmatrix}, \tag{4.7}$$

where $\underline{F} = (F_0, F_1, F_2, F_3)$ is the first row of $\mathfrak{F}$. Further, $M^s$ is in particular a symmetric matrix but generally it is an arbitrary matrix (see system (4.13) without electricity).

*Let the following remark be allowed:* If $M^s = F_0\mathrm{Id}$, then

$$\mathrm{j}^s = -\partial_t F + \mathrm{div}_x M^s = -\partial_t F + \nabla_x F_0 \,.$$

Moreover, if $F$ is a 4-gradient, i.e. $\underline{F} = \underline{\nabla}\phi$, then it applies $\partial_t F = \nabla_x F_0$ hence $\mathrm{j}^s = 0$.

**Magnetic and electric case.**

We consider the general antisymmetric situation, that is, the magnetic and electric case with matrix $\mathfrak{H}$ satisfying (equivalent to (2.6))

***General Maxwell-Lorentz aether relations:***

$$\widetilde{\mathfrak{H}} = \bar{\varepsilon}_o \mathfrak{E} - \widetilde{\mathfrak{P}} \quad \text{or} \quad \mathfrak{H} = \frac{1}{\bar{\mu}_0}\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}} - \mathfrak{P} \quad \text{with} \quad \bar{\varepsilon}_0\bar{\mu}_0\mathrm{c}^2 = 1 \tag{4.8}$$

$$\mathfrak{E} =: \begin{bmatrix} 0 & -E^{\mathrm{T}} \\ E & \mathcal{R}(B) \end{bmatrix}, \quad \widetilde{\mathfrak{P}} =: \begin{bmatrix} 0 & -P^{\mathrm{T}} \\ P & \mathcal{R}(M^a) \end{bmatrix}, \quad \mathfrak{P} := \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{P}}\mathrm{G}^{\mathrm{T}},$$

which means the following well-known formulas

***Maxwell-Lorentz aether relations:***

$$D = \bar{\varepsilon}_0 E + P\,, \quad H = \frac{1}{\bar{\mu}_0} B - M^a\,, \quad \bar{\varepsilon}_0\bar{\mu}_0\mathrm{c}^2 = 1\,. \tag{4.9}$$

There are four fields in these relations, two are determined by $\mathfrak{E}$, which are the electric field $E$ and the magnetic flux density $B$, and two $kg$-based fields are determined by $\mathfrak{P}$, which are the polarization $P$ and the magnetization $M^a$.

**4.1 Lemma.** Show that from (4.1) the equations (4.7), (4.8) and (4.9) follow.

*Proof.* It is for $\mathfrak{F}$ and $\widetilde{\mathfrak{H}}$

$$\begin{aligned} \mathfrak{F} &= \frac{1}{\mathrm{c}^2}\mathrm{G_c}^{-1}\widetilde{\mathfrak{F}}\mathrm{G_c}^{-\mathrm{T}} \\ &= \frac{1}{\mathrm{c}^2}\begin{bmatrix} -\mathrm{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}\begin{bmatrix} \frac{F_0}{\mathrm{c}^2} & -F^{\mathrm{T}} \\ -F & M^s \end{bmatrix}\begin{bmatrix} -\mathrm{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix} = \begin{bmatrix} F_0 & F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2}M^s \end{bmatrix}, \\ \widetilde{\mathfrak{H}} &= \frac{1}{\mathrm{c}^2}\mathrm{G_c}^{-1}\mathfrak{H}\mathrm{G_c}^{-\mathrm{T}} \\ &= \frac{1}{\mathrm{c}^2}\begin{bmatrix} -\mathrm{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}\begin{bmatrix} 0 & D^{\mathrm{T}} \\ -D & \mathcal{R}(H) \end{bmatrix}\begin{bmatrix} -\mathrm{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix} = \begin{bmatrix} 0 & -D^{\mathrm{T}} \\ D & \frac{1}{\mathrm{c}^2}\mathcal{R}(H) \end{bmatrix}. \end{aligned}$$

The definition $\widetilde{\mathfrak{H}} = \bar{\varepsilon}_0\mathfrak{E} - \widetilde{\mathfrak{P}}$ implies in all frames since $\bar{\varepsilon}_0\bar{\mu}_0\mathrm{c}^2 = 1$

$$\mathfrak{H} = \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{H}}\mathrm{G}^{\mathrm{T}} = \bar{\varepsilon}_0\mathrm{c}^2\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}} - \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{P}}\mathrm{G}^{\mathrm{T}} = \frac{1}{\bar{\mu}_0}\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}} - \mathfrak{P}$$

by $\mathfrak{P} = \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{P}}\mathrm{G}^{\mathrm{T}}$. This, by the way, gives in Lorentz frames

$$\widetilde{\mathfrak{P}} = \frac{1}{\mathrm{c}^2}\mathrm{G_c}^{-1}\mathfrak{P}\mathrm{G_c}^{-\mathrm{T}} = \begin{bmatrix} 0 & P^{\mathrm{T}} \\ -P & \frac{1}{\mathrm{c}^2}\mathcal{R}(M^a) \end{bmatrix}.$$

Writing down the relation $\widetilde{\mathfrak{H}} = \bar{\varepsilon}_0\mathfrak{E} - \widetilde{\mathfrak{P}}$, that is

$$\begin{bmatrix} 0 & -D^{\mathrm{T}} \\ D & \frac{1}{\mathrm{c}^2}\mathcal{R}(H) \end{bmatrix} = \begin{bmatrix} 0 & -\bar{\varepsilon}_0 E^{\mathrm{T}} \\ \bar{\varepsilon}_0 E & \bar{\varepsilon}_0\mathcal{R}(B) \end{bmatrix} - \begin{bmatrix} 0 & P^{\mathrm{T}} \\ -P & \frac{1}{\mathrm{c}^2}\mathcal{R}(M^a) \end{bmatrix},$$

gives the Maxwell-Lorentz aether relations. □

**Gravity and electric case.**

Now conditions are stated for $\mathfrak{F}$ and $\mathfrak{E}$, which imply the local existence of physical potentials. For both fields these conditions are experimentally verified.

**4.2 Assumption.** The following is assumed for $\mathfrak{F}$ and $\mathfrak{E}$:

(1) For $\mathfrak{F}$ defined in (4.7) it is required

$$\boxed{\partial_i \underline{F}_j = \partial_j \underline{F}_i \text{ for } i, j \geq 0.} \tag{4.10}$$

This implies that locally there is a ***Newtonian gravitational potential*** $\phi^g$ (according to the lemma of Poincaré), that is, $\underline{F}_i = \partial_i\phi^g$ for $i \geq 0$, or

$$\underline{F} = \underline{\nabla}\phi^g .$$

(2) Let for $\mathfrak{E}$ in (4.8) ***Faraday's law of induction*** in the version (see [10: §26 (26,5)])

$$\boxed{\partial_l\mathfrak{E}_{jk} + \partial_j\mathfrak{E}_{kl} + \partial_k\mathfrak{E}_{lj} = 0 \text{ for } j, k, l \geq 0} \tag{4.11}$$

be satisfied. Then locally there is a vector potential $\underline{A}$ (according to the lemma of Poincaré) such that

$$\mathfrak{E} = \underline{\nabla}\,\underline{A} - (\underline{\nabla}\,\underline{A})^{\mathrm{T}},$$

which also can be written as $\mathfrak{E} = (\underline{\mathrm{D}}\,\underline{A})^{\mathrm{T}} - \underline{\mathrm{D}}\,\underline{A}$. See also (6.1).

These conditions are necessary for the proof in Section 5.

**4.3 Remark.** The induction law of Faraday, that is equation (4.11), means in a Lorentz frame that

$$\mathrm{div}B = 0\,, \quad \partial_t B + \mathrm{rot}_x E = 0\,.$$

*Proof.* There are at most four equations of (4.11) which are independent of each other. They are

$$\begin{aligned}
(j,k,l) = (0,1,2): &\quad -\partial_2 E_1 + \partial_0 B_3 + \partial_1 E_2 = 0\,,\\
(j,k,l) = (0,1,3): &\quad -\partial_3 E_1 - \partial_0 B_2 + \partial_1 E_3 = 0\,,\\
(j,k,l) = (0,2,3): &\quad -\partial_3 E_2 + \partial_0 B_1 + \partial_2 E_3 = 0\,,\\
(j,k,l) = (1,2,3): &\quad \partial_3 B_3 + \partial_1 B_1 + \partial_2 B_2 = 0\,.
\end{aligned}$$

This is equivalent to the statement. □

**4.4 Remark.** It should be said that one has the following possibilities for assumptions on $\mathfrak{F}$ which contain Newton's principle 4.2(1).

(1) One can demand from $\mathfrak{F}$ that

$$\partial_j \mathfrak{F}_{ki} = \partial_i \mathfrak{F}_{kj} \text{ for } i,j,k = 0,1,2,3. \tag{4.12}$$

This would (according to the lemma of Poincaré) ensure the local existence of a vector potential $\Psi$, i.e. $\mathfrak{F}_{ki} = \partial_i \Psi_k$ for $i,k = 0,1,2,3$, hence

$$\mathfrak{F} = \underline{\mathrm{D}}\Psi\,.$$

This satisfies 4.2(1) with the potential $\phi^g := \Psi_0$.

(2) The property (4.12), with the assumption that $\mathfrak{F}$ is symmetric, is used in the proof of Section 5 to show the formula 5.9 for the gravitational case. Besides this it follows from (4.12) that locally a vector potential $\Psi$ exists with $\mathfrak{F}_{ki} = \partial_i \Psi_k$. This symmetry then implies that $\partial_0 \Psi_k = \partial_k \phi^g$ for all $k \geq 1$. Here $\phi^g := \Psi_0$. This seems to be an assumption on gravity in the absence of magnetism.

(3) Assuming that there is a vector potential $\Psi$ with

$$\mathfrak{F} = \frac{1}{2}\big(\underline{\mathrm{D}}\Psi + (\underline{\mathrm{D}}\Psi)^{\mathrm{T}}\big)$$

then $\mathfrak{F}$ is automatically symmetric. If we assume $\partial_0 \Psi_j = \partial_j \Psi_0$ for $j \geq 1$, i.e. the symmetry of the first row to the first column (as in (4.7)), then it follows

$$\underline{F}_j = \mathfrak{F}_{0j} = \partial_j \Psi_0 = \partial_j \phi^g \qquad (\text{with } \phi^g := \Psi_0)$$

that is the Newtonian potential in 4.2(1). However under this assumption the proof of in Section 5 is not done.

**Gravity and magnetism.**

Without considering the electrical phenomena the equations in (4.6) have the form

$$\begin{aligned}
&\frac{1}{\mathrm{c}^2}\partial_t F_0 - \mathrm{div}_x(F - P) = \varrho\,,\\
&-\partial_t(F + P) + \mathrm{div}_x M = \mathrm{j}\,, \quad M := M^s - \mathcal{R}(M^a)\,.
\end{aligned} \tag{4.13}$$

These are the equations in the pure magnetic case together with gravitation, i.e. omitting electricity. The matrix $M := M^s - \mathcal{R}(M^a)$ seems to play the special role. This could be relevant if one e.g. considers magnetic domains (Weiss domains).

# 5 Newton-Lorentz force

We now are in a general frame and study the forces that $\mathfrak{M}$ exerts in the mass-momentum equation

$$\underline{\mathrm{div}}(\varrho \underline{v}\underline{v}^{\mathrm{T}} + \underline{\Pi}) = \widetilde{\underline{\mathbf{f}}}\,, \tag{5.1}$$

The mass-momentum equation has the property that the test function is a covariant vector. This is the case (see [1 : Equ. (5.8)] with $Z = \underline{\mathrm{D}}Y$) if the $4\times 4$ tensor $T := \varrho\underline{v}\underline{v}^{\mathrm{T}} + \underline{\Pi}$ is a contravariant tensor, i.e. $T$ and the 4-force $\widetilde{\underline{\mathbf{f}}}$ fulfill the transformation formulas

$$\begin{aligned} T_{ij}\circ Y &= \sum_{\bar{i},\bar{j}\geq 0} Y_{i'\bar{i}} Y_{j'\bar{j}}\, T^*_{\bar{i}\bar{j}}\,, \\ \widetilde{\underline{\mathbf{f}}}_i\circ Y &= \sum_{\bar{i},\bar{j}\geq 0} Y_{i'\bar{i}\bar{j}}\, T^*_{\bar{i}\bar{j}} + \sum_{\bar{i}\geq 0} Y_{i'\bar{i}}\, \widetilde{\underline{\mathbf{f}}}^*_{\bar{i}}\,. \end{aligned} \tag{5.2}$$

Here $\varrho$ is an objective scalar, and the 4-velocity $\underline{v}$ is a contravariant vector, i.e.

$$\underline{v}_i\circ Y = \sum_{\bar{i}\geq 0} Y_{i'\bar{i}}\, \underline{v}^*_{\bar{i}}\,,$$

so that $\varrho\underline{v}\,\underline{v}^{\mathrm{T}}$, as it should be, is a contravariant tensor. We mention, that in a Lorentz frame the 4-velocity is $\underline{v} = (1, v)$ and therefore the 4-momentum is $\varrho\underline{v} = (\varrho, \varrho v)$, similar to equation (4.5) which is $\underline{\mathrm{j}} = (\varrho, \mathrm{j})$. In the general frame we write for the 4-force $\underline{\mathbf{f}}$ the following representation

$$\widetilde{\underline{\mathbf{f}}} = \underline{\mathbf{f}}_{NL} + \widetilde{\underline{\mathbf{f}}}_0\,, \tag{5.3}$$

where $\underline{\mathbf{f}}_{NL}$ consists of the Newton-Lorentz forces, which are those forces that interest us. Here $\widetilde{\underline{\mathbf{f}}}_0$ has the same transformation formula as $\widetilde{\underline{\mathbf{f}}}$, and thus contains the fictitious forces (for example, the term with Coriolis forces), and $\underline{\mathbf{f}}_{NL}$ is a contravariant vector.

**5.1 Classical Newton-Lorentz force.** The classical version is given by

$$\underline{\mathbf{f}}_{NL} = \begin{bmatrix} \mathcal{O}(\frac{1}{\mathrm{c}^2}) \\ \mathbf{f}_{NL} \end{bmatrix} \quad \text{with}$$

$$\mathbf{f}_{NL} = \mathfrak{g}\varrho^s F + \bar{\varrho}^a E + \bar{\mathrm{j}}^a \times B + \mathcal{O}(\frac{1}{\mathrm{c}^2})\,.$$

Here the terms are as they occur in classical formulas, in particular in Magnetohydrodynamics (MHD), see for example [5 : 4 The MHD model], where because of the identity $\varepsilon_0\mu_0\mathrm{c}^2 = 1$ usually also $\bar{\varrho}^a E$ is neglected.

*Proof.* Here $\mathfrak{g} = 4\pi G$ and $G = 6.67384\cdot 10^{-11}\frac{m^3}{kg\,s^2}$ is the gravitational constant. In $\mathbf{f}_{NL}$ the factor in front of the electrodynamical terms is set to $\mathbf{1}$, which is due to be in accordance with the literature, see e.g. [5 : 4 The MHD model, Equ.(4.7)]. Later we will see that this means $\mathfrak{g}\mathrm{k}_0^2\varepsilon_0 = \mathbf{1}$, which should be satisfied. Further, according to $\bar{\varrho}^a$ and $\bar{\mathrm{j}}^a$ we have from

Section 7 the following dimensions

$$\begin{aligned}
&\bar\varrho^a \left[\frac{As}{m^3}\right], \quad E\left[\frac{kg\cdot m}{As\cdot s^2}\right] \quad \Longrightarrow \quad \bar\varrho^a E\left[\frac{kg}{m^2 s^2}\right],\\
&\bar{\mathrm{j}}^a \left[\frac{A}{m^2}\right], \quad B\left[\frac{kg}{As^2}\right] \quad \Longrightarrow \quad \bar{\mathrm{j}}^a\times B\left[\frac{kg}{m^2 s^2}\right],\\
&\mathfrak{g}\left[\frac{m^3}{kg\cdot s^2}\right], \quad \varrho^s\left[\frac{kg}{m^3}\right], \quad F\left[\frac{kg}{m^2}\right] \quad \Longrightarrow \quad \mathfrak{g}\varrho^s F\left[\frac{kg}{m^2 s^2}\right].
\end{aligned}$$

Here $\bar\varrho^a$ is the charge density and $\bar{\mathrm{j}}^a$ the current density, both based on $As$. The $kg$-based quantities are $\varrho^a = \mathrm{k}_0\bar\varrho^a$ and $\mathrm{j}^a = \mathrm{k}_0\bar{\mathrm{j}}^a$, see 7.3 and 7.4. The classical result in this form also shows up in 5.2. □

**5.2 Relativistic Newton-Lorentz force.** Without magnetization and polarization the relativistic version is given by

$$\boxed{\underline{\mathbf{f}}_{NL} = \mathfrak{g}\mathrm{G}\big(\mathfrak{F}\,\mathrm{j}^s + \widetilde{\mathfrak{H}}\,\mathrm{j}^a\big)\,.} \tag{5.4}$$

This is the 4-force which arises in 5.3. In a Lorentz frame this is equal to the formula

$$\underline{\mathbf{f}}_{NL} = \mathfrak{g}\begin{bmatrix} -\dfrac{\varrho^s}{\mathrm{c}^2}F_0 - \dfrac{1}{\mathrm{c}^2}F\bullet\mathrm{j}^s \\ \varrho^s F + \dfrac{1}{\mathrm{c}^2}M^s\mathrm{j}^s\end{bmatrix} + \mathfrak{g}\bar\varepsilon_0\begin{bmatrix}\dfrac{1}{\mathrm{c}^2}E\bullet\mathrm{j}^a \\ \varrho^a E + \mathrm{j}^a\times B\end{bmatrix}$$

if $\mathfrak{F}$ satisfies (5.10) and if $\widetilde{\mathfrak{H}} = \bar\varepsilon_0\mathfrak{E}$ satisfies 4.2(2). Equivalently we write

$$\underline{\mathbf{f}}_{NL} = \mathfrak{g}\begin{bmatrix} 0 \\ \varrho^s F + \bar\varepsilon_0(\varrho^a E + \mathrm{j}^a\times B)\end{bmatrix} + \frac{\mathfrak{g}}{\mathrm{c}^2}\begin{bmatrix} -\varrho^s F_0 - F\bullet\mathrm{j}^s + \bar\varepsilon_0 E\bullet\mathrm{j}^a \\ M^s\mathrm{j}^s\end{bmatrix},$$

what you can compare with the formula in 5.1.

*Proof.* Equation (5.4) follows from Theorem 5.3, which has been shown under the assumption that $\mathfrak{F}$ satisfies (5.10), which implies the local existence of Newton's gravitational potential in 4.2(1), and $\widetilde{\mathfrak{H}} = \bar\varepsilon_0\mathfrak{E}$ satisfies Faraday's induction principle 4.2(2), which gives the existence of $B$ and $E$ in 4.3. In (4.7) we defined

$$\mathfrak{F} = \begin{bmatrix} F_0 & F^{\mathrm{T}} \\ F & \dfrac{1}{\mathrm{c}^2}M^s\end{bmatrix}, \qquad \widetilde{\mathfrak{H}} = \begin{bmatrix} 0 & -D^{\mathrm{T}} \\ D & \dfrac{1}{\mathrm{c}^2}\mathcal{R}(H)\end{bmatrix},$$

so that $\underline{\mathbf{f}}_{NL} = \mathfrak{g}\big(\mathrm{G}\mathfrak{F}\,\mathrm{j}^s + \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a\big)$ implies

$$\begin{aligned}
\underline{\mathbf{f}}_{NL} &= \mathfrak{g}\begin{bmatrix} -\dfrac{1}{\mathrm{c}^2}F_0 & -\dfrac{1}{\mathrm{c}^2}F^{\mathrm{T}} \\ F & \dfrac{1}{\mathrm{c}^2}M^s\end{bmatrix}\begin{bmatrix}\varrho^s \\ \mathrm{j}^s\end{bmatrix} + \mathfrak{g}\begin{bmatrix} 0 & \dfrac{1}{\mathrm{c}^2}D^{\mathrm{T}} \\ D & \dfrac{1}{\mathrm{c}^2}\mathcal{R}(H)\end{bmatrix}\begin{bmatrix}\varrho^a \\ \mathrm{j}^a\end{bmatrix}\\
&= \mathfrak{g}\begin{bmatrix} -\dfrac{\varrho^s}{\mathrm{c}^2}F_0 - \dfrac{1}{\mathrm{c}^2}F\bullet\mathrm{j}^s \\ \varrho^s F + \dfrac{1}{\mathrm{c}^2}M^s\mathrm{j}^s\end{bmatrix} + \mathfrak{g}\begin{bmatrix}\dfrac{1}{\mathrm{c}^2}D\bullet\mathrm{j}^a \\ \varrho^a D + \dfrac{1}{\mathrm{c}^2}\mathrm{j}^a\times H\end{bmatrix}
\end{aligned}$$

since $\mathcal{R}(H)\mathrm{j}^a = \mathrm{j}^a \times H$. Now, if neglecting polarization and magnetization $\widetilde{\mathfrak{H}} = \bar{\varepsilon}_0 \mathfrak{E}$ or $D = \bar{\varepsilon}_0 E$ and $B = \bar{\mu}_0 H$ by (4.9), we obtain finally

$$\mathfrak{g}\begin{bmatrix} \frac{1}{\mathrm{c}^2} D\bullet \mathrm{j}^a \\ \varrho^a D + \frac{1}{\mathrm{c}^2}\mathrm{j}^a \times H \end{bmatrix} = \mathfrak{g}\begin{bmatrix} \frac{\bar{\varepsilon}_0}{\mathrm{c}^2} E\bullet \mathrm{j}^a \\ \bar{\varepsilon}_0 \varrho^a E + \frac{1}{\bar{\mu}_0 \mathrm{c}^2}\mathrm{j}^a \times B \end{bmatrix} = \mathfrak{g}\bar{\varepsilon}_0 \begin{bmatrix} \frac{1}{\mathrm{c}^2} E\bullet \mathrm{j}^a \\ \varrho^a E + \mathrm{j}^a \times B \end{bmatrix}.$$

This is the assertion. To compare it with 5.1 we use $\varrho^a = \mathrm{k}_0 \bar{\varrho}^a$ and $\mathrm{j}^a = \mathrm{k}_0 \bar{\mathrm{j}}^a$, and set $\mathbf{1} = \mathfrak{g}\bar{\varepsilon}_0 \mathrm{k}_0 = \mathfrak{g}\mathrm{k}_0^2 \varepsilon_0$, if this is compatible with measurements. □

We want to regard the Newton-Lorentz force $\underline{\mathbf{f}}_{NL}$ as an "internal force", i.e. we want to write this force $\underline{\mathbf{f}}_{NL}$ with a term $\underline{\Pi}_{NL}$ under the 4-divergence, that is,

$$\underline{\mathrm{div}}\, \underline{\Pi}_{NL} + \underline{\mathbf{f}}_{NL} = \widetilde{\underline{\mathbf{f}}}_1 , \tag{5.5}$$

where $\underline{\Pi}_{NL}$ is a contravariant tensor and the vector $\widetilde{\underline{\mathbf{f}}}_1$ denotes fictitious forces. Thus (5.1) becomes the differential equation

$$\underline{\mathrm{div}}(\varrho \underline{v}\underline{v}^{\mathrm{T}} + \underline{\Pi}_{NL} + \underline{\Pi}) = \widetilde{\underline{\mathbf{f}}}_1 + \widetilde{\underline{\mathbf{f}}}_0 , \tag{5.6}$$

with general force $\widetilde{\underline{\mathbf{f}}}_1 + \widetilde{\underline{\mathbf{f}}}_0$ containing fictitious terms.

We now derive the form of the force $\underline{\mathbf{f}}_{NL}$ which is given in (5.4). We do this through the proof of (5.5) with a general formula of the matrix $\underline{\Pi}_{NL}$. This result is the main theorem of this section. We assume that we are in a Lorentz frame and that magnetization as well as polarization is set to zero. We summarize the general formulas for the matrix $\mathfrak{M}$

$$\begin{gathered} \mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H} , \quad \widetilde{\mathfrak{F}} = \mathrm{c}^2 \mathrm{G}\mathfrak{F}\mathrm{G}^{\mathrm{T}} , \quad \mathfrak{H} = \mathrm{c}^2 \mathrm{G}\widetilde{\mathfrak{H}}\mathrm{G}^{\mathrm{T}} , \\ \mathfrak{M} = \mathrm{c}^2 \mathrm{G}\mathfrak{N}\mathrm{G}^{\mathrm{T}} \quad \text{implies} \quad \mathfrak{N} = \mathfrak{F} + \widetilde{\mathfrak{H}} . \end{gathered} \tag{5.7}$$

which are important for the result.

**5.3 Theorem.** If (5.7) is true and $\widetilde{\mathfrak{H}} = \bar{\varepsilon}_0 \mathfrak{E}$ (no polarization and magnetization) then (5.5) holds with

$$\boxed{\begin{gathered} \underline{\Pi}_{NL} = \mathfrak{g}\lambda_{\mathfrak{M}} \mathrm{G} - \frac{\mathfrak{g}}{\mathrm{c}^2}(\mathfrak{M}\mathrm{G}^{-1}\mathfrak{M})^{\mathrm{S}} , \\ \underline{\mathbf{f}}_{NL} = \frac{\mathfrak{g}}{\mathrm{c}^2}\big(\widetilde{\mathfrak{F}}\mathrm{G}^{-1}\mathrm{j}^s + \mathfrak{H}\mathrm{G}^{-1}\mathrm{j}^a\big) = \mathfrak{g}\big(\mathrm{G}\mathfrak{F}\,\mathrm{j}^s + \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a\big), \\ \lambda_{\mathfrak{M}} = \lambda^s + \lambda^a , \quad \lambda^s = \frac{\mathrm{c}^2}{2}(\mathrm{G}\mathfrak{F})^{\mathrm{T}} \mathbin{:} (\mathrm{G}\mathfrak{F}) , \quad \lambda^a = \frac{\mathrm{c}^2}{4}(\mathrm{G}\widetilde{\mathfrak{H}})^{\mathrm{T}} \mathbin{:} (\mathrm{G}\widetilde{\mathfrak{H}}) . \end{gathered}}$$

This is true under the assumption that we are in the Lorentz case, and that $\mathfrak{F}$ and $\mathfrak{E}$ can be written as vector potential, see in 4.2(1) the Newtonian gravitational potential and in 4.2(2) the Faraday induction law, and that for $\mathfrak{F}$ in fact (5.10) is satisfied.

*Proof.* We have to show that

$$\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}(\mathfrak{M}\mathrm{G}^{-1}\mathfrak{M})^{\mathrm{S}} - \lambda_{\mathfrak{M}}\mathrm{G}\Big) = \mathrm{G}\mathfrak{F}\,\widetilde{\mathrm{j}}^s + \mathrm{G}\widetilde{\mathfrak{H}}\,\widetilde{\mathrm{j}}^a . \tag{5.8}$$

Now, from 3.2 we know that $\widetilde{\mathrm{j}}^a = \mathrm{j}^a$ and $\widetilde{\mathrm{j}}^s$ is $\mathrm{j}^s$ plus a term depending on the matrix $\mathfrak{M}^{\mathrm{S}}$. This additional term in (5.8) is subsumed in the term $\widetilde{\underline{\mathbf{f}}}_1$ of equation (5.5). Therefore on the right side of (5.8) the term

$$\frac{1}{\mathfrak{g}}\underline{\mathbf{f}}_{NL} := \mathrm{G}\mathfrak{F}\,\mathrm{j}^s + \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a$$

is leftover, and therefore (5.5) is verified. To prove (5.8) we compute

$$\begin{aligned}&\frac{1}{\mathrm{c}^2}(\mathfrak{M}\mathrm{G}^{-1}\mathfrak{M})^{\mathrm{S}} = \frac{1}{2\mathrm{c}^2}\big(\mathfrak{M}\mathrm{G}^{-1}\mathfrak{M} + \mathfrak{M}^{\mathrm{T}}\mathrm{G}^{-1}\mathfrak{M}^{\mathrm{T}}\big)\\ &= \frac{1}{2\mathrm{c}^2}\big((\mathfrak{M}^{\mathrm{S}} + \mathfrak{M}^{\mathrm{A}})\mathrm{G}^{-1}(\mathfrak{M}^{\mathrm{S}} + \mathfrak{M}^{\mathrm{A}}) + (\mathfrak{M}^{\mathrm{S}} - \mathfrak{M}^{\mathrm{A}})\mathrm{G}^{-1}(\mathfrak{M}^{\mathrm{S}} - \mathfrak{M}^{\mathrm{A}})\big)\\ &= \frac{1}{\mathrm{c}^2}\big(\mathfrak{M}^{\mathrm{S}}\mathrm{G}^{-1}\mathfrak{M}^{\mathrm{S}} + \mathfrak{M}^{\mathrm{A}}\mathrm{G}^{-1}\mathfrak{M}^{\mathrm{A}}\big)\,.\end{aligned}$$

Therefore it is enough to show

$$\begin{aligned}\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\mathfrak{M}^{\mathrm{S}}\mathrm{G}^{-1}\mathfrak{M}^{\mathrm{S}} - \lambda^s\mathrm{G}\Big) &= \mathrm{G}\mathfrak{F}\,\widetilde{\mathrm{j}}^s\,,\\ \underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\mathfrak{M}^{\mathrm{A}}\mathrm{G}^{-1}\mathfrak{M}^{\mathrm{A}} - \lambda^a\mathrm{G}\Big) &= \mathrm{G}\widetilde{\mathfrak{H}}\,\widetilde{\mathrm{j}}^a\,.\end{aligned}$$

The first equation is shown in (5.11), and the second in (5.13). □

That these are quite general formulas is shown in the following lemma, which we will use in the second proof of the electric part.

**5.4 General lemma.** It holds for each matrix $\underline{M}$

$$\begin{aligned}\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\underline{M}\mathrm{G}^{-1}\underline{M}\Big) &= \mathrm{G}N\widetilde{\mathrm{j}} + \mathscr{R}(N)\,,\\ \mathscr{R}_i(N) := \mathrm{c}^2\sum_{jk}\big(\partial_j(\mathrm{G}N)_{ik}\big)(\mathrm{G}N\mathrm{G})_{kj} &= \sum_{jk}\big(\partial_j(\mathrm{G}N)_{ik}\big)\underline{M}_{kj}\,,\end{aligned}$$

where the matrix $N$ is defined by $\underline{M} := \mathrm{c}^2\mathrm{G}N\mathrm{G}^{\mathrm{T}}$, and $\widetilde{\mathrm{j}} := \underline{\mathrm{div}}\,\underline{M}$.
*Remark:* Therefore only the remainder $\mathscr{R}$ has to be computed.

*Proof.* Due to the product rule for the $i$-th component it applies

$$\begin{aligned}&\sum_j \partial_j\Big(\frac{1}{\mathrm{c}^2}\underline{M}\mathrm{G}^{-1}\underline{M}\Big)_{ij} = \frac{1}{\mathrm{c}^2}\sum_{jk}\partial_j\big((\underline{M}\mathrm{G}^{-1})_{ik}\underline{M}_{kj}\big)\\ &= \sum_{jk}\partial_j\big((\mathrm{G}N)_{ik}\underline{M}_{kj}\big) \quad (\text{wegen } \underline{M}\mathrm{G}^{-1} = \mathrm{c}^2\mathrm{G}N)\\ &= \sum_{jk}(\mathrm{G}N)_{ik}\partial_j\underline{M}_{kj} + \sum_{jk}\big(\partial_j(\mathrm{G}N)_{ik}\big)\underline{M}_{kj}\\ &= \sum_k(\mathrm{G}N)_{ik}(\underline{\mathrm{div}}\,\underline{M})_k + \sum_{jk}\big(\partial_j(\mathrm{G}N)_{ik}\big)\underline{M}_{kj}\,.\end{aligned}$$

That is the claim. □

We now prove the equation (5.8) separately for gravitational and for electrical phenomena.

**Gravity part:** In the gravitational case it is $\mathfrak{M} = \widetilde{\mathfrak{F}} + \mathfrak{H}$ with

$$\widetilde{\mathfrak{F}} = \mathrm{c}^2 \mathrm{G} \mathfrak{F} G^{\mathrm{T}}\,, \quad \mathfrak{F} = (\mathfrak{F}_{ij})_{ij} = \begin{bmatrix} F_0 & F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M^s \end{bmatrix},$$

where $\underline{F} = (F_0, F)$ and $F = (F_1, F_2, F_3)$. In the following $M := M^s$ is a matrix that later is assumed to be symmetric.

**5.5 Lemma.** With $\widetilde{\mathfrak{M}} := \widetilde{\mathfrak{F}}$ we get

$$\frac{1}{\mathrm{c}^2} \widetilde{\mathfrak{M}} \mathrm{G}^{-1} \widetilde{\mathfrak{M}} = \begin{bmatrix} -\frac{F_0^2}{\mathrm{c}^4} & \frac{F_0 F^{\mathrm{T}}}{\mathrm{c}^2} \\ \frac{F_0 F}{\mathrm{c}^2} & -F F^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \frac{|F|^2}{\mathrm{c}^2} & -\frac{F^{\mathrm{T}} M}{\mathrm{c}^2} \\ -\frac{M F}{\mathrm{c}^2} & \frac{M^2}{\mathrm{c}^2} \end{bmatrix}.$$

*Proof.* It is

$$\mathrm{G}\mathfrak{F} = \begin{bmatrix} -\frac{1}{\mathrm{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} F_0 & F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{c}^2} F_0 & -\frac{1}{\mathrm{c}^2} F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M \end{bmatrix},$$

and therefore

$$\begin{aligned} \widetilde{\mathfrak{M}} = \mathrm{c}^2 \mathrm{G} \mathfrak{F} \mathrm{G} &= \mathrm{c}^2 \begin{bmatrix} -\frac{1}{\mathrm{c}^2} F_0 & -\frac{1}{\mathrm{c}^2} F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M \end{bmatrix} \begin{bmatrix} -\frac{1}{\mathrm{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \\ &= \mathrm{c}^2 \begin{bmatrix} \frac{1}{\mathrm{c}^4} F_0 & -\frac{1}{\mathrm{c}^2} F^{\mathrm{T}} \\ -\frac{1}{\mathrm{c}^2} F & \frac{1}{\mathrm{c}^2} M \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathrm{c}^2} F_0 & -F^{\mathrm{T}} \\ -F & M \end{bmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{1}{\mathrm{c}^2} \widetilde{\mathfrak{M}} \mathrm{G}^{-1} \widetilde{\mathfrak{M}} = (\mathrm{G}\mathfrak{F}) \widetilde{\mathfrak{M}} &= \begin{bmatrix} -\frac{1}{\mathrm{c}^2} F_0 & -\frac{1}{\mathrm{c}^2} F^{\mathrm{T}} \\ F & \frac{1}{\mathrm{c}^2} M \end{bmatrix} \begin{bmatrix} \frac{1}{\mathrm{c}^2} F_0 & -F^{\mathrm{T}} \\ -F & M \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\mathrm{c}^4} F_0^2 + \frac{1}{\mathrm{c}^2} |F|^2 & \frac{1}{\mathrm{c}^2} F_0 F^{\mathrm{T}} - \frac{1}{\mathrm{c}^2} F^{\mathrm{T}} M \\ \frac{1}{\mathrm{c}^2} F_0 F - \frac{1}{\mathrm{c}^2} M F & -F F^{\mathrm{T}} + \frac{1}{\mathrm{c}^2} M^2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{F_0^2}{\mathrm{c}^4} & \frac{F_0 F^{\mathrm{T}}}{\mathrm{c}^2} \\ \frac{F_0 F}{\mathrm{c}^2} & -F F^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \frac{|F|^2}{\mathrm{c}^2} & -\frac{F^{\mathrm{T}} M}{\mathrm{c}^2} \\ -\frac{M F}{\mathrm{c}^2} & \frac{M^2}{\mathrm{c}^2} \end{bmatrix}. \end{aligned}$$

□

**5.6 Lemma.** The first matrix on the right-hand side of Lemma 5.5 is

$$\begin{bmatrix} -\frac{F_0^2}{\mathrm{c}^4} & \frac{F_0 F^{\mathrm{T}}}{\mathrm{c}^2} \\ \frac{F_0 F}{\mathrm{c}^2} & -F\, F^{\mathrm{T}} \end{bmatrix} = -\mathrm{G}\underline{F} \otimes \mathrm{G}\underline{F}.$$

*Proof.* It is

$$\begin{bmatrix} \frac{1}{c^4}F_0^2 & -\frac{1}{c^2}F_0F^{\mathrm{T}} \\ -\frac{1}{c^2}F_0F & F\,F^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{c^2}F_0 \\ F \end{bmatrix} \otimes \begin{bmatrix} -\frac{1}{c^2}F_0 \\ F \end{bmatrix} = \mathrm{G}\underline{F}\otimes\mathrm{G}\underline{F}\,.$$

□

For the following lemma we assume

$$\partial_j \underline{F}_k = \partial_k \underline{F}_j \text{ for all } j,k = 0,1,2,3\,. \tag{5.9}$$

**5.7 Lemma.** If the assumption (5.9) is satisfied then

$$\underline{\mathrm{div}}(\mathrm{G}\underline{F}\otimes\mathrm{G}\underline{F} - \lambda_1^s\mathrm{G}) = -\varrho^s\,\mathrm{G}\underline{F}\,,$$
$$\text{where } \lambda_1^s := \frac{1}{2}\underline{F}\bullet\mathrm{G}\underline{F} \text{ and } \varrho^s := \underline{\mathrm{div}}(-\mathrm{G}\underline{F}).$$

*Remember:* This contains the classical formula

$$\mathrm{div}\big(F\otimes F - \lambda\mathrm{Id}\big) = -\varrho F\,,$$

where $\lambda := \frac{1}{2}|F|^2$ and $\varrho = \mathrm{div}(-F)$. Its true if $F = \nabla\phi^g$.

*Proof of the classical formula.* Because of $\partial_j F_k = \partial_k F_j$ for $j,k \geq 1$ the $k$-th component of $\mathrm{div}\big(F\otimes F\big)$ is

$$\sum_{j\geq 1}\partial_j(F\otimes F)_{kj} = \sum_{j\geq 1}\partial_j(F_kF_j) = F_k\sum_{j\geq 1}\partial_jF_j + \sum_{j\geq 1}F_j\partial_jF_k$$
$$= F_k\,\mathrm{div}F + \sum_{j\geq 1}F_j\partial_kF_j = F_k\,\mathrm{div}F + \sum_{j\geq 1}\partial_k\big(\frac{1}{2}F_j^2\big) = -\varrho F_k + \partial_k\lambda\,,$$

if $\lambda = \frac{1}{2}|F|^2$. Therefore $\mathrm{div}(F\otimes F) - \nabla\lambda = -\varrho F$. □

*Proof.* Because of Lemma 5.6 we have to show

$$\underline{\mathrm{div}}\begin{bmatrix} \frac{F_0^2}{c^4} + \frac{\lambda_1^s}{c^2} & -\frac{F_0F^{\mathrm{T}}}{c^2} \\ -\frac{F_0F}{c^2} & F\otimes F - \lambda_1^s\mathrm{Id} \end{bmatrix} = \begin{bmatrix} \frac{1}{c^2}\varrho^sF_0 \\ -\varrho^sF \end{bmatrix}.$$

It is $\lambda_1^s = \frac{1}{2}\underline{F}\bullet\mathrm{G}\underline{F} = -\frac{1}{2c^2}F_0^2 + \frac{1}{2}|F|^2$ and therefore

$$\underline{\mathrm{div}}\big(\frac{1}{c^2}F_0^2 + \lambda_1^s, -F_0F\big) = \underline{\mathrm{div}}\big(\frac{1}{2c^2}F_0^2 + \frac{1}{2}|F|^2, -F_0F\big)$$
$$= \partial_t\big(\frac{1}{2c^2}F_0^2 + \frac{1}{2}|F|^2\big) - \mathrm{div}F_0F$$
$$= \frac{1}{c^2}F_0\partial_tF_0 + F\bullet\partial_tF - F\bullet\nabla F_0 - F_0\,\mathrm{div}F \qquad (\text{it is } \partial_tF = \nabla F_0)$$
$$= F_0\big(\frac{1}{c^2}\partial_tF_0 - \mathrm{div}F\big) = \varrho^sF_0\,.$$

For the $k$-th component of $\underline{\operatorname{div}}(-\frac{1}{c^2}F_0F, F\otimes F - \lambda_1^s\mathrm{Id})$ it holds

$$\begin{aligned}
&\underline{\operatorname{div}}(-\frac{1}{c^2}F_0F_k, F_kF - \lambda_1^s\mathbf{e}_k) = -\frac{1}{c^2}\partial_t(F_0F_k) + \operatorname{div}(F_kF) - \partial_k\lambda_1^s \\
&= \underbrace{\big(-\frac{1}{c^2}\partial_tF_0 + \operatorname{div}F\big)}_{=-\varrho^s} F_k - \frac{1}{c^2}F_0 \underbrace{\partial_tF_k}_{=\partial_kF_0} + \underbrace{\nabla F_k}_{=\partial_kF} \bullet F - \partial_k\lambda_1^s \\
&= -\varrho^sF_k + \frac{1}{2}\partial_k\big(-\frac{1}{c^2}F_0^2 + |F|^2\big) - \partial_k\lambda_1^s = -\varrho^sF_k
\end{aligned}$$

by the choice of $\lambda_1^s$. □

**5.8 Lemma.** For the second matrix on the right-hand side of Lemma 5.5 we get

$$\underline{\operatorname{div}}\begin{bmatrix} |F|^2 & -F^{\mathrm{T}}M \\ -MF & M^2 \end{bmatrix} = \begin{bmatrix} -\mathrm{j}^s\bullet F \\ M\mathrm{j}^s \end{bmatrix} + \sum_{i\geq 1}\begin{bmatrix} F_i\partial_tF_i - \sum_{j\geq 1} M_{ij}\partial_{x_j}F_i \\ \Big(-\partial_tM_{ki}\cdot F_i + \sum_{j\geq 1}\partial_{x_j}M_{ki}\cdot M_{ij}\Big)_{k\geq 1} \end{bmatrix},$$

where $\mathrm{j}^s := -\partial_tF + \operatorname{div}M$.

*Proof.* The upper line of the 4-divergence is

$$\begin{aligned}
&\underline{\operatorname{div}}\,[\,|F|^2,\ -F^{\mathrm{T}}M\,] = \partial_t(|F|^2) - \operatorname{div}(M^{\mathrm{T}}F) \\
&= \sum_l \Big(\partial_t(F_l^2) - \sum_j \partial_{x_j}(M_{lj}F_l)\Big) \\
&= \sum_l \Big(\big(\partial_tF_l - \sum_j \partial_{x_j}M_{lj}\big)F_l + F_l\partial_tF_l - \sum_j M_{lj}\partial_{x_j}F_l\Big) \\
&= -\mathrm{j}^s\bullet F + \sum_l \Big(F_l\partial_tF_l - \sum_j M_{lj}\partial_{x_j}F_l\Big),
\end{aligned}$$

and the $k$-th component of $\underline{\operatorname{div}}\,[\,-MF\,,\ M^2\,]$ is

$$\begin{aligned}
&= \sum_l \Big(-\partial_t(M_{kl}F_l) + \sum_j \partial_{x_j}(M_{kl}M_{lj})\Big) \\
&= \sum_l M_{kl}\big(-\partial_tF_l + \sum_j \partial_{x_j}M_{lj}\big) + \sum_l \big(-\partial_tM_{kl}\cdot F_l + \sum_j \partial_{x_j}M_{kl}\cdot M_{lj}\big) \\
&= M\mathrm{j}^s + \sum_l \big(-\partial_tM_{kl}\cdot F_l + \sum_j \partial_{x_j}M_{kl}\cdot M_{lj}\big).
\end{aligned}$$

□

We now make the assumption

$$\partial_j\mathfrak{F}_{kl} = \partial_k\mathfrak{F}_{jl} \text{ for all } j,k,l = 0,1,2,3. \tag{5.10}$$

This for $l = 0$ is the condition (5.9) where $\mathfrak{F}_{0l} = F_l$.

**5.9 Lemma.** If $\mathfrak{F}$ is symmetric and (5.10) applies, it follows

$$\sum_{l\geq 1}\begin{bmatrix} F_l\partial_t F_l - \sum_{j\geq 1} M_{lj}\partial_{x_j} F_l \\ \Big( -\partial_t M_{kl}\cdot F_l + \sum_{j\geq 1}\partial_{x_j} M_{kl}\cdot M_{lj}\Big)_{k\geq 1}\end{bmatrix} = \mathrm{c}^4\,\underline{\mathrm{div}}(\lambda_2^s \mathrm{G})$$

with

$$\lambda_2^s := \frac{1}{2}\sum_{l\geq 1}\Big(-\frac{|\mathfrak{F}_{0l}|^2}{\mathrm{c}^2} + \sum_{j\geq 1}|\mathfrak{F}_{jl}|^2\Big) = \frac{1}{2}\sum_{l\geq 1}\left(\mathfrak{F}_{kl}\right)_{k\geq 0}\bullet \mathrm{G}\left(\mathfrak{F}_{kl}\right)_{k\geq 0}\,.$$

*Proof.* Let $\mathfrak{F} = (F_{kl})_{k,l}$, then

$$\begin{aligned} F_l\partial_t F_l - \sum_j M_{lj}\partial_{x_j}F_l &= F_{0l}\partial_t F_{0l} - \mathrm{c}^2\sum_j F_{lj}\partial_{x_j}F_{0l} \\ &= F_{0l}\partial_t F_{0l} - \mathrm{c}^2\sum_j F_{lj}\partial_t F_{jl} \qquad (\text{since } \partial_j F_{0l} = \partial_t F_{jl}) \\ &= \frac{1}{2}\big(\partial_t|F_{0l}|^2 - \mathrm{c}^2\sum_j \partial_t|F_{jl}|^2\big) \qquad (\text{since } F_{lj} = F_{jl}) \\ &= -\frac{\mathrm{c}^2}{2}\partial_t\Big(-\frac{1}{\mathrm{c}^2}|F_{0l}|^2 + \sum_j |F_{jl}|^2\Big)\,, \end{aligned}$$

and

$$\begin{aligned} -\partial_t M_{kl}\cdot F_l + \sum_j \partial_{x_j}M_{kl}\cdot M_{lj} &= -\mathrm{c}^2\partial_t F_{kl}\cdot F_{0l} + \mathrm{c}^4\sum_j \partial_{x_j}F_{kl}\cdot F_{lj} \\ &= -\mathrm{c}^2\partial_t F_{kl}\cdot F_{0l} + \mathrm{c}^4\sum_j \partial_{x_j}F_{kl}\cdot F_{jl} \qquad (\text{since } F_{lj} = F_{jl}) \\ &= -\mathrm{c}^2\partial_{x_k} F_{0l}\cdot F_{0l} + \mathrm{c}^4\sum_j \partial_{x_k}F_{jl}\cdot F_{jl} \qquad (\text{since } \partial_j F_{kl} = \partial_k F_{jl}) \\ &= \frac{\mathrm{c}^4}{2}\partial_{x_k}\Big(-\frac{1}{\mathrm{c}^2}|F_{0l}|^2 + \sum_j |F_{jl}|^2\Big)\,. \end{aligned}$$

Hence we obtain

$$\begin{bmatrix} F_l\partial_t F_l - \sum_j M_{jl}\partial_{x_j} F_l \\ \Big( -\partial_t M_{kl}\cdot F_l + \sum_j\partial_{x_j} M_{kl}\cdot M_{lj}\Big)_{k}\end{bmatrix} = \begin{bmatrix} -\mathrm{c}^2\partial_t \\ \mathrm{c}^4\nabla_x \end{bmatrix}\Big(-\frac{|F_{0l}|^2}{2\mathrm{c}^2} + \sum_j \frac{|F_{jl}|^2}{2}\Big)\,,$$

the assertion. □

Altogether it follows with $\widetilde{\mathfrak{M}} := \widetilde{\mathfrak{F}}$ in the Lorentz case under the assumptions made above

$$\boxed{\begin{gathered} \underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}} - \lambda^s\mathrm{G}\Big) = \mathrm{G}\mathfrak{F}\,\widetilde{\mathfrak{j}}^s\,, \\ \widetilde{\mathfrak{M}} := \widetilde{\mathfrak{F}} = \mathrm{c}^2\mathrm{G}\mathfrak{F}\mathrm{G}\,, \\ \lambda^s := \frac{\mathrm{c}^2}{2}(\mathrm{G}\mathfrak{F})^{\mathrm{T}} \,\boldsymbol{:}\, (\mathrm{G}\mathfrak{F})\,, \quad \widetilde{\mathfrak{j}}^s := \underline{\mathrm{div}}\,\widetilde{\mathfrak{M}}\,. \end{gathered}} \tag{5.11}$$

Because, according to 5.5 and 5.6, it is

$$\frac{1}{c^2}\widetilde{\mathfrak{M}}G^{-1}\widetilde{\mathfrak{M}} = -G\underline{F}\otimes G\underline{F} + \frac{1}{c^2}\begin{bmatrix} |F|^2 & -(M^sF)^{\mathrm{T}} \\ -M^sF & (M^s)^2 \end{bmatrix}$$

and, according to Lemma 5.7 and Lemma 5.8 and 5.9 it is

$$\begin{aligned}
&\underline{\mathrm{div}}\big(\frac{1}{c^2}\widetilde{\mathfrak{M}}G^{-1}\widetilde{\mathfrak{M}}\big) \\
&= -\underline{\mathrm{div}}(\lambda_1^s G) + \varrho^s\, G\underline{F} + \frac{1}{c^2}\Big(c^4\,\underline{\mathrm{div}}(\lambda_2^s G) + \begin{bmatrix} -j^s\bullet F \\ M^s j^s \end{bmatrix}\Big) \\
&= \underline{\mathrm{div}}\big((-\lambda_1^s + c^2\lambda_2^s)G\big) + \varrho^s \begin{bmatrix} -\frac{1}{c^2}F_0 \\ F \end{bmatrix} + \frac{1}{c^2}\begin{bmatrix} -j^s\bullet F \\ M^s j^s \end{bmatrix} \\
&= \underline{\mathrm{div}}\big((-\lambda_1^s + c^2\lambda_2^s)G\big) + \begin{bmatrix} -\frac{1}{c^2}(\varrho^s F_0 + j^s\bullet F) \\ \varrho^s F + \frac{M^s}{c^2} j^s \end{bmatrix}.
\end{aligned}$$

Now

$$\begin{bmatrix} -\frac{1}{c^2}(\varrho^s F_0 + j^s\bullet F) \\ \varrho^s F + \frac{M^s}{c^2} j^s \end{bmatrix} = G\begin{bmatrix} \varrho^s F_0 + j^s\bullet F \\ \varrho^s F + \frac{M^s}{c^2} j^s \end{bmatrix} = G\mathfrak{F}\widetilde{j}^s\,, \quad \widetilde{j}^s := \begin{bmatrix} \varrho^s \\ j^s \end{bmatrix},$$

and

$$\begin{aligned}
\lambda^s &:= -\lambda_1^s + c^2\lambda_2^s = -\frac{1}{2}\underline{F}\bullet G\underline{F} + \frac{c^2}{2}\sum_{l\geq 1} F_{\bullet l}\bullet G F_{\bullet l} \\
&= \frac{c^2}{2}\Big(-\frac{1}{c^2}\sum_{i,j\geq 0} F_{i0}G_{ij}F_{j0} + \sum_{i,j\geq 0\,;\,l\geq 1} F_{il}G_{ij}F_{jl}\Big) \\
&= \frac{c^2}{2}\sum_{i,j,k,l\geq 0} F_{ik}G_{kl}G_{ij}F_{jl} = \frac{c^2}{2}(\mathfrak{F}G)\mathbin{:}(G\mathfrak{F})\,.
\end{aligned}$$

This shows (5.11).

**Electric part:** In the electrical case, when no magnetization and polarization is present, we have, for the antisymmetric part of $\mathfrak{M}$,

$$\mathfrak{H} = c^2 G\widetilde{\mathfrak{H}}G\,, \quad \widetilde{\mathfrak{H}} = \bar{\varepsilon}_0\mathfrak{E}\,,$$

where $\varepsilon_0$ is the electric field constant and $\bar{\varepsilon}_0 = k_0\varepsilon_0$ the *kg*-based version, see Section 7. We have the fundamental equation $\bar{\varepsilon}_0\bar{\mu}_0 c^2 = \varepsilon_0\mu_0 c^2 = 1$, if respectively $\mu_0 = k_0\bar{\mu}_0$. In particular,

$$\mathfrak{H} =: \begin{bmatrix} 0 & D^{\mathrm{T}} \\ -D & \mathcal{R}(H) \end{bmatrix}, \quad \mathfrak{E} =: \begin{bmatrix} 0 & -E^{\mathrm{T}} \\ E & \mathcal{R}(B) \end{bmatrix},$$

$$\begin{bmatrix} 0 & -D^{\mathrm{T}} \\ D & \frac{1}{c^2}\mathcal{R}(H) \end{bmatrix} = \widetilde{\mathfrak{H}} = \bar{\varepsilon}_0\mathfrak{E} = \begin{bmatrix} 0 & -\bar{\varepsilon}_0 E^{\mathrm{T}} \\ \bar{\varepsilon}_0 E & \bar{\varepsilon}_0\mathcal{R}(B) \end{bmatrix},$$

thus the known representations $D = \bar\varepsilon_0 E$ and $B = \bar\mu_0 H$. Now we suppose that the Maxwell equations are satisfied, that is, the antisymmetric part of (3.1), which is Ampère's circuital law, and Faraday's induction law 4.2(2), which is described in the Lorentz case in 4.3, which is

$$\begin{aligned} \mathrm{div}_x D = \varrho^a \,, &\qquad -\partial_t D + \mathrm{rot}_x H = \mathrm{j}^a \,, \\ \mathrm{div}_x B = 0 \,, &\qquad \partial_t B + \mathrm{rot}_x E = 0 \,. \end{aligned}$$

Setting now

$$D = \bar\varepsilon_0 E \,, \quad H = \frac{1}{\bar\mu_0} B \,, \quad \bar\varrho^a := \frac{1}{\mathrm{k}_0}\varrho^a \,, \quad \bar{\mathrm{j}}^a := \frac{1}{\mathrm{k}_0}\mathrm{j}^a \,,$$

the Maxwell equations become

$$\begin{aligned} \varepsilon_0 \,\mathrm{div}_x E = \bar\varrho^a \,, &\qquad -\varepsilon_0 \partial_t E + \frac{1}{\mu_0}\mathrm{rot}_x B = \bar{\mathrm{j}}^a \,, \\ \mathrm{div}_x B = 0 \,, &\qquad \partial_t B + \mathrm{rot}_x E = 0 \,. \end{aligned}$$

We begin with the following theorem (see [1 : VI Theorem 7.3]), where the second formula is the Theorem of Poynting.

**5.10 Theorem.** In the Lorentz case we have

$$\begin{aligned} \bar\varrho^a E + \bar{\mathrm{j}}^a \times B &= -\partial_t\big(\varepsilon_0 E \times B\big) \\ &\quad + \mathrm{div}_x\Big(\big(\varepsilon_0 E\otimes E + \frac{1}{\mu_0} B\otimes B\big) - \big(\frac{\varepsilon_0}{2}|E|^2 + \frac{1}{2\mu_0}|B|^2\big)\mathrm{Id}\Big) \,, \\ -\bar{\mathrm{j}}^a \bullet E &= \partial_t\big(\frac{\varepsilon_0}{2}|E|^2 + \frac{1}{2\mu_0}|B|^2\big) + \mathrm{div}_x\big(\frac{1}{\mu_0} E\times B\big) \,. \end{aligned}$$

*Proof.* In the Lorentz case the Maxwell equations are

$$\begin{aligned} &(I)\ \ \mathrm{div}_x B = 0 &&(III)\ \ \varepsilon_0\,\mathrm{div}_x E = \bar\varrho^a \\ &(II)\ \ \partial_t B + \mathrm{rot}_x E = 0 &&(IV)\ \ -\varepsilon_0 \partial_t E + \frac{1}{\mu_0}\mathrm{rot}_x B = \bar{\mathrm{j}}^a \,. \end{aligned}$$

Multiply

$$\begin{aligned} &(I)\ \text{ by } \frac{1}{\mu_0} B &&(III)\ \text{ by } E \\ &(II)\ \text{ by } \times(\varepsilon_0 E) &&(IV)\ \text{ by } \times B \,, \end{aligned}$$

sum up and get the assertion. Indeed, the sum is

$$\begin{aligned} &\bar\varrho^a E + \bar{\mathrm{j}}^a \times B = -\varepsilon_0(\partial_t E)\times B + \varepsilon_0(\partial_t B)\times E \\ &+\frac{1}{\mu_0}(\mathrm{rot}_x B)\times B + \varepsilon_0(\mathrm{rot}_x E)\times E + \frac{1}{\mu_0}(\mathrm{div}_x B)B + \varepsilon_0(\mathrm{div}_x E)E \\ &= -\varepsilon_0\partial_t(E\times B) + \frac{1}{\mu_0}\,\mathrm{div}_x P_B + \varepsilon_0\,\mathrm{div}_x P_E \,, \end{aligned}$$

where 5.11(1) was used. This proves the first formula. For the second we compute

$$\partial_t\big(\frac{\varepsilon_0}{2}|E|^2+\frac{1}{2\mu_0}|B|^2\big)=E\bullet\partial_t(\varepsilon_0 E)+\frac{1}{\mu_0}B\bullet\partial_t B$$
$$=E\bullet(\frac{1}{\mu_0}\mathrm{rot}_x B-\bar{\mathrm{j}}^a)-\frac{1}{\mu_0}B\bullet\mathrm{rot}_x E=-\frac{1}{\mu_0}\,\mathrm{div}_x\big(E\times B\big)-\bar{\mathrm{j}}^a\bullet E\,.$$

This is the second formula. ☐

**5.11 Auxiliary lemma.** Let $w$ be a vector valued function.

(1) If $P_w:=w\otimes w-\frac{1}{2}|w|^2\mathrm{Id}$ then

$$\mathrm{div}P_w=(\,\mathrm{div}w)\,w+(\mathrm{rot}\,w)\times w\,.$$

(2) $\mathcal{R}(w)\mathcal{R}(w)=w\otimes w-|w|^2\mathrm{Id}$.

*Proof* (1). The $k$-th component of $\mathrm{div}P_w$ is

$$(\,\mathrm{div}P_w)_k=\sum_j\partial_j(w_k w_j)-\sum_j w_j\partial_k w_j$$
$$=w_k\sum_j\partial_j w_j+\sum_l(\partial_l w_k-\partial_k w_l)\cdot w_l$$

and the $k$-th component of $(\mathrm{rot}\,w)\times w$ is, when $(k,i,j)$ is cyclic,

$$\big((\mathrm{rot}\,w)\times w\big)_k=(\mathrm{rot}\,w)_i w_j-(\mathrm{rot}\,w)_j w_i$$
$$=(\partial_j w_k-\partial_k w_j)w_j-(\partial_k w_i-\partial_i w_k)w_i$$
$$=(\partial_j w_k-\partial_k w_j)w_j+(\partial_i w_k-\partial_k w_i)w_i=\sum_l(\partial_l w_k-\partial_k w_l)\cdot w_l\,.$$

☐

*Proof* (2).

$$\mathcal{R}(w)\mathcal{R}(w)=\begin{bmatrix}0&w_3&-w_2\\-w_3&0&w_1\\w_2&-w_1&0\end{bmatrix}^2$$
$$=\begin{bmatrix}-w_3^2-w_2^2&w_1w_2&w_1w_3\\w_1w_2&-w_3^2-w_1^2&w_2w_3\\w_1w_3&w_2w_3&-w_2^2-w_1^2\end{bmatrix}=w\otimes w-|w|^2\mathrm{Id}\,.$$

☐

We rewrite the equations in 5.10 as an equation in $\mathbb{R}^4$.

**5.12 Lemma.** From 5.10 we have

$$\begin{bmatrix}\frac{1}{\mathrm{c}^2}\bar{\mathrm{j}}^a\bullet E\\ \bar{\varrho}^a E+\bar{\mathrm{j}}^a\times B\end{bmatrix}=\frac{1}{\mu_0}\underline{\mathrm{div}}\begin{bmatrix}-\frac{\nu}{\mathrm{c}^2}&-\frac{1}{\mathrm{c}^2}(E\times B)^{\mathrm{T}}\\-\frac{1}{\mathrm{c}^2}E\times B&\frac{1}{\mathrm{c}^2}E\otimes E+B\otimes B-\nu\mathrm{Id}\end{bmatrix},$$

where $\nu:=\frac{1}{2\mathrm{c}^2}|E|^2+\frac{1}{2}|B|^2$. *Remark:* It is $\bar{\mathrm{j}}^a\times B=\mathcal{R}(B)\bar{\mathrm{j}}^a$.

*Proof.* Define $\nu$ as in the assertion. Since $\varepsilon_0\mu_0\mathrm{c}^2 = 1$ from 5.10

$$\begin{aligned}\frac{1}{\mathrm{c}^2}\bar{\mathrm{j}}^a\bullet E &= -\frac{1}{\mathrm{c}^2}\partial_t\big(\frac{\varepsilon_0}{2}|E|^2 + \frac{1}{2\mu_0}|B|^2\big) - \frac{1}{\mathrm{c}^2}\operatorname{div}_x\big(\frac{1}{\mu_0}E\times B\big)\\ &= \frac{1}{\mu_0}\partial_t\big(-\frac{1}{\mathrm{c}^2}\nu\big) - \frac{1}{\mu_0}\operatorname{div}_x\big(\frac{1}{\mathrm{c}^2}E\times B\big)\\ \bar{\varrho}^a E + \bar{\mathrm{j}}^a\times B &= \partial_t\big(-\varepsilon_0 E\times B\big) + \operatorname{div}_x\big(\varepsilon_0 P_E + P_B\big)\\ &= \frac{1}{\mu_0}\underline{\operatorname{div}}\big(-\frac{1}{\mathrm{c}^2}E\times B, \frac{1}{\mathrm{c}^2}P_E + P_B\big)\,,\end{aligned}$$

and $\frac{1}{\mathrm{c}^2}P_E + P_B = \frac{1}{\mathrm{c}^2}E\otimes E + B\otimes B - \nu\mathrm{Id}$. □

**5.13 Theorem.** It is

$$\mathrm{G}\mathfrak{E}\,\mathrm{j}^a = \frac{1}{\bar{\mu}_0}\underline{\operatorname{div}}(\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda_{\mathfrak{E}}\mathrm{G})\,.$$

Here $\lambda_{\mathfrak{E}} := \frac{1}{2}\big(\frac{1}{\mathrm{c}^2}|E|^2 - |B|^2\big)$.

*Proof.* Multiplying the result of 5.12 with $\mathrm{k}_0$ we obtain

$$\begin{bmatrix}\frac{1}{\mathrm{c}^2}\mathrm{j}^a\bullet E\\ \varrho^a E + \mathrm{j}^a\times B\end{bmatrix} = \frac{1}{\bar{\mu}_0}\underline{\operatorname{div}}\begin{bmatrix}-\frac{\nu}{\mathrm{c}^2} & -\frac{1}{\mathrm{c}^2}(E\times B)^{\mathrm{T}}\\ -\frac{1}{\mathrm{c}^2}E\times B & \frac{1}{\mathrm{c}^2}E\otimes E + B\otimes B - \nu\mathrm{Id}\end{bmatrix}. \tag{5.12}$$

The matrix is $\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda_{\mathfrak{E}}\mathrm{G}$ and the left side is $\mathrm{G}\mathfrak{E}\,\mathrm{j}^a$, as we will now show. It is

$$\mathrm{G}\mathfrak{E} = \begin{bmatrix}0 & \frac{1}{\mathrm{c}^2}E^{\mathrm{T}}\\ E & \mathcal{R}(B)\end{bmatrix}, \quad \mathrm{G}\mathfrak{E}\,\mathrm{j}^a = \mathrm{G}\mathfrak{E}\begin{bmatrix}\varrho^a\\ \mathrm{j}^a\end{bmatrix} = \begin{bmatrix}\frac{1}{\mathrm{c}^2}\mathrm{j}^a\bullet E\\ \varrho^a E + \mathrm{j}^a\times B\end{bmatrix}.$$

Moreover, since $E^{\mathrm{T}}\mathcal{R}(B) = -(\mathcal{R}(B)E)^{\mathrm{T}}$, we have

$$\begin{aligned}\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} &= \begin{bmatrix}0 & \frac{1}{\mathrm{c}^2}E^{\mathrm{T}}\\ E & \mathcal{R}(B)\end{bmatrix}\begin{bmatrix}0 & \frac{1}{\mathrm{c}^2}E^{\mathrm{T}}\\ -\frac{1}{\mathrm{c}^2}E & \mathcal{R}(B)\end{bmatrix}\\ &= \begin{bmatrix}-\frac{1}{\mathrm{c}^4}|E|^2 & -\frac{1}{\mathrm{c}^2}(\mathcal{R}(B)E)^{\mathrm{T}}\\ -\frac{1}{\mathrm{c}^2}\mathcal{R}(B)E & \frac{1}{\mathrm{c}^2}E\otimes E + \mathcal{R}(B)\mathcal{R}(B)\end{bmatrix},\end{aligned}$$

therefore with arbitrary $\lambda_{\mathfrak{E}}$

$$\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda_{\mathfrak{E}}\mathrm{G} = \begin{bmatrix}-\frac{1}{\mathrm{c}^2}\big(\frac{1}{\mathrm{c}^2}|E|^2 - \lambda_{\mathfrak{E}}\big) & -\frac{1}{\mathrm{c}^2}(\mathcal{R}(B)E)^{\mathrm{T}}\\ -\frac{1}{\mathrm{c}^2}\mathcal{R}(B)E & \frac{1}{\mathrm{c}^2}E\otimes E + B\otimes B - (\lambda_{\mathfrak{E}} + |B|^2)\mathrm{Id}\end{bmatrix}.$$

This matrix is the same as the right-hand side of (5.12) if

$$\frac{1}{\mathrm{c}^2}|E|^2 - \lambda_{\mathfrak{E}} = \nu \quad \text{and} \quad \lambda_{\mathfrak{E}} + |B|^2 = \nu\,.$$

This can be satisfied if $\lambda_{\mathfrak{E}} := \frac{1}{2}\big(\frac{1}{\mathrm{c}^2}|E|^2 - |B|^2\big)$. □

Altogether it follows with $\widetilde{\mathfrak{M}} := \mathfrak{H}$ in the Lorentz case under the assumptions made above

$$\boxed{\begin{aligned} &\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}} - \lambda^a\mathrm{G}\Big) = \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a\,,\\ &\widetilde{\mathfrak{M}} := \mathfrak{H} = \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{H}}\mathrm{G} = \mathrm{c}^2\bar{\varepsilon}_0\mathrm{G}\mathfrak{E}\mathrm{G}\,,\\ &\lambda^a = \frac{\mathrm{c}^2}{4}(\mathrm{G}\widetilde{\mathfrak{H}})^{\mathrm{T}} \,\mathbf{:}\, (\mathrm{G}\widetilde{\mathfrak{H}})\,,\quad \mathrm{j}^a = \underline{\mathrm{div}}\,\widetilde{\mathfrak{M}}\,. \end{aligned}} \tag{5.13}$$

Because it holds for every $\lambda$

$$\begin{aligned} &\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}} - \lambda\mathrm{G} = \mathrm{c}^2\bar{\varepsilon}_0^2\,\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda\mathrm{G}\\ &= \mathrm{c}^2\bar{\varepsilon}_0^2\big(\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda_{\mathfrak{E}}\mathrm{G}\big) \quad \text{if } \lambda = \mathrm{c}^2\bar{\varepsilon}_0^2\lambda_{\mathfrak{E}},\\ &= \bar{\varepsilon}_0\cdot\frac{1}{\bar{\mu}_0}\big(\mathrm{G}\mathfrak{E}\mathrm{G}\mathfrak{E}\mathrm{G} - \lambda_{\mathfrak{E}}\mathrm{G}\big)\,, \end{aligned}$$

since $\mathrm{c}^2\bar{\varepsilon}_0\bar{\mu}_0 = 1$, and therefore because of 5.13

$$\begin{aligned} &\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}} - \lambda\mathrm{G}\Big) = \bar{\varepsilon}_0\mathrm{G}\mathfrak{E}\,\mathrm{j}^a = \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a\,, \text{ if}\\ &\lambda = \mathrm{c}^2\bar{\varepsilon}_0^2\lambda_{\mathfrak{E}} = \frac{\mathrm{c}^2}{2}\Big(\frac{1}{\mathrm{c}^2}|\bar{\varepsilon}_0 E|^2 - |\bar{\varepsilon}_0 B|^2\Big) = \frac{\mathrm{c}^2}{2}\Big(\frac{|D|^2}{\mathrm{c}^2} - \Big|\frac{H}{\mathrm{c}^2}\Big|^2\Big)\\ &= \frac{\mathrm{c}^2}{4}\begin{bmatrix} 0 & D^{\mathrm{T}}\\ \frac{1}{\mathrm{c}^2}D & -\mathcal{R}(\frac{1}{\mathrm{c}^2}H)\end{bmatrix} \,\mathbf{:}\, \begin{bmatrix} 0 & \frac{1}{\mathrm{c}^2}D^{\mathrm{T}}\\ D & \mathcal{R}(\frac{1}{\mathrm{c}^2}H)\end{bmatrix} = \frac{\mathrm{c}^2}{4}(\mathrm{G}\widetilde{\mathfrak{H}})^{\mathrm{T}} \,\mathbf{:}\, (\mathrm{G}\widetilde{\mathfrak{H}}) = \lambda^a\,. \end{aligned}$$

**Independent proof in electrical case.**
With $\widetilde{\mathfrak{M}} := \mathfrak{M}^{\mathrm{A}} = \mathfrak{H} = \mathrm{c}^2\mathrm{G}\widetilde{\mathfrak{H}}\mathrm{G}^{\mathrm{T}}$ applies to the antisymmetric part due to 5.4

$$\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}}\Big) = \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a + \mathscr{R}(\widetilde{\mathfrak{H}})\,.$$

If we set in the remaining $N := \widetilde{\mathfrak{H}}$ so we apply

**5.14 Theorem.** If $N = (N_{kl})_{k,l\geq 0}$ is antisymmetric and satisfies

$$\partial_j N_{kl} + \partial_k N_{lj} + \partial_l N_{jk} = 0 \text{ for all } j,k,l \geq 0,$$

then it holds in theorem 5.4

$$\mathscr{R}(N) = \underline{\mathrm{div}}(\nu\mathrm{G})\,,\quad \nu = \frac{\mathrm{c}^2}{4}(\mathrm{G}N)^{\mathrm{T}} \,\mathbf{:}\, (\mathrm{G}N)\,.$$

The assumptions for $N$ are equivalent to those for $\mathfrak{E}$ in 4.2, thus

$$\underline{\mathrm{div}}\Big(\frac{1}{\mathrm{c}^2}\widetilde{\mathfrak{M}}\mathrm{G}^{-1}\widetilde{\mathfrak{M}} - \nu\mathrm{G}\Big) = \mathrm{G}\widetilde{\mathfrak{H}}\,\mathrm{j}^a\,.$$

This is the same result as in (5.13).

*Proof in Lorentz's case.* It is especially (in the Lorentz case $\lambda_0 = -\frac{1}{\mathrm{c}^2}$, $\lambda_m = 1$ for $m \geq 1$)

$$\mathrm{G}_{ij} = \lambda_i \delta_{ij}\,, \quad \lambda_i = \text{const.}$$

For the $i$-th component of $\mathscr{R}$, $i = 0, 1, 2, 3$, we have by definition

$$\begin{aligned}
&\frac{1}{\mathrm{c}^2}\mathscr{R}_i(N) = \sum_{jk} \partial_j (\mathrm{G}N)_{ik} \cdot (\mathrm{G}N\mathrm{G})_{kj} \\
&= -\sum_{jkl} \partial_j (N\mathrm{G})_{ki} \mathrm{G}_{kl} (N\mathrm{G})_{lj} = -\sum_{jk} \partial_j (N_{ki}\lambda_i) \lambda_k (N_{kj}\lambda_j)\,,
\end{aligned}$$

hence

$$\begin{aligned}
&-\frac{1}{\mathrm{c}^2}\mathscr{R}_i(N) = \lambda_i \sum_{jk} \lambda_j \lambda_k \partial_j N_{ki} \cdot N_{kj} \\
&= -\lambda_i \sum_{jk} \lambda_j \lambda_k (\partial_i N_{jk} + \partial_k N_{ij}) N_{kj} \\
&\qquad\qquad (\text{it is } N_{kj} = -N_{jk} \text{ and } N_{ij} = -N_{ji}) \\
&= \lambda_i \sum_{jk} \lambda_j \lambda_k \partial_i N_{jk} \cdot N_{jk} - \lambda_i \sum_{jk} \lambda_j \lambda_k \partial_k N_{ji} \cdot N_{jk} \\
&= \lambda_i \sum_{jk} \lambda_j \lambda_k \partial_i \frac{N_{jk}^2}{2} - \lambda_i \sum_{jk} \lambda_j \lambda_k \partial_k N_{ji} \cdot N_{jk} \\
&= \partial_i \Big( \lambda_i \sum_{jk} \lambda_j \lambda_k \frac{N_{jk}^2}{2} \Big) - \underbrace{\sum_{jk} \partial_k (N_{ji}\lambda_i) \lambda_j (N_{jk}\lambda_k)}_{= -\frac{1}{\mathrm{c}^2}\mathscr{R}_i(N)}\,.
\end{aligned}$$

It follows

$$\mathscr{R}_i(N) = -\frac{\mathrm{c}^2}{2} \partial_i \Big( \lambda_i \sum_{jk} \lambda_j \lambda_k \frac{N_{jk}^2}{2} \Big)\,.$$

If now

$$\nu = -\frac{\mathrm{c}^2}{4} \sum_{jk} \lambda_j \lambda_k N_{jk}^2 = \frac{\mathrm{c}^2}{4} \sum_{ij} (\mathrm{G}N)_{ij} (\mathrm{G}N)_{ji} = \frac{\mathrm{c}^2}{4} (\mathrm{G}N)^{\mathrm{T}} \boldsymbol{:} (\mathrm{G}N)$$

we get

$$\mathscr{R}_i(N) = \partial_i (\lambda_i \nu) = \sum_j \partial_j (\nu \mathrm{G}_{ij}) = (\underline{\operatorname{div}}(\nu \mathrm{G}))_i\,.$$

□

# 6 Vector potential

The assumptions in 4.2 are so, that there exist (locally) a gravitational potential $\phi^g$ and a vector potential $\underline{A}$ with

$$\underline{F} = \underline{\nabla}\phi^g\,, \quad \mathfrak{E} = \underline{\nabla}\,\underline{A} - (\underline{\nabla}\,\underline{A})^{\mathrm{T}}\,,$$

where we take the Lorenz gauge condition $\underline{\mathrm{div}}(\mathrm{G}\underline{A}) = 0$ for $\underline{A}$, which is the same for all observers. In a Lorentz frame this means

$$\underline{A} = (-\phi^e, A_1, A_2, A_3)\,, \quad \frac{1}{\mathrm{c}^2}\partial_t\phi^e + \mathrm{div}_x A = \underline{\mathrm{div}}(\mathrm{G}\underline{A}) = 0\,. \tag{6.1}$$

If $\mathfrak{E}$ is expressed by $E$ and $B$ as in (4.8)

$$\begin{bmatrix} 0 & -E^{\mathrm{T}} \\ E & \mathcal{R}(B) \end{bmatrix} = \mathfrak{E} = \underline{\nabla}\,\underline{A} - (\underline{\nabla}\,\underline{A})^{\mathrm{T}} = \begin{bmatrix} 0 & (\partial_0 A - \nabla A_0)^{\mathrm{T}} \\ \nabla A_0 - \partial_0 A & (\partial_i A_j - \partial_j A_i)_{i,j\ge 1} \end{bmatrix},$$

then $E = \nabla A_0 - \partial_0 A = -(\nabla\phi^e + \partial_t A)$ and $\mathcal{R}(B) = (\partial_i A_j - \partial_j A_i)_{i,j\ge1} = \nabla_x A - (\nabla_x A)^{\mathrm{T}}$. Therefore the above equations for $\underline{F}$ and $\mathfrak{E}$ become

$$\begin{aligned} F_0 = \partial_t\phi^g\,, &\quad F = \nabla\phi^g\,, \\ E = -(\nabla_x\phi^e + \partial_t A)\,, &\quad B = \mathrm{rot}\,_x A\,. \end{aligned} \tag{6.2}$$

The existence of the scalar potential came from the time dependent version of Newton's equation and the vector potential was equivalent to Faraday's law of induction. Therefore, if we plug (6.2) in the definition of $\mathfrak{M}$, the equations (3.1) become the full system of differential equations in Lorentz frames. Let us do this.

From the representation for $\mathfrak{M}$ in (4.6) and the Maxwell-Lorentz aether relations in (4.9) we obtain

$$\begin{aligned} \mathfrak{M} &= \begin{bmatrix} \frac{1}{\mathrm{c}^2}F_0 & -(F-D)^{\mathrm{T}} \\ -(F+D) & M^s + \mathcal{R}(H) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\mathrm{c}^2}F_0 & -F^{\mathrm{T}} \\ -F & M^s \end{bmatrix} + \begin{bmatrix} 0 & P^{\mathrm{T}} \\ -P & -\mathcal{R}(M^a) \end{bmatrix} + \begin{bmatrix} 0 & \bar\varepsilon_0 E^{\mathrm{T}} \\ -\bar\varepsilon_0 E & \frac{1}{\bar\mu_0}\mathcal{R}(B) \end{bmatrix}, \end{aligned}$$

where in detail with Newton's gravitational potential 4.2(1) reads

$$\begin{aligned} \begin{bmatrix} \frac{1}{\mathrm{c}^2}F_0 & -F^{\mathrm{T}} \\ -F & M^s \end{bmatrix} &= \begin{bmatrix} \frac{1}{\mathrm{c}^2}\partial_t\phi^g & -(\nabla_x\phi^g)^{\mathrm{T}} \\ -\nabla_x\phi^g & M^s \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\mathrm{c}^2}\partial_t\phi^g & -(\nabla_x\phi^g)^{\mathrm{T}} \\ 0 & M^s - \partial_t\phi^g\mathrm{Id} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\nabla_x\phi^g & \partial_t\phi^g\mathrm{Id} \end{bmatrix}, \\ \begin{bmatrix} 0 & \bar\varepsilon_0 E^{\mathrm{T}} \\ -\bar\varepsilon_0 E & \frac{1}{\bar\mu_0}\mathcal{R}(B) \end{bmatrix} &= \begin{bmatrix} 0 & -\bar\varepsilon_0(\nabla_x\phi^e + \partial_t A)^{\mathrm{T}} \\ \bar\varepsilon_0(\nabla_x\phi^e + \partial_t A) & \frac{1}{\bar\mu_0}(\nabla_x A - (\nabla_x A)^{\mathrm{T}}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\bar\varepsilon_0}{\mathrm{c}^2}\partial_t\phi^e & -\bar\varepsilon_0(\nabla_x\phi^e)^{\mathrm{T}} \\ \bar\varepsilon_0\partial_t A & -\frac{1}{\bar\mu_0}(\nabla_x A)^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} -\frac{\bar\varepsilon_0}{\mathrm{c}^2}\partial_t\phi^e & -\bar\varepsilon_0(\partial_t A)^{\mathrm{T}} \\ \bar\varepsilon_0\nabla_x\phi^e & \frac{1}{\bar\mu_0}\nabla_x A \end{bmatrix}, \end{aligned}$$

so that the matrix $\mathfrak{M}$ becomes

$$\mathfrak{M} = \begin{bmatrix} \frac{1}{c^2}\partial_t\big(\phi^g + \bar{\varepsilon}_0\phi^e\big) & -\big(\nabla_x\phi^g + \bar{\varepsilon}_0\nabla_x\phi^e\big)^{\mathrm{T}} \\ \bar{\varepsilon}_0\partial_t A & -\frac{1}{\bar{\mu}_0}(\nabla_x A)^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0 & P^{\mathrm{T}} \\ -P & M^s - \partial_t\phi^g\mathrm{Id} - \mathcal{R}(M^a) \end{bmatrix} + \mathfrak{O}$$

with

$$\mathfrak{O} := \begin{bmatrix} -\frac{\bar{\varepsilon}_0}{c^2}\partial_t\phi^e & -\bar{\varepsilon}_0(\partial_t A)^{\mathrm{T}} \\ \bar{\varepsilon}_0\nabla_x\phi^e - \nabla_x\phi^g & \frac{1}{\bar{\mu}_0}\nabla_x A + \partial_t\phi^g\mathrm{Id} \end{bmatrix}.$$

We obtain here a form of the matrix $\mathfrak{M}$ which leads to the well-known wave equations of the related vector potentials. To be concrete the matrix has the form

$$\begin{gathered} \boldsymbol{\phi} := \phi^g + \bar{\varepsilon}_0\phi^e\,, \quad M_0 := M^s - \partial_t\phi^g\mathrm{Id} - \mathcal{R}(M^a)\,, \\ \mathfrak{M} = \begin{bmatrix} \frac{1}{c^2}\partial_t\boldsymbol{\phi} & -(\nabla_x\boldsymbol{\phi} - P)^{\mathrm{T}} \\ \frac{1}{\bar{\mu}_0 c^2}\partial_t A - P & M_0 - \frac{1}{\bar{\mu}_0}(\nabla_x A)^{\mathrm{T}} \end{bmatrix} + \mathfrak{O}\,, \end{gathered} \tag{6.3}$$

with the following theorem.

**6.1 Wave equations.** The matrix $\mathfrak{M}$ has the form (6.3), where because of the gauge condition the matrix $\mathfrak{O}$ is divergence-free. Hence neglecting the Coriolis term the general system (3.1), that is, $\underline{\mathrm{div}}\,\mathfrak{M} = \widetilde{\mathrm{j}} = (\varrho, \mathrm{j})$, becomes

$$\begin{gathered} \frac{1}{c^2}\partial_t^2\boldsymbol{\phi} + \mathrm{div}_x\big(-\nabla_x\boldsymbol{\phi} + P\big) = \varrho\,, \\ \frac{1}{\bar{\mu}_0 c^2}\partial_t^2 A - \partial_t P + \mathrm{div}_x\big(M_0 - \frac{1}{\bar{\mu}_0}\mathrm{D}_x A\big) = \mathrm{j}\,. \end{gathered} \tag{6.4}$$

Therefore, if $P = 0$ and $M_0 = 0$ we have two wave equations.

*Proof.* We have

$$\underline{\mathrm{div}}\mathfrak{O} = \underline{\mathrm{div}}\begin{bmatrix} -\frac{\bar{\varepsilon}_0}{c^2}\partial_t\phi^e & -\bar{\varepsilon}_0(\partial_t A)^{\mathrm{T}} \\ \bar{\varepsilon}_0\nabla_x\phi^e - \nabla_x\phi^g & \frac{1}{\bar{\mu}_0}\nabla_x A + \partial_t\phi^g\mathrm{Id} \end{bmatrix} = \begin{bmatrix} -\bar{\varepsilon}_0\partial_t \\ \frac{1}{\bar{\mu}_0}\nabla_x \end{bmatrix}\Big(\frac{1}{c^2}\partial_t\phi^e + \mathrm{div}_x A\Big)$$

which vanishes since the Lorenz condition has been postulated. □

Here the potential $\boldsymbol{\phi} = \phi^g + \bar{\varepsilon}_0\phi^e$ leads possibly to a new interpretation of gravity. One has to compare this approach with measurements, which are related to the study of the interaction of gravitation and electromagnetic effects or the interaction of gravitation and magnetization.

# 7 Dimension

The combination of the $kg$-based gravitational quantities and the $As$-based electrodynamical quantities needs some explanation. Essential is the conversion factor $\mathrm{k}_0$ with dimension $kg/As$ which we write as

$$\mathrm{k}_0 \left[\frac{kg}{As}\right] , \tag{7.1}$$

and which will appear in 7.4. We assume coordinates $y = (t, x) \in \mathbb{R}^4$. The generic system of 4 equations in these coordinates reads

$$\begin{aligned} \partial_t\big(\square\,[*]\big) + \mathrm{div}_x\big(\square \left[\frac{*\,m}{s}\right]\big) &= \square \left[\frac{*}{s}\right] , \\ \partial_t\big(\square \left[\frac{*\,m}{s}\right]\big) + \mathrm{div}_x\big(\square \left[\frac{*\,m^2}{s^2}\right]\big) &= \square \left[\frac{*\,m}{s^2}\right] . \end{aligned} \tag{7.2}$$

Since $\partial_t$ is equipped with dimension $\left[\frac{1}{s}\right]$ and $\mathrm{div}_x$ with dimension $\left[\frac{1}{m}\right]$, the system is consistent for every fixed choice of $[*]$. Hence, if in this system one quantity $\square$ with units is specified, then the units of all other quantities $\square$ are defined.

First, consider the mass-momentum system $\underline{\mathrm{div}}(\varrho \underline{v}\underline{v}^{\mathrm{T}} + \underline{\Pi}) = \underline{\widetilde{\mathbf{f}}}$ which in a special case means

$$\underline{v} = \begin{bmatrix} 1 \\ v \end{bmatrix} , \quad \underline{\Pi} = \begin{bmatrix} 0 & 0 \\ 0 & \Pi \end{bmatrix} , \quad \underline{\widetilde{\mathbf{f}}} = \begin{bmatrix} \mathbf{r} \\ \mathbf{f} \end{bmatrix} .$$

**7.1 Mass-momentum system.** In Lorentz case (7.2) reads

$$\begin{aligned} \partial_t\big(\varrho\,[*]\big) + \mathrm{div}_x\big(\varrho v \left[\frac{*\,m}{s}\right]\big) &= \mathbf{r} \left[\frac{*}{s}\right] , \\ \partial_t\big(\varrho v \left[\frac{*\,m}{s}\right]\big) + \mathrm{div}_x\big((\varrho v v^{\mathrm{T}} + \Pi) \left[\frac{*\,m^2}{s^2}\right]\big) &= \mathbf{f} \left[\frac{*\,m}{s^2}\right] , \end{aligned} \tag{7.3}$$

where

$$* = \frac{kg}{m^3} , \quad \text{hence} \quad \varrho \left[\frac{kg}{m^3}\right] , \quad v \left[\frac{m}{s}\right] ,$$

and thus

$$\mathbf{r} \left[\frac{kg}{m^3 s}\right] , \quad \Pi \left[\frac{kg}{m\,s^2}\right] , \quad \mathbf{f} \left[\frac{kg}{m^2 s^2}\right] .$$

Now we consider the equation $\underline{\mathrm{div}}\mathfrak{M} = \widetilde{\mathrm{j}}$ and we take seperately the cases of the symmetric and the antisymmetric part. First the case of gravity, that is Newton's law in the relativistic version.

**7.2 Newton's law of gravitation.** With $\underline{F} = (F_0, F_1, F_2, F_3) = \underline{\nabla}\phi^g$ the equations in (4.6) read in the form (7.2)

$$\begin{aligned} \partial_t\big(\frac{F_0}{\mathrm{c}^2}\,[*]\big) + \mathrm{div}_x\big(-F \left[\frac{*\,m}{s}\right]\big) &= \varrho^s \left[\frac{*}{s}\right] , \\ \partial_t\big(-F \left[\frac{*\,m}{s}\right]\big) + \mathrm{div}_x\big(M^s \left[\frac{*\,m^2}{s^2}\right]\big) &= \mathrm{j}^s \left[\frac{*\,m}{s^2}\right] . \end{aligned}$$

It is

$$\varrho^s \left[\frac{kg}{m^3}\right] = \left[\frac{*}{s}\right] , \quad \text{that is} \quad * = \frac{kg\, s}{m^3}$$

and therefore

$$\frac{F_0}{\mathrm{c}^2}\,[*] = \left[\frac{kg\, s}{m^3}\right] , \quad \text{thus} \quad F_0 \left[\frac{kg}{m\, s}\right] , \quad F \left[\frac{*\, m}{s}\right] = \left[\frac{kg}{m^2}\right] .$$

This is consistent with

$$\phi^g \left[\frac{kg}{m}\right] , \quad \text{hence} \quad F_0 = \partial_t \phi^g \left[\frac{kg}{m\, s}\right] , \quad F = \nabla \phi^g \left[\frac{kg}{m^2}\right] .$$

Finally

$$M^s \left[\frac{kg}{m\, s}\right] , \quad \mathrm{j}^s \left[\frac{kg}{m^2 s}\right] .$$

Now to the magnetic part.

**7.3 Magnetism kg-based.** The equations (7.2) are, because of (4.6),

$$\begin{aligned} \partial_t\big(0\,[*]\big) + \mathrm{div}_x\Big(D\left[\frac{*\, m}{s}\right]\Big) &= \varrho^a \left[\frac{*}{s}\right] , \\ \partial_t\Big(-D\left[\frac{*\, m}{s}\right]\Big) + \mathrm{div}_x\Big(\mathcal{R}(H)\left[\frac{*\, m^2}{s^2}\right]\Big) &= \mathrm{j}^a \left[\frac{*\, m}{s^2}\right] . \end{aligned} \tag{7.4}$$

We know

$$\varrho^a \left[\frac{kg}{m^3}\right] , \quad \text{hence} \quad * = \frac{kg\, s}{m^3} ,$$

and with this it is known

$$D \left[\frac{kg}{m^2}\right], \quad H \left[\frac{kg}{m\, s}\right], \quad \varrho^a \left[\frac{kg}{m^2}\right], \quad \mathrm{j}^a \left[\frac{kg}{m^2 s}\right] .$$

The purpose of the constant $\mathrm{k}_0$ is to transform the quantities from the $As$-based version into the $kg$-based version.

**7.4 Ampère's circuital law *As*-based.** We now set

$$\mathfrak{H} = \mathrm{k}_0 \bar{\mathfrak{H}} \quad \text{i.e.} \quad \begin{bmatrix} 0 & D \\ -D & \mathcal{R}(H) \end{bmatrix} = \mathrm{k}_0 \begin{bmatrix} 0 & \bar{D} \\ -\bar{D} & \mathcal{R}(\bar{H}) \end{bmatrix} ,$$

$$D = \mathrm{k}_0 \bar{D} , \quad H = \mathrm{k}_0 \bar{H} , \quad \varrho^a = \mathrm{k}_0 \bar{\varrho}^a , \quad \mathrm{j}^a = \mathrm{k}_0 \bar{\mathrm{j}}^a .$$

This implies, because of (7.1), the well-known dimensions

$$\bar{D} \left[\frac{As}{m^2}\right], \quad \bar{H} \left[\frac{A}{m}\right], \quad \bar{\varrho}^a \left[\frac{As}{m^3}\right], \quad \bar{\mathrm{j}}^a \left[\frac{A}{m^2}\right],$$

or with $* = \dfrac{A\,s^2}{m^3}$

$$\begin{aligned} \partial_t\big(0\,[*]\big) + \operatorname{div}_x\Big(\bar D\left[\frac{*\,m}{s}\right]\Big) &= \bar\varrho^a\left[\frac{*}{s}\right], \\ \partial_t\Big(-\bar D\left[\frac{*\,m}{s}\right]\Big) + \operatorname{div}_x\Big(\mathcal{R}(\bar H)\left[\frac{*\,m^2}{s^2}\right]\Big) &= \bar{\mathrm{j}}^a\left[\frac{*\,m}{s^2}\right]. \end{aligned} \tag{7.5}$$

With the constants $\varepsilon_0$ and $\mu_0$ we set

$$\varepsilon_0 = 8.854187812\cdot 10^{-12}\frac{F}{m}\,, \quad \bar\varepsilon_0 = \mathrm{k}_0\varepsilon_0\,,$$

$$\varepsilon_0 E = \bar D - \bar P\,, \quad \text{thus} \quad D = \bar\varepsilon_0 E + P\,, \quad \text{and} \quad \varrho^{el} := \frac{1}{\varepsilon_0}\bar\varrho^a = \frac{1}{\bar\varepsilon_0}\varrho^a\,,$$

$$\mu_0 = 4\pi\cdot 10^{-7}\frac{H}{m}\,, \quad \bar\mu_0 = \frac{\mu_0}{\mathrm{k}_0}\,, \quad \bar\varepsilon_0\bar\mu_0 = \varepsilon_0\mu_0 = \frac{1}{\mathrm{c}^2}\,,$$

$$\mu_0(\bar H + \bar M^a) = B\,, \quad \text{thus} \quad B = \bar\mu_0(H + M^a)\,, \quad \text{and} \quad \mathrm{j}^{el} := \mu_0\bar{\mathrm{j}}^a = \bar\mu_0\mathrm{j}^a\,.$$

It is with different units

$$\varepsilon_0\left[\frac{F}{m}\right] = \left[\frac{As}{V\,m}\right] = \left[\frac{(As)^2}{kg\,m}\Big(\frac{s}{m}\Big)^2\right], \quad \mu_0\left[\frac{H}{m}\right] = \left[\frac{N}{A^2}\right] = \left[\frac{kg\,m}{(As)^2}\right],$$

and (7.5) becomes without polarization and magnetization

$$\begin{aligned} \operatorname{div}_x E &= \varrho^{el}\,, \\ -\frac{1}{\mathrm{c}^2}\partial_t E + \operatorname{rot}_x B &= \mathrm{j}^{el}\,, \end{aligned}$$

where

$$E\left[\frac{N}{As}\right] = \left[\frac{kg}{As}\frac{m}{s^2}\right], \quad B\,[T] = \left[\frac{kg}{As\cdot s}\right].$$

The only difference in the mass densities is that in connection with Ampère's law one uses $As$ (ampere seconds) whereas in connection with the mass-momentum system one uses $kg$ (kilogram) or $g$ (gram).

# References


[1] H.W. Alt: *Mathematical Continuum Mechanics.* Script of the Lecture at the TUM München 2011-2019

[2] Hans Wilhelm Alt: *Relativistic equations for the Chapman-Enskog hierarchy.* Advances in Mathematical Sciences and Applications (AMSA), Vol. 25, pp.131-179. 2016

[3] G. Duvaut, J.L. Lions: *Inequalities in Mechanics and Physics.* Grundlehren der mathematischen Wissenschaften. Springer 1976

[4] A.S. Eddington: *A generalisation of Weyl's theory of the electromagnetic and gravitational fields.* Proc. R. Soc. London, Ser. A, 99, 104-122. 1921

[5] Hans Goedbloed, Rony Keppens, Stefaan Poedts: *Magnetohydrodynamics of Laboratory and Astrophysical Plasmas.* Cambridge University Press 2019

[6] Hubert F. M. Goenner: *On the History of Unified Field Theories.* Living Rev. Relativity, 7, (2004), 2.

[7] Hubert F. M. Goenner: *On the History of Unified Field Theories. Part II. (ca. 1930 - ca. 1965).* Living Rev. Relativity, 17, (2014), 5.

[8] Oliver Heaviside: *A Gravitational and Electromagnetic Analogy.* [Part I, The Electrician, 31, 281-282 (1893)] [Part II, The Electrician, 31, 359 (1893)].

[9] J.D. Jackson: *Classical Electrodynamics.* 3rd edition. John Wiley & Sons 1999

[10] L.D. Landau, E.M. Lifschitz: *Lehrbuch der theoretischen Physik. Band II. Klassische Feldtheorie.* 12. Auflage. Akademie-Verlag Berlin 1992

[11] Bahram Mashhoon: *Gravitoelectromagnetism: A Brief Review.* University of Missouri-Columbia 2008

[12] Jose G. Vargas, Douglas G. Torr: *The Cartan ± Einstein Unification with Teleparallelism and the Discrepant Measurements of Newton ' s Constant G.* Foundations of Physics, Vol . 29, No. 2. 1999