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Quasilinear Elliptic-Parabolic Differential Equations*

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0. Introduction

We consider the following initial boundary value problem for a system of quasilinear elliptic-parabolic differential equations

$$\partial_t b^j(u) - \nabla \cdot a^j(b(u), \nabla u) = f^j(b(u)) \quad \text{in }]0,T[\times\Omega,\ j=1,\dots,m, \tag{0.1}$$

$$b(u) = b^0 \quad \text{on } \{0\}\times\Omega, \tag{0.2}$$

$$\begin{aligned} u = u^D \quad &\text{on }]0,T[\times\Gamma, \\ a^j(b(u), \nabla u)\cdot\nu = 0 \quad &\text{on }]0,T[\times(\partial\Omega\setminus\Gamma),\ j=1,\dots,m. \end{aligned} \tag{0.3}$$

The structure conditions are the ellipticity of a and the (weak) monotonicity of b, and b has to be a subgradient in case $m>1$. First we treat the case that b is continuous, and later (Sect. 4) we include Stefan problems, that is, we allow b to have jumps. The special cases of an elliptic equation with time as parameter, that is, $b(z)=0$, and the standard parabolic equation, that is, $b(z)=z$ are included. Some special single equations of mixed elliptic and parabolic type are given in the following.

The gas flow through a porous medium is described by the equation

$$\partial_t u = \Delta u^m \quad \text{with } m>1.$$

This can be transformed into

$$\partial_t b(u) = \Delta u \quad \text{with } b(z) := \max(z,0)^{1/m} \quad \text{or} \quad (\operatorname{sign} z)|z|^{1/m},$$

which falls in the class of equations we consider. Solutions to this equation have been studied intensively, e.g., by Oleinik, Kalashnikov and Yui-Lin' [20] and by Aronson [5]. It is known that these equations differ considerably from

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parabolic ones, since they allow solutions with compact support. The reason is that the function b is not Lipschitz continuous.

Another architypical example is

$$\partial_t \max(u, 0) = \Delta u$$

with constant negative Dirichlet data and, say, positive initial data. It turns out that the solution becomes stationary after a finite time. In other words, the parabolic region collapses after a finite time t_0, and at this time the solution jumps to the stationary data discontinuously. Nevertheless, one expects that $b(u) = \max(u, 0)$ is continuous, and indeed we show that $\partial_t b(u)$ is at least in $L^2(]0, T[\times\Omega)$ (see 2.2, 2.3). An explicit solution, for example, is given by

$$u(t, x) = \begin{cases} s(t)\,\eta\left(\frac{x}{s(t)}\,\xi\right) & \text{for } |x| \leqq s(t), \\ -\xi\eta'(-\xi)(|x| - s(t)) & \text{for } |x| > s(t) \end{cases}$$

with $s(t) = 2\xi\sqrt{t_0 - t}$, where η is the solution of

$$\eta''(y) - 2y\eta'(y) + 2\eta(y) = 0$$

with $\eta(0) > 0$, $\eta'(0) = 0$, and positive zero $\xi \approx 0.92414$, that is

$$\eta(y) = 1 - \sum_{i=1}^{\infty} \frac{y^{2i}}{i!(2i-1)}.$$

A large field of application for our results is nonsteady filtration. For one incompressible fluid in a porous medium one has to solve the equation

$$\partial_t \theta(p) = \nabla \cdot (k(\theta(p))(\nabla p + e)), \tag{0.4}$$

where p is the unknown pressure, θ the water content, k the conductivity of the porous medium, and $-e$ the direction of gravity. The typical form of the functions θ and k is given in Fig. 1.

The Kirchhoff transformation

$$u := \int_0^p k(\theta(s))\,ds$$

leads to a differential equation of the form

$$\partial_t b(u) = \nabla \cdot (\nabla u + k(b(u))\,e),$$

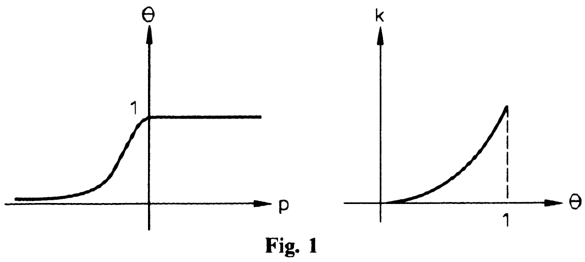

Fig. 1

where the function b behaves like θ. In special cases (without gravity) this already has been considered by Hornung [17] using the theory of monotone operators, and by van Duyn and Peletier [15] using a parabolic regularization.

The flow of several fluids through a porous medium leads to a system of equations of type (0.4), but the nonuniform ellipticity leads to special problems, which are dealt with for instance in the forthcoming papers [2, 18].

Another powerful tool for the investigation of parabolic-elliptic equations is L^1-contractions, see Benilan [7-9] and Crandall [13]. With this method equations of the type

$$\partial_t b(u) - \nabla(a(\nabla u) + e(u)) = f$$

can be handled provided e is Lipschitz continuous and a is sublinear elliptic.

Our approach is based on the natural topology induced by the energy integral

$$\int_0^T \int_\Omega (\Psi(b(u)) + |\nabla u|^r),$$

where

$$\Psi(z) := \sup_{\sigma \in \mathbb{R}} \int_0^\sigma (z - b(s))\, ds.$$

The second term of this integral is convex in u, and the first term convex in $b(u)$, since Ψ is a convex function. In Sect. 1 we prove the existence of a weak solution via a priori estimates on these two terms and in addition on

$$(b(u(t)) - b(u(t-h)))(u(t) - u(t-h))$$

in order to control the time dependence. As approximating problems we use backward time differences, but it is possible to use also parabolic regularization in the case of a single equation (see [3]). We should remark that we obtain the strong convergence of the approximating solutions with respect to the above topology.

Our method is not restricted to differential equations as in (0.1). It is possible to give an abstract version of such initial boundary value problems by replacing the spaces $L^\infty(0, T; L^1(\Omega))$ and $H^{1,r}(\Omega)$ by two Banach spaces related to each other in an appropriate way.

In Sect. 2 we deal with regularity and uniqueness questions. If b is Lipschitz continuous, we can prove an L^2-estimate on $\partial_t b(u)$ for the approximate solutions, hence at least one regular solution exists (2.3). Beyond that we prove the uniqueness of these solutions, which is based on a maximum principle (2.2), therefore strictly limited to second order equations. However, in the case of a linear operator one can prove uniqueness in the class of weak solutions (2.4).

Variational inequalities for elliptic-parabolic equations are treated in Sect. 3. We prove two existence theorems, one in the class of regular functions for time independent constraints, and another one for time dependent obstacle problems in a weak formulation. An elliptic-parabolic problem with a time dependent inequality at the boundary is treated in [4].

1. Existence

1.1. Assumptions on the Data

1) $\Omega \subset \mathbb{R}^n$ is open, bounded, and connected with Lipschitz boundary, $\Gamma \subset \partial\Omega$ is measurable with $H^{n-1}(\Gamma) > 0$, and $0 < T < \infty$.

2) b is a monotone vector field and a continuous gradient, that is, there is a convex C^1 function $\Phi: \mathbb{R}^m \to \mathbb{R}$ with $b = \nabla\Phi$. We can assume that $b(0) = 0$. Define

$$\Psi(z) = \Psi_b(z) := \sup_{\sigma \in \mathbb{R}^m} \int_0^1 (z - b(s\sigma)) \cdot \sigma \, ds$$
$$= \sup_{\sigma \in \mathbb{R}^m} (z \cdot \sigma - \Phi(\sigma) + \Phi(0)).$$

The convexity of Φ then implies that

$$B(z) := \Psi(b(z)) = b(z) \cdot z - \Phi(z) + \Phi(0)$$
$$= \int_0^1 (b(z) - b(sz)) \cdot z \, ds = \int_0^z (b(z) - b(s)) \, ds,$$

3) $a(b(z), p)$ is continuous in z and p and elliptic in the sense that

$$(a(z, p_1) - a(z, p_2)) \cdot (p_1 - p_2) \geqq c|p_1 - p_2|^r$$

with $1 < r < \infty$, and $f(b(z))$ is continuous in z.

4) The following growth condition is satisfied:

$$|a(b(z), p)| + |f(b(z))| \leqq c(1 + B(z)^{(r-1)/r} + |p|^{r-1}).$$

In general, the coefficients a, b, f may also depend on t and x, which will be discussed at the end of this section (see 1.10).

1.2. Remark. The convexity of Φ implies

$$B(z) - B(z_0) \geqq (b(z) - b(z_0)) \cdot z_0$$

for all $z, z_0 \in \mathbb{R}^m$. If we choose $\sigma = \frac{1}{\delta |b(z)|} b(z)$ in the above definition we see that

$$B(z) \geqq \int_0^1 (b(z) - b(s\sigma)) \cdot \sigma \, ds \geqq \frac{1}{\delta}(|b(z)| - \sup_{|\sigma'| \leqq \frac{1}{\delta}} |b(\sigma')|)$$

so that

$$|b(z)| \leqq \delta B(z) + \sup_{|\sigma| \leqq \frac{1}{\delta}} |b(\sigma)|.$$

1.3. Boundary and Initial Data

1) We assume that u^D is in $L^r(0, T; H^{1,r}(\Omega))$ and in $L^\infty(]0, T[\times \Omega)$, and we define

$$V := \{v \in H^{1,r}(\Omega) / v = 0 \text{ on } \Gamma\}.$$

2) We assume $\Psi(b^0)\in L^1(\Omega)$, and that b^0 maps into the range of b. Therefore there is a measurable function u^0 with $b^0=b(u^0)$.

1.4. Weak Solution. Assume 1.3. We call $u\in u^D+L^r(0,T;V)$ a *weak solution* of the initial boundary value problem (0.1)–(0.3), if the following two properties are fulfilled:

1) $b(u)\in L^\infty(0,T;L^1(\Omega))$ and $\partial_t b(u)\in L^{r^*}(0,T;V^*)$ with initial values b^0, that is,

$$\int_0^T \langle \partial_t b(u), \zeta\rangle + \int_0^T\int_\Omega (b(u)-b^0)\,\partial_t\zeta = 0$$

for every test function $\zeta\in L^r(0,T;V)\cap H^{1,1}(0,T;L^\infty(\Omega))$ with $\zeta(T)=0$.

2) $a(b(u),\nabla u)$, $f(b(u))\in L^{r^*}(]0,T[\times\Omega)$ and u satisfies the differential equation, that is,

$$\int_0^T \langle \partial_t b(u), \zeta\rangle + \int_0^T\int_\Omega a(b(u),\nabla u)\cdot\nabla\zeta = \int_0^T\int_\Omega f(b(u))\,\zeta$$

for every $\zeta\in L^r(0,T;V)$.

The main tool for proving the existence of a weak solution is to give an energy estimate, which follows, if we can justify the formula

$$\int_0^T \partial_t b(u)\cdot u = B(u(T)) - B(u^0).$$

This is done in the following lemma (see also Remark 1.6).

1.5. Lemma. *Suppose* 1.3 *is satisfied with* $\partial_t u^D\in L^1(0,T;L^\infty(\Omega))$. *If* $u\in u^D+L^r(0,T;V)$ *fulfills* 1.4.1), *then*

$$B(u)\in L^\infty(0,T;L^1(\Omega)),$$

and for almost all t the following formula holds

$$\int_\Omega B(u(t)) - \int_\Omega B(u^0) = \int_0^t\int_\Omega \langle \partial_t b(u), u-u^D\rangle$$
$$-\int_0^t\int_\Omega (b(u)-b(u^0))\,\partial_t u^D + \int_\Omega (b(u(t))-b(u^0))\,u^D(t).$$

Proof. We have for almost all $t>0$ pointwise in Ω

$$B(u(t))-B(u(t-h)) \leqq (b(u(t))-b(u(t-h)))\cdot u(t)$$

and for $t>h$

$$B(u(t))-B(u(t-h)) \geqq (b(u(t))-b(u(t-h)))\cdot u(t-h).$$

Here $u(t):=u^0$ for $-h<t<0$, therefore $b(u(t-h))=b^0$ if $t<h$. We multiply the first inequality with $\lambda_\varepsilon(t)$ and the second with $\lambda_\varepsilon(t-h)$, where

$$\lambda_\varepsilon := \min\left(1, \frac{1}{\varepsilon|u|}\right).$$

Now we can integrate over Ω and obtain for the first inequality

$$\int_\Omega \lambda_\varepsilon(t)(B(u(t))-B(u(t-h)))\leqq \int_\Omega (b(u(t))-b(u(t-h)))\,\lambda_\varepsilon(t)\,u(t)$$
$$=\langle b(u(t))-b(u(t-h)), (\lambda_\varepsilon u)(t)-u^D(t)\rangle+\int_\Omega (b(u(t))-b(u(t-h)))\,u^D(t),$$

provided we choose ε with $\varepsilon u^D \leqq 1$. Letting $\varepsilon\to 0$ and integrating t from 0 to τ we get

$$\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega B(u)-\int_\Omega B(u^0)\leqq \int_0^\tau \langle \partial_t^{-h} b(u), u-u^D\rangle - \int_0^{\tau-h}\int_\Omega (b(u)-b(u^0))\cdot\partial_t^h u^D$$
$$+\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega (b(u)-b(u^0))\cdot u^D.$$

Similarly the second inequality yields integrating t from h to τ

$$\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega B(u)-\frac{1}{h}\int_0^h\int_\Omega B(u)\geqq \int_0^{\tau-h}\langle \partial_t^h b(u), u-u^D\rangle-\int_h^\tau\int_\Omega (b(u)-b(u^0))\cdot\partial_t^{-h}u^D$$
$$+\frac{1}{h}\int_{\tau-h}^{\tau}\int_\Omega (b(u)-b(u^0))\cdot u^D-\frac{1}{h}\int_0^h\int_\Omega (b(u)-b(u^0))\cdot u^D.$$

As $h\to 0$ the first three terms on the right sides converge to the desired limits for almost all τ. The same is true for the first term on the left, hence the first inequality shows that

$$B(u)\in L^\infty(0, T; L^1(\Omega)).$$

Therefore in order to prove the Lemma we have to show that

$$\frac{1}{h}\int_0^h\int_\Omega (B(u)-B(u^0)-(b(u)-b(u^0))\cdot u^D)$$

becomes non-negative in the limit $h\downarrow 0$. First we observe that

$$\int_E b(u(t))\to 0$$

uniformly in t as $E\subset\Omega$ goes to zero in measure. This allows us to substitute u^D by a $C_0^\infty(\Omega)$ function. We could use the same argument for the term $B(u)-B(u^0)$ provided u^0 is bounded. Since in general it is not, we approximate B by

$$B^R(u^0):=\sup_{|\sigma|\leqq R}\int_0^\sigma (b(u^0)-b(s))\,ds.$$

Then $B^R(u^0)\uparrow B(u^0)$ almost everywhere as $R\uparrow\infty$, but since $B(u^0)\in L^1(\Omega)$ this convergence is also in $L^1(\Omega)$. Next let us choose functions $v_R\in L^\infty(\Omega)$ with

$$\|b(u^0)-v_R\|_{L^1(\Omega)}\leqq\frac{1}{R^2},$$

and then functions $u_R(t) \in L^\infty(\Omega)$ with

$$|u_R| \leqq R \quad \text{and} \quad u_R - u^D \in C_0^\infty(\Omega)$$

(where R should be larger than the supremum of $|u^D|$) such that the L^1-norm of

$$\sup_{|\sigma| \leqq R} \int_0^\sigma (v_R - b(s))\,ds - \int_0^{u_R(t)} (v_R - b(s))\,ds$$

is less than $\frac{1}{R}$ for every t. Then we conclude that for $R \uparrow \infty$ and uniformly in t

$$\begin{aligned}
\int_\Omega (B(u(t)) - B(u^0)) &\leftarrow \int_\Omega (B(u(t)) - B^R(u^0)) \\
&\geqq \int_\Omega \left(\int_0^{u_R(t)} (b(u(t)) - b(s))\,ds - \int_0^{u_R(t)} (v_R - b(s))\,ds \right) - \frac{2}{R} \\
&= \int_\Omega (b(u(t)) - v_R) \cdot u_R(t) - \frac{2}{R} \\
&\geqq \int_\Omega (b(u(t)) - b(u^0)) \cdot u_R(t) - \frac{3}{R},
\end{aligned}$$

and we infer that

$$\begin{aligned}
\limsup_{h \downarrow 0} \frac{1}{h} \int_0^h \int_\Omega (B(u) - B(u^0) - (b(u) - b(u^0)) \cdot u^D) \\
\geqq \lim_{R \uparrow \infty} \limsup_{h \downarrow 0} \frac{1}{h} \int_0^h \int_\Omega (b(u) - b(u^0)) \cdot (u_R - u^D) \\
= \lim_{R \uparrow \infty} \limsup_{h \downarrow 0} \int_0^h \left\langle \partial_t b(u(t)), \frac{1}{h} \int_t^h (u_R - u^D) \right\rangle dt = 0.
\end{aligned}$$

1.6. Remark. Under additional assumptions on b Lemma 1.5 holds for more general Dirichlet data u^D. If, for example, b satisfies a growth condition

$$|b(z)| \leqq C \cdot (1 + |z|^\alpha)$$

with $0 \leqq \alpha < r$, and if $b^0 \in L^{r/\alpha}(\Omega)$, then one needs only

$$u^D \in L^r(0, T; H^{1,r}(\Omega)) \cap H^{1,r_\alpha}(0, T; L^{r_\alpha}(\Omega)),$$

where $r_\alpha := \left(\frac{r}{\alpha}\right)^*$.

The interest of the following modification of Lemma 1.5 lies in the fact that it can be applied also when the Dirichlet data jump in time. The solution of the heat equation then can be used as continuation of the Dirichlet data in the interior. The statement is:

Suppose b is Lipschitz continuous, and assume 1.3 is satisfied with $\partial_t u^D \in L^{r^*}(0, T; V^*)$ such that

$$\frac{1}{h} \int_0^{T-h} \int_\Omega (u^D(t+h) - u^D(t))(b(u^D(t+h)) - b(u^D(t))\,dt$$

tends to zero for $h\downarrow 0$, and such that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^h\int_\Omega\int_{u^D}^{u^0}(b(u^0)-b(s))\,ds$$

exists and is finite. If $u\in u^D+L^r(0,T;V)$ satisfies 1.4.1) then

$$B(u)\in L^\infty(0,T;L^1(\Omega)),$$

and for almost all t the following formula holds:

$$\int_\Omega\int_{u^D(t)}^{u(t)}(b(u(t))-b(s))\,ds=\int_0^t\int_\Omega\langle\partial_t b(u),u-u^D\rangle$$
$$-\int_0^t\int_\Omega\langle\partial_t u^D,b(u)-b(u^D)\rangle+\lim_{h\downarrow 0}\frac{1}{h}\int_0^h\int_\Omega\int_{u^D}^{u^0}(b(u^0)-b(s))\,ds.$$

1.7. Existence Theorem. *Suppose the data satisfy* 1.1 *and* 1.3, *and assume that either* $\partial_t u^D\in L^1(0,T;L^\infty(\Omega))$ *or that the assumptions in* 1.6 *are fulfilled. Then there is a weak solution.*

Proof. We will perform the proof only in the first case, since in the second case one has only to apply 1.6 instead of 1.5.

The plan of the proof is the following: We approximate the differential equation by time discretization, that is, we replace $\partial_t b(u)$ by the backward difference quotient $\partial_t^{-h}b(u)$. Thus we arrive at elliptic problems, which can be solved by a Galerkin procedure.

For this we choose linearly independent functions $e_i\in V\cap L^\infty$, such that the subspace spanned by these functions is dense in V. We are looking for a function

$$u_{hm}(t,x)=u_h^D(t,x)+\sum_{i=1}^m\alpha_{hmi}(t)\,e_i(x)$$

with $\alpha_{hmi}\in L^\infty(]0,T[)$ such that for almost all t in $]0,T[$ the equality

$$\int_\Omega\partial_t^{-h}b(u_{hm}(t))\,\zeta+\int_\Omega a(b(u_{hm}(t)),\ \nabla u_{hm}(t))\,\nabla\zeta-\int_\Omega f(b(u_{hm}(t)))\,\zeta=0$$

holds for all test functions $\zeta\in V_m:=\operatorname{span}\{e_i,\dots,e_m\}$, where the initial data are given by

$$u_{hm}(t):=u_h^0(t)\qquad\text{for }-h<t<0.$$

(One could also write $b(u_{hm}(t-h))$ instead of $b(u_{hm}(t))$ in the a and f terms.)

We choose the approximating initial and boundary data such that u_h^0 is bounded, for example,

$$u_h^0:=\min\left(1,\frac{1}{h|u^0|}\right)u^0,$$

and such that u_h^D is time independent in each interval $](k-1)h,kh[$, for example

$$u_h^D(t,x):=\frac{1}{h}\int_{(k-1)h}^{kh}u^D(s,x)\,ds\qquad\text{for }(k-1)h\leqq t\leqq kh,$$

where for simplicity it is assumed that $\frac{T}{h}$ is an integer. This choice of u_h^D implies that we can determine $u_{hm}(t)$ inductively for $t \in](k-1)h, kh[$ as a solution of an elliptic problem. In fact, if $u_{hm}(t-h)$ is known, the left side of the above equation defines a continuous mapping ϕ_{hm}: $\mathbb{R}^m \to \mathbb{R}^m$, where the m parameters are the unknown coefficients of $u_{hm}(t)$. Since $B(z) \leqq z b(z)$ and

$$b(u_h^D + z)\, u_h^D \leqq \delta (u_h^D + z)\, b(u_h^D + z) + C(\delta),$$

this mapping fulfils the following estimate, where we use the notation $v = \sum_{i=1}^{m} \alpha_i e_i$:

$$\begin{aligned}\phi_{hm}(\alpha) \cdot \alpha &\geqq c \int_\Omega |\nabla v|^r + \frac{1}{h} \int_\Omega (b(u_h^D(t) + v) - b(u_{hm}(t-h)))\, v \\ &\quad - C \int_\Omega (1 + B(u_h^D + v) + |\nabla u_h^D|^r) \\ &\geqq c \int_\Omega |\nabla v|^r + \left(\frac{1}{2h} - C\right) \int_\Omega (u_h^D(t) + v)\, b(u_h^D(t) + v) \\ &\quad - C(h)\,(1 + \int_\Omega |b(u_{hm}(t-h))|^{r/(r-1)}).\end{aligned}$$

If h is small enough independent of m, the second term is non-negative, hence ϕ_{hm} has a zero, that is, $u_{hm}(t)$ exists.

We prove the convergence of the function u_{hm} as $(h, m) \to (0, \infty)$ in three steps. First we obtain an a priori estimate by multiplying the equation with $u_{hm} - u_h^D$. Then we control the time dependence by multiplying with time differences $\partial_t^{kh} u_{hm}$, which results in the compactness of the functions $b(u_{hm})$ in L^1. Finally we prove the strong convergence of u_{hm} by multiplying the equation with $u_{hm} - u$, where u is the weak limit obtained in the first step.

In the first step, we test with $\zeta = u_{hm}(t) - u_h^D(t)$ and integrate over t from 0 to τ. The parabolic part we estimate as follows:

$$\text{(1.7.1)} \qquad \begin{aligned}\int_0^\tau \int_\Omega \partial_t^{-h} b(u_{hm})\,(u_{hm} - u_h^D) &\geqq \frac{1}{h} \int_{\tau-h}^{\tau} \int_\Omega B(u_{hm}) - \int_\Omega B(u_h^0) \\ + \int_0^\tau \int_\Omega (b(u_{hm}) - b(u_h^0))\, \partial_t^h u_h^D &- \frac{1}{h} \int_{\tau-h}^{\tau} \int_\Omega (b(u_{hm}(t)) - b(u_h^0))\, u_h^D(t+h)\, dt.\end{aligned}$$

For the elliptic part we get the usual estimates. Hence using the assumption on u^D and Gronwall's argument, we obtain the a priori estimate

$$\sup_{0 \leqq t \leqq T} \int_\Omega B(u_{hm}(t)) + \int_0^T \int_\Omega |\nabla u_{hm}|^r \leqq C.$$

It follows that $u_{hm} \to u$ weakly in $L^r(0, T; V)$ for a subsequence $(h, m) \to (0, \infty)$.

Now let $k \in \mathbb{N}$ and use as test function

$$\zeta(t) := \partial_t^{kh}(u_{hm} - u_h^D)(\tau) \qquad \text{for } jh \leqq t \leqq (j+k)h$$

with $(j-1)h \leqq \tau \leqq jh$ and $1 \leqq j \leqq \frac{T}{h} - k$. Integrating over τ we obtain using the above energy estimate

$$\int_0^{T-kh} (b(u_{hm}(\tau+kh)) - b(u_{hm}(\tau)))(u_{hm}(\tau+kh) - u_{hm}(\tau))\,d\tau \leqq Ckh.$$

But since u_{hm} is a step function in time we see that this estimate is also satisfied if we replace kh by any positive number. This implies that $b(u_{hm})$ converge to $b(u)$ in $L^1(]0,T[\times\Omega)$ for a subsequence, which will be proved in Lemma 1.9 below. This lemma also shows that $B(u)\in L^\infty(0,T;L^1(\Omega))$.

The a priori estimate furthermore implies that there are functionals λ_{hm} bounded in $L^{r^*}(0,T;V^*)$ such that

$$\int_0^T \langle \lambda_{hm}, \zeta\rangle = \int_0^T \int_\Omega \partial_t^{-h} b(u_{hm})\,\zeta = -\int_0^{T-h} \int_\Omega (b(u_{hm}) - b(u_h^0))\,\partial_t^h \zeta$$

for $\zeta\in L^r(0,T;V_m)$ with $\zeta(t)=0$ for $t>T-h$. Hence for a subsequence $\lambda_m\to\lambda$ weakly in $L^{r^*}(0,T;V^*)$. We conclude that $b(u)$ satisfies 1.4.1), that is, $\lambda=\partial_t b(u)$, therefore 1.5 can be applied to u.

In order to prove the strong convergence of ∇u_{hm} take

$$\zeta = u_{hm} - (u_h^D + v_{hm})$$

as test function in the interval $]0,t_h[$, where $t_h = k_h h$ with $(k_h-1)h < t < k_h h$ for given $t\in]0,T[$, and where $v_{hm}\in L^r(0,T;V_m)$ are approximations of $u-u^D$ in $L^r(0,T;V)$ and time independent in each interval $](k-1)h, kh[$. We get

$$\text{(1.7.2)}\qquad \int_0^t \langle \partial_t^{-h} b(u_{hm}), \zeta\rangle + c\int_0^t \int_\Omega |\nabla\zeta|^r$$
$$\leqq -\int_0^t \int_\Omega a(b(u_{hm}), \nabla(u_h^D + v_{hm}))\,\nabla\zeta + \int_0^t \int_\Omega f(b(u_{hm}))\,\zeta.$$

First let us consider the parabolic term

$$\int_0^t \langle \partial_t^{-h} b(u_{hm}), \zeta\rangle = \int_0^t \langle \partial_t^{-h} b(u_{hm}), u_{hm} - u_h^D\rangle - \int_0^t \langle \partial_t^{-h} b(u), u - u^D\rangle + o(1),$$

where the Landau symbol $o(1)$ as usual denotes any term converging to zero as $h\to 0$ and $m\to\infty$. Using (1.7.1) on the first integral on the right and Lemma 1.5 on the second we get for almost all t the relationship

$$\int_0^t \langle \partial_t^{-h} b(u_{hm}), \zeta\rangle \geqq \frac{1}{h}\int_{t-h}^t \int_\Omega B(u_{hm}) - \int_\Omega B(u(t)) + o(1).$$

Now let us estimate the right side of (1.7.2). The Cauchy inequality and the weak convergence of ζ to 0 allow us to estimate

$$\left|\int_0^t \int_\Omega a(b(u_{hm}), \nabla(u_h^D + v_{hm}))\nabla\zeta\right|$$
$$\leqq \delta \int_0^t \int_\Omega |\nabla\zeta|^r + C_\delta \int_0^t \int_\Omega | a(b(u_{hm}), \nabla(u_h^D + v_h^m)) - a(b(u), \nabla u)|^{r^*} + o(1).$$

We can estimate the second integral on the right for example by

$$\int_0^t \int_\Omega \left| a(b(u), \nabla u) - a(b(u_{hm}), \nabla(u_h^D + v_{hm})) \min\left(1, \frac{1+B(u)}{1+B(u_{hm})}\right)^{\frac{1}{r^*}}\right|^{r^*}$$
$$+ \int_0^t \int_\Omega \left| a(b(u_{hm}), \nabla(u_h^D + v_{hm})) \left(1 - \min\left(1, \frac{1+B(u)}{1+B(u_{hm})}\right)\right)^{\frac{1}{r^*}}\right|^{r^*}.$$

Using the growth condition on a and the pointwise convergence of $b(u_{hm})$ we see that the first integral goes to zero by the Lebesgue dominated convergence theorem. The second integral on the right is estimated by

$$\int_0^t \int_\Omega |\nabla(u_h^D + v_{hm})|^r \left(1 - \min\left(1, \frac{1+B(u)}{1+B(u_{hm})}\right)^{\frac{1}{r^*}}\right)^{r^*}$$
$$+ \int_0^t \int_\Omega \max(0, (1+B(u_{hm}))^{\frac{1}{r^*}} - (1+B(u))^{\frac{1}{r^*}})^{r^*}.$$

The first term tends to zero as before, and the second integral is less than

$$\int_0^t \int_\Omega \max(0, B(u_{hm}) - B(u)) \leqq \int_0^t \int_\Omega (B(u_{hm}) - B(u)) + o(1)$$

since $B(u_{hm}) \to B(u)$ almost everywhere.

Since the f term can be handled similar to the elliptic part a, we see putting everything together that for almost all t

$$\int_\Omega (B(u_{hm}(t)) - B(u(t))) + c\int_0^t \int_\Omega |\nabla(u_{hm}) - u)|^r \leqq C \int_0^t \int_\Omega (B(u_{hm}) - B(u)) + o(1).$$

Then a Gronwall argument applied to the non-negative bounded function

$$\phi(t) := \limsup_{\substack{h\to 0\\ m\to\infty}} \int_\Omega (B(u_{hm}(t)) - B(u(t)))$$

yields for $t < T$

$$\nabla u_{hm} \to \nabla u \quad \text{strongly in } L^r(]0, t[\times\Omega).$$

Therefore

$$a(b(u_{hm}), \nabla u_{hm}) \to a(b(u), \nabla u),$$
$$f(b(u_{hm})) \to f(b(u))$$

almost everywhere, hence weakly in $L^{r^*}(]0, t[\times\Omega)$, which shows that u is a weak solution.

1.8. Lemma. *If two mappings v_1 and v_2 in $H^{1,r}(\Omega)$ satisfy the estimates*

$$\|v_i\|_{H^{1,r}(\Omega)} \leqq M, \ \|B(v_i)\|_{L^1(\Omega)} \leqq M, \qquad i=1,2$$

and

$$\int_\Omega (b(v_2)-b(v_1))(v_2-v_1) \leqq \delta,$$

then

$$\int_\Omega |b(v_2)-b(v_1)| \leqq \omega_M(\delta)$$

with continuous functions ω_M satisfying $\omega_M(0)=0$.

Proof. If not, we find sequences $v_{1\delta}, v_{2\delta}$ satisfying the above estimates with $\delta \downarrow 0$ and

$$\|b(v_{2\delta})-b(v_{1\delta})\|_{L^1(\Omega)} \geqq \kappa > 0.$$

Since $v_{i\delta}$ are bounded in $H^{1,r}(\Omega)$ we may extract a subsequence converging to v_i almost everywhere. Then $b(v_{i\delta}) \to b(v_i)$ almost everywhere, and from 1.2 we conclude that this convergence is in $L^1(\Omega)$, which implies

$$\|b(v_2)-b(v_1)\|_{L^1(\Omega)} \geqq \kappa.$$

On the other hand, using the notation $P_R(z) = \min\left(1, \frac{R}{z}\right) z$ we obtain similarly for $\delta \to 0$

$$\begin{aligned}\int_\Omega (b(v_2)-b(v_1)) \cdot P_R(v_2-v_1) &\leftarrow \int_\Omega (b(v_{2\delta})-b(v_{1\delta})) \cdot P_R(v_{2\delta}-v_{1\delta}) \\ &\leqq \int_\Omega (b(v_{2\delta})-b(v_{1\delta})) \cdot (v_{2\delta}-v_{1\delta}) \leqq \delta \to 0,\end{aligned}$$

that is,

$$(b(v_2)-b(v_1)) \cdot (v_2-v_1) = 0$$

almost everywhere in Ω. Then for $0<\theta<1$

$$\begin{aligned}0 &\leqq (b(v_1+\theta(v_2-v_1))-b(v_1)) \cdot (v_2-v_1) \\ &\leqq (b(v_2)-b(v_1)) \cdot (v_2-v_1) = 0,\end{aligned}$$

which tells us that the potential Φ is linear along the line segment between v_1 and v_2. We conclude that for all $z \in \mathbb{R}^m$

$$\begin{aligned}\Phi(v_2+z)-\Phi(v_2) &= \Phi(v_2+z)-\Phi(v_1)-\nabla\Phi(v_1)\cdot(v_2-v_1) \\ &\geqq \nabla\Phi(v_1)\cdot(v_2+z-v_1)-\nabla\Phi(v_1)\cdot(v_2-v_1) = b(v_1)\cdot z.\end{aligned}$$

This implies $b(v_2) = \nabla\Phi(v_2) = b(v_1)$, which is a contradiction to the above construction.

1.9. Lemma. *Suppose u_ε converge weakly in $L^r(0,T;H^{1,r}(\Omega))$ to u with the estimates*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega (b(u_\varepsilon(t+h))-b(u_\varepsilon(t)))(u_\varepsilon(t+h)-u_\varepsilon(t))\,dt \leqq C$$

and

$$\int_\Omega B(u_\varepsilon(t)) \leqq C \qquad \text{for } 0<t<T.$$

Then $b(u_\varepsilon)\to b(u)$ *in* $L^1(]0,T[\times\Omega)$ *and* $B(u_\varepsilon)\to B(u)$ *almost everywhere.*

Proof. We have to show that the functions $b(u_\varepsilon)$ are relative compact in $L^1(]0,T[\times\Omega)$, for if β is any cluster value we conclude $\beta=b(u)$ by the usual monotonicity argument as follows. Using the notation in the proof of 1.8 we obtain for $v\in L^r(0,T;H^{1,r}(\Omega))$

$$0\leqq\int_0^T\int_\Omega P_R(b(v)-b(u_\varepsilon))\cdot(v-u_\varepsilon)\to\int_0^T\int_\Omega P_R(b(v)-\beta)\cdot(v-u).$$

Replacing v by $u+\delta v$ we obtain for $\delta\downarrow 0$

$$0\leqq\int_0^T\int_\Omega P_R(b(u+\delta v)-\beta)\cdot v\to\int_0^T\int_\Omega P_R(b(u)-\beta)\cdot v,$$

which implies $b(u)=\beta$.

The first step in proving the compactness is to show that

$$\int_0^{T-h}\int_\Omega |b(u_\varepsilon(t+h))-b(u_\varepsilon(t))|\,dt\to 0$$

as $h\downarrow 0$ uniformly in ε. For this we consider for large M the sets

$$\begin{aligned}E:=\{t\in\,]0,T-h[\,/\,&\|u_\varepsilon(t+h)\|_{H^{1,r}(\Omega)}+\|u_\varepsilon(t)\|_{H^{1,r}(\Omega)}+\|u^D(t)\|_{H^{1,r}(\Omega)}\\&+\frac{1}{h}\int_\Omega(b(u_\varepsilon(t+h))-b(u_\varepsilon(t)))\cdot(u_\varepsilon(t+h)-u_\varepsilon(t))>M\}.\end{aligned}$$

Since the integral over t of the expression in this definition is bounded independent of ε, we have

$$\mathfrak{L}^1(E)\leqq\frac{C}{M},$$

and by 1.8 we have for $t\in\,]0,T-h[\,\backslash E$

$$\int_\Omega |b(u_\varepsilon(t+h))-b(u_\varepsilon(t))|\leqq\omega_M(hM),$$

and therefore using 1.2

$$\int_0^{T-h}\int_\Omega |b(u_\varepsilon(t+h))-b(u_\varepsilon(t))|\leqq T(\omega_M(hM)+C\cdot\delta)+C_\delta\cdot\frac{C}{M},$$

which by an appropriate choice of δ, M, and h is the desired estimate. Next we will approximate $b(u_\varepsilon)$ by step functions in time. Using the notation

$$v_\varepsilon(t):=\begin{Bmatrix}u_\varepsilon(t), & \text{if } t\notin E\\ 0, & \text{if } t\in E\end{Bmatrix}$$

we see that

$$\begin{aligned}
&\frac{1}{h}\int_0^h\int_0^T\int_\Omega\left|b(u_\varepsilon(t))-\sum_{i=1}^{T/h} b(v_\varepsilon((i-1)h+s))\,\chi_{](i-1)h,ih[}(t)\right|dt\,ds\\
&=\frac{1}{h}\sum_{i=1}^{T/h}\int_{(i-1)h}^{ih}\int_{(i-1)h}^{ih}\int_\Omega|b(u_\varepsilon(t))-b(v_\varepsilon(s))|\,ds\,dt\\
&\leqq\frac{1}{h}\int_{-h}^{h}\int_{\max(0,-s)}^{\min(T,T-s)}\int_\Omega|b(u_\varepsilon(t))-b(v_\varepsilon(t+s))|\,dt\,ds\\
&\leqq\sup_{|s|\leqq h}\int_{\max(0,-s)}^{\min(T,T-s)}\int_\Omega|b(u_\varepsilon(t))-b(u_\varepsilon(t+s))|\,dt\,ds+\int_E\int_\Omega|b(u_\varepsilon(t))|\,dt,
\end{aligned}$$

which by the above estimates is small uniformly in ε, if M is large and h small. Hence for all ε we find values $s_\varepsilon\in]0,h[$ such that

$$\int_0^T\int_\Omega\left|b(u_\varepsilon)-\sum_{i=1}^{T/h} b(v_\varepsilon((i-1)h+s_\varepsilon))\,\chi_{](i-1)h,ih[}\right|$$

is small. Thus it remains to prove the compactness of the functions

$$b(v_\varepsilon((i-1)h+s_\varepsilon))\in L^1(\Omega)$$

for given M and h, which is now a time independent argument. The definition of E implies that $v_\varepsilon((i-1)h+s_\varepsilon)$ are bounded in $H^{1,r}(\Omega)$, hence there is a $v\in H^{1,r}(\Omega)$ such that for a subsequence $v_\varepsilon((i-1)h+s_\varepsilon)\to v$ almost everywhere. Hence also $b(v_\varepsilon((i-1)h+s_\varepsilon))\to b(v)$ almost everywhere, but then also in $L^1(\Omega)$ by 1.2. This proves the lemma.

1.10. Generalizations. All arguments remain the same, if we assume that the coefficients a, b, and f depend on x. Also time dependence in the elliptic part, that is, for a and f, makes no difference, except that in 1.7 we have to use approximations a_h and f_h, which are piecewise constant in time, and for example defined as u_h^D in 1.7. If we want time dependence of b we need appropriate assumptions on additional terms in the proof caused by the parabolic part. Two possibilities were already given in 1.5 and 1.6, for the data there can be transformed into

$$\begin{aligned}
\tilde{b}(t,x,z)&:=b(z+u^D(t,x)),\\
\tilde{u}^D(t,x,z)&:=0,
\end{aligned}$$

that is, into homogeneous boundary data but time dependent $\tilde{b}$. It is not hard to specify assumptions including these two examples, for which the procedure in 1.7 still works. A more essential generalization is treated in Sect. 4.

2. Regularity and Uniqueness

For a single equation we can prove the uniqueness of the solution in the class of regular solutions, by which we mean solutions u such that $\partial_t b(u)$ is a function (see 2.2). But we are not able to prove the regularity of all weak solutions even under regularity conditions on the data. The reason is that $\partial_t b(u)$ is not a monotone term, hence the usual procedure of differentiating the equation does not lead to a priori estimates. To get a priori estimates we test the weak equation with $\partial_t(u-u^D)$ instead. Hence we get the existence of at least one regular solution for systems of variational type (see 2.3).

In the special case of a linear elliptic part one can prove the uniqueness in the class of weak solutions (see 2.4).

The uniqueness result in 2.2 is formulated as a comparison theorem. Therefore we define as usual

2.1. Definition. We call $u \in L^r(0,T;H^{1,r}(\Omega))$ a *subsolution* (*supersolution*), if $u \leqq (\geqq) u^D$ on $]0,T[\times\Gamma$, and if 1.4 is fulfilled with $=$replaced by $\geqq(\leqq)$ for all test functions ζ with $\zeta(0)\leqq 0$ in 1.4.1 and $\zeta \geqq 0$ in 1.4.2.

2.2. Comparison Theorem. *Suppose that the data have the continuity property*

$$|a(b(z_2),p)-a(b(z_1),p)| \leqq C\cdot|z_2-z_1|^{(r-1)/r}\cdot(1+B(z_1)^{(r-1)/r} + B(z_2)^{(r-1)/r}+|p|^{r-1}),$$
$$f(z_2)-f(z_1)\leqq C(z_2-z_1) \quad \text{for } z_2>z_1.$$

If u_- is a subsolution and u_+ a supersolution such that $B(u_\pm)$ and

$$\partial_t(b(u_-)-b(u_+))$$

are in $L^1(]0,T[\times\Omega)$, then $u_-\leqq u_+$.

Proof. For small $\delta>0$ let

$$\psi_\delta(z) := \min\left(1, \max\left(\frac{z}{\delta}, 0\right)\right)$$

and set $\zeta := \psi_\delta(u_- - u_+)$ in the time interval $]0,t[$ as test function in the inequality 1.4.2 for u_- and u_+. Then

$$\int_0^t\int_\Omega \partial_t(b(u_-)-b(u_+))\,\psi_\delta(u_- - u_+) + \frac{c}{\delta}\int_0^t\int_\Omega \chi_{\{0<u_- - u_+<\delta\}}|\nabla(u_- - u_+)|^r$$
$$\leqq \frac{C}{\delta}\int_0^t\int_\Omega \chi_{\{0<u_- - u_+<\delta\}}|a(b(u_-),\nabla u_+)-a(b(u_+),\nabla u_+)|^{r^*}$$
$$+\int_0^t\int_\Omega \chi_{\{u_- - u_+>0\}}\max(0, f(b(u_-))-f(b(u_+)))$$
$$\leqq C\int_0^t\int_\Omega \chi_{\{0<u_- - u_+<\delta\}}(1+B(u_-)+B(u_+)+|\nabla u_+|^r)$$
$$+C\int_0^t\int_\Omega \max(b(u_-)-b(u_+),0).$$

The first term on the right tends to zero as $\delta\to 0$, and the parabolic term on the left converges to

$$\int_0^t\int_\Omega \chi_{\{u_->u_+\}}\partial_t(b(u_-)-b(u_+)) = \int_0^t\int_\Omega \partial_t \max(b(u_-)-b(u_+),0)$$
$$=\int_\Omega \max(b(u_-(t))-b(u_+(t)),0)$$

by the inequality in 1.4.1 for u_- and u_+. Then Gronwall's lemma yields $b(u_-)\leqq b(u_+)$, in particular $b(u_-)=b(u_+)$ in the set $\{u_- > u_+\}$. If we go with this information in the first estimate above we see that everything cancels except the elliptic term on the left. Thus we obtain $\nabla(u_- - u_+)=0$ in $\{0<u_- - u_+ <\delta\}$, in other words $\max(0, \min(u_- - u_+, \delta))=\text{const}$. This implies $u_- \leqq u_+$, since it is true on $]0, T[\times\Gamma$.

In the remainder of this section we consider systems. First let us prove the existence of a regular solution.

2.3. Theorem. *Assume* 1.1 *with* $r\geqq 2$, *and assume* b *is Lipschitz continuous. Suppose there is a continuous function* A *of* $b(z)$ *and* p *such that* $\nabla_p A=a$, *and such that* $\nabla_z A$ *and* $\nabla_z f$ *are measurable with*

$$|\nabla_z A(b(z), p)|^2+|\nabla_z f(b(z))|^2+|A(b(z), 0)|\leqq C(1+B(z)+|p|^r).$$

Moreover for the boundary and initial data assume $u^D\in H^{1,r}(0, T; H^{1,r}(\Omega))$ *and* $b^0=b(u^0)$ *for some* u^0 *with* $u^0-u^D(0)\in V$. *Then there is a weak solution* u *with* $\partial_t b(u)\in L^2(]0, T[\times\Omega)$.

Proof. In fact, we will show that the solution constructed in 1.7 is regular. For this we have to prove an a priori estimate for the approximating solutions u_{hm}, where we have to choose the initial data $u^0_{hm}\in u^D_h(0)+V_m$ such that they are bounded and converge to u^0 in $H^{1,r}(\Omega)$. We test the equation for u (let us write u instead of u_{hm}) with $\partial_t^{-h}(u-u_h^D)$ in the time interval $]0, t_h[$, where t_h is a multiple of h converging to t. We get

$$\int_0^{t_h}\int_\Omega \partial_t^{-h} b(u)\, \partial_t^{-h}(u-u_h^D)+\int_0^{t_h}\int_\Omega a(b(u), \nabla u)\, \partial_t^{-h}\nabla(u-u^D)=\int_0^{t_h}\int_\Omega f(b(u))\, \partial_t^{-h}(u-u_h^D).$$

In order to handle the parabolic term we note that

$$\partial_t^{-h} b(u)\cdot\partial_t^{-h} u\geqq c\,|\partial_t^{-h} b(u)|^2.$$

In fact this is equivalent to

$$(z_1-z_2)\cdot\int_0^1 \nabla b(t z_1+(1-t) z_2)\, dt\cdot(z_1-z_2)\geqq c\left|\int_0^1 \nabla b(t z_1+(1-t) z_2)\, dt\cdot(z_1-z_2)\right|^2,$$

which easily follows from the fact that $\nabla b(z)$ is a positive semi-definite symmetric matrix uniformly bounded in z. We conclude that

$$\partial_t^{-h} b(u)\cdot\partial_t^{-h}(u-u_h^D)\geqq c|\partial_t^{-h} b(u)|^2-C\,|\partial_t^{-h} u_h^D|^2.$$

Integrating the elliptic part we obtain

$$\begin{aligned}\int_0^{t_h}\int_\Omega a(b(u), \nabla u)\, \partial_t^{-h}\nabla u &\geqq \int_\Omega A(b(u(t_h)), \nabla u(t_h))-\int_\Omega A(b(u^0_{hm}), \nabla u^0_{hm})\\ &\quad -\frac{C}{h}\int_0^{t_h}\int_\Omega |A(b(u(t)), \nabla u(t-h))-A(b(u(t-h)), \nabla u(t-h))|\, dt\\ &\geqq \int_\Omega A(b(u(t_h)), \nabla u(t_h))-\int_\Omega A(b(u^0_{hm}), \nabla u^0_{hm})\end{aligned}$$

$$-C\int_0^{t_h}\int_\Omega (1+B(u(t))+B(u(t-h))+|\nabla u(t-h)|^r)^{\frac{1}{2}}\cdot|\partial_t^{-h}b(u(t))|\,dt$$

$$\geq \int_\Omega A(b(u(t_h)),\nabla u(t_h))-\delta\int_0^{t_h}\int_\Omega|\partial_t^{-h}b(u)|^2$$

$$-C_\delta\left(1+\int_\Omega(\Psi(b(u_{hm}^0))+|\nabla u_{hm}^0|^r)+\int_0^{t_h}\int_\Omega(\Psi(b(u))+|\nabla u|^r)\right).$$

Moreover

$$\left|\int_0^{t_h}\int_\Omega a(b(u),\nabla u)\,\partial_t^{-h}\nabla u_h^D\right|\leq C\int_0^{t_h}\int_\Omega(1+\Psi(b(u))+|\nabla u|^r+|\partial_t^{-h}\nabla u_h^D|^r),$$

and by partial integration

$$\int_0^{t_h}\int_\Omega f(b(u))\,\partial_t^{-h}(u-u_h^D)\leq\int_\Omega f(b(u(t_h)))(u-u_h^D)(t_h)+\delta\int_0^{t_h}\int_\Omega|\partial_t^{-h}b(u)|^2$$

$$+C_\delta\left(1+\int_\Omega(\Psi(b(u_{hm}^0))+|u_{hm}^0-u_h^D(0)|^r)+\int_{-h}^{t_h}\int_\Omega(\Psi(b(u))+|\nabla u|^r+|\nabla u_h^D|^r)\right).$$

Since

$$A(b(z),p)\geq A(b(z),0)+a(b(z),0)\cdot p+c|p|^r\geq c|p|^r-C(1+B(z)),$$

we conclude using the a priori estimate for u in the proof of 1.7

$$\int_0^{t_h}\int_\Omega|\partial_t^{-h}b(u)|^2+\int_\Omega|\nabla u(t_h)|^r$$

$$\leq C\left(1+\int_0^{t_h}\int_\Omega\Psi(b(u))+\int_\Omega(\Psi(b(u(t_h)))+\Psi(b(u_{hm}^0)))\right).$$

From this estimate the theorem follows.

Now we will prove a uniqueness result in the case that $a(z,p)$ is linear in p. This is connected to the theory of monotone operators, since formally an equation of the form

$$\partial_t b(u)=Au$$

is equivalent to

$$\partial_t A^{-1}v=b^{-1}(v),$$

which now is a monotone evolution equation (see [17]). Since usually this approach is used only for time independent operators A, in our theorem the time dependence is stated explicitly.

2.4. Theorem. *Suppose* 1.1 *with* $r=2$ *and*

$$a(t,x,b(z),p)=A(t,x)\,p+e(b(z)),$$

where $A(t,x)$ *is a symmetric matrix and measurable in* t *and* x *such that for some* $\alpha>0$

$$A-\alpha I \quad \textit{and} \quad A+\alpha\partial_t A$$

are positive definite. Moreover assume that

$$|e(b(z_2))-e(b(z_1))|^2+|f(b(z_2))-f(b(z_2))|^2 \leqq C(b(z_2)-b(z_1))(z_2-z_1).$$

Then there is at most one weak solution.

Proof. Suppose u_1 and u_2 are two weak solutions. Then

$$\beta := b(u_2)-b(u_1) \in L^2(0, T; V^*)$$

by 1.4.1, hence there is a function $v \in L^2(0, T; V)$ such that

$$\int_0^T \int_\Omega \nabla v A \nabla \zeta = \int_0^T \langle \beta, \zeta \rangle$$

for all $\zeta \in L^2(0, T; V)$. Then

$$\begin{aligned}
&2\int_h^{\tau+h} \langle \partial_t^{-h}\beta, v\rangle + \frac{1}{h}\int_0^h \langle \beta, v\rangle \\
&= \int_0^\tau \langle \partial_t^h \beta(t), v(t+h)\rangle\,dt - \int_0^\tau \langle \beta, \partial_t^h v\rangle + \frac{1}{h}\int_\tau^{\tau+h} \langle \beta, v\rangle \\
&= \int_h^{\tau+h}\int_\Omega \nabla v \partial_t^{-h} A \nabla v + \frac{1}{h}\int_\tau^{\tau+h}\int_\Omega \nabla v A \nabla v \\
&\quad + \frac{1}{h}\int_0^\tau \int_\Omega (\nabla v(t+h)-\nabla v(t))\,A(t)(\nabla v(t+h)-\nabla v(t))\,dt.
\end{aligned}$$

Letting $h \to 0$ we obtain for almost all τ

$$\int_0^\tau \langle \partial_t \beta, v\rangle = \frac{1}{2}\left(\int_0^\tau \int_\Omega \nabla v \partial_t A \nabla v + \int_\Omega \nabla v(\tau)\,A(\tau)\,\nabla v(\tau)\right).$$

On the other hand we have

$$\int_0^\tau \int_\Omega \nabla v A \nabla (u_2-u_1) = \int_0^\tau \langle \beta, u_2-u_1\rangle = \int_0^\tau \int_\Omega (b(u_2)-b(u_1))(u_2-u_1).$$

Hence using v as test function in the weak differential equation 1.4.2 we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \nabla v(\tau)\,A(\tau)\,\nabla v(\tau) + \int_0^\tau \int_\Omega (b(u_2)-b(u_1))(u_2-u_1) \\
&= \int_0^\tau \int_\Omega ((f(b(u_2))-f(b(u_1)))\,v - (e(b(u_2))-e(b(u_1)))\,\nabla v - \tfrac{1}{2}\nabla v \partial_t A \nabla v) \\
&\leqq \delta \int_0^\tau \int_\Omega (b(u_2)-b(u_1))(u_2-u_1) + \tfrac{1}{2}\int_0^\tau \int_\Omega \nabla v (C_\delta I - \partial_t A)\,\nabla v.
\end{aligned}$$

Then the assertion follows from Gronwall's lemma.

3. Variational Inequalities

Here we prove two existence theorems for variational inequalities of elliptic-parabolic differential equations. For the first one we assume that the constraint is time independent and therefore given by a closed convex set in $H^{1,r}(\Omega)$. For this situation we prove the existence of a regular solution (see 3.1) under the assumptions in 2.3. Under the general assumptions in Sect. 1 we can prove the existence of a weak solution only for equations with special constraints in $L^r(\Omega)$, namely for obstacle problems, but here we can include a large class of time dependent obstacles (see 3.2).

3.1. Theorem. *Suppose the data r, a, b, and f satisfy the assumptions in Theorem 2.3, and suppose the initial data $b^0 = b(u^0)$ satisfy $u^0 \in K$, where K is a closed convex set in $H^{1,r}(\Omega)$ satisfying $\|v\|_{L^r(\Omega)} \leqq C(1+\|\nabla v\|_{L^r(\Omega)})$ for $v \in K$. Then there is a function $u \in L^r(0,T;K)$ with $\partial_t b(u) \in L^2(]0,T[\times\Omega)$ satisfying the initial condition $b(u(0)) = b(u^0)$ and solving the variational inequality*

$$\int_0^T \int_\Omega \partial_t b(u)(v-u) + \int_0^T \int_\Omega a(b(u), \nabla u)\nabla(v-u) \geqq \int_0^T \int_\Omega f(b(u))(v-u)$$

for all $v \in L^r(0,T;K)$.

Proof. We will use the method of penalization. For this denote by P the projection onto K defined by

$$\langle j(u-Pu), Pu - v\rangle \geqq 0 \qquad \text{for } v \in K.$$

Here j means the dual mapping defined by

$$\langle j(u), v\rangle = \int_\Omega (|u|^{r-2}u\zeta + |\nabla u|^{r-2}\nabla u \cdot \nabla\zeta).$$

For positive small ε we are looking for a solution u_ε of the penalized equation

$$\partial_t b(u_\varepsilon) + \frac{1}{\varepsilon} j(u_\varepsilon - Pu_\varepsilon) - \nabla \cdot a(b(u_\varepsilon), \nabla u_\varepsilon) = f(b(u_\varepsilon)).$$

Since we will prove the existence of a regular solution of the variational inequality, we have to give an a priori bound for $\partial_t b(u_\varepsilon)$, and therefore to combine the existence proof in Sect 1 with the proof of 2.3. We may proceed as before with the exception of the additional monotone penalty term. Therefore we have to check the sign of this term for the various test functions used in the proof of Theorems 1.7 and 2.3. We start with approximating solutions $u_{\varepsilon hm} \in L^r(0,T;V_m)$, which are step functions in time satisfying the identity

$$\int_0^T \int_\Omega \partial_t^{-h} b(u_{\varepsilon hm})\zeta + \frac{1}{\varepsilon}\int_0^T \langle j(u_{\varepsilon hm} - P(u_{\varepsilon hm})), \zeta\rangle + \int_0^T \int_\Omega a(b(u_{\varepsilon hm}), \nabla u_{\varepsilon hm})\nabla\zeta = \int_0^T \int_\Omega f(b(u_{\varepsilon hm}))\zeta$$

for all functions $\zeta \in L^r(0,T;V_m)$ with $b(u_{\varepsilon hm}(t)) = b(u^0)$ for $-h < t < 0$. Here V_m is a finite dimensional subspace approximating $V = H^{1,r}(\Omega)$. For the energy estimate we take (let us write u instead of $u_{\varepsilon hm}$) $\zeta = u - u^0$ as test function. Since

$$\langle j(u-Pu), u-u^0\rangle = \|u-Pu\|_{H^{1,r}}^r + \langle j(u-Pu), Pu-u^0\rangle,$$

where both terms on the right are non-negative, this estimate is independent of ε (and of course as in 1.7 independent of h and m). The second step is to prove compactness by an a priori bound for $\partial_t^{-h} b(u)$ similarly as in 2.3. For this we test with $\zeta = \partial_t^{-h} u$ in the time interval $]0, t_h[$. Then the penalty term becomes

$$\frac{1}{h}\int_0^{t_h} \langle j(u-Pu)(t), u(t)-u(t-h)\rangle\, dt$$
$$=\frac{1}{h}\int_0^{t_h} \langle j(u-Pu)(t), Pu(t)-Pu(t-h)\rangle\, dt$$
$$+\frac{1}{h}\int_0^{t_h} \langle j(u-Pu)(t), (u-Pu)(t)-(u-Pu)(t-h)\rangle\, dt.$$

The first term is non-negative by definition of P and the second term by convexity is

$$\geqq \frac{1}{hr}\int_0^{t_h} (\|(u-Pu)(t)\|_{H^{1,r}}^r - \|(u-Pu)(t-h)\|_{H^{1,r}}^r)\, dt$$
$$=\frac{1}{hr}\int_{t_h-h}^{t_h} \|u-Pu\|_{H^{1,r}}^r \to \frac{1}{r}\|(u-Pu)(t_h)\|_{H^{1,r}}^r$$

for $h \to 0$. Here we used the fact that $u^0 \in K$. For the parabolic part we have

$$\int_0^{t_h}\int_\Omega \partial_t^{-h} b(u)\,\partial_t^{-h} u \geqq c \int_0^{t_h}\int_\Omega |\partial_t^{-h} b(u)|^2,$$

since b is Lipschitz continuous. Since

$$\int_0^{t_h}\int_\Omega a(b(u), \nabla u)\,\partial_t^{-h}\nabla u \geqq \frac{1}{h}\int_0^{t_h}\int_\Omega (A(b(u(t)), \nabla u(t)) - A(b(u(t)), \nabla u(t-h)))\, dt$$

the elliptic part can be handled similarly as in 2.3. We thus obtain an a priori estimate for

$$\int_0^T |\partial_t^{-h} b(u)|^2 + \sup_{0\leqq t\leqq T}\left(\int_\Omega A(b(u(t)), \nabla u(t)) + \frac{1}{\varepsilon}\|(u-Pu)(t)\|_{H^{1,r}}^r\right)$$

independent of ε, h, and m. To prove the strong convergence of $u_{\varepsilon hm}$ to its weak limit u_ε we take $\zeta = u_{\varepsilon hm} - u_\varepsilon$ as test function in $]0, t_h[$. Because of the monotonicity the penalty term becomes

$$\int_0^{t_h} \langle j(u_{\varepsilon hm} - Pu_{\varepsilon hm}), u_{\varepsilon hm} - u_\varepsilon\rangle \geqq \int_0^{t_h} \langle j(u_\varepsilon - Pu_\varepsilon), u_{\varepsilon hm} - u_\varepsilon\rangle \to 0$$

for $(h,m)\to(0,\infty)$. This proves the existence of a solution u_ε of the penalized problem with the above a priori estimates. These estimates in particular imply that for a subsequence $u_\varepsilon \to u$ weakly, $b(u_\varepsilon)\to b(u)$ strongly, and $\|u_\varepsilon - Pu\|_{L^\infty(0,T;H^{1,r}(\Omega))} \to 0$, that is, $u(t)\in K$ for almost all t. For the strong convergence of u_ε to u we take as always $\zeta = u_\varepsilon - u$ as test function. Since $u(t)\in K$ we have

$$\langle j(u_\varepsilon - Pu_\varepsilon), u_\varepsilon - u\rangle \geqq \langle j(u_\varepsilon - Pu_\varepsilon), Pu_\varepsilon - u\rangle \geqq 0,$$

hence the strong convergence follows as before. Then one derives the variational inequality as usual.

Now we prove another existence theorem for the time dependent obstacle problem. We cannot show in general $\partial_t b(u) \in L^{r^*}(0, T; H^{1,r}(\Omega)^*)$ in that case. And so the penalty method seems not to be the most appropriate here.

3.2. Theorem. *Let the conditions on a, b, f, b^0 be as in the existence Theorem* 1.7. *Suppose two obstacles $\psi_- \leqq \psi_+$ given with $\psi_\pm \in L^r(0, T; H^{1,r}(\Omega)) \cap L^\infty(]0, T[\times\Omega)$ and $\partial_t \psi_\pm \in L^1(]0, T[\times\Omega)$ and $\psi_-(0) \leqq u^0 \leqq \psi_+(0)$. Then there exists a solution $u \in L^r(0, T; H^{1,r}(\Omega))$ with $\psi_- \leqq u \leqq \psi_+$ of the variational inequality*

$$[\partial_t b(u), \alpha(v-u)] + \int_0^T \int_\Omega (a(b(u), \nabla u) \cdot \nabla(\alpha(v-u)) - f(b(u))\alpha(v-u)) \geqq 0$$

for all non-negative $\alpha \in C^1([0, T])$ with $\alpha(T) = 0$, and for all $v \in L^r(0, T; H^{1,r}(\Omega))$ with $\partial_t v \in L^1(]0, T[\times\Omega)$ and $\psi_- \leqq v \leqq \psi_+$.

Here the brackets on the left are to be understood formally as

$$[\partial_t b(u), \alpha(v-u)] := \int_0^T \int_\Omega B(u)\partial_t \alpha - \int_0^T \int_\Omega b(u)\partial_t(\alpha v) + \int_\Omega B(u^0)\alpha(0) \\ - \int_\Omega b(u^0(x)) v(0, x) \alpha(0)\, dx.$$

Proof. We approximate the inequality by time discretization. Thus the discrete variational inequality reads

$$\int_\Omega \partial_t^{-h} b(u_h(t))(v-u_h)(t) + \int_\Omega (a(b(u_h(t-h)), \nabla u_h(t)) \cdot \nabla(v-u_h)(t) \\ - f(b(u_h(t-h)))(v-u_h)(t)) \geqq 0$$

with $\psi_-^h \leqq u_h$, $v \leqq \psi_+^h$. As initial condition we pose

$$b(u_h(t)) = b^0 \quad \text{for } t < 0$$

and the constraint we approximate by

$$\psi_-^h(t) := \frac{1}{h} \int_{ih}^{(i+1)h} \psi_- \quad \text{and} \quad \psi_+^h(t) := \frac{1}{h} \int_{ih}^{(i+1)h} \psi_+ \qquad \text{for } ih \leqq t \leqq (i+1)h.$$

Setting $v = \psi_-^h$ we get an estimate on

$$\int_0^T \int_\Omega |\nabla u_h|^r.$$

We cannot expect an estimate for $\partial_t^{-h} b(u_h)$. The reason is that not all $C_0^\infty(\Omega)$ functions are allowed as test functions. But multiplying those functions with a function of u_h which measures the distance to the constraint we are able to get an estimate on the time derivative of another monotone function of u.

To make this construction we take

$$v(t, x) = u_h(t, x) \pm g(u_h(t, x), t, x)\, \varphi(t, x)$$

where φ is any test function in $L^r(0, T; H^{1,r}(\Omega))$ with $|\varphi| \leqq 1$. The function g (depending on h) is defined, for example, by

$$g(z, t, x) := \hat{g}(z - \psi_-^h(t, x))\, \hat{g}(\psi_+^h(t, x) - z),$$

$$\hat{g}(z) = \begin{cases} \dfrac{z^2}{1+z^2} & \text{for } z \geqq 0, \\ 0 & \text{for } z \leqq 0. \end{cases}$$

We get

$$\begin{aligned} \partial_t^{-h} b(u_h)(t)\, g(u_h)(t) = \nabla \cdot (g(u_h)(t)\, a(b(u_h(t-h), \nabla u_h(t))) \\ + (f(b(u_h(t-h))) - \nabla(g(u_h))(t) \cdot a(b(u_h(t-h)), \nabla u_h(t))) \end{aligned}$$

in the dual of $L^r(0, T; H^{1,r}(\Omega)) \cap L^\infty(]0, T[\times \Omega)$. If we define the function

$$G(z, t, x) := b(z)\, g(z, t, x) - \int_0^z b(s)\, g'(s, t, x)\, ds,$$

which is monotone in z, we have the identity

$$\begin{aligned} \partial_\tau^{-h} G(u_h(\tau, x), t, x) = \partial_\tau^{-h} b(u_h(\tau, x))\, g(u_h(\tau, x), t, x) \\ - \int_{u_h(\tau-h)}^{u_h(\tau)} \frac{b(s) - b(u_h(\tau-h))}{h}\, g'(s, t, x)\, ds. \end{aligned}$$

The difficulty in getting an estimate for $\partial_t^{-h} G(u_h)$ is now reduced to finding an estimate for the second term on the right hand side in $L^1(]h, T[\times \Omega)$. So choosing as a test function in the discrete variational inequality

$$v(t) = \min(\psi_+^h(t), \max(u_h(t-h), \psi_-^h(t))),$$

we obtain a bound on

$$\begin{aligned} &\int_\Omega \partial_t^{-h} b(u_h(t))(u_h(t) - u_h(t-h)) \\ &\qquad \leqq C(1 + \int_\Omega (|\nabla u_h(t)|^r + |\nabla u_h(t-h)|^r + |\nabla \psi_\pm(t)|^r + |\partial_t^{-h} \psi_\pm|)). \end{aligned}$$

Since

$$\begin{aligned} &\left| \int_{u_h(t-h)}^{u_h(t)} \frac{b(s) - b(u_h(t-h))}{h}\, g'(s, t, x)\, ds \right| \\ &\qquad \leqq \partial_t^{-h} b(u_h(t))(u_h(t) - u_h(t-h)) \end{aligned}$$

we finally can conclude that

$$\frac{1}{kh}\left(G(u_h(t, x), t, x) - G(u_h(t - kh, x), t, x)\right)$$

is bounded in the dual of $L^r(kh, T; H^{1,r}(\Omega)) \cap L^\infty(]kh, T[\times \Omega)$. Multiplying these difference quotients with time differences $u_h(t) - u_h(t - kh)$ we see as in the proof of the existence Theorem 1.7 that $G(u_h)$ converges almost everywhere and so also $b(u_h)$.

In order to prove strong convergence of u_h let v be as in the statement of the theorem. Take

$$v_h = \max(\psi^h_-, \min(\psi^h_+, v))$$

as test function for u_h and integrate from 0 to t. For the parabolic part we get

$$\begin{aligned}\int_0^t \int_\Omega \partial_t^{-h} b(u_h)(u_h - v_h) &\geqq \frac{1}{h}\int_{t-h}^t \int_\Omega B(u_h) - \int_\Omega B(u^0)\\ &\quad - \left(\frac{1}{h}\int_{t-h}^t \int_\Omega b(u_h)\,v_h - \frac{1}{h}\int_0^h \int_\Omega b(u^0)\,v_h\right) + \int_0^{t-h}\int_\Omega b(u_h)\,\partial_t^h v_h\\ &= [\partial_t b(u), \chi_{]0,t[\times\Omega}(u-v)] + o(1)\end{aligned}$$

as $h \to 0$ for almost all t. The elliptic part is estimated by

$$\geqq c\int_0^t \int_\Omega |\nabla(u_h - v_h)|^r + \int_0^t \int_\Omega (a(b(u), \nabla v)\,\nabla(u-v) - f(b(u))(u-v)) + o(1),$$

therefore

$$\begin{aligned}&[\partial_t b(u), \chi_{]0,t[\times\Omega}(u-v)] + \int_0^t \int_\Omega (a(b(u), \nabla v)\,\nabla(u-v) - f(b(u))(u-v))\\ &\quad + c\limsup_{h\to 0}\int_0^t \int_\Omega |\nabla(u_h - v)|^r \leqq 0.\end{aligned}$$

As v we use the following approximation of u:

$$v_{\tau\varepsilon} := \max(\psi_-, \min(\psi_+, u_{\tau\varepsilon}))$$

for $h>0$, $0<\tau<h$, $0<\varepsilon<h$, where

$$u_{\tau\varepsilon}(t) := u(ih-\tau) + \max\left(0, 1 - \frac{(i+1)h-\tau-t}{\varepsilon}\right)(u((i+1)h-\tau) - u(ih-\tau))$$

for $ih-\tau<t<(i+1)h-\tau$ and $0 \leqq i \leqq k := \frac{T}{h}$. If the initial data u^0 are not in $H^{1,r}(\Omega)$ we replace in this definition $u(t)$ for $t<0$ by functions $u^\delta \in H^{1,r}(\Omega)$ such that $\psi_-(0) \leqq u^\delta \leqq \psi_+(0)$ and

$$\int_\Omega |B(u^\delta) - B(u^0)| + \int_\Omega |(b(u^\delta) - b(u^0))\,u^\delta| \leqq \delta,$$

and let finally $\delta \to 0$. Since $v_{\tau\varepsilon}(t) = u(t)$ for $t = ih - \tau$, we obtain for small ε

$$\begin{aligned}[\partial_t b(u), \chi_{]0,T-\tau[\times\Omega}(u - v_{\tau\varepsilon})] &\geqq -\delta + \int_\Omega (B(u(T-\tau)) - B(u^\delta))\\ &\quad - \int_\Omega (b(u(T-\tau))\,u(t-\tau) - b(u^\delta)\,u^\delta) + \int_0^{T-\tau}\int_0 b(u)\,\partial_t v_{\tau\varepsilon}\\ &\geqq -\delta + \sum_{i=0}^{k-1}\int_\Omega b(u((i+1)h-\tau))(u(ih-\tau) - u((i+1)h-\tau))\\ &\quad + \int_0^{T-\tau}\int_\Omega b(u)\,\partial_t v_{\tau\varepsilon}.\end{aligned}$$

Now for almost all τ almost everywhere in Ω

$$\int_{ih-\tau}^{(i+1)h-\tau} b(u)\,\partial_t v_{\tau\varepsilon} \geqq - \int_{ih-\tau}^{(i+1)h-\tau} |b(u)-b(u((i+1)h-\tau))|\,|\partial_t \psi_\pm| + b(u((i+1)h-\tau))(u((i+1)h-\tau)-u(ih-\tau))+o(1)$$

as $\varepsilon \to 0$, which yields

$$\liminf_{\varepsilon\to 0} [\partial_t b(u), \chi_{]0,T-\tau[\times\Omega}(u-v_{\tau\varepsilon})] \geqq -\delta - \sum_{i=0}^{k-1} \int_{ih-\tau}^{(i+1)h-\tau} \int_\Omega |b(u)-b(u((i+1)h-\tau))|\,|\partial_t \psi_\pm|.$$

The elliptic part converges for $\varepsilon\to 0$, since $v_{\tau\varepsilon}\to v_\tau$ in $L^r(0,T;H^{1,r}(\Omega))$ (for δ fixed). Then integrating over τ and letting $h\to 0$ we obtain

$$\lim_{h'\to 0} \frac{1}{h} \int_0^h \int_0^{T-\tau} \int_\Omega |\nabla(u_{h'}-v_\tau)|^r\, d\tau \leqq \delta + o(1),$$

and therefore for $0<t_1<t_2<T$

$$\lim_{h'\to 0} \int_{t_1}^{t_2} \int_\Omega |\nabla(u_{h'}-u)| = 0.$$

That the limit fulfils the variational inequality is seen as usually.

3.3. Remark. For simplicity we did not include the case of additional Dirichlet data in 3.2. But this is no problem, since the proof remains exactly the same, if we replace $H^{1,r}(\Omega)$ by

$$V := \{v\in H^{1,r}(\Omega)/v=0 \text{ on } \Gamma\},$$

where Γ is as in 1.1.1, that is, if in addition we consider time independent Dirichlet data. Of course, we then have to assume that $\psi_-(t)\leqq 0\leqq \psi_+(t)$ on Γ for all t.

4. Stefan Problems

In this section we allow the monotone vector field b to be non-continuous. This is the case for the well known class of Stefan problems, for which b usually consists of a continuous function and a step function. The main difference to the previous existence proofs then is that we get only the weak convergence of the parabolic part. Therefore in this paper the nonlinearities are allowed to depend only on the continuous part of b. Let us describe the situation more carefully.

4.1. Assumptions. We assume that b is a monotone subgradient, that is, there is a convex function $\Phi: \mathbb{R}^m \to \mathbb{R}$ such that

$$\xi\in b(z) \quad \text{if and only if} \quad \Phi(\sigma)-\Phi(z) \geqq \xi\cdot(\sigma-z) \quad \text{for all } \sigma\in\mathbb{R}^m.$$

The elliptic part is given by $a(b_s(z), b_c(z), p)$ and $f(b_s(z), b_c(z))$ with continuous mappings a and f satisfying the ellipticity condition 1.1.3 in p. Here $b = b_s + b_c$ with a continuous monotone gradient field b_c and a monotone subgradient b_s. We assume that a and f do not depend on the jumps of b_s, which means that, for example, for f

$$\xi_1, \xi_2 \in b_s(z) \quad \text{implies} \quad f(\xi_1, b_c(z)) = f(\xi_2, b_c(z)).$$

This justifies the notation $f(b_s(z), b_c(z))$. The growth condition now is

$$|a(\tilde{z}, b_c(z), p)| + |f(\tilde{z}, b_c(z))| \leqq C(1 + B_c(z)^{(r-1)/r} + |p|^{r-1}).$$

Sometimes we will write $f(b(z))$ instead of $f(b_s(z), b_c(z))$, which is not correct but simplifies the formulas. Because of technical details in the compactness argument (see 4.4) we need the additional growth condition

$$|\xi| \leqq C(1 + |z|^{r-1}) \quad \text{for } \xi \in b_s(z).$$

First we summarize some basic properties of the function Φ.

4.2. Remark. The function Ψ_b is now defined by

$$\Psi_b(z) := \sup_{\sigma \in \mathbb{R}^m} (z \cdot \sigma - \Phi(\sigma) + \Phi(0))$$

and satisfies

$$\Psi_b(\xi) = \xi \cdot z - \Phi(z) \quad \text{for } \xi \in b(z).$$

The function B is not uniquely determined. Later we will use the notation $B(u) = \beta \cdot u - \Phi(u)$ for a given function $\beta \in b(u)$. Then as before

$$|\beta| \leqq \delta B(u) + \delta \sup_{|\sigma| \leqq \frac{1}{\delta}} (\Phi(\sigma) - \Phi(0)).$$

In the existence proof (4.5) we shall approximate b by smooth vector fields $b_\varepsilon = \nabla \Phi_\varepsilon$, where Φ_ε is given by

$$\Phi_\varepsilon(z) := \int_{\mathbb{R}^m} \Phi_s(z - c)\, \phi_\varepsilon(\sigma)\, d\sigma + \varepsilon |z|^r + \Phi_c(z).$$

In this definition ϕ_ε is a smooth symmetric mollifier with support in $B_\varepsilon(0)$. It is easy to see that $|b_{s\varepsilon}(z)| \leqq C(1 + |z|^{r-1})$ uniformly in ε.

Our existence result for Stefan problems is based on the technique in Sect. 1 and the following additional observation.

4.3. Lemma. *Suppose $u_\varepsilon \to u$ weakly in $L^r(0, T; H^{1,r}(\Omega))$ and $b_{s\varepsilon}(u_\varepsilon) \to \beta_s$ weakly in $L^1(]0, T[\times \Omega)$ such that*

$$\int_D (b_{s\varepsilon}(u_\varepsilon \circ \tau) - b_{s\varepsilon}(u_\varepsilon)) \cdot (u_\varepsilon \circ \tau - u_\varepsilon) \to 0$$

uniformly in ε for subsets $D \subset\subset]0, T[\times \Omega$ and for translations $\tau(t, x) = (t + h, x)$ with $h \to 0$. Then $\beta_s \in b_s(u)$.

Proof. For every $v \in L^r(]0,T[\times\Omega)$ we compute pointwise

$$\begin{aligned}\Phi_{s\varepsilon}(v)-\Phi_{s\varepsilon}(u_\varepsilon)-b_{s\varepsilon}(u_\varepsilon\circ\tau)\cdot(v-u_\varepsilon)\\ \geqq \Phi_{s\varepsilon}(u_\varepsilon\circ\tau)-\Phi_{s\varepsilon}(u_\varepsilon)-b_{s\varepsilon}(u_\varepsilon\circ\tau)\cdot(u_\varepsilon\circ\tau-u_\varepsilon)\\ \geqq -(b_{s\varepsilon}(u_\varepsilon\circ\tau)-b_{s\varepsilon}(u_\varepsilon))\cdot(u_\varepsilon\circ\tau-u_\varepsilon).\end{aligned}$$

Since $\Phi_{s\varepsilon}(z)\leqq C(1+|z|^r)$ uniformly in ε we see that $\Phi_{s\varepsilon}(v)\to\Phi_s(v)$ in $L^1(]0,T[\times\Omega)$ as $\varepsilon\to 0$. Moreover for any $\beta\in b_s(u)$ we have

$$\int_D(\Phi_{s\varepsilon}(u_\varepsilon)-\Phi_s(u))=\int_D\int_{\mathbb{R}^m}(\Phi_s(u_\varepsilon-\sigma)-\Phi_s(u))\,\phi_\varepsilon(\sigma)\,d\sigma\geqq\int_D\beta\cdot(u_\varepsilon-u)\to 0,$$

which shows

$$\int_D\Phi_s(u)\leqq\liminf_{\varepsilon\to 0}\int_D\Phi_{s\varepsilon}(u_\varepsilon).$$

We obtain that

$$\int_D(\Phi_s(v)-\Phi_s(u)-b_{s\varepsilon}(u_\varepsilon\circ\tau)\cdot(v-u_\varepsilon))\geqq-\omega(\varepsilon)-\omega(h)$$

with a continuous function ω satisfying $\omega(0)=0$. Therefore for small $\delta>0$ using mollifiers $\varphi_\delta(h,\xi)=\varphi_\delta^1(h)\,\varphi_\delta^2(\xi)$ we obtain the estimate

$$\int_D((\Phi_s(v)-\Phi_s(u))*\varphi_\delta^2-((b_{s\varepsilon}(u_\varepsilon)*\varphi_\delta^1)(v-u_\varepsilon))*\varphi_\delta^2)\geqq-\omega(\varepsilon)-\tilde\omega(\delta).$$

Since u_ε are bounded in $L^r(0,T;H^{1,r}(\Omega))$ we see by the growth condition on b_s that uniformly for $|\xi|\leqq\delta$

$$\begin{aligned}&\Big|\int_D b_{s\varepsilon}(u_\varepsilon)*\varphi_\delta^1(t,x+\xi)\,((v-u_\varepsilon)(t,x+\xi)-(v-u_\varepsilon)(t,x))\,dx\,dt\Big|\\ &\leqq C\Big(\int_D|(v-u_\varepsilon)(t,x+\xi)-(v-u_\varepsilon)(t,x)|^r\Big)^{\frac{1}{r}}\leqq C\cdot\delta.\end{aligned}$$

Also $(\Phi_s(v)-\Phi_s(u))*\varphi_\delta^2$ converges to $\Phi_s(v)-\Phi_s(u)$ in $L^1(D)$. Therefore the mollified function $b_{s\varepsilon}(u_\varepsilon)*\varphi_\delta$ satisfies the estimate

$$\int_D(\Phi_s(v)-\Phi_s(u)-(b_{s\varepsilon}(u_\varepsilon)*\varphi_\delta)\cdot(v-u_\varepsilon))\geqq-\omega(\varepsilon)-\tilde{\tilde\omega}(\delta).$$

By assumption we know that $b_{s\varepsilon}(u_\varepsilon)*\varphi_\delta\to\beta_s*\varphi_\delta$ in $L^{r^*}(D)$ as $\varepsilon\to 0$ for fixed δ. Moreover the growth condition on b_s implies that $\beta_s\in L^{r^*}(D)$, and therefore $\beta_s*\varphi_\delta\to\beta_s$ in $L^{r^*}(D)$ as $\delta\to 0$. We obtain

$$\int_D(\Phi_s(v)-\Phi_s(u)-\beta_s\circ(v-u))\geqq 0,$$

that is, $\beta_s\in b_s(u)$.

While this lemma handles the parabolic part, the next lemma deals with the nonlinearities. Since they do not depend on the discontinuities of b, we can prove the following version of the compactness result in 1.9.

4.4. Lemma. *Suppose $u_\varepsilon \to u$ weakly in $L^r(0,T;\, H^{1,r}(\Omega))$ satisfying the estimates*

$$\frac{1}{h}\int_0^{T-h}\int_\Omega (b_\varepsilon(u_\varepsilon(t+h)) - b_\varepsilon(u_\varepsilon(t)))\cdot(u_\varepsilon(t+h)-u_\varepsilon(t))\,dt \leqq C,$$

and

$$\int_\Omega B_\varepsilon(u_\varepsilon(t)) \leqq C \quad \text{for } 0<t<T.$$

Then for a subsequence

$$b_\varepsilon(u_\varepsilon) \to \beta \in b(u) \qquad \text{weakly in } L^1(]0,T[\times\Omega),$$

and the functions

$$a(b_{s\varepsilon}(u_\varepsilon), b_{c\varepsilon}(u_\varepsilon), \nabla u) \quad \text{and} \quad f(b_{s\varepsilon}(u_\varepsilon), b_{c\varepsilon}(u_\varepsilon))$$

converge almost everywhere in Ω.

Proof. It was shown in 1.9 that $b_{c\varepsilon}(u_\varepsilon)\to b_c(u)$ in $L^1(]0,T[\times\Omega)$. Using the growth condition on b_s we see that for $D\subset\subset]0,T[\times\Omega$ and small h

$$\begin{aligned}\int_0^T\int_D &(b_{s\varepsilon}(u_\varepsilon(x+he)) - b_{s\varepsilon}(u_\varepsilon(x)))\cdot(u_\varepsilon(x+he)-u_\varepsilon(x))\,dx\\ &\leqq C\left(\int_0^T\int_\Omega |u_\varepsilon(x+he)-u_\varepsilon(x)|^r\,dx\right)^{1/r} \leqq C\cdot h.\end{aligned}$$

Moreover $b_{s\varepsilon}(u_\varepsilon)$ are bounded in $L^{r^*}(]0,T[\times\Omega)$, hence for a subsequence $b_{s\varepsilon}(u_\varepsilon)\to\beta_s$ weakly in this space. Then $\beta_s\in b_s(u)$ by 4.3, that is,

$$\beta := \beta_s + b_c(u) \in b(u).$$

In order to obtain the pointwise convergence of the nonlinearities we shall argue similarly as in 1.9. For this we define

$$a^R := \min\left(1, \frac{R}{|a|}\right) a,$$

and we choose compact subsets $S\subset]0,T[\times\Omega$ with small measure of its complement such that $\nabla u\in C^0(S)$. For large M define E as in 1.9 but with b replaced by b_ε. Then for $t\in]0,T-h[\setminus E$ we have an estimate

$$\int_{S(t)} |a^R(b_\varepsilon(u_\varepsilon(t+h)), \nabla u(t)) - a^R(b_\varepsilon(u_\varepsilon(t)), \nabla u(t))| \leqq \omega_{R,M}(hM) \tag{4.4.1}$$

with some continuous function $\omega_{R,M}$ satisfying $\omega_{R,M}(0)=0$. If not, we would find sequences t_ε and functions $v_{1\varepsilon}, v_{2\varepsilon}$ bounded in $H^{1,r}(\Omega)$, such that

$$\int_\Omega (b_\varepsilon(v_{2\varepsilon}) - b_\varepsilon(v_{1\varepsilon}))\cdot(v_{2\varepsilon}-v_{1\varepsilon}) \to 0,$$

but

$$\int_{S(t_\varepsilon)} |a^R(b_\varepsilon(v_{2\varepsilon}), \nabla u(t_\varepsilon)) - a^R(b_\varepsilon(v_{1\varepsilon}), \nabla u(t_\varepsilon))| \geqq \kappa > 0.$$

Let us choose a subsequence such that $t_\varepsilon \to t_0$, $v_{i\varepsilon} \to v_i$ almost everywhere in Ω, and $\chi_{S(t_\varepsilon)} \to \chi_0$ weakly star in $L^\infty(\Omega)$. Now take $x \in \Omega$. If $x \in S(t_\varepsilon)$ for a subsequence then $x \in S(t_0)$ and $\nabla u(t_\varepsilon, x) \to \nabla u(t_0, x)$. Also $v_{i\varepsilon}(x) \to v_i(x)$, hence $b_\varepsilon(v_{i\varepsilon}(x))$ is bounded. For any cluster value ξ of this sequence we must have $\xi \in b(v_i(x))$ and therefore

$$a^R(b_\varepsilon(v_{i\varepsilon}(x)), \nabla u(t_\varepsilon, x)) \to a^R(\xi, \nabla u(t_0, x)) = a^R(b(v_i(x)), \nabla u(t_0, x)).$$

We conclude that

$$\int_\Omega \chi_0 \, |a^R(b(v_2), \nabla u(t_0)) - a^R(b(v_1), \nabla u(t_0))| \geqq \kappa > 0.$$

On the other hand it follows from 1.8 that $b_c(v_2) = b_c(v_1)$, and since $b_{s\varepsilon}(v_{i\varepsilon})$ are bounded in $L^*(\Omega)$, we can assume that $b_{s\varepsilon}(v_{i\varepsilon}) \to \beta_i$ weakly in $L^*(\Omega)$. We conclude

$$\begin{aligned}\int_\Omega (\beta_2 - \beta_1) \cdot (v_2 - v_1) &\leftarrow \int_\Omega (b_{s\varepsilon}(v_{2\varepsilon}) - b_{s\varepsilon}(v_{1\varepsilon})) \cdot (v_{2\varepsilon} - v_{1\varepsilon}) \\ &\leqq \int_\Omega (b_\varepsilon(v_{2\varepsilon}) - b_\varepsilon(v_{1\varepsilon})) \cdot (v_{2\varepsilon} - v_{1\varepsilon}) \to 0,\end{aligned}$$

that is, $(\beta_2 - \beta_1) \cdot (v_2 - v_1) = 0$ almost everywhere in Ω.

By 4.3 (here v_i are time independent) we know that $\beta_i \in b(v_i)$. We conclude that for $0 \leqq \theta \leqq 1$

$$\begin{aligned}\Phi(v_1 + \theta(v_2 - v_1)) - \Phi(v_1) - \theta \beta_1 \cdot (v_2 - v_1) \\ &= \int_0^\theta (b(v_1 + s(v_2 - v_1)) - \beta_1) \cdot (v_2 - v_1) \, ds \\ &\leqq \theta(\beta_2 - \beta_1) \cdot (v_2 - v_1) = 0,\end{aligned}$$

consequently,

$$\Phi(v_2 + z) - \Phi(v_2) = \Phi(v_2 + z) - \Phi(v_1) - \beta_1 \circ (v_2 - v_1) \geqq \beta_1 \cdot z,$$

for all $z \in \mathbb{R}^m$, that is, $\beta_1 \in b(v_2)$; in other words, $b(v_1) \cap b(v_2) \neq \varnothing$. This implies

$$a^R(b(v_2), \nabla u(t_0)) = a^R(b(v_1), \nabla u(t_0)),$$

a contradiction to the above construction. Thus the estimate (4.4.1) is proved.

Now we proceed as in the proof of 1.9 and conclude that for small h (and v_ε defined as in 1.9)

$$\frac{1}{h} \int_0^h \int_0^T \int_\Omega \left| a^R(b_\varepsilon(u_\varepsilon(t)), \nabla u(t)) - \sum_{i=1}^{T/h} a^R(b_\varepsilon(v_\varepsilon((i-1)h + s)), \nabla u(t)) \, \chi(t)_{](i-1)h, ih[} \right| dt \, ds$$

is small uniformly in ε. Since a^R does not depend on the jumps of b_s we see as in 1.9 that for fixed M, h, i, and s the functions

$$(t, x) \to a^R(b_\varepsilon(v_\varepsilon((i-1)h + s, x), \nabla u(t, x)) \, \chi_{](i-1)h, ih[}(t)$$

are compact in $L^1(]0,T[\times\Omega)$. Together with the estimate just proved this shows that the functions $a^R(b_\varepsilon(u_\varepsilon), \nabla u)$ are compact in $L^1(]0,T[\times\Omega)$, but then also the functions $a(b_\varepsilon(u_\varepsilon), \nabla u)$, for they are bounded in $L^{r^*}(]0,T[\times\Omega)$ by the growth condition in 4.1, which implies

$$\int_\Omega |a(b_\varepsilon(u_\varepsilon), \nabla u) - a^R(b_\varepsilon(u_\varepsilon), \nabla u)| \leqq C\cdot R^{-1/(r-1)}.$$

Thus the proof of 4.4 is complete.

Now we combine the previous existence results with the special observations in this section and obtain

4.5. Existence Theorem. *Suppose the data satisfy* 4.1 *and the remaining assumptions in* 1.7, *except that the initial data are given by* $\beta^0 \in b(u^0)$ *for some* u^0 *with* $B(u^0) := \beta^0\cdot u^0 - \Phi(u^0) \in L^1(\Omega)$. *Then there is a weak solution of the Stefan problem, that is, a function* $u \in u^D + L^r(0,T;V)$ *and a function* $\beta \in b(u)$ *with* $\partial_t\beta \in L^{r^*}(0,T;V^*)$ *and initial values* β^0 (*in the sense of* 1.4.1), *such that in the latter space the differential equation*

$$\partial_t\beta - \nabla\cdot a(b_s(u), b_c(u), \nabla u) = f(b_s(u), b_c(u))$$

is satisfied.

Remark. We remind the reader that the notation $a(b_s(u), b_c(u), \nabla u)$ and the same for f does not mean, that a or f may depend on the jumps explicitly.

Proof. Denote by u_ε a solution as obtained in 1.7 for the smooth vector field b_ε defined in 4.2. Since b_ε is bijective we can choose β^0 as initial data. Then, if $b_\varepsilon(u_\varepsilon^0) = \beta^0$, we have for example

$$\beta_\varepsilon(u_\varepsilon^0) - B(u^0) = \beta^0\cdot(u_\varepsilon^0 - u^0) + \Phi(u^0) - \Phi_\varepsilon(u_\varepsilon^0) \leqq \Phi(u_\varepsilon^0) - \Phi_\varepsilon(u_\varepsilon^0) \leqq 0.$$

Multiplying the equation with $u_\varepsilon - u^D$ and with the differences we obtain as usual the estimates in 4.4, hence for a subsequence we have the convergence properties stated in that lemma. Now we want to prove that u_ε converges strongly to u. Multiplying the equation as usual with $u_\varepsilon - u$ we obtain as in 1.7

$$\int_0^t \langle \partial_t b_\varepsilon(u_\varepsilon), u_\varepsilon - u^D\rangle - \int_0^t \langle \partial_t b_\varepsilon(u_\varepsilon), u - u^D\rangle + c\int_0^t\int_\Omega |\nabla(u-u^D)|^r$$
$$\leqq \int_0^t\int_\Omega (-a(b_\varepsilon(u_\varepsilon), \nabla u)\,\nabla(u_\varepsilon - u) + f(b_\varepsilon(u_\varepsilon))\,(u_\varepsilon - u)).$$

By the energy estimate $\partial_t b_\varepsilon(u_\varepsilon)$ are bounded functionals in $L^{r^*}(0,T;V^*)$. We conclude that for a subsequence $\partial_t b_\varepsilon(u_\varepsilon) \to \partial_t\beta$ weakly in this space and that β has initial values β^0 in the sense of 1.4.1. Moreover, using the function

$$B(u) := \beta\cdot u - \Phi(u), \qquad \text{where } \beta\in b(u),$$

we can prove Lemma 1.5 also for β. This and the fact that $B_\varepsilon(u_\varepsilon^0) \leqq B(u^0)$ implies that the parabolic part in the above inequality is

$$\geqq \int_\Omega (B_\varepsilon(u_\varepsilon(t)) - B(u(t))) + R_\varepsilon(t)$$

with a function R_ε converging weakly to zero. The elliptic part on the right of the above inequality can be handled as in Sect. 1 using the results of 4.4, that is, up to terms which become small for small ε, the right side can be estimated by

$$\int_0^t \int_\Omega \max(B_c(u_\varepsilon(t)) - B_c(u(t)), 0).$$

For the new B_s terms we have

$$\int_0^t \int_\Omega (B_{s\varepsilon}(u_\varepsilon) - B_s(u)) \geqq \int_0^t \int_\Omega (b_{s\varepsilon}(u_\varepsilon) - \beta_s) \cdot u + \int_0^t \int_\Omega (\Phi_s(u) - \Phi_{s\varepsilon}(u)),$$

which tends to zero as $\varepsilon \to 0$. Thus we obtain the strong convergence and therefore the existence of a weak solution as in the proof of 1.7.

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Note. [3] is an earlier version of this paper containing more details. [2], [4], [18] are articles in preparation based on this paper. Our approach is a new one. The parts of the proofs which are similar to earlier papers are repeated here.

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