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The behavior of the free boundary for the dam problem

The Behavior of the Free Boundary for the Dam Problem (*).

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1. - Introduction.

In this paper we study the behavior of the free boundary for the dam problem near the given boundaries to reservoirs, atmosphere, and impervious layers, where we restrict ourselves to the two dimensional homogeneous case.

We start with a solution of the problem as it was obtained in [3] by approximating the free boundary problem by models for saturated-unsaturated flow through porous media. That is, a pair u, γ is a solution of the dam problem, if

(1.1) $u \in M(u_0)$ with $u \geqslant 0$, and $\gamma \in L^\infty(\Omega)$ with $\chi_{\{u>0\}} \leqslant \gamma \leqslant 1$.

(1.2) For every $v \in M(u_0)$

$$\int_\Omega \nabla(v - u)(\nabla u + \gamma e) \geqslant 0 .$$

Here Ω is an open bounded connected set in $\mathbb{R}^2$ with Lipschitz boundary and denotes the porous medium. On the boundary of Ω three relatively open sets are given, $S_{res}, S_{air}, S_{imp}$, which denote respectively the boundary to reservoirs, to atmosphere, and the impervious part of the boundary (fig. 1). It is assumed that these three sets are disjoint and that the boundary $\partial\Omega$ is the union of their closures. On S_{res} and S_{air} the pressure is given by a function $u^0 \in C^{0,1}(\mathbb{R}^2)$, which is non-negative on S_{res} and zero on S_{air} (which means that we do not consider the case of a capillary fringe). The set of
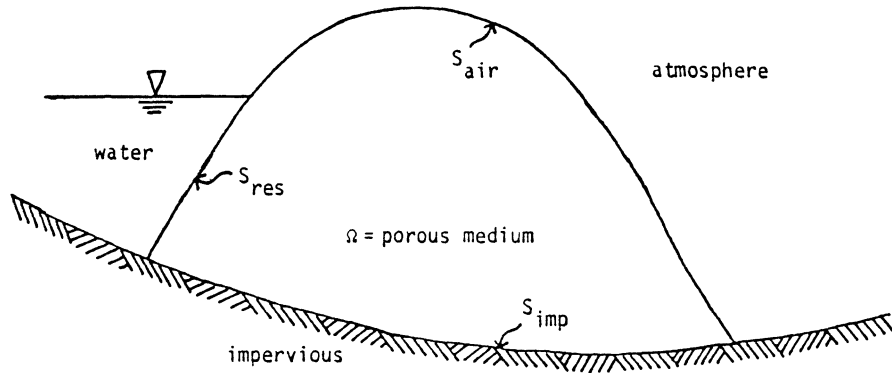
Fig.1

admissible function in (1.1) and (1.2) is defined by

$$M(u^0) := \{v \in H^{1,2}(\Omega) : v = u_0 \text{ on } S_{res} \text{ and } v \leqslant 0 \text{ on } S_{air}\} .$$

In (1.2) e denotes the vector $(0, 1)$.

Formally, the variational inequality in (1.2) is equivalent to the differential equation

$$\nabla \cdot v = 0 \quad \text{in } \Omega ,$$

where $v = - (\nabla u + \gamma e)$, and the boundary conditions

$$u = u_0 \quad \text{on } S_{res} \cup S_{air} ,$$

$$v \cdot \nu = 0 \quad \text{on } S_{imp} ,$$

$$v \cdot \nu \geqslant 0 \quad \text{on } S_{air} .$$

We are interested in the behavior of the free boundary $\Omega \cap \partial\{u > 0\}$ at points of detachment from the fixed boundary and we will show that the free boundary behaves as indicated in fig. 2-4 (see also [10], 2.2.1). Our procedure is as follows.

First we summarize the basic properties of the solution related to the Lipschitz continuity up to the boundary $\partial\Omega$ (section 2). This allows us to consider linear blow-ups (2.5), which are the basic tool in order to obtain the local results in sections 5-8. We will see that u is Lipschitz continuous

in $\bar{\Omega}$, except at the separation points between reservoirs and atmosphere in the case of overflow, and except points at which the compatibility condition is violated.

In section 4 we prove that $\gamma = \chi_{\{u>0\}}$, which says, that in the case of a homogeneous medium the solution consists only of saturated regions where $u > 0$ and dry regions where $\gamma = 0$. For general inhomogeneous media this is not true (see [3] Beispiel 4.6). The comparison lemma in section 3 together with our local results about the free boundary leads to a uniqueness proof (9.3, see [9, 13]), which shows that other different approaches gave the same solution ([1, 4, 5, 12, 14], and for further references [6]).

The regularity of the free boundary in the interior was proved in [2], and results about the qualitative behavior were obtained in [7, 8, 11]. In [7, 11] global arguments also gave results about the tangent of the free boundary at endpoints in the special case of a rectangular dam. In [8] it was proved by local methods that the free boundary is tangential to the fixed boundary at the atmosphere on the top. In our paper we use only local methods in order to study the behavior of the free boundary near $\partial\Omega$.

In section 5 we deal with the free boundary near points on the surface of reservoirs. Using a class of linear super- and subsolutions we prove that there are three possibilities: overflow, a horizontal free boundary, or a right angle between free and fixed boundary (fig. 2).
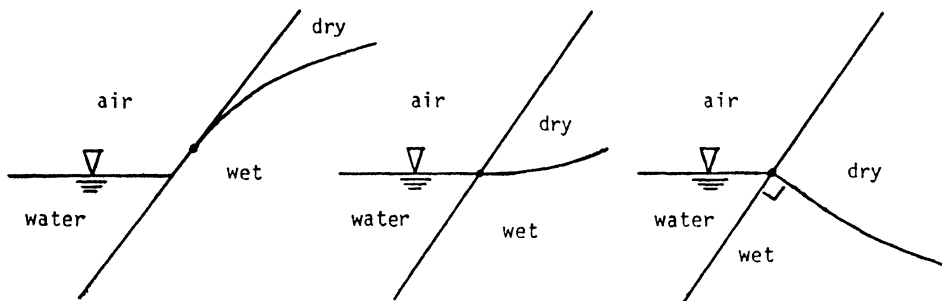
Fig. 2.1   Fig. 2.2   Fig. 2.3

In section 6 we study the free boundary near points at the atmosphere. We show that for points on the lower part of the porous medium (vertical points of $\partial\Omega$ included) the free boundary has a vertical tangent (Fig. 3). On the upper part of the porous medium the free boundary is always tangential to $\partial\Omega$, which was proved in [8].

In section 7 we consider free boundary points near the impervious part of $\partial\Omega$. Using blow-up techniques we prove that the free boundary is tan-
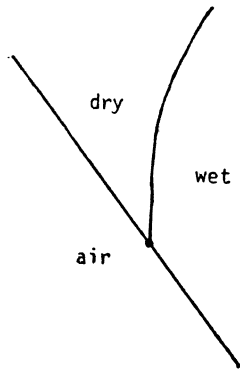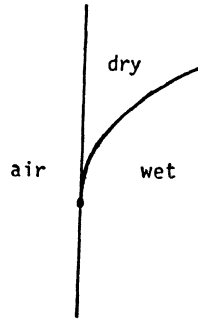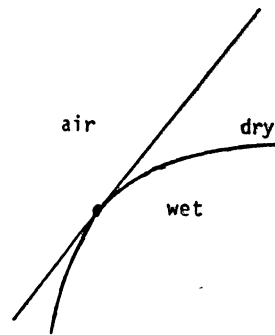
Fig.3.1   Fig.3.2   Fig.3.3

gential to $\partial\Omega$ or horizontal, where the case of a non horizontal tangent appears only at points on the upper part of the porous medium (fig. 4). Changing the porous medium afterwards one can construct examples for fig. 3.3 and fig. 4.3, where the set $\partial\Omega \cap \overline{\{u > 0\}}$ is an arbitrary closed subset of $\partial\Omega$.
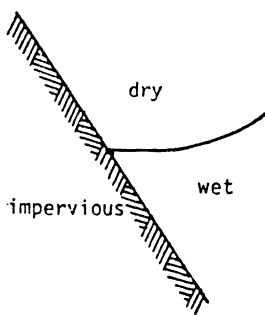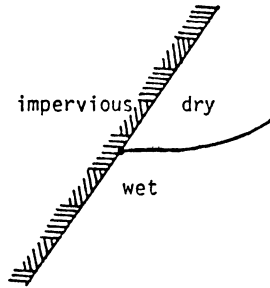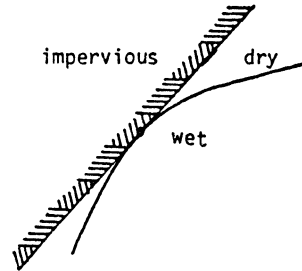
Fig.4.1   Fig.4.2   Fig.4.3

Whenever the porous medium lies above its boundary near the point of interest, we need the assumption that there is some dry neighborhood above this point in order to start with our local arguments. However, under suitable conditions on the global data this assumption can be verified (see 4.6, 9.2). Local counterexamples are given in 5.2.2.), 6.5. In all cases we assume that the fixed boundary $\partial\Omega$ is of class $C^{1,1}$ near the detachment point. The only interesting remaining case are separation points between atmosphere and impervious parts, which is treated in section 8, where we allow the fixed boundary to have two different tangents at such points.

Although we deal with homogeneous porous media, it is clear, that the local results also can be obtained for inhomogeneous unisotropic media under certain regularity assumptions on the permeability. In the unisotropic case this would change only with the direction of the possible tangents for the free boundary, which can be computed formally. However, if there are jumps in the permeability, the situation becomes different. In general, the free boundary is no longer a smooth graph (see [3], Beispiel 4.6), and it would be of interest to study the local behavior of the solution at least near free boundary points in the interior. Another important generalization is the three dimensional case, for which we have limit curves instead of limit points of the free boundary on $\partial\Omega$. However, we think that some of our techniques could be used in order to prove that at almost all points of such a curve the behavior is as for the two dimensional problem, if we look at a vertical section to $\partial\Omega$.

In section 9 we give global applications of the local results in the previous sections. First we show that the solution is unique up to ground water lakes, where we assume that the boundary sets consist of a finite number of smooth curves. Then we study the stability of the local behavior of the free boundary with respect to small perturbations of the geometry of the dam. Using this we can construct examples which show that actually all cases in fig. 2-4 are possible.

2. – Basic remarks.

Since our techniques in section 5-8 are of local nature, it is convenient to define

2.1 LOCAL SOLUTIONS. *If B is, for example, a ball, we call $u \in H^{1,2}(B)$ and $\gamma \in L^{\infty}(B)$ a local solution in $\Omega \cap B$, if u is a solution in this region as in (1.1)-(1.2) with respect to its values on ∂B as Dirichlet data, that is*

$$u \geqslant 0 \quad and \quad \chi_{\{u>0\}} \leqslant \gamma \leqslant 1$$

with boundary values

$$u = u^0 \quad on \quad (S_{\text{res}} \cup S_{\text{air}}) \cap B\,,$$

and for test functions $\zeta \in \overset{\circ}{H}{}^{1,2}(B)$ with $\zeta = 0$ on $S_{\text{res}} \cap B$ and $\zeta \leqslant 0$ on $S_{\text{air}} \cap B$ the inequality

$$\int_{\Omega \cap B} \nabla\zeta(\nabla u + \gamma e) \geqslant 0$$

holds.

The maximum regularity for solutions u locally in Ω is the Lipschitz continuity, which was proved in [3], Satz 3.6. In the special case of harmonic functions this proof is more transparent, and therefore let us repeat it here.

2.2 LEMMA. *If u, γ is a local solution, and if a ball $B_r(x) \subset \Omega \cap B \cap \{u > 0\}$ touches the free boundary $\Omega \cap B \cap \partial\{u > 0\}$, then*

$$u(x) \leqslant Cr$$

with a universal constant C.

PROOF. We can assume $\overline{B_r(x)} \subset \Omega \cap B$. For small $\delta > 0$ let v be the harmonic function in $D := B_{(1+\delta)r}(x) \backslash B_{r/2}(x)$ with

$$v = \lambda := \inf_{\partial B_{r/2}(x)} u \qquad \text{on } \partial B_{r/2}(x),$$

$$v = 0 \qquad\qquad \text{on } \partial B_{(1+\delta)r}(x).$$

Then [3], Satz 3.3 (see proof of 3.4) implies

$$\int_{D \cap \{u > 0\}} |\nabla(v - u)|^2 \leqslant - \int_{D \cap \{u = 0\}} |\nabla v|(1 - |\nabla v|).$$

Since D contains a free boundary point, u is not harmonic in D, hence the left side is positive. We conclude ($|\nabla v|$ is a monotone function of the distance from x)

$$|\nabla v| < 1 \qquad \text{on } \partial R_{(1+\delta)r}(x).$$

On the other hand

$$|\nabla v| \geqslant c\, \frac{\lambda}{r} \geqslant c\, \frac{u(x)}{r}$$

using Harnack's inequality.

2.3 REMARK. 2.2. implies (see [3], Satz 3.7) that u is Lipschitz continuous locally in Ω. If $D \subset\subset \Omega$ is connected and contains a free boundary point, the supremum and the Lipschitz constant of u in D depend only on the geometry of Ω and D.

PROOF. Let $\varkappa := \mathrm{dist}(D, \partial\Omega)$ and $x_0 \in D \cap \partial\{u > 0\}$. Then for $x \in D$ we find balls $B_\varkappa(x_j) \subset \Omega$, $j = 1, ..., k$, with $x_k = x$ and $x_{j-1} \in B_{\varkappa/4}(x_j)$, where k depends only on $\varkappa$ and the diameter of D. Then by 2.2

$$u(x) \leqslant C\varkappa, \qquad \text{if } B_\varkappa(x_j) \cap \partial\{u > 0\} \text{ is non empty}.$$

and by Harnack's inequality

$$u(x_j) \leqslant C u(x_{j-1}) , \quad \text{if } B_\varkappa(x_j) \cap \partial\{u > 0\} \text{ is empty} .$$

We get

$$u(x) \leqslant C\big(\varkappa, \operatorname{diam}(D)\big) .$$

Later we will use the Lipschitz continuity of u up to the boundary. Therefore let us prove another version of 2.2 including the fixed boundary.

2.4 BASIC LEMMA. *Let u be a local solution in $\Omega \cap B_R$, and assume that $\Omega \cap B_R$ and $\partial\Omega \cap B_R$ are connected, and that the curvature of $\partial\Omega \cap B_R$ does not exceed $1/R$. If $\Omega \cap B_{R/4}$ contains a free boundary point, then*

1) *for small $\delta > 0$*

$$u \leqslant C(\delta) R \quad in \ \{x \in R_{R/4}/\operatorname{dist}(x, \partial\Omega) \geqslant \delta R\} .$$

2) $u \leqslant CR + \sup_{\partial\Omega \cap B_{3R/4}} u \ in \ \Omega \cap B_{R/4}$,

and

$$\inf_{\partial\Omega \cap B_{3R/4}} u \leqslant CR .$$

3) *if $\partial\Omega \cap B_R \subset S_{\mathrm{imp}}$, then*

$$u \leqslant CR \quad in \ \Omega \cap B_{R/4} .$$

Here C is a universal constant.

PROOF. By homogeneity we can assume $R = 4$. Let $x \in \Omega \cap B_1$ with $u(x) > 0$ and $B_r(x)$ the maximum ball contained in $B_4 \backslash (\Omega \cap \partial\{u > 0\})$. Then $B_{2r}(x)$ contains a free boundary point x_f, and because of the assumption on the curvature of $\partial\Omega$, we have $x_f \in B_{\delta r}(x_0) \subset \Omega$ for some point x_0. Then, if $B_{\delta r}(x)$ is contained in Ω, we conclude as in 2.3

$$u(x) \leqslant C(\delta) r \leqslant C(\delta) ,$$

which, in particular, proves 1). If $B_{\delta r(x)}$ is not contained in Ω we have (now let $\delta = 1/4$)

$$x \in \Omega \cap B_{r/4}(x_1) \subset \Omega \cap B_{r/2}(x_1) \subset \Omega \cap \{u > 0\}$$

for some point $x_1 \in \partial\Omega$, and as in the first case we can estimate

$$u(x_2) \leqslant C , \quad \text{where } x_2 = x_1 - \frac{r}{4}\nu(x_1) .$$

Now let P_x be the Poisson kernel of $B_{r/2}(x_1) \cap \Omega$ with pole at x. By the assumptions on $\partial\Omega$ we have

$$C \geqslant u(x_2) = \int\limits_{\partial(B_{r/2}(x_1) \cap \Omega)} u P_{x_2} d\mathcal{H}^1 \geqslant \inf_{B_{r/2}(x_1) \cap \partial\Omega} u \int\limits_{B_{r/2}(x_1) \cap \partial\Omega} P_{x_2} d\mathcal{H}^1 \geqslant c \inf_{B_s \cap \partial\Omega} u ,$$

which proves the second inequality in 2). Moreover

$$u(x) = \int\limits_{\partial(B_{r/2}(x_1) \cap \Omega)} u P_x d\mathcal{H}^1 \leqslant \sup_{B_{r/2}(x_1) \cap \partial\Omega} u + \int\limits_{\Omega \cap \partial B_{r/2}(x_1)} u P_x d\mathcal{H}^1$$

and by Harnack's inequality the second term can be estimated by

$$\int\limits_{\Omega \cap \partial B_{r/2}(x_1)} u P_{x_2} d\mathcal{H}^1 \leqslant u(x_2) \leqslant C ,$$

which is the first estimate in 2). For 3) use the Green-Neumann function G_x, that is,

$$u(x) = \int\limits_{\partial B_{r/2}(x_1) \cap \Omega} u \partial_{-\nu} G_x d\mathcal{H}^1 - \int\limits_{B_{r/2}(x_1) \cap \partial\Omega} G_x e \cdot \nu \, d\mathcal{H}^1 ,$$

Using Harnack's inequality again we see that

$$u(x) \leqslant C u(x_2) + C r \leqslant C .$$

The local behavior of a solution u, γ near a point $x_0 \in \bar{\Omega}$ is described by the blow-up limits at this point. These blow-ups have the advantage that they are globally defined, and that they involve only the data at the point x_0, hence they have analytic boundary data and satisfy an analytic differential equation, even in the case that we start with a more general equation for u. Moreover, many free boundary problems are closed with respect to blow-up limits, therefore these limits are global solutions. In general the definition of this limit has to take into account the homogeneity properties of the special problem. In this paper we are interested in linear blow-ups at the boundary.

2.5 BLOW-UP LIMITS. *Let* u, γ *be a Lipschitz continuous local solution in* $\Omega \cap B_R$ *and assume that*

1) $0 \in \partial\Omega$ *with* $u(0) = 0$,

2) Ω *has normal* ν^* *at* 0,

3) *near* 0 *the sets* $\partial\Omega \cap \{\pm x \cdot (i\nu_*) > 0\}$ *belong to* S_{res}, S_{air}, *or* S_{imp}.

(*By continuation we can assume that* u *and* γ *are defined in* B_R, *preserving the Lipschitz constant for* u). *Consider the blow-up sequence*

$$u_r(x) := \frac{1}{r} u(rx) \qquad and \qquad \gamma_r(x) := \gamma(rx) \qquad for \ |x| < \frac{R}{r}.$$

Then there are functions $u_* \in C^{0,1}(\mathbf{R}^2)$ *and* $\gamma_* \in L^\infty(\mathbf{R}^2)$ *such that for a subsequence*

$$\nabla u_r \to \nabla u_* \qquad weakly \ star \ in \ L^\infty_{\text{loc}}(\mathbf{R}^2),$$

$$u_r \to u_* \qquad in \ C^{0,\alpha}_{\text{loc}}(\mathbf{R}^2) \ for \ 0 < \alpha < 1,$$

$$\gamma_r \to \gamma_* \qquad weakly \ star \ in \ L^\infty_{\text{loc}}(\mathbf{R}^2).$$

u_*, γ_* *is a global solution in* $\Omega_* := \{x \cdot \nu_* < 0\}$ (*that is, a local solution in* $\Omega_* \cap B_R$ *for every* R) *with respect to boundary data induced by* 3).

PROOF. u_r are uniformly Lipschitz continuous and γ_r are uniformly bounded, hence we have the above convergence properties. To see that u_*, γ_* is a solution, denote by S^r_{res}, S^*_{res}, etc. the corresponding boundary sets of $\Omega_r := (1/r)\Omega$ and Ω^*. By 2) and 3) every function $\zeta \in H^{1,2}(\mathbf{R}^2)$ with compact support and $\zeta = 0$ on S^*_{res}, $\zeta \leqslant 0$ on S^*_{air} can be approximated strongly in $H^{1,2}(\mathbf{R}^2)$ by functions ζ_r with these properties on S^r_{air}, S^r_{res}. Then

$$0 \leqslant \int_\Omega \frac{1}{r^2} \nabla \zeta_r \left(\frac{x}{r}\right) (\nabla u(x) + \gamma(x)e) \, dx = \int_{\Omega_r} \nabla \zeta_r (\nabla u_r + \gamma_r e) \to \int_{\Omega_*} \nabla \zeta (\nabla u_* + \gamma_* e).$$

2.6 DEFINITION. *In general one considers blow-up sequences* u_r *with respect to balls* $B_r(z_r)$ *defined by*

$$u_r(x) = \frac{1}{r} u(z_r + rx)$$

for an arbitrary sequence $z_r \in \bar{\Omega} \cap \{u = 0\}$. *If the boundary sets of the blow-up domains* $\Omega_r := (1/r)(\Omega - z_r)$ *converge in an appropriate way, one has the same statements as above.*

3. – Comparison lemmas.

The general definition for super- and subsolutions is stated in 3.1, but because of technical details near the fixed boundary we need some additional regularity properties in order to prove the comparison lemma 3.2. However, using the results of sections 4-8 this lemma implies that under certain conditions on the data the solution is unique (9.3). Another comparison lemma, which can be used for certain explicit subsolutions, is stated in 3.4.

3.1 DEFINITION. *Let* $u \in H^{1,2}(\Omega)$ *with* $u \geqslant 0$ *and* $\gamma \in L^{\infty}(\Omega)$ *with* $\chi_{\{u>0\}} \leqslant$ $\leqslant \gamma \leqslant 1$. *We call* u, γ *a*

1) *supersolution, if* $u \geqslant u_0$ *on* S_{res}, *and*

$$\int_{\Omega} \nabla\zeta(\nabla u + \gamma e) \geqslant 0$$

for every non-negative $\zeta \in H^{1,2}(\Omega)$ *vanishing on* $S_{\text{res}} \cup S_{\text{air}}$;

2) *subsolution, if* $u \leqslant u^0$ *on* $S_{\text{res}} \cup S_{\text{air}}$, *and*

$$\int_{\Omega} \nabla\zeta(\nabla u + \gamma e) \leqslant 0$$

for every non-negative $\zeta \in H^{1,2}(\Omega)$ *vanishing on* S_{res}.

These definitions are such that u, γ is a solution if and only if it is a super- and a subsolution.

3.2 COMPARISON LEMMA FROM ABOVE. *Suppose* u, γ *is a solution, and* $v, \chi_{\{v>0\}}$ *is a supersolution positive in a neighborhood of* S_{res} *with the additional property that* $\overline{\Omega \cap \partial\{v > 0\}}$ *consists of Lipschitz graphs in vertical direction locally in* Ω *and a set* $E \subset \partial\Omega$ *with*

$$\int_{\Omega \cap B_{\varrho}(E)} (1 + |\nabla u| + |\nabla v|) = o(\varrho) \quad \text{for } \varrho \to 0 .$$

Then $u \leqslant v$ *on connected components of* $\Omega \cap \{v > 0\}$ *touching* S_{res}, *and* $\gamma = 0$ *above these components.*

PROOF. For $\varrho > 0$ consider the cut-off function

$$\eta_{\varrho}(x) := \min\left(1, \max\left(0, \frac{\text{dist}(x, E)}{\varrho} - 1\right)\right),$$

and for $\delta > 0$ define

$$\psi_\delta(s) := \max\left(0, \min\left(\frac{s}{\delta}, 1\right)\right),$$

and for $\varepsilon > 0$ let

$$d_\varepsilon(x) := \min\left(\frac{\text{dist}\,(x, \Omega \cap \{v > 0\})}{\varepsilon}, 1\right).$$

Then $\eta_\varrho d_\varepsilon(\psi_\delta(u) - 1) - \eta_\varrho \psi_\delta(u - v)$ is an admissible test function for u, γ in (1.2), hence

$$\int_\Omega \nabla\big(\eta_\varrho d_\varepsilon(\psi_\delta(u) - 1) - \eta_\varrho \psi_\delta(u - v)\big)(\nabla u + \gamma e) \geqslant 0 \,,$$

and $\eta_\varrho \psi_\delta(u - v)$ is an admissible test function for v in 3.1.1), hence

$$\int_\Omega \nabla(\eta_\varrho \psi_\delta(u - v))(\nabla v + \chi_{\{v > 0\}} e) \geqslant 0 \,.$$

Adding these integrals we obtain

$$0 \leqslant \int_\Omega \nabla(\eta_\varrho \psi_\delta(u - v))(\nabla(v - u) - (\gamma - \chi_{\{v > 0\}})e) - \int_\Omega \nabla\big(\eta_\varrho d_\varepsilon(1 - \psi_\delta(u))\big)(\nabla u + \gamma e),$$

therefore

$$\int_{\Omega \cap \{v > 0\}} \eta_\varrho \psi_\delta'(u - v)|\nabla(u - v)|^2 \leqslant \int_{\Omega \cap \{v = 0\}} \eta_\varrho(1 - d_\varepsilon)\nabla(1 - \psi_\delta(u))(\nabla u + \gamma e) -$$

$$-\int_\Omega \eta_\varrho(1 - \psi_\delta(u))\nabla d_\varepsilon(\nabla u + \gamma e) + 3\int_\Omega |\nabla \eta_\varrho|(1 + |\nabla u| + |\nabla v|) \,.$$

By assumption the last integral converges to zero for $\varrho \to 0$ independent of ε and δ. Clearly the first term tends to zero as $\varepsilon \downarrow 0$. Therefore we have to show that the middle term becomes non-positive in the limit $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, which is an estimate near

$$\Sigma_\varrho := \Omega \cap \partial\{v > 0\} \backslash B_\varrho(E) \,.$$

Since locally $\Omega \cap \{v > 0\}$ is a subgraph in vertical direction, d_ε increases with height near Σ_ϱ. Therefore for small ε one part of the second integral above is

$$= - \int_{B_\varepsilon(\Sigma_\varrho) \cap \{u = 0\}} \eta_\varrho \nabla d_\varepsilon(\gamma e) \leqslant 0 \,,$$

and using 2.3 we have for the second part

$$= - \int_{B_\varepsilon(\Sigma_\varrho) \cap \{u>0\}} \eta_\varrho(1 - \psi_\delta(u))\nabla d_\varepsilon(\nabla u + e) \leqslant C_\varrho \int_{B_\varepsilon(\Sigma_\varrho) \cap \{0<u<\delta\}} |\nabla d_\varepsilon| \,.$$

Since Σ_ϱ is a Lipschitz graph by assumption and $\Omega \cap \{u>0\}$ a subgraph in vertical direction (see 4.1), and since $d_\varepsilon = 0$ below Σ_ϱ, this can be estimated by

$$\leqslant C_\varrho \mathcal{H}^1\big(\partial \{v>0\} \cap B_{C_\varrho\varepsilon}(\Sigma_\varrho) \cap \{0<u<\delta + C_\varrho\varepsilon\}\big)$$

which tends to zero as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$.

We thus proved that

$$\frac{1}{\delta} \int_{\Omega \cap \{v>0\} \cap \{0<u-v<\delta\}} \eta_\varrho|\nabla(u-v)|^2 \to 0 \qquad \text{for } \delta \leqslant \delta(\varrho) \text{ and } \varrho \to 0 \,.$$

From this the lemma follows easily. Let D be a connected component of $\Omega \cap \{v>0\}$ containing a given point $x_0 \in S_\mathrm{res}$ as boundary point. Then v is positive in $\Omega \cap B_r(x_0)$ for some $r > 0$. If we replace v in $\Omega \cap B_r(x_0)$ by the harmonic function with same boundary values, we get a new super-solution, hence we can apply the above estimate for the new v in the following way. If

$$w := \max(u - v, 0) \,,$$

we have for $\zeta \in C_0^\infty(B_r(x_0))$ and $\delta \downarrow 0$

$$\int_{\Omega \cap B_r(x_0)} \nabla\zeta\nabla w \leftarrow \int_{\Omega \cap B_r(x_0)} \psi_\delta(w)\nabla\zeta\nabla w =$$

$$= \int_{\Omega \cap B_r(x)} \nabla(\psi_\delta(w)\zeta)\nabla(u-v) - \int_{\Omega \cap B_r(x)} \zeta\nabla\psi_\delta(w)\nabla \,.$$

The first term vanishes, since v is harmonic in $\Omega \cap B_r(x_0)$ and $\psi_\delta(w)\zeta = 0$ in $\{u = 0\}$. The second term is

$$\geqslant - C\frac{1}{\delta} \int_{\Omega \cap B_r(x_0) \cap \{0<w<\delta\}} |\nabla w|^2 \to 0 \,.$$

This shows that the function $w \in H^{1,2}(B_r(x_0))$ with $w = 0$ in $B_r(x_0)\setminus\Omega$ is harmonic in $B_r(x_0)$, therefore zero in $B_r(x_0)$. By continuation the same argument implies $w = 0$ in D. Since then $u = 0$ and therefore $\partial_h\gamma = 0$ above $\Omega \cap \partial D$, one estimate above shows that for compact curves $\Sigma = \mathrm{graph}\, g \subset$

$\subset \Omega \cap \partial D$

$$0 = \lim_{\varepsilon \downarrow 0} \int_{B_\varepsilon(\Sigma)} \nabla d_\varepsilon(\gamma e) = \lim_{\varepsilon \downarrow 0} \int_{B_\varepsilon(\Sigma) \cap \{v=0\}} \gamma \partial_h d_\varepsilon = \int \gamma(y, g(y)) \, dy \ .$$

This completes the proof of the lemma.

3.3 REMARK. In the definition of supersolution no overflow condition appears, and the comparison lemma says, that the solution is the minimal supersolution. This result was already obtained in [1] by construction. Here u is called minimal if for every supersolution v we have $u \leqslant v$ in Ω except in certain ground water lakes (see [1], 4. and the uniqueness theorem 9.3 in this paper).

The following comparison Lemma is not sharp, but easier to prove.

3.4 LEMMA. *Suppose u, γ is a solution, and $v, \chi_{\{v>0\}}$ is a subsolution with $\nabla v \cdot (\nabla v + e) \geqslant 0$ in $\Omega \cap \{v > 0\}$. Then $u \geqslant v$ in every connected component of $\Omega \cap \{u > 0\}$ touching a segment of S_{res}.*

PROOF (see [3], Satz 3.3). We have

$$0 = \int_\Omega \nabla \max(v - u, 0)(\nabla u + \gamma e) \ ,$$

that is,

$$\int_{\Omega \cap \{u>0\}} |\nabla \max(v - u, 0)|^2 = \int_\Omega \nabla \max(v - u, 0)(\nabla v + \chi_{\{v>0\}} e)$$
$$- \int_{\Omega \cap \{u=0\}} \nabla v \cdot (\nabla v + (1 - \gamma)e) \ ..$$

The first term is non-positive, since v is a subsolution, and the second integral is non-negative, since by assumption

$$\nabla v \cdot (\nabla v + e) \geqslant 0 \ ,$$

that is,

$$\nabla v \notin B_{1/2}\left(-\frac{e}{2}\right) \supset B_{(1-\gamma)/2}\left(-\frac{(1-\gamma)e}{2}\right).$$

4. – The free boundary in the interior.

If u, γ is a solution for inhomogeneous media with permeability a, then $\nabla \gamma \cdot (ae) \leqslant 0$, provided $\nabla \cdot (ae) \geqslant 0$ ([3], Satz 4.4), which immediately gives 4.1

in our case. Moreover this condition on the permeability implies $\gamma = \chi_{\{u>0\}}$, which was proved in [3], 4.1-4.4. Here we will detail the proof for two dimensional homogeneous media. Knowing this, one can apply the regularity results in [2], that is, the free boundary in the interior consists of analytic curves.

4.1 REMARK. If $\{y_0\} \times [h_0, h_1] \subset \Omega$, then $u(y_0, h_1) > 0$ implies $u(y_0, h_0) > 0$.

4.2 LEMMA. If $\{y_0\} \times [h_0 - \varepsilon, h_1 + \varepsilon] \subset \Omega \cap \{u = 0\}$, then $u(y, h) = o(y - y_0)$ for $y \to y_0$ uniformly for $h_0 \leqslant h \leqslant h_1$.

PROOF. Define $\Gamma := \{y_0\} \times [h_0, h_1]$ and

$$l := \limsup_{(v,h)\to\Gamma} \frac{u(y, h)}{|y - y_0|} .$$

Choose a sequence $(y_r, h_r) \to \Gamma$ with $r = |y_r - y_0|$, such that

$$\frac{1}{r} u(y_r, h_r) \to l .$$

We can assume that $y_r > y_0$. Since u is locally Lipschitz continuous (2.3) l is finite, and for a subsequence the blow-up sequence u_r with respect to the balls $B_r((y_0, h_r))$ converges to a limit u_*, which satisfies

$$\Delta u_* \geqslant 0 \quad \text{in } \mathbb{R}^2 ,$$

$$u_*(y, h) \leqslant ly \quad \text{for } y \geqslant 0 ,$$

$$u_*(1, 0) = l .$$

By the strong maximum principle this implies

$$u_*(y, h) = ly \quad \text{for } y \geqslant 0 .$$

Since u_*, γ_* is a solution we obtain $l = 0$ by the following argument. For non-negative functions $\zeta \in C_0^\infty(\mathbb{R}^2)$ and $d_\delta(y, h) := \max(\min(1 + y/\delta, 1), 0)$ we have

$$l \int\limits_{\{v=0\}} \zeta \, d\mathcal{K}^1 = - \int\limits_{\{v>0\}} \nabla\zeta(\nabla u_* + e) = \int\limits_{\{v<0\}} \nabla(d_\delta\zeta)(\nabla u_* + \gamma_* e) =$$

$$= \int\limits_{\{v<0\}} d_\delta \nabla\zeta(\nabla u_* + \gamma_* e) + \frac{1}{\delta} \int\limits_{\{-\delta<v<0\}} \zeta\partial_v u_* .$$

The first term tends to zero for $\delta \downarrow 0$, and since $u^* = 0$ on $\{y = 0\}$ the second equals

$$-\frac{1}{\delta} \int\limits_{\{-\delta < v < 0\}} u_* \partial_v \zeta - \frac{1}{\delta} \int\limits_{\{v = -\delta\}} u_* \zeta \, d\mathcal{H}^1,$$

where the first integral is of order δ^2 by the Lipschitz continuity of u_* and the second integral is non-negative. This proves $l \leqslant 0$.

4.3 SEPARATION LEMMA. *If* $\{y_0\} \times]h_0, h_1[\subset \Omega \cap \{u = 0\}$, *then there is no flux through this line, that is, from both sides this line can be considered as impervious boundary.*

PROOF. Let $\zeta \in C_0^\infty(\Omega)$ with support in a small neighborhood of a point on this line, and define

$$d_\delta(y, h) := \min\left(\max\left(\frac{y - y_0}{\delta}, 0\right), 1\right).$$

Then similar as in the previous proof we have for $\delta \downarrow 0$

$$\int\limits_{\{v > v_0\}} \nabla\zeta(\nabla u + \gamma e) \leftarrow \int\limits_{\Omega} d_\delta \nabla\zeta(\nabla u + \gamma e) =$$

$$= -\int\limits_{\Omega} \zeta \nabla d_\delta(\nabla u + \gamma e) = -\frac{1}{\delta} \int\limits_{\{0 < v - v_0 < \delta\}} \zeta \partial_v u = \frac{1}{\delta} \int\limits_{\{0 < v - v_0 < \delta\}} u \partial_v \zeta - \frac{1}{\delta} \int\limits_{\{v = v_0 + \delta\}} \zeta u \, d\mathcal{H}^1,$$

which tends to zero by 4.2.

Using 4.2 and 4.3 we can prove that γ is a characteristic function.

4.4 THEOREM. *Suppose* $\{y_0\} \times [h_0, h_1[\subset \Omega \cap \{u = 0\}$ *with* $(y_0, h_1) \in \partial\Omega \setminus \overline{S_{res}}$, *such that in a neighborhood of this point* $\partial\Omega$ *is a Lipschitz graph in vertical direction. Then* $\gamma = 0$ *in a neighborhood of* $\{y_0\} \times]h_0, h_1[$.

PROOF. Let $h_0 < h_2 < h_1$. If u would be positive near (y_0, h_2) on one side of the line $\{y = y_0\}$, Hopf's principle would contradict 4.2. Therefore we find values $y_- < y_0 < y_+$ near y_0 with $u(y_\pm, h_2) = 0$.

Then $u = 0$ above these points by 4.1, and 4.3 implies that u, γ is a solution in

$$\tilde{\Omega} := \{(y, h) \in \Omega : y_- < y < y_+ \text{ and } h > h_2\}$$

with boundary sets

$$\tilde{S}_{res} = \{(y, h_2) : y_- < y < y_+\},$$
$$\tilde{S}_{air} = S_{air},$$
$$\tilde{S}_{imp} = S_{imp} \cup \{(y, h) \in \Omega : y = y_+ \text{ or } y_- \text{ and } h > h_2\}.$$

Since $u \leqslant C\delta$ on $\tilde{S}_{\text{res}}$, $\delta := y_+ - y_-$, we can apply the comparison lemma 3.2 to the supersolution

$$v(y, h) := \max \left(C\delta - (h - h_2), 0 \right),$$

and we obtain

$$u(y, h) = 0 \quad \text{for } y_- \leqslant y \leqslant y_+ \text{ and } h \geqslant h_2 + C\delta .$$

Then also $\gamma = 0$ in this region, which follows immediately if we take $\zeta(y, h) = \min \left(C\delta - (h - h_2), 0 \right)$ as test function for the solution u, γ in $\tilde{\Omega}$.

4.5 COROLLARY *Let* $x_0 = (y_0, h_0) \in \Omega \cap \partial\{u > 0\}$ *and* $x_1 = (y_0, h_1)$ *the first point on* $\partial\Omega$ *above* x_0. *If* x_1 *satisfies the conditions in 4.4, then* $\Omega \cap \partial\{u > 0\}$ *is an analytic graph in vertical direction near* x_0 *and* $\gamma = 0$ *above this curve.*

PROOF. If we apply 4.4 to all points of $\Omega \cap \{u = 0\}$ near x_0 we get that $\gamma = \chi_{\{u>0\}}$ in a neighborhood of x_0 and above this neighborhood. Moreover $\gamma = 0$ in a neighborhood of x_1. For this situation the regularity of the free boundary near x_0 was proved in [2].

Example 6.5 shows that we need global assumptions to be sure that every free boundary point is regular.

4.6 THEOREM. *If* ∂S_{res} *consists of a finite number of points, and if* $u_0 > 0$ *on* S_{res}, *then* $\gamma = \chi_{\{u>0\}}$ *and locally* $\Omega \cap \partial\{u > 0\}$ *is an analytic graph in vertical direction.*

PROOF. Let $x_0 = (y_0, h_0)$ be a point a little bit above a given free boundary point. As in the proof of 4.4 we find a sequence $y_k \downarrow y_0$ with $u(y_k, h_0) = 0$. Let (y_k, h_k) be the first point in $\partial\Omega$ above (y_k, h_0), and choose a subsequence such that $(y_k, h_k) \to x$. Denote the polygon with corners (y_k, h_k), (y_k, h_0), (y_{k+1}, h_0), and (y_{k+1}, h_{k+1}) by Γ_k and let Ω_k be the connected component of $\Omega \backslash \Gamma_k$ containing the lower part of the small strip enclosed by Γ_k. By the above choice of the subsequence and since $\partial\Omega$ is Lipschitz we infer that for large k the set Ω_k is bounded by Γ_k and a curve $\Sigma_k \subset \partial\Omega$, and that (y_k, h_k) and (y_{k+1}, h_{k+1}) are the only common points of Σ_k and Γ_k. Moreover for large k and $\tilde{k} > k + 1$ the intersection $\Sigma_k \cap \Sigma_{\tilde{k}}$ is empty. Since ∂S_{res} is finite we have two possibilities:

1. *case*: $\Sigma_k \subset S_{\text{res}}$ for some k. Then $u > 0$ on Σ_k and 4.1 yields $u > 0$ in Ω_k, which together with the Hopf principle contradicts 4.2.

2. *case*: $E_k \subset \partial\Omega \backslash \overline{S_{res}}$ for all large k. Then we conclude as in 4.4 that $u(y, h) = 0$ for $(y, h) \in \Omega_k$ with $h \geqslant h_0 + C(y_k - y_0)$ and also $\gamma = 0$ in this region. Since this is true for all large k, and since the same argument holds from the left side, we proved that $\gamma = 0$ in a neighborhood of $\{y_0\} \times \,]h_0, h_0 + \varepsilon]$ for some $\varepsilon > 0$. Then the assertion follows as in 4.5.

The following statement is similar to 4.4 and can be interpreted as a nondegeneracy property. The proof also applies to higher dimensions.

4.7 LEMMA. *Let $Q = \,]-1, 1[\times \,]0, \infty[$ and suppose that $Q \cap \Omega$ is contained in $\{h < h_1\}$, $Q \cap \overline{S_{res}}$ is empty, and $Q \cap \overline{S_{air}} \cap \overline{S_{imp}}$ is finite. Moreover, assume that $Q \cap \partial\Omega$ is a Lipschitz graph in vertical direction. Then for $0 < h_0 < h_1$ there is a small constant $c = c(h_0, Q \cap \partial\Omega) > 0$, such that $u \leqslant c$ in $\Omega \cap \partial Q$ implies $\gamma = 0$ in a neighborhood of $(0, h_0)$.*

PROOF. Define $D := Q \cap \{h < g(y)\}$, where $g:[-1, 1] \to \,]0, \infty[$ is a smooth function with $g(\pm 1) = 2h_1$, $g'(\pm 1) = 0$, and $g(0) < h_0$. Consider the harmonic function v in D with boundary values $v = 0$ on graph g and $v(y, h) = 2h_1 - h$ for $(y, h) \in \partial D \backslash$graph g. There is an $\varepsilon > 0$, such that εv is a local supersolution in $\Omega \cap Q$. Since $\varepsilon v \geqslant c > 0$ on $\Omega \cap \partial Q$, the assertion follows from lemma 3.2 applied to the domain $Q \cap \Omega$ and the boundary sets $\tilde{S}_{res} = \Omega \cap \partial Q$, $\tilde{S}_{air} = S_{air}$, $\tilde{S}_{imp} = S_{imp}$. In order to verify the conditions in the comparison lemma choose g such that $E := \partial\Omega \cap$ graph g consists of a finite number of points z belonging to S_{air} or S_{imp}. At such points ∇u satisfies a Morrey condition

$$\int\limits_{\Omega \cap B_\varrho(z)} |\nabla u|^2 = O(\varrho^{2\alpha})$$

for some $\alpha > 0$.

5. – The free boundary near reservoirs.

We consider the free boundary near a point on the fixed boundary, which lies on the surface of a reservoir.

5.1 ASSUMPTIONS. *u, γ is a local solution in $\Omega \cap B_R$ for some $R > 0$ and*

1) $\partial\Omega \cap B_R$ *is a $C^{1,1}$ curve with curvature less then $\varkappa_0$, contains the origin, and has tangent $\exp[i\sigma_0]$ at the origin, where $0 < \sigma_0 < \pi$.*

2) $\partial\Omega \cap B_R \cap \{h < 0\}$ *is contained in S_{res} with Dirichlet data*

$$u(y, h) = -h \,.$$

3) $\partial\Omega \cap B_R \cap \{h > 0\}$ *is contained in* S_{air} *with Dirichlet data*

$$u(y, h) = 0 .$$

5.2 REMARKS 1) Since we are interested in local properties of the free boundary, we can choose R such that

$$B_R \cap \Omega = B_R \cap \{y > Y(h)\}$$

with a $C^{1,1}$ function Y satisfying

$$Y(0) = 0 , \qquad Y'(0) = \cot \sigma_0 , \qquad |Y''| \leqslant \varkappa_0 < \frac{1}{R} .$$

2) We should remark, that in general the assumptions in 5.1 give no information about the behavior of the free boundary at the origin. For example, if $\sigma_0 > \pi/2$ and $0 < \sigma < \sigma_0 - \pi/2$, then u^σ, γ^σ is a local solution in $\{y > (\cot \sigma_0)h\} \cap B_R$, where u^σ is the linear function in 5.5 and

$$\gamma^\sigma := 1 - \frac{\sin \sigma_0}{\sin (\sigma_0 - \sigma) \cos \sigma} .$$

Therefore in the case $\sigma_0 \geqslant \pi/2$ we need the additional assumption, that $\gamma = \chi_{\{u>0\}}$ in a neighborhood of some point above the origin. But then (see 4.6) we can choose R small enough, such that this is fulfilled in $\Omega \cap B_R$.

3) The special data in 5.1.2) are not essential for our techniques, but since they are the standard ones for the dam problem, for simplicity we restrict ourselves to this case.

Our procedure is as follows. In the case that u is positive in a neighborhood of the origin, we have overflow in this neighborhood. Otherwise the origin is a limit point of the free boundary, and we show (5.3) that this is equivalent to the Lipshitz continuity of u near the origin. Therefore we are allowed to consider blow-up functions, which are studied in 5.6. Using these results we can go back to the original function u and obtain the desired result (5.7).

5.3 OVERFLOW LEMMA. *If u is as in 5.1 with the properties in 5.2.1), then either $u > 0$ in $\Omega \cap B_\varepsilon$ for some $\varepsilon > 0$ or $|\nabla u| \leqslant C$ in $\Omega \cap B_{R/8}$ with a universal constant C.*

PROOF. If u is positive in a neighborhood of the origin, then

$$u\left(\exp\left[i\sigma_0\right]z\right) = -\frac{\sin\sigma_0}{\pi}\operatorname{Re}\left(iz\log z\right) + O(z)$$

for $z \to 0$, that is, u is not Lipschitz continuous. Otherwise 0 is a limit point of the free boundary. Then for $x \in \Omega \cap B_{R/8}$ with $u(x) > 0$ let $B_r(x)$ be the maximum ball not intersecting $\Omega \cap \partial\{u > 0\}$. Since $B_r(x)$ does not contain the origin, we have to distinguish between the following three cases:

1. *case*: $B_{r/4}(x) \subset \Omega$. Then u is harmonic in this ball and we get from 2.4.1) (applied to $\Omega \cap B_{5r}(x)$)

$$|\nabla u(x)| \leqslant \frac{C}{r} \sup_{B_{r/8}(x)} u \leqslant C .$$

2. *case*: $x \in B_{r/4}(x_1)$ for some $x_1 \in S_{\mathrm{res}}$. Then u is harmonic in $\Omega \cap B_{r/2}(x_1)$ and $\partial\Omega \cap B_{r/2}(x_1) \subset S_{\mathrm{res}}$. The first estimate in 2.4.2) (again applied to $\Omega \cap B_{5r}(x)$) implies

$$u < C\left(r + |x_1 \cdot e|\right) \quad \text{in } \Omega \cap B_{r/2}(x_1) ,$$

and from the second inequality we obtain

$$Cr \geqslant \inf_{\partial\Omega \cap B_{4r}(x)} u \geqslant |x_1 \cdot e| - Cr .$$

Therefore the harmonic function $w(x) := u(x) + x \cdot e$ in $\Omega \cap B_{r/2}(x_1)$ satisfies

$$|w| \leqslant Cr \quad \text{in } \Omega \cap B_{r/2}(x_1) ,$$

$$w = 0 \quad \text{on } \partial\Omega \cap B_{r/2}(x_1) ,$$

and by well known estimates we obtain

$$|\nabla w(x)| \leqslant C .$$

3. *case*: $x \in B_{r/4}(x_1)$ for some $x_1 \in S_{\mathrm{air}}$. As in the 2. case we conclude (now the Dirichlet data on $\partial\Omega \cap B_{r/2}(x_1)$ are zero)

$$|\nabla u(x)| \leqslant C .$$

5.4 REMARK. The length of overflow can be estimated from below in the following sense. There is a universal constant C, such that the inequality

$u(x) \geqslant C|x|$ for some $x \in \Omega \cap B_{R/4}$ implies $u > 0$ in $\Omega \cap B_{|x|}$. This follows from 2.4.2).

In the case of a linear fixed boundary there is a class of linear super- and subsolutions, which give us the essential information about the behavior of the free boundary.

5.5 LINEAR SOLUTIONS. For $\sigma_0 - \pi < \sigma < \sigma_0$ define

$$u^\sigma(x) := \frac{\sin \sigma}{\sin (\sigma - \sigma)} \exp\left[i\left(\sigma - \frac{\pi}{2}\right)\right] \cdot x,$$

if positive, and $u^\sigma(x) = 0$ otherwise.
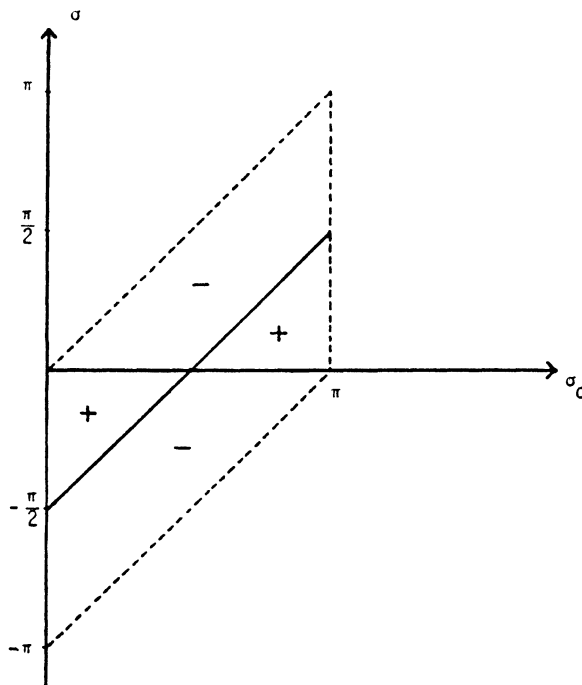
Fig.5

Fig. 5 shows the regions of the parameters σ_0 and σ for which u^σ is a super-and subsolution in the half space

$$\Omega_* := \left\{x/x \cdot \exp\left[i\left(\sigma_0 - \frac{\pi}{2}\right)\right] > 0\right\},$$

that is,

$$\underset{(-)}{+}(\nabla u^\sigma + e) \cdot \nu \geqslant 0 \quad \text{on } \Omega_* \cap \partial\{u^\sigma > 0\}.$$

Therefore u^σ is a solution, if and only if $\sigma = 0$ (horizontal free boundary) or $\sigma = \sigma_0 - \pi/2$ (right angle). u^0 is the solution at rest.

Therefore it is natural to define

$$\sigma_- := \min\left(0, \sigma_0 - \frac{\pi}{2}\right) \quad \text{and} \quad \sigma_+ := \max\left(0, \sigma_0 - \frac{\pi}{2}\right),$$

and to state the following theorem, which we will prove using the comparison lemmas 3.2 and 3.4.

5.6 THEOREM. *Assume $\partial\Omega$ is linear near the origin and u, γ is a local solution as in 5.1. Then the following statements hold.*

1) *If $\exp[i\sigma]$ is any tangent vector of the free boundary at 0, then $\sigma \geqslant \sigma_-$.*

2) *If $u \leqslant u^{\sigma_+}$ with $\gamma = \chi_{\{u>0\}}$ (relevant only for $\sigma_0 \geqslant \pi/2$), then either $u = u^{\sigma_+}$ or the free boundary has the tangent $\exp[i\sigma_-]$ at 0 and u^{σ_-} is the unique blow-up function.*

3) *If $u \geqslant u^{\sigma_+}$, then either $u = u^{\sigma_+}$ or we have overflow, that is, $u > 0$ in a neighborhood of 0.*

PROOF OF 3). For $r \leqslant R$ define

$$\sigma_*(r) := \sup\{\sigma: u \geqslant u^\sigma \text{ in } \Omega \cap B_r\}$$

$$\sigma_* := \lim_{r \downarrow 0} \sigma_*(r).$$

Let us assume that there is no overflow. Then 5.3 says that u is Lipschitz continuous in $B_{R/8}$, hence by 2.5 we can choose a sequence $r \downarrow 0$ such that u_r, γ_r converges to a blow-up limit u_*, γ_*. Then $u_* \geqslant u^{\sigma_*}$ and $\sigma_* < \sigma_0$, and since $u(0, h) = 0$ for $h \geqslant 0$, we have $\sigma^* \leqslant \pi/2$.

Now assume that $u_*(x_0) > 0$ for some $x_0 \in \Omega \cap \partial\{u^{\sigma_*} > 0\}$. Then by the strong maximum principle $u_* > u^{\sigma_*}$ in $\Omega \cap \{u^{\sigma_*} > 0\}$, and since $u_r \to u_*$ uniformly we must have

$$u_r \geqslant u^{\sigma_* + \varepsilon} \quad \text{in } B_\varepsilon(x_0)$$

for some $\varepsilon > 0$ and small r. Moreover for the point $x_1 \in S_{\text{res}} \cap \partial B_{r_0}$, where $r_0 = |x_0|$, we have by the Hopf principle (ν normal of $\partial\Omega$)

$$\partial_{-\nu} u_*(x_1) > \partial_{-\nu} u^{\sigma_*},$$

and since $u_r \to u_*$ in C^1 near x_1, we also must have

$$u_r \geqslant u^{\sigma_* + \varepsilon} \quad \text{in} \quad \Omega \cap B_\varepsilon(x_1)$$

for some $\varepsilon > 0$ and small r. On the arc Σ_ε of ∂B_{r_0} joining $B_\varepsilon(x_0)$ and $B_\varepsilon(x_1)$ we have $u_* > u^{\sigma_*}$, therefore also

$$u_r \geqslant u^{\sigma_* + \varepsilon^1} \quad \text{on} \quad \Sigma_\varepsilon$$

for some ε^1 and small r. Altogether we see that

$$u_r \geqslant u^{\sigma_* + \varepsilon} \quad \text{on} \quad \partial(\Omega \cap B_{r_0})$$

for some ε and small r. Since $u^{\sigma_* + \varepsilon}$ is a subsolution, we obtain from the comparison lemma 3.4 applied to the domain $\tilde{\Omega} = \tilde{\Omega} \cap B_{r_0}$ and the boundary sets $\tilde{S}_{\text{res}} = \tilde{S}_{\text{res}} \cup (\Omega \cap \partial B_{r_0})$ and $\tilde{S}_{\text{air}} = S_{\text{air}}$

$$u_r \geqslant u^{\sigma_* + \varepsilon} \quad \text{in} \quad \Omega \cap B_{r_0},$$

that is, $\sigma(r) \geqslant \sigma_* + \varepsilon$, a contradiction to the definition of σ_*.

Therefore we must have

$$u_* = 0 \quad \text{on} \quad \partial\{u^{\sigma_*} > 0\},$$

and we will show, that this implies

$$u_* = u^{\sigma_*} \quad \text{and} \quad \sigma_* = \sigma_+ .$$

In the case $\sigma_* < \pi/2$ we conclude (see 4.1) $u_* = 0$ in $\{u^{\sigma_*} = 0\} \cap \{y \geqslant 0\}$ and $\partial_h \gamma_* = 0$ in this region. Therefore on $\partial\{u^{\sigma_*} > 0\}$ (γ_* is defined from above)

$$(\nabla u_* + e) \cdot \nu = \gamma_* e \cdot \nu \geqslant 0 .$$

(By 4.2 the case $\sigma_* = \pi/2 < \sigma_0$ cannot occur.) This proves, that on the line $\partial\{u^{\sigma_*} > 0\}$

$$\partial_{-\nu}(u_* - u^{\sigma_*}) \leqslant (\nabla u^{\sigma_*} + e) \cdot \nu .$$

The right side is non-positive, since $\sigma_* \geqslant \sigma_+$ by our assumption in the statement of the theorem, and the left side is non-negative since $u_* \geqslant u^{\sigma_*}$. Therefore $\sigma_* = \sigma_+$ and $\partial_{-\nu}(u_* - u^{\sigma_*}) = 0$ on the above line. Since the Cauchy problem has a unique solution, we conclude $u_* = u^{\sigma_+}$.

If we now apply the same arguments to u and σ_+ instead of u_* and σ_* we obtain $u = u^{\sigma_+}$.

PROOF OF 1). Define σ_* as before, where $\sigma_*(r) := \sigma_0 - \pi$, if the set in the definition is empty. We will argue as in the proof of 3), and therefore let u_*, γ_* be a blow-up limit, which exists by 5.3, since there is nothing to prove in the overflow case.

First we see that the Lipschitz continuity immediately implies $\sigma_* > \sigma_0 - \pi$. Then we can apply the arguments in the proof of 3) and obtain

$$0 \leqslant \partial_{-\nu}(u^* - u^{\sigma_*}) \leqslant (\nabla u^{\sigma_*} + e) \cdot \nu$$

on the line $\partial\{u^{\sigma_*} > 0\}$, which implies $\sigma_* \geqslant \sigma_-$.

PROOF OF 2). Define

$$\sigma^*(r) := \inf\{\sigma : u \leqslant u^{\sigma} \text{ in } \Omega \cap B_r\}$$

$$\sigma^* := \lim_{r \downarrow 0} \sigma^*(r).$$

Since $u \leqslant u^{\sigma_+}$ we have no overflow, and therefore by 5.3 we can choose a blow-up sequence u_r, γ_r converging to u_*, γ_*.

Then $u^{\sigma_-} \leqslant u_* \leqslant u^{\sigma_*}$ (we proved $\sigma_* \geqslant \sigma_-$) and $\sigma_- \leqslant \sigma^* \leqslant \sigma_+$. If $\sigma^* = \sigma_-$, the free boundary has the desired tangent. Therefore let $\sigma^* > \sigma_-$ (hence $\sigma_0 \neq \pi/2$). We will prove $u = u^{\sigma_+}$ as in the proof of 3), but now we argue from above.

Assume there is a free boundary point $x_0 \in \{u^{\sigma_*} > 0\} \cap \{u^{\sigma_-} = 0\}$ of u_*. Then for a subsequence $r \downarrow 0$ there are free boundary points $x_r = (y_r, h_r)$ of u_r converging to x_0, since otherwise u_* would be harmonic in a neighborhood of x_0. Then $u_r(y_r, h) = 0$ for $h \geqslant h_r$ and similar as in the proof of 3) we conclude that for some $\varepsilon > 0$ and small r in the above subsequence

$$u_r \leqslant u^{\sigma^* - \varepsilon} \quad \text{on } \partial(\Omega \cap Q_r),$$

where

$$Q_r := \{(y, h) : |y| < y_r, h > -2h_r\}.$$

(In the case $\sigma_0 > \pi/2$ cut Q_r in the region where by assumption $\gamma_r = 0$.) Now we apply the comparison lemma from above (3.2) to the domain $\tilde{\Omega}_r = \Omega \cap Q_r$ and the boundary sets $\tilde{S}_{res} = S_{res} \cup (\partial Q_r \cap \{u^{\sigma^* - \varepsilon} > 0\})$ and $\tilde{S}_{air} = S_{air}$ or $S_{air} \cup (\partial Q_r \setminus \overline{\{u^{\sigma^* - \varepsilon} > 0\}})$. This can be done, since the separation lemma (4.3) and the fact that $\gamma_r = 0$ near the top of Q_r in the case

$\sigma_0 > \pi/2$ ensure that u_r is a solution with respect to the data defined above. We obtain $u_r \leqslant u^{\sigma^*-\varepsilon}$ in $\bar{\Omega}$, a contradiction to the definition of σ^*.

Therefore u_* is harmonic in $\{u^{\sigma^*} > 0\}$ and zero on $\partial\{u^{\sigma^*} > 0\}$, which implies $u_* = 0$ and $\partial_h\gamma_* = 0$ in $\{u^{\sigma^*} = 0\}$. But since $\gamma_r = \chi_{\{u_r>0\}} = 0$ in $\{u^{\sigma^+} = 0\}$ near 0 by the assumption $u \leqslant u^{\sigma^+}$ and 4.5 we see that $\gamma_* = \chi_{\{u_*>0\}}$ and therefore

$$0 \leqslant \partial_{-\nu}(u^{\sigma^*} - u_*) = -(\nabla u^{\sigma^*} + e) \cdot \nu \leqslant 0$$

on the line $\partial\{u^{\sigma^*} > 0\}$, which as in the proof of 3) yields

$$u_* = u^{\sigma^*} \quad \text{and} \quad \sigma^* = \sigma_+ .$$

If we apply the same argument to u and σ_+ instead of u_* and σ^*, we obtain $u = u^{\sigma^+}$.

5.7 THEOREM. *Let u, γ be a local solution as in 5.1, such that 0 is a limit point of the free boundary, and in the case $\sigma_0 \geqslant \pi/2$ assume that $\gamma = 0$ in a neighborhood of some point above 0. Define*

$$\tau := \liminf_{r \downarrow 0}\{\sigma \in]\sigma_0 - \pi, \sigma_0[\,|\, r\exp[i\sigma] \in \partial\{u > 0\}\} .$$

Then the following is true.

1) $\tau \geqslant \sigma_-$.

2) *If $\sigma_- \leqslant \tau < \sigma_+$, then $\tau = \sigma_-$ and the free boundary has the tangent $\exp[i\sigma_-]$ at 0.*

3) *If $\tau \geqslant \sigma_+$, the $\tau = \sigma_+$ and the free boundary has the tangent $\exp[i\sigma_+]$ at 0.*

PROOF OF 1). We proceed as in the proof of 5.6.1), but since $\partial\Omega$ is not a straight line we have to change the definition of the functions u^σ. For small $r > 0$ let $\sigma_r \in]0, \pi[$ be the maximum value for which

$$\{-s\exp[i\sigma_r]: 0 \leqslant s \leqslant r\} \subset \mathbf{R}^2\setminus\Omega ,$$

and define $u^{r\sigma}$ as u^σ in 5.5, but with σ_0 replaced by σ_r. Then $u^{r\sigma}$ are subsolutions for $\sigma_r - \pi < \sigma < \min(\sigma_r - \pi/2, 0)$, and $\sigma_r \uparrow \sigma_0$ as $r \downarrow 0$. Using $u^{r\sigma}$ in $\Omega \cap B_r$ instead of u^σ in the proof of 5.6.1) we obtain the assertion. (For 1) the assumption on γ is not needed.).

PROOF OF 3). Since 0 is a limit point of the free boundary, u is Lipschitz continuous (5.3), and blow-up limits u_*, γ_* exist. We will show, that every blow-up limit equals u^{σ_+}. For $\varepsilon > 0$ consider the domain

$$D_\varepsilon := \{r \exp[i\theta] \in \Omega : r > 0 \text{ and } \theta < \sigma_+ - \varepsilon\} .$$

By assumption u is harmonic in $B_r \cap D_\varepsilon$ for small r. Since D_ε has an angle less than π at 0 we conclude

$$u(x) \geqslant u^{\sigma_+ - \varepsilon}(x) - o(x) \quad \text{for } x \to 0 ,$$

which implies $u_* \geqslant u^{\sigma_+ - \varepsilon}$. Since ε was arbitrary we get $u_* \geqslant u^{\sigma_+}$ and 5.6.3) gives $u_* = u^{\sigma_+}$.

This shows that

$$\frac{1}{r} \operatorname{dist}\left(r \exp[i\sigma_+], \partial\{u > 0\}\right) \to 0 \quad \text{for } r \to 0 .$$

If not, there would be an $\varepsilon > 0$ and a sequence $r \downarrow 0$ such that $u > 0$ in $B_{\varepsilon r}(r \exp[i\sigma_+])$, hence the corresponding blow-up would be harmonic in $B_\varepsilon(\exp[i\sigma_+])$, a contradiction. Therefore we find a sequence of free boundary points $(y_k, h_k) \to 0$ with

$$h_k < y_k \tan \sigma_+ + o(y_k) ,$$

and since u is Lipschitz continuous we have

$$\sup_{x \in \partial\tilde{\Omega}_k} \frac{u(x)}{|x|} \to 0 \quad \text{for } k \to \infty,$$

where

$$\tilde{\Omega}_k = \{(y, h) : 0 < y < y_k \text{ and } y \tan \sigma_+ < h < g(y)\} .$$

Here $(y, g(y))$ is the first point on $\partial\Omega$ above $(y, 0)$ or a point in the neighborhood, in which by assumption $\gamma = 0$. For given $\varepsilon > 0$ consider the function

$$v_\varepsilon(x) := \max\left(0, \cos(\varepsilon + \sigma_+) \exp[i(\varepsilon + \sigma_+ - \pi/2)] \cdot x\right) .$$

Then $u \leqslant v_\varepsilon$ on $\partial\tilde{\Omega}_k$ for large k and using the separation lemma 4.3 we can apply the comparison lemma 3.2 to $\tilde{\Omega}_k$ with boundary sets $\tilde{S}_{\text{res}} = \partial\tilde{\Omega}_k \cap \{v_\varepsilon > 0\}$ and $\tilde{S}_{\text{air}} = \partial\tilde{\Omega}_k \backslash \overline{\{v_\varepsilon > 0\}}$. We obtain that the free boundary of u is contained in $\overline{\{v_\varepsilon > 0\}}$ near the origin, which proves 3), since ε was arbitrary.

PROOF OF 2). First let us assume that the corresponding lim sup in the statement of 5.7 does not exceed σ_+, which will be proved at the end. Then similar as in the proof of 1) let $\sigma_r \in]0, \pi[$ be the minimal value for which

$$\{- s \exp [i\sigma_r]: 0 \leqslant s \leqslant r\} \subset \bar{\Omega}$$

and define $u^{r\sigma}$ as u^σ in 5.5 with σ_0 replaced by σ_r. Then $u^{r\sigma}$ are supersolutions at 0 for $\min (\sigma_r - \pi/2, 0) < \sigma < \max (\sigma_r - \pi/2, 0)$, and $\sigma_r \downarrow \sigma_0$ as $r \downarrow 0$. Since $\tau < \sigma_+$, we find for some $\varepsilon > 0$ a sequence of points $x_r = r \exp [i(\sigma_+ - \varepsilon)] \to 0$ with $u(x_r) = 0$. If u_* denotes a blow-up limit with respect to this sequence, we have $u_*(\exp [i(\sigma_+ - \varepsilon)]) = 0$, and because of our additional assumption on the free boundary we know

$$u(x) \leqslant u^{\sigma_+}(x) + o(x) \quad \text{for } x \to 0 .$$

This shows $u_* \leqslant u^{\sigma_+}$ and $u_* \neq u^{\sigma_+}$. Then 5.6.2) shows that u^{σ_-} is the unique blow-up limit of u_* at 0, that is,

$$u_* \leqslant u^\sigma \quad \text{in } \Omega_* \cap B_s$$

for given $\sigma > \sigma_-$ and small s. Then comparing u with the supersolutions $u^{\sigma r}$ we obtain the result as in the proof of 5.6.2).

To complete the proof let $\varepsilon > 0$ be small and v_ε as above, and choose r small enough such that $v_\varepsilon(y, h) < - h$ on $S_{\text{res}} \cap \bar{B}_r$. Assume that the corresponding lim sup in the statement of the theorem is larger than $\sigma_+ + \varepsilon$. Then since $\tau \leqslant \sigma_+$ the set

$$\partial\{v_\varepsilon > 0\} \backslash \overline{\{u > 0\}}$$

contains infinite many connected components converging to zero, such that u is positive somewhere between them. Moreover we can choose a monotone sequence I_k of them containing points $x_k = (y_k, h_k)$ with

$$u(y_k, h_k - \varepsilon y_k) = 0 .$$

Let us consider the function $w := v_\varepsilon - u$, and let D_k be the connected component of

$$B_r \cap \{v_\varepsilon > 0\} \cap \{w > 0\}$$

containing I_k as a part of the boundary.

If two of these components are equal, say $D_{k_1} = D_{k_2}$, we can choose a simple curve Γ in this component joining x_{k_1} and x_{k_2}. Denote by Q the

domain bounded by Γ and the vertical rays starting from x_{k_1} and x_{k_2} (in the case $\sigma_0 > \pi/2$ (we have $\sigma_0 \neq \pi/2$) only up to the neighborhood, where $\gamma = 0$ by assumption). Then $u \leqslant v_\varepsilon$ on $\partial(\Omega \cap Q)$ and we can apply the comparison lemma 3.2 to the domain $\tilde{\Omega} = \Omega \cap Q$ with boundary sets $\tilde{S}_{res} =$
$= \partial\tilde{\Omega} \cap \{v_\varepsilon > 0\}$ and $\tilde{S}_{air} = S_{air} \cup (\partial\tilde{\Omega} \setminus \overline{\{v_\varepsilon > 0\}})$. We obtain $u \leqslant v_\varepsilon$ in $\tilde{\Omega}$ a contradiction to the construction of the segments I_k. Now pick a maximum z_k of the continuous function w in $\overline{D}_k$. Then $w(z_k) \geqslant w(y_k, h_k - \varepsilon y_k) > 0$, and since $w \leqslant 0$ on $B_r \cap \partial D_k$ and $\Delta w = 0$ in $D_k \setminus \partial\{u > 0\}$ we have two possibilities, $z_k \in D_k \cap \partial\{u > 0\}$ and $z_k \in \partial B_r \cap \tilde{D}_k$.

In the first case (see [8], Lemma 3.2) we have by the Hopf principle at this point (by 4.5 the free boundary is smooth)

$$0 < \nabla w \cdot \nu = (\nabla v_\varepsilon + e) \cdot \nu .$$

Since $w = v_\varepsilon$ on $\partial\{u > 0\}$ the point z_k is a maximum for v_ε on $\partial\{u > 0\}$, that is, $\nu = \nu_\varepsilon$, where ν_ε is the normal of $\partial\{v_\varepsilon > 0\}$. But $(\nabla v_\varepsilon + e) \cdot \nu_\varepsilon = 0$, a contradiction. Therefore $z_k \in \partial B_r \cap \overline{D}_k$. Since there was no connection between the components D_k, we find points $z_k^1 \in \partial B_r \cap \partial D_k$ between z_k and z_{k+1}, hence $w(z_k^1) = 0$. Choose a subsequence such that $z_k^1 \to z^1$. Since by construction $w < 0$ at the point of $\partial B_r \cap S_{res}$ we conclude $z^1 \in \Omega \cap \{v_\varepsilon > 0\}$, and since $w(z^1) = 0$ we must have $u(z^1) > 0$. Therefore w is harmonic in a neighborhood of z^1, hence $\{w > 0\}$ has a finite number of components near z^1. Since $z_k^1 \in \partial D_k$ converge to z^1, this cannot be.

6. – The free boundary near the atmosphere.

In this section we study the local behavior of the free boundary near boundary points to the air. We show that the free boundary is vertical or tangential.

6.1 ASSUMPTION. *u, γ is a local solution in $\Omega \cap B_R$ for some $R > 0$ and*

1) *$\partial\Omega \cap B_R$ is a $C^{1,1}$ curve, contains the origin, and has normal $\nu_0 = \exp[i(\sigma_0 + \pi/2)]$ at the origin, where $0 \leqslant \sigma_0 \leqslant \pi$.*

2) *$\partial\Omega \cap B_R$ is contained in S_{air}.*

3) *The origin is a limit point of the free boundary.*

4) *If $\pi/2 \leqslant \sigma_0 \leqslant \pi$ we assume that $\gamma = 0$ in a neighborhood of some point in $\Omega \cap B_R$ above the origin.*

6.2 REMARK. We can choose R small enough, such that $\Omega \cap B_R$ is a small deformation of a half ball, and such that (using 4.6) $\gamma = \chi_{\{u>0\}}$ in $\Omega \cap B_R$. If $\sigma_0 > \pi/2$ we can assume that $\gamma = 0$ in a neighborhood of $\{0\} \times]0, R[$.

6.3 LEMMA. $|\nabla u| \leqslant C$ in $\Omega \cap B_{R/2}$ with a universal constant C.

PROOF. As in 5.3 (1. and 3. case).

In the case $0 \leqslant \sigma_0 < \pi/2$ the free boundary is always tangential to the fixed boundary. This was proved in [8], Theorem 2.1, and is also a special case of 8.4.

6.4 THEOREM. For $0 \leqslant \sigma_0 < \pi/2$ the free boundary is tangential at 0.

6.5 REMARK. For $\sigma_0 \geqslant \pi/2$ the additional assumption 6.1.4) is necessary in order to get the local results. As counterexample consider the domain

$$\Omega_{r,\varepsilon} := \left\{ (y, h) \in \mathbf{R}^2 : h > 0 \text{ and } h < 1 - \varepsilon \left(1 - \frac{|y|}{r} \right) \text{ for } |y| \leqslant r \right\}$$

with boundary sets $S_{\text{res}} = \partial \Omega_{r,\varepsilon} \cap \{0 < h < 1\}$, $S_{\text{air}} = \partial \Omega_{r,\varepsilon} \setminus \overline{S_{\text{res}}}$, and $S_{\text{imp}} = \emptyset$, and Dirichlet data $u^0 = 0$ on S_{air} and

$$u^0(y, h) = 1 - h \quad \text{for } (y, h) \in S_{\text{res}} .$$

Let $u_{r,\varepsilon}, \gamma_{r,\varepsilon}$ be a solution constructed in [3]. As in [3], Beispiel 4.6 we compare $u_{r,\varepsilon}$ with linear functions

$$v(y, h) = \frac{1}{1 + s^2} \max (0, \pm sy - h + c_s)$$

with

$$c_s = 1 + rs + 2\varepsilon s^2 .$$

The comparison lemma 3.2 implies

$$\{u_{r,\varepsilon} > 0\} \subset \left\{ 0 < h < 1 - \frac{1}{8\varepsilon} \max (|y| - r, 0)^2 \right\} \subset \{|y| < r + 2\sqrt{2\varepsilon}\} .$$

Moreover, using an appropriate continuation of u^0 as test function v in 1.2 we see that for $\varepsilon \leqslant 1/2$

$$\int_{\Omega_{r,\varepsilon}} |\nabla u_{r,\varepsilon}|^2 \leqslant C \cdot (r + 2\sqrt{2\varepsilon}) .$$

A sequence of such functions gives a local solution for which 6.1.4) is not satisfied (see fig. 6).
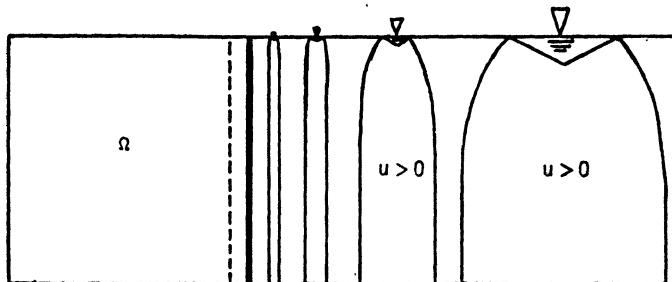

Fig.6

6.6 THEOREM. *If* $\pi/2 < \sigma_0 \leqslant \pi$, *then the free boundary is vertical at* 0.

PROOF. Let R as in 6.2 and $\partial\Omega =$ graph g_0 near 0 with $g_0(0) = 0$ and $g_0'(0) = \tan\sigma_0$. Then $\gamma = 0$ in a neighborhood of $\{0\} \times]0, R[$. Moreover, if we allow σ_0 to be any value between $\pi/2$ and $3\pi/2$, we can assume that $u = 0$ in $\{y < 0\}$.

First we see that $u > 0$ in Ω near $(y, g_0(y))$ for small $y > 0$. Otherwise u is a non-negative subharmonic function in

$$\{(y, h) : 0 < y < y_0 \text{ and } g_0(y) < h < R/2\}$$

vanishing on its boundary for some $y_0 > 0$. We conclude $u = 0$ in this domain, that is, 0 cannot be a limit point for the free boundary, a contradiction to 6.1.3). This shows that in a small neighborhood

$$\Omega \cap \{u > 0\} = \{(y, h)/y > 0 \text{ and } g_0(y) < h < g(y)\} \ ,$$

where g is a continuous function with $g(y) > g_0(y)$ for $y > 0$ and $g(0) = 0$. Another consequence of the subharmonicity of u is that u can be estimated by the harmonic function v in $\Omega \cap B_R \cap \{y > 0\}$ with boundary values u, hence

$$u(x) \leqslant v(x) = o(x) \quad \text{for } x \to 0 \ ,$$

and on $\partial\Omega$

$$\partial_{-\nu} u(y, g_0(y)) \leqslant \partial_{-\nu} v(y, g_0(y)) \to 0 \quad \text{for } y \downarrow 0 \ .$$

Now let $0 < \varepsilon < r$ small and define $D_{\varepsilon,r} := B_R \cap \{\varepsilon < y < r\}$. Using the

fact that $\gamma = 0$ near the top of $D_{\varepsilon,r}$ we can take

$$\zeta(x) = \pm \min\left(1, \frac{1}{\delta} \operatorname{dist}\left(x, \mathbb{R}^2 \backslash (D_{\varepsilon,r} \cap \Omega)\right)\right)$$

as test function in 2.1. Letting $\delta \downarrow 0$ we obtain for almost all ε and r

$$0 = \int\limits_{\Omega \cap \partial D_{\varepsilon,r}} (\nabla u + \gamma e) \cdot \nu \, d\mathcal{H}^1 + \int\limits_{D_{\varepsilon,r} \cap \partial \Omega} (\nabla u + e) \cdot \nu \, d\mathcal{H}^1 =$$

$$= \int\limits_{g_0(r)}^{g(r)} \partial_\nu u(r, h) \, dh - \int\limits_{g_0(\varepsilon)}^{g(\varepsilon)} \partial_\nu u(\varepsilon, h) \, dh + \int\limits_{\varepsilon}^{r} \left(\nabla u(y, g_0(y)) + e\right) \cdot (g_0'(y), -1) \, dy .$$

Since $g(0) = 0$ and since u is Lipschitz continuous we deduce for $\varepsilon \downarrow 0$

$$0 = \int\limits_{g_0(r)}^{g(r)} \partial_\nu u(r, h) \, dh + \int\limits_{0}^{r} \left(\nabla u(y, g_0(y)) + e\right) \cdot (g_0'(y), -1) \, dy .$$

It was shown above that $\nabla u(y, g_0(y)) \to 0$ for $y \downarrow 0$. Thus we obtain

$$\lim_{r \downarrow 0} \frac{1}{r} \int\limits_{g_0(r)}^{g(r)} \partial_\nu u(r, h) \, dh = 1 .$$

Since ∇u is bounded, this implies in particular

$$l := \liminf_{r \downarrow 0} \frac{g(r) - g_0(r)}{r} > 0 .$$

If the statement of the theorem is not true, there is a sequence $r_k \downarrow 0$ with

$$\frac{g(r_k) - g_0(r_k)}{r_k} \to l < \infty ,$$

hence for $\delta > 0$ and large k

$$\frac{1}{2} \leqslant \frac{1}{r_k} \int\limits_{g_0(r_k)}^{g(r_k)} \partial_\nu u(r_k, h) \, dh \leqslant \int\limits_{\delta + \tan \sigma}^{l - \delta + \tan \sigma_0} \partial_\nu u(r_k, r_k s) \, ds + C\delta .$$

If δ is small enough, this implies that there are points $x_k = (r_k, r_k s_k)$ with $0 < s_k - \delta - \tan \sigma_0 < l$ and

$$\partial_\nu u(x_k) \geqslant c .$$

But u is harmonic in $B_{c\delta r_k}(x_k)$ for large k. Hence Poisson's formula yields

$$|\partial_y u(x_k)| \leqslant \frac{C}{r_k} \fint_{\partial B_{c\delta r_k}(x_k)} u\, d\mathcal{H}^1 \to 0$$

for $k \to 0$, since u decays faster than linearly. This is a contradiction.

The last case is

6.7 THEOREM. *For* $\sigma_0 = \pi/2$ *the free boundary is vertical at* 0, *and* $\{y > 0\}$ *is the tangent cone of* $\Omega \cap \{u > 0\}$ *at* 0.

PROOF. First we prove that for some large $a > 0$

$$u(y, -ay) > 0 \quad \text{for small } y > 0 .$$

For this assume $u(y_0, -ay_0) = 0$ and choose h_0 such that (y_0, h_0) lies in the neighborhood where $\gamma = 0$ by 6.1.4). Consider the domain

$$\tilde{\Omega} := \{(y, h) \in \Omega : y < y_0 \text{ and } -ay_0 < h < h_0\} .$$

Then $u \leqslant Cy_0$ at the bottom of $\tilde{\Omega}$ by 6.3, and therefore

$$u(y, h) \leqslant v(y, h) := \max\big((C - a)y_0 - h, 0\big) \quad \text{for } (y, h) \in \partial\tilde{\Omega} .$$

Using the separation lemma 4.3 we can apply the comparison lemma 3.2 to $\tilde{\Omega}$ with boundary sets $\tilde{S}_{\text{res}} = \partial\tilde{\Omega} \cap \{h = -ay_0\}$ and $\tilde{S}_{\text{air}} = \partial\tilde{\Omega} \setminus \tilde{S}_{\text{res}}$, and we obtain $u(y, h) = 0$ for $(y, h) \in \tilde{\Omega}$ with $h \geqslant (C - a)y_0$, which is a contradiction to 6.1.3) if $a > C$.

Now assume that the theorem is not true. Then

$$-a \leqslant l := \liminf_{r \downarrow 0} \frac{g(r)}{r} < \infty ,$$

where g is defined as in the proof of 6.6. Furthermore $u(x) = o(x)$ for $x \to 0$ by [8], Proposition 2.1. Since u is harmonic in $B_{cy}((y, -2ay))$ for small $y > 0$ and some $c > 0$, we conclude using Poisson's formula

$$\nabla u(y, -2ay) \to 0 \quad \text{for } y \downarrow 0 .$$

Then we get a contradiction as in the proof of 6.6.

7. – The free boundary near the impervious boundary.

In this section we will show, that at points of the impervious part of the fixed boundary the free boundary is horizontal or tangential.

7.1 ASSUMPTIONS. *u, γ is a local solution in $\Omega \cap B_R$ for some $R > 0$ and*

1) *$\partial\Omega \cap B_R$ is a $C^{1,1}$ curve, contains the origin, and has normal $\nu_0 =$*
    $= \exp[i(\sigma_0 + \pi/2)]$ at the origin $(0 \leqslant \sigma_0 \leqslant \pi)$.

2) *$\partial\Omega \cap B_R$ is contained in S_{imp}.*

3) *The origin is a limit point of the free boundary.*

4) *If $\pi/2 \leqslant \sigma_0 \leqslant \pi$, we assume that $\gamma = 0$ in a neighborhood of some point in $\Omega \cap B_R$ above the origin.*

7.2 REMARK. We can choose R small enough as in 6.2.

First let us prove the Lipschitz continuity of u.

7.3 LEMMA. *If u is as in 7.1, then*

$$|\nabla u| \leqslant C \quad in \ \Omega \cap B_{R/2}$$

with a universal constant C.

PROOF. We can assume that u has the properties in 7.2. For $x \in \Omega \cap B_{R/8}$ with $u(x) > 0$ let $B_r(x)$ be the maximum ball not intersecting $\Omega \cap \partial\{u > 0\}$. If $B_{r/4}(x) \subset \Omega$ we get the estimate as in 5.3. Otherwise $x \in B_{r/4}(x_1)$ for some $x_1 \in S_{\mathrm{imp}}$. Then 2.4.3) (applied to $\Omega \cap B_{5r}(x)$) says

$$u \leqslant Cr \quad in \ B_{r/2}(x_1) .$$

Therefore the harmonic function $w(x) := u(x) + (x - x_1)\cdot e$ in $\Omega \cap B_{r/2}(x_1)$ satisfies

$$|w| \leqslant Cr \quad in \ \Omega \cap B_{r/2}(x_1) ,$$

$$\partial_\nu w = 0 \quad on \ \partial\Omega \cap B_{r/2}(x_1) ,$$

and by well known estimates we obtain $|\nabla w(x)| \leqslant C$.

If the free boundary is not tangential to the fixed boundary at 0, we will show that it becomes horizontal, which in terms of u means

$$u(x) = -x\cdot e + o(x) \quad for \ x \to 0 .$$

The main difference to the situation in sections 5 and 6 is that here at the point of interest the velocity vector becomes zero. Our proof is based on the following observation.

7.4 REMARK. Formally ∇u satisfies

$$\left|\nabla u + \frac{e}{2}\right| = \frac{1}{2} \quad \text{on} \quad \Omega \cap \partial\{u > 0\},$$

$$\begin{aligned} (\nabla u + e)\cdot v &= 0 && \text{on } S_{\text{imp}}, \\ (\nabla u + e)\cdot(iv) &= 0 && \text{on } S_{\text{res}} \end{aligned} \quad \text{(if } u^0 \text{ as in 5.1),}$$

$$\left.\begin{aligned} \nabla u\cdot(iv) &= 0 \\ (\nabla u + e)\cdot v &\leqslant 0 \end{aligned}\right\} \text{on } S_{\text{air}}.$$

Here we are interested only in the first two equalities, which say that in the case of a linear fixed boundary ∇u maps the set $\partial(\Omega \cap \{u > 0\}) \cap B_R$ into the boundary of the set

$$\Lambda := \overline{B_{\frac{1}{2}}\left(-\frac{e}{2}\right)} \cup \{-e + \tau : \tau\cdot v = 0\},$$
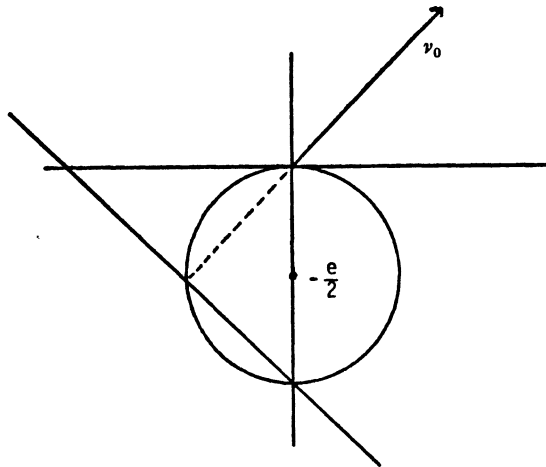
(see Fig. 7).

Fig.7

Moreover ∇u is bounded by 7.3. Therefore it is natural to prove that the values $\nabla u(x)$ tend to Λ for $x \to 0$. This implies that for blow up limits ∇u_*

maps into $\overline{B_{\frac{1}{2}}(-e/2)}$, which forces the free boundary of u_* to be concave to the fluid (7.7). Knowing this we can prove the statements in 7.8-7.12.

7.5 LEMMA. *If u is as in 7.1, then* dist $(\nabla u(x), \Lambda) \to 0$ *as* $x \to 0$.

PROOF. If not, there is a sequence $x_k \to 0$ in $\Omega \cap \{u > 0\}$ with $\nabla u(x_k) \in \mathbb{R}^2 \backslash B_\varepsilon(\Lambda)$ for some $\varepsilon > 0$. We can choose x_k such that $|\nabla u(x_k) + e/2|$ converges to the limit superior l of all such values, which is finite since u is Lipschitz continuous. Let $B_{r_k}(x_k)$ be the maximum ball contained in $\Omega \cap \{u > 0\}$ touching the boundary of this set at $z_k \in \partial B_{r_k}(x_k)$. Then for a subsequence the blow-up functions with respect to $B_{r_k}(z_k)$, that is,

$$u_k(x) := \frac{1}{r_k}\left(u(z_k + r_k x) - u(z_k)\right)$$

converge locally uniformly to a function u_*, and

$$\frac{z_k - x_k}{r_k} \to \nu_* \,.$$

Then u_* is harmonic in $B_1(-\nu_*)$ and $\nabla u_k \to \nabla u_*$ uniformly in $B_{\frac{1}{2}}(-\nu_*)$, hence $|\nabla u_*(-\nu_*) + e/2| = l$ and $\nabla u_*(-\nu_*) \in \mathbb{R}^2 \backslash B_\varepsilon(\Lambda)$. If $\nabla u_* \neq$ const in $B_1(-\nu_*)$, then ∇u_* is an open mapping, which contradicts the definition of l, since each neighborhood of $\nabla u_*(-\nu_*)$ contains points outside $B_\varepsilon(\Lambda) \cup$ $\cup \overline{B_1(-e/2)}$. Therefore u_* must be linear in the connected component D of the set where u_* is harmonic containing $B_1(-\nu_*)$. Since $u_*(0) = 0$ we conclude

$$u_*(x) = -x \cdot \nu_* \quad \text{for } x \in D \,.$$

Now the domains $\Omega_k = (1/r_k)(\Omega - z_k)$ converge to Ω_*, which is either the entire plane or a half plane with normal ν_0. Hence we have the following three cases.

1. *case.* $B_\varrho \subset \Omega_*$ for some $\varrho > 0$. Then the points z_k were free boundary points and the blow-up sequence converges to a local solution in B_ϱ (see 2.5). Therefore (see 4.1) $\nu_* \cdot e > 0$ and $\partial_h \gamma_* = 0$ in $\{x \in B_\varrho | x \cdot \nu_* > 0\}$. Then the free boundary condition

$$(\nabla u_* + (1 - \gamma_*)e) \cdot \nu_* = 0 \quad \text{on } \{x \in B_\varrho : x \cdot \nu_* = 0\}$$

implies

$$\nabla u_*(-\nu_*) = \nabla u_* \in B_{\frac{1}{2}}\left(-\frac{e}{2}\right) \subset \Lambda \,,$$

a contradiction.

2. *case*. $\Omega_* = \{x \cdot \nu_* < 0\}$ and $u(z_k) \geqslant c r_k$. Then u_* is harmonic in $B_\varrho \cap \Omega_*$ for some $\varrho > 0$, therefore

$$(\nabla u_* + e) \cdot \nu^* = 0 \quad \text{on } \partial \Omega_* ,$$

that is, $\nabla u_*(-\nu_*) = \nabla u_* \in \Lambda$, again a contradiction.

3. *case*. $\Omega_* = \{x \cdot \nu_* < 0\}$ and $u(z_k) = o(r_h)$. Then the blow-up limit u_*, γ_* is a local weak solution by 2.5. If u_* has a free boundary in $B_\varrho \cap \Omega_*$ we argue as in the first case, otherwise we argue as in the second case.

7.6 LEMMA. *Let* u_r, γ_r *be a blow-up sequence at the origin converging to* u_*, γ_*. *Then*

1) $\Omega_* \cap \partial \{u_* > 0\}$ *consists of analytic curves and* $\gamma_* = 0$ *above these curves.*

2) *If* $(y_1, h_1), (y_2, h_2) \in \Omega_* \cap \partial \{u_* = 0\}$, *then for given* $\varepsilon > 0$ *and large* r *we have*

$$\gamma_r(y, h) = 0 \quad \text{for } y_1 + \varepsilon \leqslant y \leqslant y_2 - \varepsilon \quad \text{and } h \geqslant \max(h_1, h_2) + C(y_2 - y_1)$$

with a universal constant C.

PROOF. For $0 \leqslant \sigma_0 < \pi/2$ the first statement follows from 4.6. Therefore let $\sigma_0 \geqslant \pi/2$ and choose $x_0 \in (y_0, h_0) \in \Omega_* \cap \partial \{u_* > 0\}$. By 4.2 either $u_* = 0$ in a left neighborhood of x_0 or there are free boundary points (y_1, h_1) with $y_1 < y_0$ near x_0. Each free boundary point of u_* is a limit of free boundary points of the blow-up sequence u_r, since otherwise u_* would be harmonic near that point. Therefore we find points (y_{1r}, h_2) and (y_{0r}, h_2) in $\{u_r = 0\}$ with $h_2 := \max(h_0, h_1) + 1$ and $y_{0r} \to y_0$, $y_{1r} \to y_1$. Consider the domain

$$\tilde{\Omega}_r := \left\{ (y, h) \,|\, y_{0r} < y < y_{1r} \text{ and } h_2 < h < \frac{1}{r} g(ry) \right\},$$

where $(ry, g(ry))$ is the first point on $\partial \Omega$ or a point in the neighborhood in 7.1.4). As in the proof of 4.4 we can apply the comparison lemma 3.2 to the function

$$v(y, h) := \max \big(C(y_{0r} - y_{1r}) - (h - h_2), 0 \big)$$

in $\tilde{\Omega}_r$, where C is the Lipschitz constant of u. Hence $\gamma_r = 0$ in $\tilde{\Omega}_r \cap \{v = 0\}$,

which in particular proves 2). Letting $r \to 0$ we obtain

$$\gamma_*(y, h) = 0 \quad \text{for } y_1 < y < y_0 \quad \text{and} \quad h > h_2 + C(y_0 - y_1) \,.$$

Since the same is true from the right side, 1) follows as in 4.5.

7.7 LEMMA. *If u_*, γ_* is any blow-up limit at the origin, then $|\nabla u_* + e/2| \leqslant \frac{1}{2}$ in $\Omega_* \cap \{u_* > 0\}$ and $\Omega_* \cap \partial\{u_* > 0\}$ consists of analytic curves, which are concave to the fluid.*

PROOF. For $x \in \Omega_* \cap \{u_* > 0\}$ we have $\nabla u_r(x) \to \nabla u^*(x)$ and therefore $\nabla u_*(x) \in \Lambda$ by 7.5.

If we assume that $\nabla u_*(x)$ lies on a ray in $\Lambda \setminus \overline{B_i(-e/2)}$, then ∇u_* must be constant in the connected component D of $\{u_* > 0\}$ containing x, since otherwise ∇u_* is an open mapping and its image would contain values outside Λ. Therefore D must be the intersection of Ω_* and a half plane, and since ∇u_* maps the free boundary into $\partial B_i(-e/2)$ by 7.6.1) and 7.4, we conclude $D = \Omega_*$. But since $u_*(0) = 0$ this implies $u_* = 0$ on $\partial\Omega_*$, hence $\nabla u_* = -\alpha\nu_0$ for some $\alpha > 0$ and $(\nabla u_* + e)\cdot\nu_0 = 0$. Therefore ∇u_* must lie on $\partial B_i(-e/2)$, which is a contradiction.

We thus proved, that ∇u_* maps $\Omega_* \cap \{u_* > 0\}$ into $\overline{B_i(-e/2)}$, and the free boundary into $\partial B_i(-e/2)$. Now it is easy to see, that the differential equation and the free boundary conditions for u_* (use 7.6.1)) imply, that on the free boundary

$$\partial_\nu^2 u_* + \partial_\tau^2 u_* = 0 \,, \quad \partial_\tau^2 u_* = \varkappa e \cdot \nu \,, \quad \partial_{\tau\nu} u_* = \varkappa e \cdot \tau \,,$$

where τ is a unit tangent and $\varkappa$ the curvature of the free boundary in ν direction. Hence

$$0 \leqslant \partial_\nu \left| \nabla u_* + \frac{e}{2} \right|^2 = \varkappa \quad \text{on } \Omega_* \cap \partial\{u_* > 0\} \,.$$

The following arguments differ in some details for different values of σ_0. Therefore let us divide the result of this section into five theorems.

7.8 THEOREM. *If $\pi/2 < \sigma_0 < \pi$, then the free boundary is horizontal at 0, and in a small neighborhood there is no fluid on the left side.*

PROOF. We divide the proof into several steps.

1. *step*: First let us prove the last statement. We have $u(0, h) = 0$ for $h \geqslant 0$ and $\gamma = 0$ in a neighborhood of some point $(0, h_0)$. For small $r > 0$

we take an approximation of the characteristic function of $]-r, 0[\times]-\infty, h_0[$ as test function for u, γ in 2.1. Using the separation lemma 4.3 we obtain for almost all r, if $\partial\Omega = \text{graph } g_0$,

$$0 = -\int_{g_0(-r)}^{h_0} \partial_y u.$$

Integrating over r we get

$$0 \geqslant \int_{g_0(-r)}^{h_0} u,$$

hence the assertion.

2. *step*: Define for $y > 0$

$$g(y) := \inf \{h: (y, h) \in \Omega \text{ and } u(y, h) = 0\}$$

and

$$l_- := \liminf_{\downarrow 0} \frac{g(y)}{y} \in [\tan \sigma_0, \infty].$$

We will show that $l_- = 0$. To prove this choose a sequence $y_k \downarrow 0$ with $g(y_k)/y_k \to l_-$ and denote by u_* a blow-up limit with respect to $r_k :=$
$:= \left|(y_k, g(y_k))\right|$. If $l_- = \tan \sigma_0$, we compare the solution u in the region

$$\tilde{\Omega} := \{(y, h) \in \Omega: 0 < y < y_k \text{ and } h < h_0\}$$

with the trivial solution

$$v(y, h) := \max \left(\frac{y_k \tan \sigma_0}{2} - h, 0\right).$$

Since u is Lipschitz continuous we see that $u(y_k, h) \leqslant v(y_k, h)$ for all h and large k. Hence the comparison lemma 3.2 yields $u \leqslant v$ in $\tilde{\Omega}$, in particular $u = 0$ in a neighborhood of 0, a contradiction to 7.1.3). Therefore $l_- > \tan \sigma_0$ and u_* must be harmonic in

$$\{(y, h) \in \Omega_*: y > 0 \text{ and } h < l_- y\}.$$

Then the boundary condition $\partial_{-\nu} u_* = e \cdot \nu \neq 0$ on $\partial\Omega_*$ implies that u_* is positive in this cone. Since u_* cannot be positive in a right neighborhood of 0 by 4.2, we conclude that 0 is a limit point for the free boundary of u_*, hence $l_- < \infty$. Then 7.6 implies that the free boundary of u_* is the graph

of a continuous function g_* with

$$g_*(y) \geqslant l_- y \quad \text{for } y \geqslant 0 ,$$

$$g_*(0) = 0 \quad \text{and} \quad g_*(y_*) = l_- y_* , \quad \text{where} \quad y_* := (1 + l_-^2)^{-\frac{1}{2}} .$$

Since the free boundary of u_* is concave to the fluid by 7.7, we conclude $g_*(y) = l_- y$ for $0 \leqslant y \leqslant y_*$. Since $u_* = 0$ and $\partial_{-\nu} u_* = y_*$ on this segment we obtain by the uniqueness of the Cauchy problem that u_* is the positive part of a linear function. Then the flux condition $(\nabla u_* + e) \cdot \nu = 0$ on $\partial \Omega_*$ yields $l_- = 0$.

3. step: Now we will show that also

$$l_+ := \limsup_{\nu \downarrow 0} \frac{g(y)}{y}$$

equals zero. Again choose a sequence $y_k \downarrow 0$ with $g(y_k)/y_k \to l_+$, and let u_* be the blow-up limit of u_{r_k} with $t_k := |(y_k, g(y_k))|$. As in the 2. step we see that near 0 the free boundary of u_* is the graph of a continuous function g_* with

$$0 \leqslant g_*(y) \leqslant l_+ y \quad \text{for } y \geqslant 0 , \quad \text{and} \quad g_*(0) = 0 .$$

Moreover, as in the proof of 7.6.2) we get that for large k

$$u_{r_k}(y, h) = 0 \quad \text{for } 0 \leqslant y \leqslant 1 \quad \text{and} \quad h \geqslant g_*(1) + C .$$

This implies $l_+ < \infty$. Therefore the free boundary points $(1/r_k)(y_k, g(y_k))$ of u_{r_k} converge to a point $(y_*, l_+ y_*) \in \Omega_* \cap \{y > 0\}$, which by 7.6 is a free boundary point for u_*, in other words, $g_*(y) = l_+ y_*$. Since the free boundary is concave to the fluid by 7.7, we conclude $g_*(y) = l_+ y$ for $y \geqslant 0$, and then $l_+ = 0$ as in the previous step.

7.9 THEOREM. *If $\sigma_0 = \pi$, then the free boundary is horizontal at 0.*

PROOF. Define g, l_-, l_+ as in the proof of 7.8. If we assume $l_- > 0$, then the arguments in the second step of the proof of 7.8 lead to a contradiction. Therefore $l_- = 0$. Now suppose that

$$\frac{u(y, h)}{y} \to 0 \quad \text{for } (y, h) \in \partial \Omega \text{ with } y \downarrow 0 .$$

Then as in the proof of 5.7.3) use the functions v_ε (here with $\sigma_+ = 0$) as comparison functions for u and conclude $l_+ = 0$.

To complete the proof assume that there is a sequence $(y_k, h_k) \in \partial\Omega$ with $y_k \downarrow 0$ and $u(y_k, h_k) \geqslant c y_k$. Let u_* be a blow-up limit at the origin with respect to $r_k = |(y_k, h_k)|$. Then $u_*(1, 0) \geqslant c > 0$, and there is a maximum value $s \in [0, 1[$ with $u_*(s, 0) = 0$. Then $(s, 0)$ is a limit point of the free boundary of u_* from the right (otherwise Hopf's principle would contradict 4.2). 7.6 then implies that $\gamma_* = 0$ in a neighborhood of some point (s, h_0). But the first step of the proof of 7.8 (applied to u_* with $\sigma_0 = \pi/2$) says that this situation is impossible.

Now we deal with the upper impervious boundary. First let us characterize blow-up limits.

7.10 LEMMA. *If* $0 \leqslant \sigma_0 < \pi/2$, *then*

1) $u(0, -r) \geqslant cr$ *for small* $r > 0$.

2) *If* u_* *is any blow-up limit at* 0, *then* $u_*(x) = -e \cdot v_0 x \cdot v_0$ *or (for* $\sigma_0 > 0$*)*

$$\Omega_* \cap \partial\{u_* > 0\} = \{(y, g_*(y)) : y > 0\}$$

with a convex function g_* *satisfying* $g_*(0) = 0$ *and* $g'_*(0) = 0$.

PROOF. Assume that $u(0, -r_k) = o(r_k)$ for a sequence $r_k \downarrow 0$, and let u_* be a corresponding blow-up limit. Then $u_*(0, h) = 0$ for $-1 \leqslant h \leqslant 0$. Hence $u_* = 0$ in a neighborhood of 0 by 4.4, and since $u_{r_k} \to u_*$ uniformly, 4.7 shows, that $u_{r_k} = 0$ in a neighborhood of 0 for large k, a contradiction to 7.1.3). Thus 1) is proved. For blow-up limits u_* at 0 we distinguish between two cases.

1. *case*: u_* is harmonic in $\Omega_* \cap B_\varepsilon$ for some ε. Then the holomorphic function $f := \exp[-i\sigma_0](\nabla u_* + i)$ defined in $\Omega_* \cap B_\varepsilon$ has real values on $\partial\Omega_* \cap B_\varepsilon$ (see 7.4), therefore has a holomorphic continuation in B_ε. Since $u_*(0) = 0$ and $u_* > 0$ in $\Omega_* \cap B_\varepsilon$ by 1), $\nabla u_*(0)$ is perpendicular to $\exp[i\sigma_0]$, that is, $f(0) = \sin \sigma_0$. If f is constant, then u_* is the desired linear function. If not, f has an expansion

$$f(z) = \sin \sigma_0 + az^m + O(z^{m+1})$$

for some $a \neq 0$ and some integer $m \geqslant 1$. Now the property $|\nabla u_* + e/2| \leqslant \frac{1}{2}$ which was proved in 7.7 says that $0 \leqslant f \leqslant \sin \sigma_0$ on $\partial\Omega_* \cap B_\varepsilon$. This means that the exponent m must be even, hence $m \geqslant 2$. We conclude that for small ε

the image $\nabla u_*(\Omega_* \cap B_\varepsilon)$ covers nearly a ball with center $\nabla u_*(0) = - e \cdot \nu_0 \, \nu_0$, whereas $\nabla u_*(\Omega_* \cap B_\varepsilon) \subset \overline{B_{\frac{1}{4}}(- e/2)}$ by 7.7, a contradiction.

2. *case*: The origin is a limit point of $\Omega_* \cap \partial \{u_* > 0\}$. By 1) the blow-up u_* is positive in some cone $\{|y| + ch < 0\}$, and therefore

$$\Omega_* \cap \{u_* > 0\} = \{h < g_*(y)\}$$

with a lower semicontinuous function $g_*: R \to R$. In addition we know from 7.7 that g_* is convex in the open set $I := \{y: g_*(y) < y \tan \sigma_0\}$. Assume there is an interval $]y_1, y_0[\subset I$ with $y_0 \notin I$. Now statement 1) applied to u_* says that for some small $c > 0$ near $(y_0, y_0 \tan \sigma_0)$

$$u_*(y_0 + y, y_0 \tan \sigma_0 + h) \geqslant c|h| \quad \text{for } |y| + ch < 0 \;.$$

Therefore the convexity of g_* implies that

$$s_0 := \lim_{\nu \uparrow \nu_0} g_*'(y)$$

is finite and larger than $\tan \sigma_0$. Moreover every blow-up limit u_{**} of u_* at $(y_0, g_*(y_0))$ satisfies

$$u_{**}(y, h) = 0 \quad \text{for } h = s_0 y < 0 \;,$$

$$u_{**}(y, h) > 0 \quad \text{for } h < s_0 y < 0 \;.$$

Then the uniqueness of the Cauchy problem at the free boundary line of u_{**} implies that $\nabla u_{**} = (s_0, -1)/(s_s^2 + 1)$, which contradicts the boundary condition of u_{**} on $\partial \Omega_*$, since $s_0 > \tan \sigma_0$. We thus proved that $I =]0, \infty[$. Then the properties of the second blow-up u_{**} of u_* at 0 imply $g_*'(0) = 0$.

A corollary of 7.10.2) is

7.11 THEOREM *If $\sigma_0 = 0$, then the free boundary is horizontal at 0.*

7.12 THEOREM. *If $0 < \sigma_0 < \pi/2$ then the free boundary is tangential to $\partial \Omega$ at 0, or it is horizontal and in a small neighborhood $u > 0$ on the left side.*

PROOF. Let $\Omega \cap \{u > 0\} = \{h < g(y)\}$. First assume that $(y, g(y)) \notin \partial \Omega$ for small $y > 0$. Since $g(y) \geqslant - C|y|$ by 7.10.1) and 7.3 we obtain similar as in the proof of 7.8 (step 2-3) that $g(y)/y \to 0$ for $y \downarrow 0$ if $\liminf g(y)/y < \tan \sigma_0$. Thus every blow-up limit u_* at 0 has the form $u_*(y, h) = \max(-h, 0)$, hence $u > 0$ on the left side in a small neighborhood of 0.

Now consider the case that $(y_k, g(y_k)) \in \partial\Omega$ for a sequence $y_k \downarrow 0$. We have to show that this implies

$$\lim_{v \downarrow 0} \frac{g(y)}{y} = \tan \sigma_0 .$$

If not, we can choose $0 < \sigma < \sigma_0$, a subsequence of y_k, and a sequence z_k with

$$\frac{g(y)}{y} \geqslant \frac{g(z_k)}{z_k} = \tan \sigma \quad \text{for } y_k \leqslant y \leqslant z_k .$$

Consider a blow-up limit u_* at 0 with respect to $r_k = z_k$ such that $y_k/r_k \to s$ with $0 \leqslant s \leqslant 1$. Since $u_* > 0$ in $\{h < - C|y|\}$ by 7.10.1) and the Lipschitz continuity of u_*, there must be a free boundary point u_* below $(1, \tan \sigma)$. Then we obtain from 7.6 that $(1, \tan \sigma)$ itself must be this boundary point and that $s < 1$. This means, if we write $\Omega_* \cap \{u_* > 0\} = \{h < g_*(y)\}$,

$$g_*(1) = \tan \sigma , \quad g_*(s) = s \tan \sigma_0 ,$$

$$g_*(y) \geqslant y \tan \sigma \quad \text{for } s < y < 1 .$$

Then 7.10.2) implies $s = 0$ and $\sigma = 0$, a contradiction.

It remains to show that the free boundary is tangential to the left, but this is a consequence of 7.10.2).

8. – Points between atmosphere and impervious part.

8.1 ASSUMPTIONS. u, γ is a local solution in $\Omega \cap B_R$ for some $R > 0$ and

1) $\partial\Omega \cap B_R$ consists of two $C^{1,1}$ curves Γ_1 and Γ_2 starting at the origin and with tangents $\exp[i\sigma_1]$ and $\exp[i\sigma_2]$ at the origin $(0 < \alpha := \sigma_2 - \sigma_1 < 2\pi)$. The two unit normals of $\partial\Omega$ at the origin are $\nu_1 := -i \exp[i\sigma_1]$ and $\nu_2 := i \exp[i\sigma_2]$.

2) Each curve Γ_1 and Γ_2 belongs to S_{imp} or S_{air}.

3) The origin is a limit point of the free boundary.

4) $\gamma = 0$ in a neighborhood of some point in $\Omega \cap B_R$ above the origin.

8.2 REMARK. As in 6.2 let us choose R small enough.

As in the previous sections the first step is to prove

8.3 LEMMA. *u is Lipschitz continuous near the origin, that is,* $|\nabla u|$ *is bounded in* $\Omega \cap B_{R/2}$.

PROOF. By the proofs of 2.4, 5.3, and 7.3 we have to show that $u(x)$ can be estimated by $|x|$ near the origin. Therefore assume that there are points $x_k \in \Omega \cap \{u > 0\}$ with $r_k := |x_k| \to 0$ and

$$m_k := \frac{u(x_k)}{r_k} \to + \infty.$$

Then it follows from the basic lemma 2.4 (replace there B_R, for example, by $B_{8r_k} \backslash B_{r_k/8}$) that the annular regions $\Omega \cap B_{4r_k} \backslash B_{r_k/4}$ contain no free boundary points for large k. Moreover, we have

$$u(x) \geqslant cm_k \operatorname{dist}(x, S_{\text{air}}) \quad \text{for } |x| = r_k,$$

if Γ_1 or Γ_2 is contained in S_{air}, and

$$u(x) \geqslant cm_k r_k \quad \text{for } |x| = r_k$$

otherwise.

We will show that this implies that 8.1.3) cannot be satisfied. Hence for $x \in \Omega \cap \partial B_{r_k}$ with $u(x) > 0$ let us consider the maximum ball $B_r(x)$ contained in $\Omega \cap \{u > 0\}$. Then $u(x) \geqslant cm_k r$, hence $B_r(x)$ cannot touch the free boundary by lemma 2.2. Hence $B_r(x)$ reaches the fixed boundary $\partial\Omega$, and this is true for all $x \in \Omega \cap \partial B_{r_k}$ and large k. In the case $\alpha \geqslant \pi$ the fixed boundary satisfies an inner ball condition. We conclude that $u > 0$ in $\Omega \cap B_{r_k}$ for large k, a contradiction.

If $\alpha < \pi$, we have to distinguish between several cases. If Γ_1 and Γ_2 belong to S_{imp}, we consider the function

$$v(x) := sx \cdot \big(\exp[i\sigma_1] + \exp[i\sigma_2] \big) ,$$

where s is a fixed large number. Then $u \geqslant v$ on $\Omega \cap \partial B_{r_k}$ for large k, and for $x \in \partial\Omega \cap B_{r_k}$ and $k \to \infty$ we have

$$\big(\nabla v(x) + e \big) \cdot v(x) \leqslant s \cos\left(\alpha + \frac{\pi}{2} \right) + 1 + o(1) ,$$

which is negative for large k if s is chosen large enough. Therefore v is a subsolution in $\Omega \cap B_{r_k}$, and since $|\nabla v| \geqslant 1$ it follows from lemma 3.4 that $u \geqslant v > 0$ in $\Omega \cap B_{r_k}$, again a contradiction. Now let us consider the case

that Γ_1 is contained in S_{air} and Γ_2 in S_{imp}. If $\pi/2 < \alpha < \pi$ we argue as follows. Choose $C^{1,1}$ continuations of Γ_1 and Γ_2 and denote by D_k the domain enclosed by Γ_1 and ∂B_{r_k} containing $\Omega \cap B_{r_k}$. Let v be the harmonic function in D_k with

$$v(x) := s \operatorname{dist}(x, \Gamma_1) \quad \text{for } x \in \partial D_k .$$

Then for $x \in S_{\text{imp}} \cap D_k$ and $k \to \infty$

$$\big(\nabla v(x) + e\big) \cdot v(x) = s \cos \alpha + e \cdot \nu_2 + o(1) ,$$

which as above leads to a contradiction. Now assume $\alpha = \pi/2$. If $e \cdot \nu_2 < 0$ we argue as before. If $e \cdot \nu_2 = 0$ we consider the function

$$v := s d_1 \cdot \left(1 + \frac{M}{2} d_1 + M d_2\right),$$

where d_1 and d_2 are the $C^{1,1}$ functions

$$d_1(x) := \operatorname{dist}(x, \Gamma_1) \quad \text{for } x \in D_k ,$$

$$d_2(x) := \begin{cases} \operatorname{dist}(x, \Gamma_2) & \text{for } x \in D_k \cap \Omega , \\ -\operatorname{dist}(x, \Gamma_2) & \text{for } x \in D_k \backslash \Omega . \end{cases}$$

Then

$$\Delta v \geqslant (M - C)s - CMsr_k > 0$$

for large k if $M > C$. Moreover

$$|\nabla v| \geqslant s\big(1 - CMr_k\big) \geqslant 1$$

for large k if $s > 1$, and on $S_{\text{imp}} \cap D_k$ we have for $k \to \infty$

$$(\nabla v + e) \cdot \nu = -e \cdot \nabla d_2 - s\big(M d_1 + (1 + M d_1)\nabla d_1 \cdot \nabla d_2\big) \leqslant$$
$$\leqslant s d_1 \cdot \big(-M + C + O(r_k)\big),$$

which is negative for large k, provided M is chosen large enough. This means that v is a subsolution, which is a contradiction.

Now let us consider the case $\alpha = \pi/2$ with $e \cdot \nu_2 > 0$. As above one verifies that the function

$$v := s d_1 \cdot \left(1 - \frac{M}{2} d_1 - M d_2\right)$$

is superharmonic in D_k with $(\nabla v + e) \cdot \nu \geqslant 0$ on $S_{\text{imp}} \cap D_k$ for large k, provided $M > C$ and $s \geqslant 1$, where the choice of k does not depend on s. In order to apply the comparison lemma 3.2 we need that $v \geqslant u$ on $\Omega \cap \partial B_{r_k}$. But since $u \leqslant C_k d_1$ on this set the desired inequality holds for large k if we choose $s \geqslant 2C_k$. We infer $u \leqslant v \leqslant 2C_k d_1$ in $\Omega \cap B_{r_k}$, a contradiction. In the case $\alpha < \pi/2$ the above harmonic function v in D_k satisfies

$$(\nabla v + e) \cdot \nu \geqslant s(\cos \alpha + o(1)) - (1 + o(1))$$

on $S_{\text{imp}} \cap D_k$ for $k \to \infty$, therefore we can use this function as a supersolution. Finally, if Γ_1 and Γ_2 belong to S_{air} with $\alpha < \pi$, a supersolution v with $v(x) = = o(|x|)$ for $x \to 0$ exists. Thus the lemma is proved.

8.4 THEOREM. *The tangent cone of the free boundary at the origin is either a line or consists of rays in directions* $\exp[i\sigma]$ *with* $\sigma = \sigma_1,\ \sigma_1 + \pi,\ \sigma_2,\ \sigma_2 + \pi,\ 0,$ $\pi/2,\ \text{or}\ \pi.$

PROOF. We will proceed as in section 7, that is, we will study the values of $\nabla u(x)$ as $x \to 0$. For this denote by A (see 7.4) the closed set consisting of the curves $\partial B_{\frac{1}{2}}(-e/2)$ and

$$\{-e + \tau : \tau \cdot \nu_j = 0\}\ ,$$

if Γ_j belongs to S_{imp}, and

$$\{-\xi \nu_j : \xi \geqslant \max(e \cdot \nu_j, 0)\}\ ,$$

if Γ_j belongs to S_{air}, and of all bounded domains enclosed by these curves. As in 7.5 one proves that the values $\nabla u(x)$ approach A as $x \to 0$. Moreover it follows as in 7.7 (first part of the proof) that the gradient of every blow-up limit at the origin (that is, with respect to balls $B_r(0)$) maps into $\overline{A}$. Our aim is to use the concavity arguments in section 7. For this we have to study the exceptional case where $\overset{\circ}{A} \backslash B_{\frac{1}{2}}(-e/2)$ is non-empty, which occurs only if, say, Γ_1 belongs to S_{air} and Γ_2 to S_{imp} with $\nu_1 \cdot \nu_2 \neq 0$ and $\nu_1 \neq \nu_2$.

Then $D := \overset{\circ}{A} \backslash \overline{B_{\frac{1}{2}}(-e/2)}$ is a simple connected domain and there is a point $z_* \in \overline{D}$ with $(z_* + e) \cdot \nu_2 = 0$ and $z_* = -\xi \nu_1$ for some $\xi > \max(e \cdot \nu_1, 0)$ (see fig. 8 for the special case ii) below). If z_* is not achieved by $\nabla u(x)$ for $x \to 0$, we conclude as before (see 7.5) that

$$\text{dist}(\nabla u(x), \mathbf{R}^2 \backslash D) \to 0$$

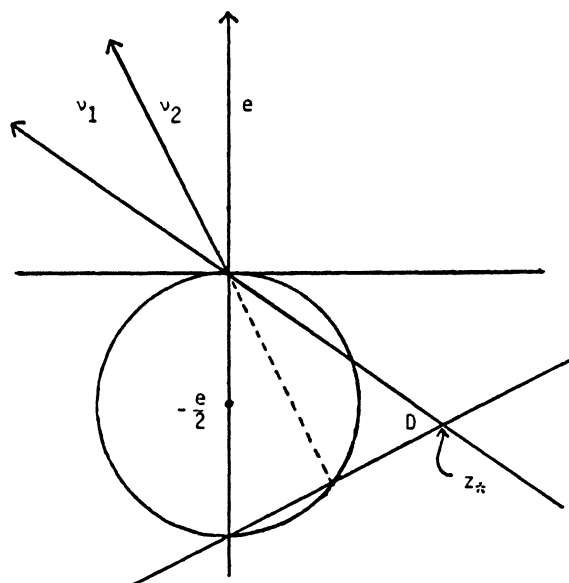for $x \to 0$, and that (see proof of 7.7) the gradient of every blow-up limit. maps into $\overline{B_4(-e/2)}$.

Fig. 8

Therefore we have to consider the case where $\nabla u(x_k) \to z_*$ for a sequence $x_k \in \Omega \cap \{u > 0\}$ converging to zero. Let $B_{r_k}(x_k)$ be the largest ball contained in $\mathbf{R}^2 \setminus (\Omega \cap \partial\{u > 0\})$. We distinguish between two cases.

If $r_k/\varrho_k \geqslant c > 0$, where $\varrho_k := |x_k|$, for a subsequence the blow-up functions,

$$u_k(x) := \frac{1}{\varrho_k} u(\varrho_k x)$$

converge to a weak solution u_*, γ_* and $x_k/\varrho_k \to x_* \in \overline{\Omega}_*$. Since the func- tions u_k are harmonic near x_* satisfying the boundary conditions on the fixed boundary they converge near x_* to u_* in $C^{1,\alpha}$, hence $\nabla u_*(x_*) = z_*$. If ∇u_* would not be constant, z_* would be an interior point of the image of ∇u_* in the case that $x_* \in \Omega_*$, which contradicts the fact that ∇u_* maps into A. If $x_* \in \partial\Omega_*$, we conclude that a small deformation of a half ball neighborhood of z_* is contained in the image of ∇u_*, again a contradiction since A forms an acute angle at z_*. We thus see that $\nabla u_* = z_*$ in the con- nected component of $\Omega_* \cap \{u_* > 0\}$ containing the intersection of Ω_* with a neighborhood of x_*. This easily implies that u_* is positive and linear in.

the intersection of Ω_* with some half plane with $u_* = 0$ on S^*_{air}, that is, $u_*(r \exp[i\sigma_1]) = 0$ for $r \geqslant 0$. Moreover, since ∇u_* maps the free boundary into $\overline{B_\frac{1}{4}(-e/2)}$, but $\nabla u_* = z_* \notin \overline{B_\frac{1}{4}(-e/2)}$, there is no free boundary, that is, u_* is linear and positive in Ω_*. Hence $\alpha \leqslant \pi$, and since $\nu_1 \cdot \nu_2 \neq 0$ and $\nu_1 \neq \nu_2$ we infer that $0 < \alpha < \pi/2$ or $\pi/2 < \alpha < \pi$. Noticing that by 8.1.3) we can assume $-(3/2)\pi < \sigma_1 < \sigma_2 < \pi/2$, we have to deal with the following three cases:

i) $0 < \alpha < \pi/2$, $|\sigma_2 + \pi| < \pi/2$, $\sigma_1 < -\pi$,

ii) $\pi/2 < \alpha < \pi$, $-\pi < \sigma_1 < -\pi/2 < \sigma_2 < \pi/2$,

iii) $\pi/2 < \alpha < \pi$, $-\pi/2 < \sigma_1 < 0 < \sigma_2 < \pi/2$.

In case i) we consider for small $\delta > 0$ the function $v \in H^{1,2}(\Omega \cap B_R)$ given by

$$v(r \exp[i\theta]) = 0 \quad \text{for } \theta < \sigma_1 + \delta,$$

$$\nabla v(r \exp[i\theta]) = (1 - s\delta) \exp[i\delta] z_* \quad \text{for } \theta > \sigma_1 + \delta,$$

where we can choose $s > 0$ such that v is a subsolution. Since $v \leqslant u$ on $\Omega \cap \partial B_{\varrho_k}$ for large k, we can apply the comparison lemma 3.4. We conclude that $\exp[i\sigma_1]$ is the only possible tangent direction of the free boundary at the origin, which proves the theorem in this particular case.

In case ii) we want to show that the free boundary is tangent to Γ_1 and Γ_2 at the origin. First let us consider the left side. Define

$$l := \liminf_{\nu \uparrow 0} \left\{ \frac{h}{|y|} : u(h, y) = 0 \right\}.$$

It follows from 7.10.1) that $l > -\infty$, and we have $l < -\tan \sigma_1$ if the free boundary would not be tangent to Γ_1. Now choose free boundary points (y_k, h_k) with $y_k \uparrow 0$ and $h_k/|y_k| \to l$, and denote by u_∞ a blow-up limit at the origin with respect to $|y_k|$. Then $u_\infty(-1, l) = 0$ and u_∞ is positive in $\{h < l|y|\}$. Therefore $(-1, l)$ is a free boundary point of u_∞ with slope $|l|$, that is, with outer normal $-\exp[i\sigma_l]$, where $\sigma_1 + \pi/2 < \sigma_l < 0$, in other words

$$\nabla u_\infty(-1, l) = -(\sin \sigma_l) \exp[i\sigma_l] \in \partial B_\frac{1}{4}\left(-\frac{e}{2}\right).$$

Since ∇u_∞ maps into $\overline{A}$, and since $\overline{A}$ coincides with $\overline{B_\frac{1}{4}(-e/2)}$ in a neighborhood of $\nabla u_\infty(-1, l)$, it follows from 7.7 that the free boundary of u_∞ is

·concave near $(-1, l)$. Therefore following the free boundary to the right the values of ∇u_∞ change counterclockwise on $\partial B_{\frac{1}{2}}(-e/2)$, hence the concavity of the free boundary is preserved. On the other hand the free boundary is not a straight line, since then ∇u_∞ would be a constant with $(\nabla u_\infty + e) \cdot \nu_2 = 0$, but $(\nabla u_\infty(-1, l) + e) \cdot \nu_2 \neq 0$. We thus see that going along the free boundary we reach a point of S^*_{air} in a non-tangential manner, hence u_∞ decays faster than linearly at this point, a contradiction to 4.7. Now let us consider the free boundary on the right side, where we have to give a different argument (not if $\sigma_2 \leqslant 0$). Assume that the origin is a limit point of the free boundary from the right. Since the above blow-up limit u_* is positive in $\Omega_* \cup S^*_{\text{imp}}$ we find for a subsequence points $\tilde{x}_\varkappa \in S_{\text{imp}}$ with

$$\varrho_\varkappa \leqslant \tilde{\varrho}_\varkappa := |\tilde{x}_\varkappa| < \varrho_{\varkappa - 1}$$

such that $u(\tilde{x}_\varkappa) = 0$ and $u(x) > 0$ for $x \in S_{\text{imp}}$ with $\varrho_\varkappa \leqslant |x| < \tilde{\varrho}_\varkappa$. Moreover $\varrho_\varkappa / \tilde{\varrho}_\varkappa \to 0$ for $\varkappa \to \infty$. Denote by u_∞ a blow-up limit with respect to $\tilde{\varrho}_\varkappa$. Then

$$u_\infty(y, h) > 0 \qquad \text{for } (y, h) \in \Omega_* \text{ with } y < \cos \sigma_2$$

and

$$u_\infty(\exp[i\sigma_2]) = 0 \, .$$

Therefore u_∞ is not a linear function, hence ∇u_∞ has an expansion

$$\nabla u_\infty(x) = z_* \Big(1 - a(\tilde{x} \exp[i\sigma_1])^{m\pi/2\alpha - 1}(1 + H(x)) \Big) \, ,$$

where $H(x) \to 0$ for $x \to 0$, and where a is a non-zero real and m an integer. Since ∇u_∞ is bounded we must have $m \geqslant 2\alpha/\pi$, and since ∇u_∞ maps into $\overline{\overset{\circ}{A}}$ we see that $m + 1 \leqslant 4\alpha/\pi$, hence $m = 2$. Moreover, since near the origin ∇u_∞ maps $\partial \Omega_*$ into $\partial \overset{\circ}{A}$, we conclude that $H = 0$. But then $u_\infty = u_*$ on S^*_{imp}, a contradiction to the fact that u_∞ has a zero on S^*_{imp} by construction.

Case iii) can be treated in a similar way.

Now we have to consider the case, that for the sequence at the beginning we have $r_k/\varrho_k \to 0$ for $k \to \infty$. Then the balls $B_{r_k}(x_k)$ touch the free boundary at certain points y_k. For a subsequence

$$\frac{y_k - x_k}{r_k} \to \nu_*$$

and the blow-up sequence

$$u_k(x) := \frac{1}{r_k} u(y_k + r_k x)$$

converges to a solution u_*, γ_* in a neighborhood of $\overline{B_1(-\nu_*)}$ with $u_*(0) = 0$. As before we see that $\nabla u_* = z_*$ in the connected component of $\Omega_* \cap$ $\cap \{u_* > 0\}$ containing $\Omega_* \cap B_1(-\nu_*)$. In the cases where $0 \in \Omega_*$ or where $0 \in \partial\Omega_*$ but ν_* is different from the outer normal of $\partial\Omega_*$ at 0, the free boundary $\Omega_* \cap \partial\{u_* > 0\}$ must be non-empty, which implies $z_* = \nabla u_* \in \overline{B_{\frac{1}{4}}(-e/2)}$, a contradiction. In the remaining case we infer

$$u_*(x) = x \cdot z_* \quad \text{for } x \in \Omega_*,$$

and z_* is a negative multiple of the outer normal of $\partial\Omega_*$ at 0. Since we had $z_* = -\xi\nu_1$ and $\nu_1 \neq \nu_2$ we see that $0 \in S^*_{\text{air}}$, which means that u_* is a strict subsolution and therefore one derives a contradiction by the following subsolution argument. For small $\varepsilon > 0$, functions $\eta \in C_0^\infty(]-\frac{1}{4}, \frac{1}{4}[)$ with $0 \leqslant \eta \leqslant 1$ and $\eta(0) = 1$, $|t| \leqslant \frac{1}{4}$, and small s consider the set

$$D_s := \{x \in B_1(0) : x_2 \leqslant -\varepsilon + s\eta(x_1 - t)\},$$

where x_1 and x_2 are the tangential and normal components of x. Then $u_k > 0$ on $\overline{D}_0$ for large k, hence there is a maximal value s_k for which $D_{s_k} \subset \{u_k > 0\}$. Assuming that D_{s_k} does not touch the fixed boundary there must be a free boundary point $\xi_k \in \partial D_{s_k} \backslash \partial B_1(0)$. Moreover for large k we have

$$u_k \geqslant v,$$

if v is the harmonic function in D_{s_k} with

$$v = 0 \qquad \partial D_{s_k} \cap B_1(0)$$

$$v = u_* - \varepsilon \quad \text{on } \partial D_{s_k} \backslash B_1(0).$$

Therefore $\nabla v(\xi_k) = \alpha \nabla u_k(\xi_k)$ with $0 \leqslant \alpha \leqslant 1$. Thus the inequality $|\nabla u_k(\xi_k) + e/2| \leqslant \frac{1}{2}$ implies $|\nabla v(\xi_k) + e/1| \leqslant \frac{1}{2}$. On the other hand, since $v \to u_*$ in $C^{1,\alpha}$ for $\varepsilon \to 0$ uniformly with respect to t and s_k (we have $s_k \leqslant C\varepsilon$) we see that

$$\left| \nabla v(\xi_k) + \frac{e}{2} \right| \geqslant \left| \nabla u_*(\xi_k) + \frac{e}{2} \right| - C\varepsilon = \left| z_* + \frac{e}{2} \right| - C\varepsilon,$$

which is larger than $\frac{1}{2}$ for small ε, and then contradicts the above estimate. We thus proved that the sets D_{s_k} touch the fixed boundary for all t and large k, that is, $B_{\frac{1}{4}}(0)$ contains no free boundary points of u_k ($D^2(s\eta)$ is bounded and the curvature of $\partial\Omega_k$ tends to zero). This fact certainly contradicts the construction of the sequence u_k.

Thus it is shown that in order to prove the theorem we can assume that $\nabla u(x)$ approaches $\overline{B_i(-e/2)}$ for $x \to 0$. In other words (see 7.6, 7.7) we may assume that for every blow-up limit u_*, γ_* we have

$$\gamma_* = \chi_{\{u_* > 0\}}, \qquad \left| \nabla u_* + \frac{e}{2} \right| \leqslant \frac{1}{2},$$

and that $\Omega_* \cap \partial\{u_* > 0\}$ consists of analytic curves concave to the fluid. Then we can complete the proof using arguments which were given in 7.8-7.12.

8.5 REMARK. The statement 8.4 could be improved by showing that actually some cases cannot occur. For instance, if $\Gamma_1 \cup \Gamma_2 \subset S_{air}$ with $\alpha < \pi$, it is clear that $u(x) = o(x)$ for $x \to 0$. If in addition $-(3/2)\pi < \sigma_1 < \sigma_2 < \pi/2$ the supersolution argument in 4.7 leads to a contradiction to 8.1.3).

9. – Global results.

As pointed out in the introduction we will prove uniqueness and stability theorems for the free boundary using the results in the previous sections.

9.1 ASSUMPTIONS. *We assume that $\partial\Omega$ consists of a finite number of $C^{1,1}$ curves belonging to S_{res}, S_{air}, or S_{imp}, with angles between 0 and 2π at its endpoints. Moreover, the number of points and segments of $\{h/(y, h) \in \partial\Omega\}$ is supposed to be bounded uniformly in y. Assume $u^0 > 0$ on S_{res} and $u^0 = 0$ on S_{air}. Suppose $u^0(x) > 0$ for points $x \in \overline{S_{res}} \cap \overline{S_{imp}}$, and that for points $x \in \overline{S_{res}} \cap \overline{S_{air}}$ the data satisfy 5.1 in some neighborhood of x.*

9.2 CONCLUSION. *If (u, γ) is a weak solution, it follows from 4.6 that $\gamma = \chi_{\{u > 0\}}$, and therefore at each point $x \in \partial\Omega$ which is limit point of the free boundary the properties in 6.1, 7.1, 8.1, respectively, are satisfied, so that we can use all the results of this paper. Moreover, 9.1 implies that there is a finite number of functions $g_i \in C^0(\mathbf{R}, \mathbf{R})$ with*

$$\Omega \cap \partial\{u > 0\} = \{(y, h) \in \Omega : h = g_i(y) \text{ for some } i\} .$$

First let us prove the following.

9.3 UNIQUENESS THEOREM. *Under the assumptions in 9.1 the solution of the dam problem is unique up to ground water reservoirs, that is, there is a solution u of the dam problem such that any other solution $\tilde{u}$ is of the*

form

$$\tilde{u}(y, h) = u(y, h) + \sum_j \max (h_j - h, 0) \chi_{D_j}$$

with certain real numbers h_j and connected components D_j of $\Omega \cap \{h < h_j\}$.

9.4 REMARK. The solution u in the statement of the theorem is the minimal solution constructed in [1]. If there exists a solution $\tilde{u}$ beside u, then there are infinitely many such solutions given by arbitrary values $h'_j < h_j$. On the other hand it is easy to state global properties of the geometry of Ω which exclude the existence of ground water reservoirs.

PROOF OF 9.3. Pick two solutions u and v. First we will separate the ground water reservoirs, which means that we will prove the following: If D is a connected component of $\Omega \cap \{v > 0\}$ such that $\overline{D} \cap S_{\text{res}}$ is empty then

$$\nabla v + e = 0 \quad \text{in } D.$$

By 9.1 the set $E := \overline{D} \cap \overline{S_{\text{res}}}$ is finite, and

$$\int_D \nabla \zeta (\nabla v + \chi_{\{v > 0\}} e) \geqslant 0$$

for all non-positive functions $\zeta \in H^{1,2}(\Omega)$ vanishing in $B_\varrho(E)$ and outside $B_\delta(\overline{D})$, if δ is small enough depending on ϱ. Thus we choose cut-off functions $\eta_\varrho \in C_0^\infty(B_\delta(\overline{D}))$ with

$$\eta_\varrho = 1 \quad \text{in } \overline{D} \backslash B_{2\varrho}(E), \quad \eta_\varrho = 0 \quad \text{in } B_\varrho(E),$$

$$|\nabla \eta_\varrho| \leqslant \frac{C}{\varrho} \quad \text{in } B_{2\varrho}(E) \cap \{v > 0\}.$$

Then we take as test function

$$\zeta(y, h) := \eta_\varrho(y, h)(h_{\min} - h - v(y, h)),$$

where $h_{\min}$ is the minimum height in D. We obtain for $\varrho \to 0$

$$0 \geqslant \int_D \eta_\varrho (\nabla v + e)(\nabla v + \chi_{\{v>0\}} e) - C \int_{\Omega \cap B_{2\varrho}(E)} |\nabla \eta_\varrho|(|\nabla v| + \chi_{\{v>0\}}) \geqslant$$

$$\geqslant \int_D \eta_\varrho |\nabla v + e|^2 - C \Big(\int_{\Omega \cap B_{2\varrho}(E)} (\nabla v|^2 + 1) \Big)^{\frac{1}{2}} \to \int_D |\nabla v + e|^2.$$

Hence it remains to show that $u \leqslant v$ in each connected component of $\{v > 0\}$ touching S_{res}. This we can prove repeating the proof of the comparison lemma 3.2 with the following slight modifications.

We let $E \subset \partial\Omega$ be the finite set consisting of all limit points of the free boundary of v, which are endpoints of the $C^{1,1}$ curves of $\partial\Omega$, or which come from connected components of $\{v > 0\}$ touching S_{res}, but do not lie on upper parts of $\partial\Omega$. Since

$$|\nabla u(x)| + |\nabla v(x)| \leqslant C(1 + \log|x - x_0|)$$

for $x_0 \in E$ (see 5.3, 6.3, 7.3, 8.3) the set E satisfies the conditions of the comparison lemma 3.2. Moreover we use the extended free boundary

$$\Sigma := \left\{(y, h) \in \overline{\{v > 0\}} : (y, h + s) \notin \Omega \text{ or } v(y, h + s) = 0 \quad \text{for small } s > 0\right\}$$

$$\Sigma := \Sigma \backslash B_\varrho(E)$$

in the proof of 3.2. Then we can proceed as in this proof except that we have to give estimates near points of Σ_ϱ which are contained in $S_{\mathrm{air}} \cup S_{\mathrm{imp}}$ on the top of $\partial\Omega$ or in S_{imp} at the bottom of $\partial\Omega$. In the latter case we can copy the proof of 3.2 since Σ_ϱ is Lipschitz near such points by the above reservoir argument. Since we did not prove that Σ_ϱ is Lipschitz near the top we use there a modified argument. Let

$$\left\{(y, g(y)) : y_1 < y < y_2\right\}$$

be a part of S_{air} on the top of the porous medium, such that

$$K := \left\{y : y_1 < y < y_2 \text{ and } (y, g(y)) \in \Sigma_\varrho\right\}$$

is a compact subset of $]y_1, y_2[$. Moreover there is a continuous function g_1 with $(y, g_1(y)) \in \Sigma_\varrho$ for $y_1 < y < y_2$. Choose open sets V with $K \subset V \subset]y_1, y_2[$ and non-negative functions $\omega \in C_0^\infty(V)$ with $\omega > 0$ on K. Then $\overline{\Omega} \cap$ $\cap$ graph $(g_1 + \omega)$ is smooth. Now we change the definition of d_ε in the proof of 3.2 replacing $\Omega \cap \{v > 0\}$ by $\{h < g_1(y) + \omega(y)\}$. Then the second integral in the proof of 3.2 can be estimated as before, and the first integral converges for $\varepsilon \to 0$ to

$$-\int_D \eta_\varrho \nabla \psi_\delta(u)(\nabla u + \gamma e),$$

where

$$D = \left\{(y, h) \in \Omega : y \in V \text{ and } 0 < h - g_1(y) < \omega(y)\right\}.$$

But this can be estimated by

$$-\int_{D}\eta_{\varrho}\,\partial_{k}\psi_{\delta}(u)\leqslant\int_{V\backslash K}\psi_{\delta}\big(u(y,g_{1}(y))\big)\,dy\;+\int_{D}\psi_{\delta}(u)\,\partial_{k}\eta_{\varrho}\leqslant\mathfrak{L}^{1}(V\backslash K)\,+\,C_{\varrho}\,\mathfrak{L}^{2}(D)\;,$$

which by an appropriate choice of V and ω is arbitrary small. Since the same argument holds on impervious parts on the top, the proof is complete.

The uniqueness of the solution of the dam problem immediately implies the stability of the solution and the free boundary in the interior of Ω with respect to small perturbations of the data. However, the behavior of the free boundary near $\partial\Omega$ in general is not stable, but we can prove the following two statements.

9.5 STABILITY OF OVERFLOW. *Suppose that* Ω^{λ}, $S_{\mathrm{res}}^{\lambda}$, *etc. for small* λ *is a* $C^{1,1}$ *perturbation of* Ω, S_{res}, *etc., and that* u_{λ}^{0} *is an* $H^{1,2}$ *perturbation of* u^{0} *satisfying the assumptions in* 9.1. *Denote by* u_{λ} *and* u *the corresponding unique solutions. If* $u > 0$ *in a neighborhood of a point* $x_{0} \in \overline{S_{\mathrm{res}}} \cap \overline{S_{\mathrm{air}}}$, *then the same is true for* u_{λ} *for small* λ.

PROOF. If not, for a sequence $\lambda \to 0$ the points x_{0}^{λ} are limit points of the free boundary of u_{λ}. Then by the overflow lemma 5.3 the functions u_{λ} are uniformly Lipschitz continuous in $\Omega^{\lambda} \cap B_{R}(x_{0}^{\lambda})$ for some $R > 0$. Hence for a subsequence u_{λ} converges to a solution, which by uniqueness must be u. Therefore u is Lipschitz continuous in $\Omega \cap B_{R}(x_{0})$, whereas by assumption u is positive in $\Omega \cap B_{R}(x_{0})$ for small R. This contradicts the overflow lemma.

9.6 STABILITY OF THE RIGHT ANGLE PROPERTY. *Assume the same as in* 9.5. *If at a point* $x_{0} \in \overline{S_{\mathrm{res}}} \cap \overline{S_{\mathrm{air}}}$ *with* $\sigma_{0} < \pi/2$ *the free boundary of* u *has tangent direction* $\exp[i\sigma^{-}]$ (σ_{0}, σ^{-} *as in* 5.1, 5.5), *then the same is true for* u_{λ} *for small* λ.

PROOF. Assume $x_{0}^{\lambda} = 0$. Since

$$u(x) \leqslant u^{\sigma^{-}}(x) + o(x)$$

for $x \to 0$ we see that for given small $\delta > 0$

$$u \leqslant (1 + \delta)u^{\sigma^{-} + \delta}$$

in $\Omega \cap B_{\varepsilon}(0)$ for some $\varepsilon > 0$. We conclude (see the proof of 5.6.3)) that for small λ

$$u_{\lambda} \leqslant (1 + 2\delta)u^{\sigma^{-} + 2\delta}$$

on $\partial(\Omega^\lambda \cap D^\lambda)$ with appropriate domains D^λ containing $B_\varepsilon(0)$ for some ε independent of λ. The comparison lemma 3.2 then yields that this estimate is satisfied in $\Omega^\lambda \cap B_\varepsilon(0)$. Defining τ^λ as in 5.7 this means

$$\tau^\lambda \leqslant \sigma_- + 2\delta < \sigma_+^\lambda$$

for small λ, if δ was chosen small enough. Then 5.7.2) gives the result.

9.7 REMARK. The statement in 9.6 is true also for $\sigma_0 > \pi/2$, provided we assume that $\partial\Omega$ is a graph in vertical direction near the first point above x_0.

9.8 EXAMPLES. Finally let us describe shortly some special global situations. First we consider the case of two reservoirs $S_{res} = S_1 \cup S_2$ with heights h_1 and h_2, say, $h_1 < h_2$. This means that

$$h_j = \sup_{x \in S_j} x \cdot e \, ,$$

$$u^0(y, h) = h_j - h \quad \text{for } (y, h) \in S_j \, .$$

If $u^j(y, h) := \max(h_j - h, 0)$, then u^2 is a supersolution, and u^1 is a subsolution, provided $S_{air} \subset \{h > h_1\}$. Then by 3.2 every solution u satisfies $u \leqslant u^2$ in the connected components of $\Omega \cap \{u^2 > 0\}$ touching S_2, and by 3.4 we have $u \geqslant u^1$ in connected components of $\Omega \cap \{u > 0\}$ touching S_1.

Consider a point $x_0 \in \bar{S}_2 \cap \overline{S_{air}}$ with height h_2 and standard normal ν_0, that is $0 < \nu_0 \cdot e < 1$. We conclude that near x_0 either $u = u^2$ or the free boundary has the right angle property (see theorem 5.7.2) and the proof of 5.6.2)).
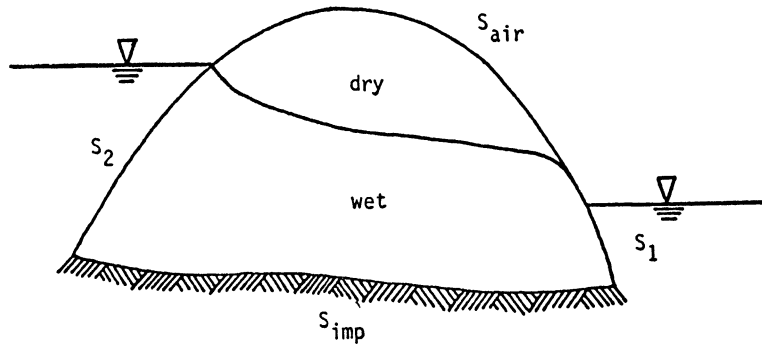

Fig.9

Similar we conclude for points $x_0 \in \bar{S}_1 \cap \overline{S_{\mathrm{air}}}$ with height h_1 and standard normal as before that either $u = u^1$ or there is no free boundary near x_0 (overflow case) (see theorem 5.7.3) and the proof of 5.6.3)). Moreover, every solution u is positive in

$$\Omega_j := \{(y, h) \in \Omega : (y, s) \in \Omega \text{ for } h < s < h' \text{ and some } h' \text{ with } (y, h') \in S_j\}.$$

Therefore, if $(\Omega \cap \{h < h_1\}) \cup \Omega_2$ is connected, we see that the free boundary always forms a right angle at the upper reservoir, and that there is always overflow near the lower reservoir. Under the additional assumption that for every height h_0 all connected components D of $\Omega \cap \{h < h_0\}$ with $\bar{D} \cap \partial\Omega \subset \overline{S_{\mathrm{imp}}}$ intersect $\{h < h_1\}$ we obtain the uniqueness of the solution from 9.3. Such a situation is plotted in fig. 9. Some cases with a non standard normal ν_0 at the lower reservoir, that is, $0 > \nu_0 \cdot e > -1$, are shown in fig. 10.1-4. It follows from the proof of Theorem 5.6 that the free boundary behaves as drawn in fig. 10.
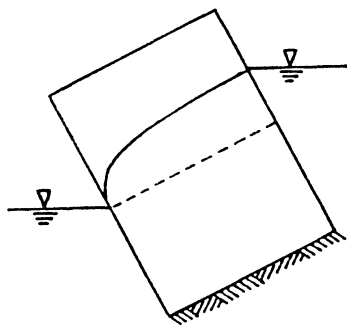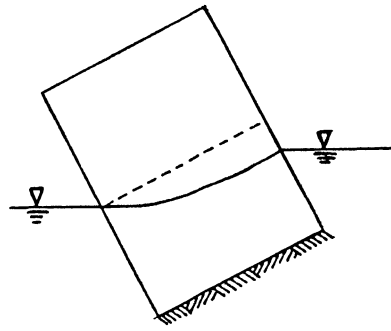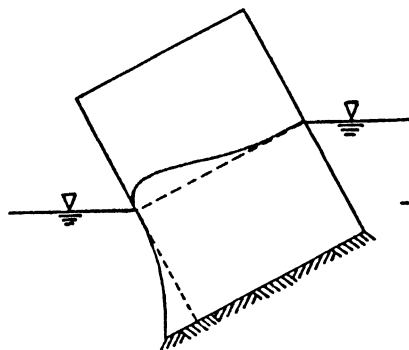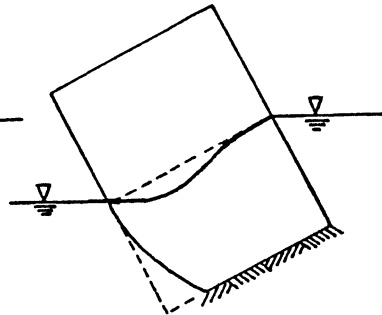

Fig.10.1

Fig.10.2

Fig.10.3

Fig.10.4

All the conclusions are the same for the highest and lowest level in the case of more than two water reservoirs.

Now let us study the case of three reservoirs with levels $h_1 < h_2 < h_3$, where we think about a shape of the porous medium as shown in fig. 11. In particular the height of the point P should be less than h_1.
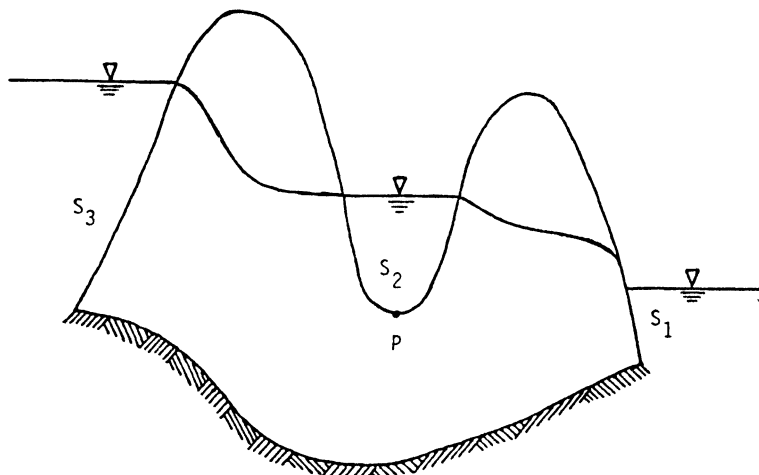

Fig. 11

Under similar assumptions as above there is a unique solution for every height h_2. If $h_2 = h_1$ we have overflow on both sides of the reservoir S_2, and by the stability result the same is true for h_2 near h_1. Similar we have the right angle property on both sides of S_2 if $h_2 = h_3$, and by the stability result the same is true for h_2 near h_3. Therefore (again a stability argument) there must be an intermediate value of h_2, for which neither overflow nor a right angle occurs on the left side of S_2. Then by Theorem 5.7 the free boundary must be horizontal there. It follows from [8] that the left part of the free boundary is monotone. In addition the right free bound-
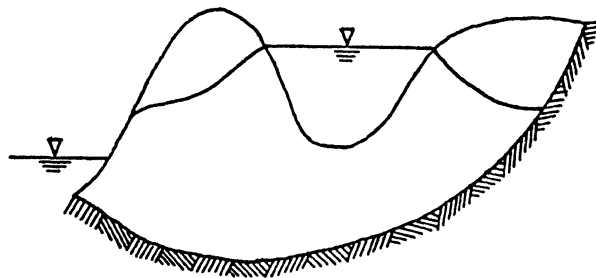

Fig. 12

ary either has overflow at S_2 and exactly one local estremum, or starts with a right angle at S_2 and is monotone. Similar one finds a value of h_2, for which the free boundary starts horizontal to the right of S_2.

Another situation including a horizontal free boundary at the impervious part is shown in fig. 12. For this example the monotonicity of the free boundary also can be proved using arguments of [8].

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