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THE ENTROPY PRINCIPLE FOR INTERFACES. FLUIDS AND SOLIDS

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Abstract. This paper is a systematic theory for interfaces of fluids and solids, based on the entropy principle of rational thermodynamics. It differs substantially from existing presentations of interface problems and provides several new aspects. Balance laws for mass, momentum, energy, and entropy are formulated as distributional equations, in this version they attain their natural form. In addition, we give a systematic study of frame indifference, and we show how this implies important structural properties of interfacial terms. Finally, we give theorems on the necessity of entropy inequalities. Properties of classical equilibrium thermodynamics and existing non-equilibrium thermodynamics are contained in the theory presented here. The content of this paper will be completed by another one, where more examples of interfacial problems, in particular those with surface tension, will be treated.

Remark. This version corrects some notational errors, which were in the original published paper.

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1 Introduction

The entropy principle, as formulated in rational thermodynamics, see for example [14], [15], is a well established method to study the structure of physical conservation laws. It is also a unique method to give an a-priori estimate, which can be used in cases one treats the differential equations mathematically.

We start with a set of physical processes $\mathcal{P}$, where for situations with a moving interface, the laws in its natural way are formulated in the sense of distributions in the physical space-time domain. The standard laws are those for mass, momentum and energy. This applies also to elasticity, where these laws are transformed into partial differential equations in the reference configuration. It also applies, if interfaces are present. Then these laws have to be understood in the sense of distributions. Indeed, this becomes obvious in cases, where one interprets interfaces as limit of thin layers.

This distributional formulation can be transformed into a strong version. It consists of differential equations in the surrounding volumes and differential equations and constraints on the interface. The corresponding analysis is contained in section 2.

In this paper we take the entropy principle from [14, section 1.3.1], which after a long history of thermodynamics showed up as an intrinsic version of this principle. In section 6 the entropy principle for processes $\mathcal{P}$ is formulated in analogy to the conversation laws in the space of distributions. That is, if (H, Ψ) are distributions denoting entropy and entropy flux, then it is postulated, see (6.1), that

$$\partial_t H + \operatorname{div}_x \Psi \geq 0$$

in the set of processes $\mathcal{P}$. In general, there are contribution of the entropy H and the entropy flux Ψ in domains and on surfaces. Therefore the rules of thermodynamic processes are formulated in different regions, and the concept of Lagrange multipliers in [14, section 1.3.1.2] is no longer usable. The exploitation of the entropy principle becomes an interesting procedure and uses its strong version, for which section 2 is a necessary tool.

In the isothermal case, the entropy principle reduces to a free energy inequality, which usually follows from the entropy inequality by erasing the heat flux.

In general, the entropy inequality and for isothermal models the free energy inequality leads to restrictions on the structure of differential equations and the structure of constitutive relations describing the class of physical processes.

Mathematically, the entropy principle for H, rewritten in terms of $-H$, and in the isothermal case the free energy principle for F, that is, see (6.3),

$$\partial_t F + \operatorname{div}_x \Phi \leq G_{\mathrm{ext}}\,,$$

serve the same purpose. The corresponding inequalities usually are taken as basic tool to prove existence of solutions. In fact, integrating these inequalities over the space domain one obtains the basic estimate for a mathematical treatment of the system. In special cases, where the arising boundary terms vanish or are nonnegative, do to the boundary conditions, the total negative entropy, respectively the total free energy, is a Liapunov functional of the system.

There is a second basic tool, which describes the class of materials under consideration. These are constitutive relations between the quantities in balance laws. In most cases these relations are given by pointwise equations for values and also derivatives of certain independent variables. These constitutive functions are subject to the principle of frame indifference, or the principle of objectivity.

In general, the principle of objectivity states, that all mathematical descriptions of physical processes have to be invariant under observer transformations. In particular, this applies to balance laws and constitutive functions, see section 4 and section 5. We explain in section 3, how to formulate and exploit objectivity (or: frame indifference) for distributional systems. Objectivity leads to transformation rules for all involved quantities and to restrictions for the constitutive functions. Because objectivity in its general version is somewhat unclear in the community, we have added a section about the force term, see section 9.

One possibility to deal with the entropy principle is to describe a class of physical processes by a set of balance laws with constraints like constitutive relations. Then, usually based on previous knowledge, one defines an entropy and an entropy flux in terms of the quantities in the balance laws, and one confirms the validity of the entropy inequality. Then one says the considered model is consistent with thermodynamics.

The procedure of rational thermodynamics is able to produce also necessity results. Again a class of physical processes is given by a set of balance laws with constitutive relations. One starts with quite general constitutive functions for entropy and entropy flux and computes the entropy production. Then one considers the entropy production as algebraic expression for values and derivatives of the involved functions, and the balance laws as algebraic constraints. The postulate that this expression has to be nonnegative leads to conditions for the coefficients of this expression. It is well known, that with this systematic procedure all formulas in classical thermodynamics can be derived.

If one deals with a new situation, the advantage of the procedure in rational thermodynamics becomes evident. The two physical principles, the entropy principle and the principle of objectivity, imply certain structures on the underlying partial differential equations. These structures are essential for the mathematical treatment of such quasilinear differential equations. Hence there is a strong influence of physical principles to the mathematical understanding of these systems.

Notation: In this paper arguments, which represent physical quantities, are denoted in their suggestive way. The same applies to arguments, which represent partial derivatives. Here the notation as argument is quite formal, a perfectly rigorous way should use another writing.

2 Analysis on interfaces

The basis for our analysis is the formulation of balance laws on moving interfaces in the sense of distributions. Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be a (local) time-space domain. In applications usually the dimension is $n = 3$, but also the cases $n = 1, 2$ can occur, if it is assumed that the physical quantities are homogeneous in the remaining directions.

For a physical quantity E, given as distribution $E \in \mathscr{D}'(D)$, together with corresponding flux components $Q^i \in \mathscr{D}'(D)$, $i = 1, \ldots, n$, and a production $F \in \mathscr{D}'(D)$, we consider the distributional equation

$$\partial_t E + \sum_{i=1}^n \partial_{x_i} Q^i = F \quad \text{in } \mathscr{D}'(D), \tag{2.1}$$

that is,

$$\langle\, \partial_t\zeta \,,\, E \,\rangle + \sum_{i=1}^n \langle\, \partial_{x_i}\zeta \,,\, Q^i \,\rangle + \langle\, \zeta \,,\, F \,\rangle = 0 \tag{2.2}$$

for all "test functions" $\zeta \in C_0^\infty(D)$. Similarly, the differential inequality

$$\partial_t E + \sum_{i=1}^n \partial_{x_i} Q^i \le F \quad \text{in } \mathscr{D}'(D) \tag{2.3}$$

means that

$$\langle\, \partial_t\zeta \,,\, E \,\rangle + \sum_{i=1}^n \langle\, \partial_{x_i}\zeta \,,\, Q^i \,\rangle + \langle\, \zeta \,,\, F \,\rangle \ge 0 \tag{2.4}$$

for all nonnegative functions $\zeta \in C_0^\infty(D)$.

In general there are three different equivalent versions of balance laws. They can be formulated

- as differential equation,
- with test functions,
- with test volumes.

The formulation with test volumes is common in physical text books, and corresponds to the transport theorem. The formulation with test functions is the distributional formulation presented in this paper. In this formulation balance laws appear in its natural form. The equivalence of the distributional formulation and the formulation as differential equation will be the content of theorem 2.4. Formally, the formulation with test volumes can be viewed as the distributional formulation with characteristic functions as test functions.

Later we shall consider distributional equations for example for mass, momentum, and energy, where the distributions will have contributions in open sets as well as on fixed or moving interfaces. For such surfaces two vector fields are important, the velocity vector and the curvature vector.

2.1 Evolving surfaces. For a given set $\Gamma \subset \mathbb{R} \times \mathbb{R}^n$ let $\Gamma_t := \{x \in \mathbb{R}^n\,;\ (t,x) \in \Gamma\}$ for $t \in \mathbb{R}$. Let $0 \le d \le n$ be an integer. Then $t \mapsto \Gamma_t$ is called a d-dimensional smooth "evolving surface" or "moving surface", if Γ is a smooth $d+1$-dimensional surface (for our purpose a C^2-surface), such that it's tangent space never is "spacelike", that is

$$T_{(t,x)}(\Gamma) \not\subset \{0\} \times \mathbb{R}^n \quad \text{for all } (t,x) \in \Gamma.$$

Then the following holds:

(1) ***Velocity vector.*** For $(t,x) \in \Gamma$ there is a unique vector $\mathbf{v}_\Gamma(t,x) \in \mathbb{R}^n$ with

$$(1, \mathbf{v}_\Gamma(t,x)) \in T_{(t,x)}(\Gamma) \quad \text{and} \quad \mathbf{v}_\Gamma(t,x) \in T_x(\Gamma_t)^\perp. \tag{2.5}$$

An equivalent characterization of $\mathbf{v}_\Gamma$ is, that for every C^1-curve $s \mapsto \gamma(s) \in \Gamma_s$ with $\gamma(t) = x$

$$\gamma'(t) - \mathbf{v}_\Gamma(t,x) \in T_x(\Gamma_t) \quad \text{and} \quad \mathbf{v}_\Gamma(t,x) \in T_x(\Gamma_t)^\perp. \tag{2.6}$$

Therefore $\mathbf{n}\bullet\gamma'(t) = \mathbf{n}\bullet\mathbf{v}_\Gamma(t,x)$ for every normal vector $\mathbf{n} \in T_x(\Gamma_t)^\perp$. Moreover

$$\begin{aligned} &T_{(t,x)}(\Gamma) = \operatorname{span}\{(1, \mathbf{v}_\Gamma(t,x))\} \oplus (\{0\} \times T_x(\Gamma_t)), \\ &T_{(t,x)}(\Gamma)^\perp = \{(-\mathbf{v}_\Gamma(t,x)\bullet\mathbf{n}, \mathbf{n})\,;\ \mathbf{n} \in T_x(\Gamma_t)^\perp\}. \end{aligned} \tag{2.7}$$

(2) ***Curvature vector.*** For $(t,x) \in \Gamma$ there is a unique vector $\kappa_\Gamma(t,x) \in T_x(\Gamma_t)^\perp \subset \mathbb{R}^n$ satisfying

$$\mathbf{n}\bullet\kappa_\Gamma(t,x) = \mathbf{n}\bullet \textstyle\sum_{k=1}^d \partial_{\tau_k(x)}\tau_k(x) \qquad \text{for } \mathbf{n} \in T_x(\Gamma_t)^\perp$$

for every local tangential orthonormal system $\{\tau_1, \ldots, \tau_d\}$ of Γ_t at x.

We mention, that $\mathbf{v}_\Gamma(t,x)$ and $\kappa_\Gamma(t,x)$ are independent of local orientations of the surface and that $\frac{1}{d}\kappa_\Gamma(t,x)$ is the "mean curvature vector" of Γ_t at x. Note, that $T_x(\Gamma_t) \subset \mathbb{R}^n$ is the tangent space of the time slice Γ_t of Γ at t, whereas $T_{(t,x)}(\Gamma) \subset \mathbb{R}^{1+n}$ is the tangent space of the entire surface $\Gamma \subset \mathbb{R}^{1+n}$.

Proof (1). By assumption there is a vector $v \in \mathbb{R}^n$ and a d-dimensional subspace $V \subset \mathbb{R}^n$ such that

$$T_{(t,x)}(\Gamma) = \{a(1,v) + (0,\tau)\,;\ a \in \mathbb{R},\ \tau \in V\}. \tag{2.8}$$

Define $\mathbf{v}_\Gamma(t,x) := v - P(v)$, where P is the orthogonal projection of $\mathbb{R}^n$ to V. Using a local parametrization of Γ around (t,x) one verifies that $V = T_x(\Gamma_t)$ and that (2.6) holds. □

Proof (2). This is standard differential geometry for surfaces in $\mathbb{R}^n$. There is a unique symmetric bilinear map $B : T_x(\Gamma_t) \times T_x(\Gamma_t) \to T_x(\Gamma_t)^\perp$ satisfying $B(\tau_1(x), \tau_2(x)) = P(\partial_{\tau_1(x)}\tau_2(x))$ for all smooth local tangential vector fields τ_1, τ_2 of Γ_t around x, where now P is the orthogonal projection of $\mathbb{R}^n$ to $T_x(\Gamma_t)^\perp$. Then $\kappa_\Gamma(t,x) := \operatorname{trace} B$. □

2.2 Surface measure. Let $\Gamma \subset \mathbb{R} \times \mathbb{R}^n$ be an evolving surface as in 2.1. In the following we shall work with the surface measure μ_Γ, defined by

$$\mu_\Gamma := (\mathrm{L}_1 \otimes \mathrm{H}_d) \llcorner \Gamma,$$

that is

$$\int_D f \,\mathrm{d}\mu_\Gamma = \int_{\mathbb{R}} \int_{\Gamma_t} f(t,x)\,\mathrm{d}\mathrm{H}_d(x)\,\mathrm{d}\mathrm{L}_1(t).$$

Here and in the following H_d denotes the d-dimensional Haussdorff measure (surface measure) and L_m the m-dimensional Lebesgue measure. Another possibility would be to use $\mathrm{H}_{d+1} \llcorner \Gamma$. One easily verifies that

$$\int_\Gamma f\,\mathrm{d}\mathrm{H}_{d+1} = \int_\Gamma \sqrt{1 + |\,\mathbf{v}_\Gamma\,|^2} \cdot f\,\mathrm{d}(\mathrm{L}_1 \otimes \mathrm{H}_d) = \int_D \sqrt{1 + |\,\mathbf{v}_\Gamma\,|^2} \cdot f\,\mathrm{d}\mu_\Gamma. \tag{2.9}$$

However, in the following we shall work with μ_Γ, since it has easier properties under observer transformations (see the proof of 5.1).

2.3 Definition. On Γ we use the following differential operators:

(1) $\partial_t^\Gamma e(t,x) := \partial_{(1,\mathbf{v}_\Gamma)} e(t,x) = (\partial_t + \mathbf{v}_\Gamma \bullet \nabla) e(t,x)$ for $e : \Gamma \to \mathrm{I\!R}$,

(2) $\nabla^\Gamma e(t,x) := \sum_{k=1}^d (\partial_{\tau_k} e(t,x)) \tau_k$ for $e : \Gamma \to \mathrm{I\!R}$,

(3) $\mathrm{div}^\Gamma q(t,x) := \sum_{k=1}^d \tau_k \bullet \partial_{\tau_k} q(t,x)$ for $q : \Gamma \to \mathrm{I\!R}^n$.

Here $\{\tau_k\,;\ k = 1, \dots, d\}$ is any orthonormal system of $T_x(\Gamma_t)$. Moreover, there are also time-space analogues, defined by

(4) $\underline{\nabla}^\Gamma e := \sum_{k=0}^d (\partial_{\underline{\tau}_k} e) \underline{\tau}_k$ for $e : \Gamma \to \mathrm{I\!R}$,

(5) $\underline{\mathrm{div}}^\Gamma q := \sum_{k=0}^d \underline{\tau}_k \bullet \partial_{\underline{\tau}_k} q$ for $q : \Gamma \to \mathrm{I\!R}^{n+1}$.

Here $\{\underline{\tau}_k\,;\ k = 0, \dots, d\}$ is any orthonormal system of $T_{(t,x)}(\Gamma)$.

With this we are able to formulate the following theorem for balance laws on evolving surfaces.

2.4 Theorem. Let $D \subset \mathrm{I\!R} \times \mathrm{I\!R}^n$ be an open set and $\Gamma \subset D$ a smooth evolving surface without boundary in D, that is $\overline{\Gamma} \cap D \subset \Gamma$. Then for smooth functions $e : \Gamma \to \mathrm{I\!R}$, $q : \Gamma \to \mathrm{I\!R}^n$, $f : \Gamma \to \mathrm{I\!R}$ the following is equivalent:

(1) ***Distributional formulation.***

$$\partial_t(e\mu_\Gamma) + \mathrm{div}(q\mu_\Gamma) = \ \text{(resp. } \leq\text{)}\ f\mu_\Gamma \quad \text{in } \mathscr{D}'(D).$$

(2) ***Strong formulation in time-space.***

$$(e,q)(t,x) \in T_{(t,x)}(\Gamma) \quad \text{for all } (t,x) \in \Gamma,$$
$$\sqrt{1 + |\,\mathbf{v}_\Gamma\,|^2} \cdot \underline{\mathrm{div}}^\Gamma \left(\frac{1}{\sqrt{1 + |\,\mathbf{v}_\Gamma\,|^2}} (e,q) \right) = \ \text{(resp. } \leq\text{)}\ f \quad \text{on } \Gamma.$$

(3) ***Strong formulation.***

$$(q - e\mathbf{v}_\Gamma)(t,x) \in T_x(\Gamma_t) \quad \text{for all } (t,x) \in \Gamma,$$
$$\partial_t^\Gamma e - e\,\kappa_\Gamma \bullet \mathbf{v}_\Gamma + \mathrm{div}^\Gamma (q - e\mathbf{v}_\Gamma) = \ \text{(resp. } \leq\text{)}\ f \quad \text{on } \Gamma.$$

Supplement: The last differential equation is equivalent to

$$\partial_t^\Gamma e + \mathrm{div}^\Gamma q = \ \text{(resp. } \leq\text{)}\ f \quad \text{on } \Gamma.$$

Note, that here the term under the divergence has a spatial normal component $e\mathbf{v}_\Gamma$.

We mention, that the strong formulations in 2.4(2) and 2.4(3) not only contain differential equations on Γ, but also the condition, that $q - e\mathbf{v}_\Gamma$ has to be a spatial tangential vector field.

Proof. By (2.4) the distributional inequality says

$$\int_D (\partial_t \zeta \cdot e + \nabla\zeta \bullet q + \zeta \cdot f) \, \mathrm{d}\mu_\Gamma \geq 0 \tag{2.10}$$

for all nonnegative functions $\zeta \in C_0^\infty(D)$. We write this as weak divergence equation in the time-space domain. Setting $\underline{\nabla} := (\partial_t, \nabla)$ and

$$w := \sqrt{1 + |\,\mathbf{v}_\Gamma\,|^2} \tag{2.11}$$

and using (2.9), equation (2.10) becomes

$$\int_\Gamma \frac{1}{w}(\underline{\nabla}\zeta \bullet (e, q) + \zeta \cdot f) \, \mathrm{dH}_{d+1} \geq 0. \tag{2.12}$$

This then also holds for all nonnegative C^1-functions ζ with compact support in D.

Now replace $\zeta \geq 0$ by $\widetilde{\zeta} = \zeta \cdot (1 + \sin(a\psi)) \geq 0$, where $a \in \mathbb{R}$ and ψ is any C^1-function vanishing on Γ. Then $\underline{\nabla}\zeta$ is replaced by

$$\underline{\nabla}\widetilde{\zeta} = (1 + \sin(a\psi))\underline{\nabla}\zeta + \zeta \cos(a\psi) a \underline{\nabla}\psi.$$

Since $\psi = 0$ on Γ, this is equal to

$$\underline{\nabla}\widetilde{\zeta} = \underline{\nabla}\zeta + \zeta a \underline{\nabla}\psi \quad \text{ on } \Gamma.$$

Then (2.12) implies

$$\begin{aligned} 0 &\leq \int_\Gamma \frac{1}{w}(\underline{\nabla}\widetilde{\zeta} \bullet (e, q) + \widetilde{\zeta} \cdot f) \, \mathrm{dH}_{d+1} \\ &= \int_\Gamma \frac{1}{w}(\underline{\nabla}\zeta \bullet (e, q) + \zeta \cdot f) \, \mathrm{dH}_{d+1} + a \int_\Gamma \frac{1}{w} \zeta \underline{\nabla}\psi \bullet (e, q) \, \mathrm{dH}_{d+1}. \end{aligned}$$

Since a is an arbitrary number, it follows that the additional a-term has to vanish, that is

$$\int_\Gamma \frac{1}{w} \zeta \underline{\nabla}\psi \bullet (e, q) \, \mathrm{dH}_{d+1} = 0.$$

Since this is true for all nonnegative ζ, we conclude that

$$\underline{\nabla}\psi \bullet (e, q) = 0 \quad \text{ on } \Gamma.$$

One can choose ψ so that $\underline{\mathbf{n}} = \underline{\nabla}\psi$ on Γ with a smooth normal field $\underline{\mathbf{n}}$ (not a unit normal), that is $\underline{\mathbf{n}}(t, x) \in T_{(t,x)}(\Gamma)^\perp$ at a given point $(t, x) \in \Gamma$. Doing so one concludes that

$$(e, q)(t, x) \in T_{(t,x)}(\Gamma).$$

This is equivalent to (see (2.7))

$$q^0(t,x) := (q - e\mathbf{v}_\Gamma)(t,x) \in T_x(\Gamma_t). \tag{2.13}$$

With this property (2.12) becomes

$$\int_\Gamma (\underline{\nabla}^\Gamma \zeta \bullet (\frac{1}{w}(e,q)) + \zeta \cdot \frac{1}{w} f)\, \mathrm{dH}_{d+1} \geq 0 \tag{2.14}$$

with a tangential vector field $\frac{1}{w}(e,q)$. Integration by parts on Γ gives that

$$\begin{aligned} 0 \leq & \int_\Gamma \zeta \cdot \left(-\underline{\mathrm{div}}^\Gamma (\frac{1}{w}(e,q)) + \frac{1}{w} f \right) \mathrm{dH}_{d+1} \\ = & \int_D \zeta \cdot \left(-w \underline{\mathrm{div}}^\Gamma (\frac{1}{w}(e,q)) + f \right) \mathrm{d}\mu_\Gamma \end{aligned}$$

for all nonnegative test functions ζ, hence

$$w \cdot \underline{\mathrm{div}}^\Gamma (\tfrac{1}{w}(e,q)) \leq f \quad \text{ on } \Gamma. \tag{2.15}$$

This proves (2). We claim that this, together with property (2.13), is equivalent to the differential inequality in (3). In fact, since by equation (2.13)

$$(e,q) = e(1,\mathbf{v}_\Gamma) + (0,q^0),$$

the left-hand side in (2.15) equals

$$= (1,\mathbf{v}_\Gamma) \bullet \underline{\nabla}^\Gamma e + e \cdot w \cdot \underline{\mathrm{div}}^\Gamma (\tfrac{1}{w}(1,\mathbf{v}_\Gamma)) + w \cdot \underline{\mathrm{div}}^\Gamma (\tfrac{1}{w}(0,q^0)).$$

Using the differential identities (2.16) and (2.17) (see lemma 2.5 below) this equals

$$= \partial_t^\Gamma e - e \cdot \kappa_\Gamma \bullet \mathbf{v}_\Gamma + \mathrm{div}^\Gamma q^0,$$

which proves (3). The supplement follows, since

$$\mathrm{div}^\Gamma (e\mathbf{v}_\Gamma) = -e \cdot \kappa_\Gamma \bullet \mathbf{v}_\Gamma$$

by the following proposition 2.6, if one sets $\mathbf{n} := e\mathbf{v}_\Gamma$ as spatial normal vector field. □

Thus it remains to show that

2.5 Lemma. Let $\Gamma \subset \mathrm{I\!R} \times \mathrm{I\!R}^n$ as in 2.1, and w as in (2.11). Then

$$w \cdot \underline{\mathrm{div}}^\Gamma (\tfrac{1}{w}(1,\mathbf{v}_\Gamma)) = -\kappa_\Gamma \bullet \mathbf{v}_\Gamma, \tag{2.16}$$

and for every spatial tangential vector field q^0, that is $q^0(t,x) \in T_x(\Gamma_t)$,

$$w \cdot \underline{\mathrm{div}}^\Gamma (\tfrac{1}{w}(0,q^0)) = \mathrm{div}^\Gamma q^0. \tag{2.17}$$

Proof. To prove (2.16), consider a local orthonormal system $\{\tau_k(t,x)\,;\ k=1,\dots,d\}$ of $T_x(\Gamma_t)$. Setting

$$\underline{\tau}_0 := \frac{1}{w}(1,\mathbf{v}_\Gamma), \quad \underline{\tau}_k := (0,\tau_k) \quad \text{for } k=1,\dots,d,$$

we obtain a local orthonormal system $\{\underline{\tau}_k(t,x)\,;\ k=0,\dots,d\}$ of $T_{(t,x)}(\Gamma)$. Then

$$w\cdot\underline{\mathrm{div}}^\Gamma(\tfrac{1}{w}(1,\mathbf{v}_\Gamma)) = w\textstyle\sum_{k=0}^d \underline{\tau}_k\bullet\partial_{\underline{\tau}_k}\underline{\tau}_0.$$

Since $\underline{\tau}_0\bullet\partial_{\underline{\tau}_0}\underline{\tau}_0 = 0$ and $\underline{\tau}_k\bullet\partial_{\underline{\tau}_k}\underline{\tau}_0 = -(\partial_{\underline{\tau}_k}\underline{\tau}_k)\bullet\underline{\tau}_0$ for $k=1,\dots,d$, this equals

$$\begin{aligned}&= -w\textstyle\sum_{k=1}^d \underline{\tau}_0\bullet\partial_{\underline{\tau}_k}\underline{\tau}_k = -(1,\mathbf{v}_\Gamma)\bullet\sum_{k=1}^d\partial_{\underline{\tau}_k}(0,\tau_k)\\ &= -\mathbf{v}_\Gamma\bullet\textstyle\sum_{k=1}^d\partial_{\tau_k}\tau_k = -\mathbf{v}_\Gamma\bullet\kappa_\Gamma\end{aligned}$$

by the properties in definition 2.1(1).

Equation (2.17) follows from the following computation for test functions $\zeta\in C_0^\infty(D)$, where we use that q^0 is a spatial tangential vector field, that is, $(0,q^0)$ is tangential to Γ:

$$\begin{aligned}&\int_D \zeta\cdot w\cdot\underline{\mathrm{div}}^\Gamma(\frac{1}{w}(0,q^0))\,\mathrm{d}\mu_\Gamma = \int_\Gamma \zeta\cdot\underline{\mathrm{div}}^\Gamma(\frac{1}{w}(0,q^0))\,\mathrm{dH}_{d+1}\\
&= -\int_\Gamma\frac{1}{w}(\underline{\nabla}^\Gamma\zeta)\bullet(0,q^0)\,\mathrm{dH}_{d+1} \quad \text{(integration by parts on }\Gamma)\\
&= -\int_\Gamma\frac{1}{w}(\nabla^\Gamma\zeta)\bullet q^0\,\mathrm{dH}_{d+1} \quad \text{(note, that }\nabla^\Gamma\text{ and }\underline{\nabla}^\Gamma\text{ are different)}\\
&= -\int_{\mathbb{R}}\left(\int_{\Gamma_t}(\nabla^\Gamma\zeta)\bullet q^0\,\mathrm{dH}_d\right)\mathrm{dL}_1(t) \quad \text{(see (2.9))}\\
&= \int_{\mathbb{R}}\left(\int_{\Gamma_t}\zeta\cdot\mathrm{div}^\Gamma(q^0)\,\mathrm{dH}_d\right)\mathrm{dL}_1(t) = \int_D\zeta\cdot\mathrm{div}^\Gamma(q^0)\,\mathrm{d}\mu_\Gamma.\end{aligned}$$

□

2.6 Proposition. Let $\Gamma\subset\mathbb{R}\times\mathbb{R}^n$ as in 2.1. Then

$$\mathrm{div}^\Gamma\mathbf{n} = -\kappa_\Gamma\bullet\mathbf{n} \tag{2.18}$$

for spatial normal vector fields $\mathbf{n}$, that is $\mathbf{n}(t,x)\in T_x(\Gamma_t)^\perp$.
Note: It is $\mathbf{n}$ a not necessary unit vector field.

Proof. We use the notation in 2.3. Then, by the product rule,

$$\mathrm{div}^\Gamma\mathbf{n} = \textstyle\sum_{k=1}^d\tau_k\bullet\partial_{\tau_k}\mathbf{n} = -\sum_{k=1}^d(\partial_{\tau_k}\tau_k)\bullet\mathbf{n} = -\kappa_\Gamma\bullet\mathbf{n},$$

since $\tau_k\bullet\mathbf{n}=0$ for $k=1,\dots,d$. □

As a consequence of 2.4 one obtains the well known transport theorem, which in text books often is taken as a starting point for the analysis of moving interfaces (see e.g. [12], [16], [10]). In this paper we shall not make use of this theorem.

In general, if $n=3$ the set Γ_t can be a surface, a line measure or a point measure. Let us consider a distributional balance law for $d=n-1$, that is for moving hypersurfaces separating two media.

2.7 Moving interfaces. We consider the local situation. Let $D \subset \mathbb{R} \times \mathbb{R}^n$ be an open set consisting of two open sets Ω^1 and Ω^2 (that is $d = n$ in 2.1) separated by a smooth evolving hypersurface Γ (that is $d = n-1$ in 2.1), in particular, $\Gamma \subset D$ has no boundary within D, that is $\overline{\Gamma} \cap D = \Gamma$. For $(t,x) \in \Gamma$ we let

$$\nu_m(t,x) \in T_x(\Gamma_t)^\perp \subset \mathbb{R}^n \quad \text{the external unit normal of } \Omega_t^m. \tag{2.19}$$

Then $\nu_1 + \nu_2 = 0$. Moreover, $\mathbf{v}_\Gamma(t,x)$ as well as $\kappa_\Gamma(t,x)$ are scalar multiples of $\nu_m(t,x)$. We denote by μ_{Ω^1}, μ_{Ω^2}, μ_Γ the corresponding measures from 2.2. Then a single balance law is an equality (resp. inequality) of the form

$$\partial_t E + \mathrm{div} Q = \text{ (resp. } \leq \text{) } F \quad \text{in } \mathscr{D}'(D) \tag{2.20}$$

with distributions given by

$$\begin{aligned} E &= \textstyle\sum_{m=1}^2 e^m \mu_{\Omega^m} + e^s \mu_\Gamma, \\ Q &= \textstyle\sum_{m=1}^2 q^m \mu_{\Omega^m} + q^s \mu_\Gamma, \\ F &= \textstyle\sum_{m=1}^2 f^m \mu_{\Omega^m} + f^s \mu_\Gamma. \end{aligned} \tag{2.21}$$

Here $e^m, q_i^m, f^m : \overline{\Omega^m} \to \mathbb{R}$ and $e^s, q_i^s, f^s : \Gamma \to \mathbb{R}$ for simplicity are assumed to be smooth functions.

The interface version of 2.4 is

2.8 Theorem. Under the assumptions in 2.7 distributional law (2.20) is equivalent to the following:

(1) For $m = 1,2$ in Ω^m:

$$\partial_t e^m + \mathrm{div} q^m = \text{ (resp. } \leq \text{) } f^m.$$

(2) For all $(t,x) \in \Gamma$:

$$(q^s - e^s \mathbf{v}_\Gamma)(t,x) \in T_x(\Gamma_t).$$

(3) On Γ:

$$\begin{aligned} &\partial_t^\Gamma e^s - e^s \, \kappa_\Gamma \bullet \mathbf{v}_\Gamma + \mathrm{div}^\Gamma (q^s - e^s \mathbf{v}_\Gamma) \\ &= \text{ (resp. } \leq \text{) } f^s + \textstyle\sum_{m=1}^2 (q^m - e^m \mathbf{v}_\Gamma) \bullet \nu_m. \end{aligned}$$

Without any interface quantities the last identity is the “Rankine-Hugoniot condition”. With interface quantities it contains “Kotchine conditions”. As in 2.4(3) the left-hand side of the differential equation in 2.8(3) equals $\partial_t^\Gamma e^s + \mathrm{div}^\Gamma q^s$. Here the flux q^s has a normal component $\mathbf{n} \bullet q^s(t,x) = e(t,x) \mathbf{n} \bullet \mathbf{v}_\Gamma(t,x)$ in direction $\mathbf{n} \in T_x(\Gamma_t)^\perp$.

Proof. By (2.4) and (2.9) the distributional inequality in (2.20) means

$$\begin{aligned}&\sum_{m=1}^{2}\int_{\Omega^m}\left(\partial_t\zeta\cdot e^m+\nabla\zeta\bullet q^m+\zeta\cdot f^m\right)\,\mathrm{dL}_{n+1}\\&+\int_{\Gamma}\left(\partial_t\zeta\cdot e^s+\nabla\zeta\bullet q^s+\zeta\cdot f^s\right)\,\mathrm{d}(\mathrm{L}_1\otimes\mathrm{H}_{n-1})\geq 0\end{aligned}\tag{2.22}$$

for all nonnegative $\zeta\in C_0^\infty(D)$. Integration by parts for the time-space domain Ω^m yields for $m=1,2$

$$\begin{aligned}&\int_{\Omega^m}\left(\partial_t\zeta\cdot e^m+\nabla\zeta\bullet q^m+\zeta\cdot f^m\right)\,\mathrm{dL}_{n+1}\\&=\int_{\Gamma}\zeta\cdot(e^m,q^m)\bullet\underline{\nu}_m\,\mathrm{dH}_n+\int_{\Omega^m}\zeta\cdot\left(-\partial_t e^m-\mathrm{div}q^m+f^m\right)\,\mathrm{dL}_{n+1}.\end{aligned}$$

Here $\underline{\nu}_m$ is the external unit normal of Ω^m as time-space domain, which by 2.1(1) satisfies

$$\underline{\nu}_m=\tfrac{1}{w}(-\mathbf{v}_\Gamma\bullet\nu_m,\nu_m),\quad w:=\sqrt{1+|\,\mathbf{v}_\Gamma\,|^2}.\tag{2.23}$$

Therefore, by (2.9),

$$\int_{\Gamma}\zeta\cdot(e^m,q^m)\bullet\underline{\nu}_m\,\mathrm{dH}_n=\int_{\Gamma}\zeta\cdot(q^m-e^m\mathbf{v}_\Gamma)\bullet\nu_m\,\mathrm{d}(\mathrm{L}_1\otimes\mathrm{H}_{n-1}).$$

Inserting this into (2.22) we get

$$\sum_{m=1}^{2}\int_{\Omega^m}\zeta\cdot g^m\,\mathrm{dL}_{n+1}+\int_{\Gamma}\left(\partial_t\zeta\cdot e^s+\nabla\zeta\bullet q^s+\zeta\cdot g^s\right)\,\mathrm{d}(\mathrm{L}_1\otimes\mathrm{H}_{n-1})\geq 0,$$

where

$$\begin{aligned}g^m&:=-\partial_t e^m-\mathrm{div}q^m+f^m,\\g^s&:=f^s+\sum_{m=1}^{2}(q^m-e^m\mathbf{v}_\Gamma)\bullet\nu_m.\end{aligned}$$

Choosing nonnegative test functions $\zeta\in C_0^\infty(\Omega^m)$ we obtain $g^m\geq 0$ in Ω^m, which shows (1). Then replace ζ by $\zeta\cdot\eta_\delta$ with a sequence of smooth functions η_δ satisfying $0\leq\eta_\delta\leq 1$, $\eta_\delta=1$ is some neighbourhood of Γ, and $\eta_\delta\searrow 0$ as $\delta\searrow 0$ pointwise in $D\setminus\Gamma$. This gives

$$\int_{\Gamma}\left(\partial_t\zeta\cdot e^s+\nabla\zeta\bullet q^s+\zeta\cdot g^s\right)\,\mathrm{d}(\mathrm{L}_1\otimes\mathrm{H}_{n-1})\geq 0.$$

Then apply 2.4. □

In the following sections we will apply these results to evaluate balance laws.

3 The principle of frame indifference

Restrictions for the description of a class of physical processes come from the "principle of objectivity", or "principle of frame indifference". It consists of the following axioms:

- The value of physical quantities depend on the observer.
- The type of a physical quantity is given by a transformation rule.
- The description of a physical process has to be independent of the observer.

The last property is "objectivity" and states, that the description of physical processes has to be the same for all observers. This applies to formulations with differential equations, see section 5, as well as to formulations with constitutive relations or constraints, see section 4. The first property is the well known "relativity", an classical example being the Doppler effect. The second property is "rationality" saying, that is is possible to describe analytically, how quantities change under observer transformations. Therefore the description of any situation in classical continuum physics has to be "frame indifferent", and this description includes everything like differential equations, constitutive relations, the domain of definition, positivity of functions.

In order to formulate the principle of objectivity, one has to specify how coordinates transform, where in this paper we restrict to classical continuum physics. A general Newtonian observer transformation of the coordinates $y = (t, x)$ into coordinates $y^* = (t^*, x^*)$ has the form

$$\begin{bmatrix} t \\ x \end{bmatrix} = y = Y(y^*) = \begin{bmatrix} t^* + \mathrm{a} \\ X(t^*, x^*) \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Q(t^*)x^* + \mathrm{b}(t^*) \end{bmatrix} \tag{3.1}$$

with $\mathrm{a} \in \mathbb{R}$ and smooth functions $\mathrm{b} : \mathbb{R} \to \mathbb{R}^n$ and $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfying $Q^{\mathrm{T}}Q = \mathrm{Id}$ and $\det Q = 1$. Then the derivative of Y is

$$\underline{\mathrm{D}}Y = (Y_{k\,'l})_{k,l=0,\dots,n} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}.$$

Here $\underline{\mathrm{D}}$ denotes the differential with respect to (t^*, x^*) and $\dot{X}$ the time derivative

$$\dot{X}(t^*, x^*) := \frac{\partial}{\partial t^*} X(t^*, x^*) = \dot{Q}(t^*)x^* + \dot{\mathrm{b}}(t^*).$$

A special observer transformation is an Euclidean transformation, which is of the form

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Qx^* + t^*\mathrm{c} + \mathrm{b} \end{bmatrix} \tag{3.2}$$

with $\mathrm{a} \in \mathbb{R}$, $\mathrm{b}, \mathrm{c} \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ satisfying $Q^{\mathrm{T}}Q = \mathrm{Id}$ and $\det Q = 1$. If Y is a Newton transformation, then for any (t_0^*, x_0^*) the approximation

$$\begin{bmatrix} t^* \\ x^* \end{bmatrix} \mapsto \begin{bmatrix} t \\ x \end{bmatrix} = Y(t_0^*, x_0^*) + \underline{\mathrm{D}}Y(t_0^*, x_0^*) \begin{bmatrix} t^* - t_0^* \\ x^* - x_0^* \end{bmatrix}$$

is an Euclidean transformation. Indeed this is a characterization of Newtonian transformations.

In principle, one can consider frame indifference with respect to a given group of transformations. However, because of its importance, we will, in this paper, always take the group of Newton transformations.

The following definitions work for any group of observer transformations. A general nonlinear "transformation rule" for a set of quantities U has the following form: Consider for one observer a set of quantities $(t,x) \mapsto U(t,x)$ arising in some description. Assume that for another observer, with a transformation

$$Y : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n, \quad (t^*,x^*) \mapsto (t,x) = Y(t^*,x^*)$$

of variables according to (3.1), these quantities are $(t^*,x^*) \mapsto U^*(t^*,x^*)$. The connection between these values are given by a, in general nonlinear, function Z_Y with

$$U(t,x) = Z_Y(t^*,x^*,U^*(t^*,x^*)) \quad \text{for all } (t,x) = Y(t^*,x^*). \tag{3.3}$$

Here consistency with the group structure of the set of observer transformations is required: If $(t^*,x^*) = Y^*(\bar{t},\bar{x})$ is another observer transformation, then $(t,x) = (Y \circ Y^*)(\bar{t},\bar{x})$, and therefore $U^*(t^*,x^*) = Z_{Y^*}(\bar{t},\bar{x},\bar{U}(\bar{t},\bar{x}))$ and $U(t,x) = Z_{Y\circ Y^*}(\bar{t},\bar{x},\bar{U}(\bar{t},\bar{x}))$ is satisfied. It then follows from (3.3) that

$$Z_{Y\circ Y^*}(\bar{t},\bar{x},\bar{U}(\bar{t},\bar{x})) = Z_Y(Y^*(\bar{t},\bar{x}), Z_{Y^*}(\bar{t},\bar{x},\bar{U}(\bar{t},\bar{x}))) \tag{3.4}$$

has to be satisfied. All explicit rules in this paper have this property.

A "physical quantity" is a family of time-space functions for all observers, where the domain of definition and also the values depend on the observer, and where the type of such a quantity is defined by a transformation rule. We give two basic examples.

First the definition of a "velocity". Let $(t,x) \mapsto v(t,x) \in \mathbb{R}^n$ be the vector field for a particular observer, and $(t^*,x^*) \mapsto v^*(t^*,x^*) \in \mathbb{R}^n$ the corresponding vector field for another observer, with (3.1) as transformation of coordinates. Then we consider the transformation rule

$$v(t,x) = \dot{X}(t^*,x^*) + Q(t^*)v^*(t^*,x^*) \quad \text{for } (t,x) = Y(t^*,x^*).$$

The short notation for this is

$$v \circ Y = \dot{X} + Qv^*. \tag{3.5}$$

This is the definition of a velocity.

The second example involves only orthogonal transformations. Let G be a k-tensor, that is a quantity

$$G = \left(G_{i_1,\dots,i_k}\right)_{i_1,\dots,i_k=1,2,\dots,n}.$$

Then G is called an "objective k-tensor", if it obeys the transformation rule

$$G_{i_1,\dots,i_k} \circ Y = \textstyle\sum_{j_1,\dots,j_k=1}^{n} \left(\prod_{l=1}^{n} Q_{i_l j_l}\right) G^*_{j_1,\dots,j_k}.$$

The following are special objective k-tensors: For $k = 0$ the function G is called an "objective scalar" satisfying $G \circ Y = G^*$, for $k = 1$ an "objective vector" satisfying $G \circ Y = QG^*$, that is $G_i \circ Y = \sum_{j=1}^{n} Q_{ij} G_j^*$, and for $k = 2$ an "objective tensor" satisfying $G \circ Y = QG^* Q^{\mathrm{T}}$, that is $G_{ik} \circ Y = \sum_{j,l=1}^{n} Q_{ij} Q_{kl} G_{jl}^*$.

Examples of objective scalars are mass, concentrations, phase fractions, pressure, surface tension, the divergence of a velocity, internal energy, and entropy. Examples of objective vectors are diffusive fluxes, velocity differences, internal forces, and heat flux. Examples of objective tensors are the stress tensor and the symmetric part of a velocity gradient.

Transformation rules for derivatives of a physical quantity are obtained by differentiating the transformation rule of this quantity. Note, that transformation rules never are subject to a physical description, that is the properties of the set $\mathcal{P}$. The only purpose of a transformation rule is the definition of a physical quantity.

4 Objectivity of constitutive relations

An important type of equations describing physical processes are constitutive relations. A particular choice of such a relation is equivalent to the selection of a class of special materials. A constitutive relation is a pointwise equation given by a constitutive function. Objectivity means, that constitutive functions have to be the same for all observers. Let us explain this for a quite general class of constitutive relations appearing in balance laws describing a class of physical processes.

Assume that the components of $(t, x) \mapsto G(t, x)$ and $(t, x) \mapsto U(t, x)$ are quantities like scalars, vectors, tensors, ore others. A "constitutive relation" between G and U is a function $\widehat{G}$, such that for all processes under consideration the identity

$$G(t, x) = \widehat{G}(U(t, x))$$

is satisfied for all (t, x) in the domain of definition.

The principle of objectivity states, that the "constitutive function" $\widehat{G}$ has to be the same for all observers. This is also called the "principle of material frame indifference". Therefore, if G^* and U^* are the same quantities for another observer with coordinates (t^*, x^*), then the equation

$$G^*(t^*, x^*) = \widehat{G}(U^*(t^*, x^*))$$

has to be satisfied in the transformed domain. Note, that here the same function $\widehat{G}$ is used. Knowing the transformation rules for G and U one obtains functional equations for the constitutive function $\widehat{G}$.

The pattern to exploit objectivity is as follows: As an example assume a general nonlinear transformation rule for (G, U) with the special property, that in the transformation of the U-variables the G-variables do not enter. That is, with certain functions Z_Y^1 and Z_Y^2

$$G(t, x) = Z_Y^1(t^*, x^*, G^*(t^*, x^*), U^*(t^*, x^*)), \quad U(t, x) = Z_Y^2(t^*, x^*, U^*(t^*, x^*))$$

for all $(t,x) = Y(t^*,x^*)$, where Y is the observer transformation. In short notation this reads

$$G\circ Y = Z_Y^1(\bullet, G^*, U^*), \quad U\circ Y = Z_Y^2(\bullet, U^*). \tag{4.1}$$

Such a rule covers nearly all examples, where usually Z_Y^1 and Z_Y^2 are (affine) linear functions in the last arguments. We have

$$G = \widehat{G}(U) \quad \text{and} \quad G^* = \widehat{G}(U^*).$$

Since the first identity is the same as $G\circ Y = \widehat{G}(U\circ Y)$ we see that

$$\widehat{G}(Z_Y^2(\bullet, U^*)) = \widehat{G}(U\circ Y) = G\circ Y = Z_Y^1(\bullet, G^*, U^*) = Z_Y^1(\bullet, \widehat{G}(U^*), U^*).$$

Therefore, for all occurring functions U^* in the considered class of physical processes and all observer transformations Y the identity

$$\widehat{G}(Z_Y^2(\bullet, U^*)) = Z_Y^1(\bullet, \widehat{G}(U^*), U^*) \tag{4.2}$$

has to be satisfied. This is a functional equations for the constitutive function $\widehat{G}$, and therefore is a restriction on this function. How to solve such functional equations in standard situations is well known, but in general can be a difficult algebraic problem.

5 Objectivity of differential equations

We consider a set of solutions of a system of distributional balance laws. On a certain time-space domain $D \subset \mathbb{R}\times\mathbb{R}^n$ the distributions are $E^j, Q_i^j, F^j \in \mathscr{D}'(D)$ for $j = 1,\dots,M$ satisfying

$$\partial_t E^j + \textstyle\sum_{i=1}^n \partial_{x_i} Q_i^j = F^j \quad \text{in } \mathscr{D}'(D) \text{ for } j = 1,\dots,M. \tag{5.1}$$

Objectivity of balance laws

First let us consider a special system with

$$E^j = e^j\mu_\Gamma, \quad Q_i^j = q_i^j\mu_\Gamma, \quad F^j = f^j\mu_\Gamma$$

for $j = 1,\dots,M$ in D with a $(d+1)$-dimensional surface $\Gamma \subset D \subset \mathbb{R}\times\mathbb{R}^n$ without boundary in D, that is $\partial\Gamma \subset \partial D$. (The classical case, in which $\Gamma = D$ is $(n+1)$-dimensional, is included as a special case.) Then the distributional equations in (5.1) read

$$\int_D \Big(\partial_t\zeta\bullet e + \sum_{i=1}^n \partial_{x_i}\zeta\bullet q_i + \zeta\bullet f\Big)\,\mathrm{d}\mu_\Gamma = 0 \quad \text{for } \zeta \in C_0^\infty(D;\mathbb{R}^M), \tag{5.2}$$

where $e = (e^j)_{j=1,\dots,M}$, $q_i = \big(q_i^j\big)_{j=1,\dots,M}$, and $f = (f^j)_{j=1,\dots,M}$.

We assume that the system is complete in the sense, that we are able to define the "physical type" of this system by a linear transformation rule for test functions of the form

$$\zeta\circ Y = Z^{-\mathrm{T}}\zeta^*, \tag{5.3}$$

that is $\zeta^* = Z^{\mathrm{T}}\zeta\circ Y$ with a given matrix Z. We mention, that this rule applies only to certain special cases of balance laws, but these cases are typical and important. The principle of objectivity requires, that the form of (5.2) is the same for all observers.

5.1 Lemma. If the physical type of system (5.2) is given by transformation rule (5.3) for test functions, then the system is objective, if

$$\begin{aligned} e\circ Y &= Ze^*, \\ q_i\circ Y &= \dot{X}_i Ze^* + \textstyle\sum_{j=1}^n Q_{ij} Z q_j^* \qquad \text{for } i = 1, \dots, n, \\ f\circ Y &= Z_{\prime 0} e^* + \textstyle\sum_{j=1}^n Z_{\prime j} q_j^* + Z f^*. \end{aligned} \tag{5.4}$$

Here $Z_{\prime 0} = Z_{\prime t^*}$ and $Z_{\prime j} = Z_{\prime x_j^*}$ are the partial derivatives of the matrix Z.

Proof. Setting $\partial_0 := \partial_t$, $q_0 := e$, and $\partial_i := \partial_{x_i}$ for $i = 1, \dots, n$, the system of balance laws (5.2) reads in distributional form

$$\int_D \Big(\sum_{k=0}^n \partial_k \zeta \bullet q_k + \zeta \bullet f \Big) \, \mathrm{d}\mu_\Gamma = 0 \qquad \text{for } \zeta \in C_0^\infty(D; \mathbb{R}^M). \tag{5.5}$$

We have to show that this form is the same for all observers. By (5.3) the transformation rule is $\zeta\circ Y = Z^{-\mathrm{T}} \zeta^*$, that is $\zeta^* = Z^{\mathrm{T}} \zeta\circ Y$, so that for $l = 0, \dots, n$

$$\partial_l \zeta^* = {Z_{\prime l}}^{\mathrm{T}} \zeta\circ Y + \textstyle\sum_{k=0}^n Y_{k\prime l} Z^{\mathrm{T}} (\partial_k \zeta)\circ Y. \tag{5.6}$$

With $D = Y(D^*)$, hence also $\Gamma = Y(\Gamma^*)$, we compute

$$\begin{aligned} &\int_{D^*} \Big(\sum_{l=0}^n \partial_l \zeta^* \bullet q_l^* + \zeta^* \bullet f^* \Big) \, \mathrm{d}\mu_{\Gamma^*} \\ &= \int_{D^*} \Big(\sum_{k=0}^n (\partial_k \zeta\circ Y) \bullet (\sum_{l=0}^n Y_{k\prime l} Z q_l^*) + (\zeta\circ Y) \bullet (Z f^* + \sum_{l=0}^n Z_{\prime l} q_l^*) \Big) \, \mathrm{d}\mu_{\Gamma^*} \\ &= \int_D \Big(\sum_{k=0}^n \partial_k \zeta \bullet (\sum_{l=0}^n Y_{k\prime l} Z q_l^*)\circ Y^{-1} + \zeta \bullet (Z f^* + \sum_{l=0}^n Z_{\prime l} q_l^*)\circ Y^{-1} \Big) \, \mathrm{d}\mu_\Gamma \ , \end{aligned}$$

since the Jacobian of the transformation of $\mathrm{d}\mu_{\Gamma^*}$ into $\mathrm{d}\mu_\Gamma$ equals 1. The last integral is of the same form as in (5.5), if the transformation rules

$$\begin{aligned} q_k\circ Y &= \textstyle\sum_{l=0}^n Y_{k\prime l} Z q_l^* \qquad \text{for } k = 0, \dots, n, \\ f\circ Y &= \textstyle\sum_{l=0}^n Z_{\prime l} q_l^* + Z f^* \end{aligned} \tag{5.7}$$

are satisfied. With $e = q_0$ this becomes (5.4). □

As examples see the mass-momentum system in section 8 and the mass-momentum-energy system in section 10.

We did not claim, that (5.4) is necessary for objectivity, since one can add terms in (5.7), for which the weak integral will be unchanged. However, in a special case this can be excluded: We mention, that (5.4) implies $e\circ Y = Ze^*$, and it has been shown in [2, Section 7], that under quite general assumptions on the class of processes this equation implies transformation rules (5.4) for all other quantities.

Objectivity of distributional laws

We adopt the above considerations to a general system of distributional equations as in (5.1). With vectorial distributions $E = (E^j)_{j=1,\dots,M}$, $Q_i = \left(Q_i^j\right)_{j=1,\dots,M}$, and $F = (F^j)_{j=1,\dots,M}$ and using the vector notation for test functions $\zeta = (\zeta^j)_{j=1,\dots,M} \in C_0^\infty(D;\mathrm{I\!R}^M)$ system (5.1) becomes

$$\langle\, \partial_t\zeta\,,\, E\,\rangle + \textstyle\sum_{i=1}^n \langle\, \partial_i\zeta\,,\, Q_i\,\rangle + \langle\, \zeta\,,\, F\,\rangle = 0 \quad \text{for all } \zeta \in C_0^\infty(D;\mathrm{I\!R}^M). \tag{5.8}$$

As above the "physical type" of this system is defined by transformation rule (5.3) for test functions, where ζ^* is the test function of the new observer, and the matrix Z usually is a differential operator in Y. Again objectivity means, that (5.8) has to have the same form for all observers.

5.2 Lemma. If the physical type of system (5.8) is given by transformation rule (5.3), then the system is objective, if

$$\begin{aligned} S_Y(E) &= ZE^*, \\ S_Y(Q_i) &= \dot{X}_i ZE^* + \textstyle\sum_{j=1}^n Q_{ij} ZQ_j^* \quad \text{for } i = 1,\dots,n, \\ S_Y(F) &= Z_{\prime 0}E^* + \textstyle\sum_{j=1}^n Z_{\prime j}Q_j^* + ZF^*\ . \end{aligned} \tag{5.9}$$

Remark: Here for a distribution $G \in \mathscr{D}'(D)$ the transformed distributions $S_Y(G) \in \mathscr{D}'(D^*)$, with $D = Y(D^*)$, is defined by $\langle\, \zeta^*\,,\, S_Y(G)\,\rangle := \langle\, \zeta^*\circ Y^{-1}\,,\, G\,\rangle$.

Proof. As above, we introduce $\partial_0 := \partial_t$ and $Q_0 := E$. Then, for an observer with *-values,

$$\begin{aligned} &\langle\, \partial_0\zeta^*\,,\, E^*\,\rangle + \textstyle\sum_{i=1}^n \langle\, \partial_i\zeta^*\,,\, Q_i^*\,\rangle + \langle\, \zeta^*\,,\, F^*\,\rangle \\ &= \textstyle\sum_{l=0}^n \langle\, \partial_l\zeta^*\,,\, Q_l^*\,\rangle + \langle\, \zeta^*\,,\, F^*\,\rangle \\ &= \textstyle\sum_{k=0}^n \langle\, (\partial_k\zeta)\circ Y\,,\, \sum_{l=0}^n Y_{k\prime l} ZQ_l^*\,\rangle + \langle\, \zeta\circ Y\,,\, ZF^* + \sum_{l=0}^n Z_{\prime l}Q_l^*\,\rangle \end{aligned}$$

using (5.6) for the test function. Now, for the original observer,

$$\begin{aligned} &\langle\, \partial_0\zeta\,,\, E\,\rangle + \textstyle\sum_{i=1}^n \langle\, \partial_i\zeta\,,\, Q_i\,\rangle + \langle\, \zeta\,,\, F\,\rangle \\ &= \textstyle\sum_{k=0}^n \langle\, \partial_k\zeta\,,\, Q_k\,\rangle + \langle\, \zeta\,,\, F\,\rangle \\ &= \textstyle\sum_{k=0}^n \langle\, (\partial_k\zeta)\circ Y\,,\, S_Y(Q_k)\,\rangle + \langle\, \zeta\circ Y\,,\, S_Y(F)\,\rangle\ . \end{aligned}$$

From this one sees that objectivity for system (5.1) is satisfied, if

$$\begin{aligned} S_Y(Q_k) &= \textstyle\sum_{l=0}^n Y_{k\prime l} ZQ_l^* \quad \text{for } k = 0,\dots,n, \\ S_Y(F) &= \textstyle\sum_{l=0}^n Z_{\prime l}Q_l^* + ZF^*. \end{aligned}$$

□

We mention that these identities are generalizations of the identities in 5.1, since for distributions, which are given on surfaces, we have an explicit form of the operator S_Y.

5.3 Surface measures. If for an evolving d-dimensional surface $\Gamma \subset D$ without boundary in D a distribution $G \in \mathscr{D}'(D)$ is given by

$$G = g\mu_\Gamma,$$

then for an observer transformation Y and $\Gamma = Y(\Gamma^*)$ the transformed distribution $S_Y(G)$ is given by

$$S_Y(G) = g\circ Y\,\mu_{\Gamma^*}.$$

Proof. The distributional equation $G = g\mu_\Gamma$ means that

$$\langle\, \zeta\,,\, G\,\rangle = \int_{\mathrm{I\!R}} \int_{\Gamma_t} \zeta(t,x) g(t,x)\,\mathrm{dH}_d(x)\,\mathrm{d}t.$$

Then the transformed distribution $S_Y(G)$ is given by

$$\begin{aligned}\langle\, \zeta^*\,,\, S_Y(G)\,\rangle &= \big\langle\, \zeta^*\circ Y^{-1}\,,\, G\,\big\rangle \\ &= \int_{\mathrm{I\!R}} \int_{\Gamma_t} \zeta^*\circ Y^{-1}(t,x) g(t,x)\,\mathrm{dH}_d(x)\,\mathrm{d}t \\ &= \int_{\mathrm{I\!R}} \int_{\Gamma^*_{t^*}} \zeta^*(t^*,x^*) g\circ Y(t^*,x^*)\,\mathrm{dH}_d(x^*)\,\mathrm{d}t^*.\end{aligned}$$

This is due to the fact (compare the proof of 5.1), that the Jacobian of the transformation equals 1. Thus $S_Y(G) = g\circ Y\,\mu_{\Gamma^*}$. □

Consequences

From the previous results we obtain the following.

5.4 Proposition. Consider system (5.1) for an evolving interface with

$$\begin{aligned} E^j &= \textstyle\sum_{m=1}^2 e^{mj}\mu_{\Omega^m} + e^{sj}\mu_\Gamma, \\ Q^j &= \textstyle\sum_{m=1}^2 q^{mj}\mu_{\Omega^m} + q^{sj}\mu_\Gamma, \\ F^j &= \textstyle\sum_{m=1}^2 f^{mj}\mu_{\Omega^m} + f^{sj}\mu_\Gamma, \end{aligned} \tag{5.10}$$

with Ω^m, $m = 1,2$, and Γ as in 2.7. Define the physical type of this system by transformation rule (5.3) for test functions. Then objectivity is satisfied, if the three sets of quantities (e^m, q^m, f^m) for $m = 1,2$ and (e^s, q^s, f^s) obey the transformation rule (5.4).

Proof. One possibility is, to apply 5.2. □

Another possibility is to apply 5.1 locally to the domains Ω^m for $m = 1,2$, and then to an equation on the surface Γ: We know from 2.8 that (5.1) with quantities as in (5.10) is equivalent to the following properties for $j = 1,\dots,M$

$$\begin{aligned} \partial_t e^{mj} + \mathrm{div} q^{mj} = f^{mj} \quad &\text{in } \Omega^m \text{ for } m = 1,2, \\ (q^{sj} - e^{sj}\mathbf{v}_\Gamma)(t,x) \in T_x(\Gamma_t) \quad &\text{for all } (t,x) \in \Gamma, \\ \partial_t^\Gamma e^{sj} - e^{sj}\ \kappa_\Gamma \bullet \mathbf{v}_\Gamma\ + \mathrm{div}^\Gamma(q^{sj} - e^{sj}\mathbf{v}_\Gamma) & \\ = f^{sj} + \textstyle\sum_{m=1}^2 (q^{mj} - e^{mj}\mathbf{v}_\Gamma)\bullet\nu_m \quad &\text{on } \Gamma. \end{aligned} \tag{5.11}$$

Moreover, by 2.4, the last two identities are equivalent to the distributional equations

$$\partial_t(e^{sj}\mu_\Gamma) + \operatorname{div}(q^{sj}\mu_\Gamma) = (f^{sj} + \textstyle\sum_{m=1}^{2}(q^{mj} - e^{mj}\mathbf{v}_\Gamma)\bullet\nu_m)\mu_\Gamma \tag{5.12}$$

for $j = 1, \ldots, M$. This system is of the form (5.1) with

$$E^j = e^{sj}\mu_\Gamma, \quad Q^j = q^{sj}\mu_\Gamma, \quad F^j = (f^{sj} + \textstyle\sum_{m=1}^{2}(q^{mj} - e^{mj}\mathbf{v}_\Gamma)\bullet\nu_m)\mu_\Gamma. \tag{5.13}$$

We mention the following:

5.5 Remark. Objectivity of (5.1) with representation (5.10) is equivalent to objectivity of (5.1) in Ω_m for $m = 1, 2$ and objectivity of (5.12) for $j = 1, \ldots, m$ (that is (5.1) with representation (5.13)).

Proof. Assume objectivity of (5.1) in the sense of 5.4. Consider the Rankine-Hugoniot term on the right hand side of (5.12). Let

$$c^k := (q^{mk} - e^{mk}\mathbf{v}_\Gamma)\bullet\nu \quad \text{for } k = 1, \ldots, M,$$

where ν is a unit normal on Γ. The assertion follows, if for $c = \big(c^k\big)_{k=1,\ldots,M}$ transformation rule $c{\circ}Y = Zc^*$ holds, where Z is the matrix of the system with representation (5.10). Now by (5.4)

$$e^m{\circ}Y = Ze^{m*}, \quad q^m{}_i{\circ}Y = \dot{X}_i Z e^{m*} + \textstyle\sum_{j=1}^{n} Q_{ij} Z q^{m*}_{\;j}$$

for $i = 1, \ldots, n$, or

$$e^{mk}{\circ}Y = \textstyle\sum_{l=1}^{M} Z_{kl}e^{ml*}, \quad q^{mk}_{\;i}{\circ}Y = \sum_{l=1}^{M} \dot{X}_i Z_{kl} e^{ml*} + \sum_{l=1}^{M}\sum_{j=1}^{n} Q_{ij} Z_{kl} q^{ml*}_{\;j}$$

for $k = 1, \ldots, M$ and $i = 1, \ldots, n$, or

$$e^{mk}{\circ}Y = \textstyle\sum_{l=1}^{M} Z_{kl}e^{ml*}, \quad q^{mk}{\circ}Y = \sum_{l=1}^{M} Z_{kl}e^{ml*}\dot{X} + \sum_{l=1}^{M} Z_{kl} Q q^{ml*}$$

for $k = 1, \ldots, M$. From this we infer using 5.6

$$\begin{aligned} c^k{\circ}Y &= q^{mk}{\circ}Y\bullet\nu{\circ}Y - e^m{\circ}Y\,\mathbf{v}_\Gamma{\circ}Y\bullet\nu{\circ}Y \\ &= \textstyle\sum_{l=1}^{M}\Big(Z_{kl}e^{ml*}\dot{X}\bullet Q\nu^* + Z_{kl}Qq^{ml*}\bullet Q\nu^* - Z_{kl}e^{ml*}(\dot{X}\bullet Q\nu^*\; Q\nu^* + Q\mathbf{v}_\Gamma{}^*)\bullet Q\nu^*\Big) \\ &= \textstyle\sum_{l=1}^{M}\big(Z_{kl}q^{ml*}\bullet\nu^* - Z_{kl}e^{ml*}\mathbf{v}_\Gamma{}^*\bullet\nu^*\big) = \sum_{l=1}^{M} Z_{kl}c^{l*}. \end{aligned}$$

This shows $c{\circ}Y = Zc^*$. □

5.6 Proposition. Let Γ be an evolving interface. Then

$$\mathbf{v}_\Gamma{\circ}Y = Q(\operatorname{Id} - P^*)\,Q^{\mathrm{T}}\dot{X} + Q\mathbf{v}_\Gamma{}^*, \quad \kappa_\Gamma{\circ}Y = Q\kappa_\Gamma^*,$$

where P^* at (t^*, x^*) denotes the orthogonal projection on the tangent space $T_{x^*}(\Gamma^*_{t^*})$. (This rules for $\mathbf{v}_\Gamma$ and κ_Γ also hold for lower dimensional evolving surfaces.) If $\Gamma := \partial\Omega$ and $\nu(t, x)$ is the outer unit normal of $\partial\Omega_t$ at x, then

$$\nu{\circ}Y = Q\nu^*, \quad \mathbf{v}_\Gamma{\circ}Y = \dot{X}\bullet Q\nu^*\; Q\nu^* + Q\mathbf{v}_\Gamma{}^*.$$

Proof. The normal $\nu(t, \bullet)$ and the curvature $\kappa_\Gamma(t, \bullet)$ are defined geometrically on Γ_t. With respect to the space variable observer transformations consist of translations and rotations. This implies that ν and κ_Γ are objective vectors.

As we shall see, $\mathbf{v}_\Gamma$ neither behaves like a velocity nor like an objective vector, its a mixture of both. Let us use the particle approach for a general moving surface. $\mathbf{v}_\Gamma$ is characterized as a normal vector satisfying for all local curves $t \mapsto \xi(t) \in \Gamma_t$ the identity

$$\mathbf{v}_\Gamma(t, \xi(t)) \bullet \nu = \dot{\xi}(t) \bullet \nu \tag{5.14}$$

for all normal vectors ν of Γ_t at $\xi(t)$. For an observer transformation we have

$$\xi(t) = X(t^*, \xi^*(t^*)), \quad t^* - t = \text{const}\,,$$

thus time derivative gives

$$\dot{\xi}(t) = \dot{X}(t^*, \xi^*(t^*)) + Q(t^*)\dot{\xi}^*(t^*),$$

hence

$$\mathbf{v}_\Gamma(t, \xi(t)) \bullet \nu = \dot{X}(t^*, \xi^*(t^*)) \bullet \nu + \dot{\xi}^*(t^*) \bullet (Q(t^*)^{\mathrm{T}} \nu).$$

Now $\nu^* := Q(t^*)^{\mathrm{T}} \nu$ is a normal vector of $\Gamma^*_{t^*}$ at $\xi^*(t^*)$. Using (5.14) we obtain in the new coordinates

$$\dot{\xi}^*(t^*) \bullet (Q(t^*)^{\mathrm{T}} \nu) = \dot{\xi}^*(t^*) \bullet \nu^* = {\mathbf{v}_\Gamma}^*(t^*, \xi^*(t^*)) \bullet \nu^*.$$

Thus for $(t, x) = Y(t^*, x^*) \in \Gamma$ and $\nu = Q(t^*)\nu^* \in T_x(\Gamma_t)^\perp$

$$\begin{aligned} \mathbf{v}_\Gamma(t, x) \bullet \nu &= \dot{X}(t^*, x^*) \bullet \nu + {\mathbf{v}_\Gamma}^*(t^*, x^*) \bullet \nu^* \\ &= \dot{X}(t^*, x^*) \bullet \nu + (Q(t^*) {\mathbf{v}_\Gamma}^*(t^*, x^*)) \bullet \nu, \end{aligned}$$

hence for $(t, x) = Y(t^*, x^*)$

$$\mathbf{v}_\Gamma(t, x) - (\dot{X}(t^*, x^*) + Q(t^*) {\mathbf{v}_\Gamma}^*(t^*, x^*)) \in T_x(\Gamma_t),$$

that is, the normal component, with respect to Γ, of $\mathbf{v}_\Gamma - (\dot{X} + Q{\mathbf{v}_\Gamma}^*) \circ Y^{-1}$ vanishes, or

$$(\mathrm{Id} - P \circ Y)(\mathbf{v}_\Gamma \circ Y - \dot{X} - Q{\mathbf{v}_\Gamma}^*) = 0,$$

where P at (t, x) denotes the orthogonal projection on $T_x(\Gamma_t)$. Since $\mathbf{v}_\Gamma$ is a normal vector (with respect to Γ) and since $Q(t^*)$ maps $T_{x^*}(\Gamma^*_{t^*})^\perp$ into $T_x(\Gamma_t)^\perp$, that is $\mathrm{Id} - P \circ Y = Q(\mathrm{Id} - P^*)\, Q^{\mathrm{T}}$, this gives

$$\mathbf{v}_\Gamma \circ Y = Q(\mathrm{Id} - P^*)\, Q^{\mathrm{T}} \dot{X} + Q{\mathbf{v}_\Gamma}^*. \tag{5.15}$$

In the special case of an interface we have $(\mathrm{Id} - P^*)(\xi) = \xi \bullet \nu^* \nu^*$ for $\xi \in \mathbb{R}^n$, thus

$$\mathbf{v}_\Gamma \circ Y = (Q^{\mathrm{T}} \dot{X}) \bullet \nu^* Q \nu^* + Q{\mathbf{v}_\Gamma}^* = \dot{X} \bullet (Q\nu^*) Q\nu^* + Q{\mathbf{v}_\Gamma}^*. \tag{5.16}$$

$\square$

6 The entropy principle

In this section we give a short description of the entropy principle. We consider a set of processes $\mathcal{P}$ consisting of solutions of a system in the time-space variable $(t, x) \in D \subset \mathbb{R} \times \mathbb{R}^n$. It is supposed to satisfy the following axiom.

Entropy principle

The "entropy principle" states, that for each process in $\mathcal{P}$ there is an entropy $H \in \mathscr{D}'(D)$ and an entropy flux $\Psi = (\Psi_1, \ldots, \Psi_n)$ with $\Psi_i \in \mathscr{D}'(D)$ satisfying the entropy inequality

$$\partial_t H + \sum_{i=1}^{n} \partial_{x_i} \Psi_i \geq 0 \quad \text{in } \mathscr{D}'(D). \tag{6.1}$$

Moreover, as a postulate, the entropy is an objective scalar.

We have formulated this axiom in the space of distributions, since this allows for physical processes including quantities on a curve, or on an surface between two media. The postulate, that the entropy is an objective scalar, is equivalent to the fact that the weak differential equation,

$$\int_D \left((\partial_t \zeta) H + \sum_{i=1}^{n} (\partial_{x_i} \zeta) \Psi_i \right) \leq 0 \quad \text{for } \zeta \in C_0^\infty(D) \text{ with } \zeta \geq 0,$$

transforms between observers with the rule

$$\zeta \circ Y = \zeta^*$$

for test functions (see (5.3) with $Z = 1$).

In addition, what is not formulated in the above definition, it is assumed that (H, Ψ) is "nontrivial" and has a "similar structure" as the processes in $\mathcal{P}$. The exact mathematical formulation of these properties is a difficult task and depends on the special situation.

As an example, one might consider a set of processes $\mathcal{P}$ consisting of solutions of a system of distributional balance laws, that is on open sets $D \subset \mathbb{R} \times \mathbb{R}^n$ the distributions are $E^j, Q_i^j, F^j \in \mathscr{D}'(D)$ for $j = 1, \ldots, M$ satisfying

$$\partial_t E^j + \sum_{i=1}^{n} \partial_{x_i} Q_i^j = F^j \quad \text{in } \mathscr{D}'(D) \text{ for } j = 1, \ldots, M. \tag{6.2}$$

Let us assume that these quantities have contributions on domains and on an interface as in (2.21). And, of course, there are constitutive relations between these quantities in the balance laws. The "principle of equipresence" formulated by Truesdell states, that all quantities should depend on the same variables. Then also the entropy H and its flux Ψ should have these contributions on domains and on an interface. Therefore, we take quantities with a structure, which is comparable to the structure of the quantities in (6.2). However, in concrete examples the dependence of the entropy usually is more restrictive than the dependence of the entropy flux. (In [3, Theorem 2] the entropy flux might contain a term involving a time derivative, although this is not true for the energy flux.)

Often physical systems are considered in the isothermal case. The isothermal situation can be obtained by a limit procedure, in which usually the heat flux is erased from the system. In this limit the temperature is constant, system (6.2) does not contain an equation for the total energy, and the entropy inequality reduces to an inequality for the "total free energy". Thus for a class of processes with a prescribed constant temperature the entropy inequality reduces to a

Free energy inequality
For each process in $\mathcal{P}$ there is a (total) free energy $F \in \mathscr{D}'(D)$ and a (total) free energy flux $\Phi = (\Phi_1, \ldots, \Phi_n)$ with $F, \Phi_i \in \mathscr{D}'(D)$ satisfying the free energy inequality

$$\partial_t F + \sum_{i=1}^{n} \partial_{x_i} \Phi_i \leq G_{\text{ext}} \quad \text{in } \mathscr{D}'(D) \tag{6.3}$$

for all processes, where $G_{\text{ext}} \in \mathscr{D}'(D)$ is a free energy production term.

In contrary to the entropy the free energy is not an objective scalar, the physical type depends on the processes in $\mathcal{P}$. Since the inequality has to be objective, the quantity

$$G - G_{\text{ext}} \leq 0 \quad \text{with} \quad G := \partial_t F + \sum_{i=1}^{n} \partial_{x_i} \Phi_i$$

has to be an objective scalar.

Thus, working with a free energy inequality, one has to determine the term G_{ext}. This requires certain knowledge about the corresponding entropy principle behind it, and about the transformation rule of the free energy production.

In the isothermal limit essentially one has to replace the energy flux, which usually is denoted by q. This procedure is quite difficult, in particular for problems with interfaces. In the standard case of one fluid one has $G_{\text{ext}} = v \bullet \mathbf{f}$ where v is the velocity of the fluid and $\mathbf{f}$ the external force.

7 Some examples

In this section we give some examples, which show, that the entropy principle, respectively the free energy principle, is valid in a broad class of physical situations. This includes

- an example from thermostatics,
- the Boltzmann equation,
- an example from molecular dynamics.

These examples stand somewhat aside our main applications in this paper, but it is worthwhile to mention them.

In 1909 Carathéodory [6] considered the equation

$$\dot{u}_0 = \sum_{i=1}^{N} \widehat{f}_i(u)\dot{u}_i, \tag{7.1}$$

proposing a property which, as is shown by him, is equivalent to the entropy principle in the thermostatic case (see also [14, Section 5.1.3] and [1]).

7.1 Processes. We denote by $\mathcal{P}_1$ the set of all vectors $u = (u_0, u_1, \ldots, u_N)$ whose components are functions of time and a local solutions of (7.1). If u_0, and u_i, f_i, $i = 1, \ldots, N$, are objective scalars, then $\mathcal{P}_1$ is objective.
Note: One can add to each observer a rigid velocity v and set $\dot{g} = \partial_t g + v \bullet \nabla g$ for every function g. If g is in addition an objective scalar, the fact that g is space independent, that is $\nabla g = 0$, is objective. Then $\dot{g} = \partial_t g$ is the time derivative.

Proof. It is $f_i = \widehat{f}_i(u)$ and since since f_i and the components of u are objectve scalars, it is obvious, that $\widehat{f}_i$ is frame indifferent. Now u depends only on time, that is

$$\nabla u_i = 0.$$

Let u_i^* be the values for another observer, since it is an objective scalar by the rule $u_i \circ Y = u_i^*$. We compute $(\mathrm{D}X)^{\mathrm{T}}(\nabla u_i) \circ Y = \nabla u_i^*$. Thus also $\nabla u_i^* = 0$. We also obtain from $u_i \circ Y = u_i^*$, that $(\dot{u}_i) \circ Y = \dot{u}_i^*$, so that the differential equation is objective. Therefore the set $\mathcal{P}_1$ is objective. □

In our framework the entropy principle reads as follows.

7.2 Lemma. Assume that the matrix

$$\Big(\widehat{f}_{i\,\prime u_0}(z)\widehat{f}_j(z) + \widehat{f}_{i\,\prime u_j}(z)\Big)_{i,j=1,\ldots,N} \tag{7.2}$$

is symmetric for z in a domain $U \subset \mathbb{R} \times \mathbb{R}^N$. Then there exist functions $z \mapsto \widehat{\eta}(z)$ satisfying

$$\widehat{\eta}_{\prime u_0}(z)\widehat{f}_i(z) + \widehat{\eta}_{\prime u_i}(z) = 0 \quad \text{for } i = 1, \ldots, N \tag{7.3}$$

and $\widehat{\eta}_{\prime u_0}(z) \neq 0$ for $z \in U$.

Proof. Consider the vector fields $w_i(z) := (\widehat{f}_i(z), \mathbf{e}_i) \in \mathbb{R}^{1+N}$, where $\{\mathbf{e}_i\,;\ i = 1, \ldots, N\}$ is the canonical basis in $\mathbb{R}^N$. The assumption (7.2) says that the Lie-bracket $[w_i, w_j] := \partial_{w_i} w_j - \partial_{w_j} w_i$ vanishes. Hence there exists a nontrivial function $\widehat{\eta}$ with $w_i(z) \bullet \nabla \widehat{\eta}(z) = 0$. A detailed proof can be found in [1, Theorem 5.7]. □

7.3 Entropy principle. Assume that the matrix in (7.2) is symmetric. Then there exists a nontrivial $\eta = \widehat{\eta}(u)$ with $\widehat{\eta}$ as in (7.3), so that

$$\dot{\eta} = 0,$$

that is, the entropy principle for $\mathcal{P}_1$ is satisfied.
Note: For the rigid velocity v introduced above we have $\dot{g} = \partial_t g + v \bullet \nabla g = \partial_t g + \mathrm{div}(gv)$ for every function g, which depends only on time. This is because $\nabla g = 0$ and $\mathrm{div} v = 0$.

Proof. It follows that

$$\begin{aligned}\dot\eta &= \eta'_{u_0}\dot u_0 + \textstyle\sum_{i=1}^N \eta'_{u_i}\dot u_i \\ &= \eta'_{u_0}\left(\dot u_0 - \textstyle\sum_{i=1}^N f_i\ \dot u_i\right) = 0\end{aligned}$$

inserting η'_{u_i} from (7.3). □

We mention, that for all functions u

$$\dot\eta = \eta'_{u_0}\left(\dot u_0 - \textstyle\sum_{i=1}^N f_i\ \dot u_i\right),$$

where the multiplier η'_{u_0}, in analogy to section 11, can be thought of the inverse absolute temperature, comprehensible with [6], since u_0 denotes the internal energy.

Our second example is Boltzmann's equation

$$\partial_t f + \textstyle\sum_{i=1}^n c_i \partial_{x_i} f + \sum_{i=1}^n \mathbf{f}_i \partial_{c_i} f = r \tag{7.4}$$

for a function $(t,x,c) \mapsto f(t,x,c) \in \mathbb{R}$, where c is the velocity variable. The variables (t,x,c) have the following transformation behaviour under an observer change:

$$\begin{bmatrix} t \\ x \\ c \end{bmatrix} = \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \\ \dot X(t^*,x^*) + Q(t^*)c^* \end{bmatrix}, \text{ where } \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Q(t^*)x^* + \mathrm{b}(t^*) \end{bmatrix}. \tag{7.5}$$

Hence the variable (t,x) transform like usual and c like a velocity. The quantity $f(t,x,c)$ is the density of atoms at (t,x) with velocity c. The rate $r(t,x,c)$, the collision production, in the classical case, is given by

$$\begin{aligned} r(t,x,c) = \int_{\mathbb{R}^n}\int_{\partial \mathrm{B}_1(0)} \Big(& f(t,x,c-q(c-c',\mathbf{k}))f(t,x,c'+q(c-c',\mathbf{k})) \\ & - f(t,x,c)f(t,x,c')\Big) \cdot w(c+c',\mathbf{k})\,\mathrm{dH}_{n-1}(\mathbf{k})\,\mathrm{d}c' \end{aligned} \tag{7.6}$$

with a weight function $w \geq 0$ and the velocity vector

$$q(c-c',\mathbf{k}) := \mathbf{k}\bullet(c-c')\ \mathbf{k}.$$

The basis for this is that a collision of two particles conserves momentum and energy. First let us clarify frame indifference.

7.4 Objectivity. Let $\mathcal{P}_2$ consist of solutions of (7.4), where f behaves like an objective scalar, that is

$$f(t,x,c) = f^*(t^*,x^*,c^*), \tag{7.7}$$

if arguments transform as in (7.5). Then $\mathcal{P}_2$ is objective, if the $\mathbf{f}$-term satisfies

$$\mathbf{f}(t,x,c) = \ddot X(t^*,x^*) + 2\dot Q(t^*)c^* + Q(t^*)\mathbf{f}^*(t^*,x^*,c^*) \tag{7.8}$$

that is, transforms like a force. A constraint, which is objective for $\mathcal{P}_2$, is

$$\mathrm{div}_c \mathbf{f} = 0. \tag{7.9}$$

Proof. The transformation of the variables (t, x, c) is given in (7.5). Then (7.7) implies that (we omit arguments)

$$\begin{aligned}
\partial_{c^*} f^* &= Q^{\mathrm{T}} \partial_c f, \\
\partial_{x^*} f^* &= Q^{\mathrm{T}} \partial_x f + \left(\mathrm{D}_{x^*} \dot{X}\right)^{\mathrm{T}} \partial_c f = Q^{\mathrm{T}} \partial_x f + \dot{Q}^{\mathrm{T}} \partial_c f, \\
\partial_{t^*} f^* &= \partial_t f + \dot{X} \bullet \partial_x f + (\ddot{X} + \dot{Q} c^*) \bullet \partial_c f.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\partial_{t^*} f^* + c^* \bullet \partial_{x^*} f^* + \mathbf{f}^* \bullet \partial_{c^*} f^* \\
&= \partial_t f + (\dot{X} + Q c^*) \bullet \partial_x f + (\ddot{X} + 2\dot{Q} c^* + Q \mathbf{f}^*) \bullet \partial_c f.
\end{aligned}$$

Since $c = \dot{X} + Qc^*$ the result follows, if $\mathbf{f} = \ddot{X} + 2\dot{Q}c^* + Q\mathbf{f}^*$, that is (7.8) is assumed.

To prove (7.9) we compute the derivative of (7.8) with respect to c^*, that is $\mathbf{f}_{\prime c} Q = 2\dot{Q} + Q\mathbf{f}^*_{\prime c^*}$ or $\mathbf{f}_{\prime c} = 2\dot{Q} Q^{\mathrm{T}} + Q\mathbf{f}^*_{\prime c^*} Q^{\mathrm{T}}$. From this it follows that $\operatorname{trace} \mathbf{f}_{\prime c} = \operatorname{trace} \mathbf{f}^*_{\prime c^*}$.

To prove objectivity of (7.6), we note that $c - c'$ as the difference of two velocities is an objective vector. Also the collision vector $\mathbf{k}$ is assumed to be an objective vector. □

Multiplying equation (7.4) with a function and integrating over the velocity space $c \in \mathbb{R}^n$ leads to the following well known lemma, provided f or ζ has a certain decay at $|c| = \infty$.

7.5 Lemma. Assume $\operatorname{div}_c \mathbf{f} = 0$. Then for all functions $(t, x, c) \mapsto \zeta(t, x, c)$

$$\begin{aligned}
&\partial_t \left(\int \zeta f \,\mathrm{d}c \right) + \sum_i \partial_{x_i} \left(\int c_i \zeta f \,\mathrm{d}c \right) \\
&= \int \zeta r \,\mathrm{d}c + \int \left(\partial_t \zeta + \sum_i c_i \partial_{x_i} \zeta + \sum_i \mathbf{f}_i \partial_{c_i} \zeta \right) f \,\mathrm{d}c,
\end{aligned}$$

provided all c-integrals exist.

Proof. The only nontrivial term is

$$\int \zeta \sum_{i=1}^n \mathbf{f}_i \partial_{c_i} f \,\mathrm{d}c = -\int \sum_{i=1}^n \partial_{c_i}(\zeta \mathbf{f}_i) f \,\mathrm{d}c = -\sum_i \int \mathbf{f}_i \partial_{c_i} \zeta \cdot f \,\mathrm{d}c$$

since $\operatorname{div}_c \mathbf{f} = 0$. □

For the collision term we prove the following

7.6 Theorem. If the collision term r satisfies (7.6), then it follows

$$
\begin{aligned}
&-\int \zeta(t,x,c)\, r(t,x,c)\, \mathrm{d}c \\
&= \int_{\mathrm{IR}^n}\int_{\mathrm{IR}^n}\int_{\partial \mathrm{B}_1(0)} \frac{w(c+c',\mathbf{k})}{4}\cdot\Big(\zeta(t,x,c-q(c-c',\mathbf{k})) \\
&\qquad +\zeta(t,x,c'+q(c-c',\mathbf{k})) - \zeta(t,x,c) - \zeta(t,x,c')\Big) \\
&\qquad \cdot\Big(f(t,x,c-q(c-c',\mathbf{k}))f(t,x,c'+q(c-c',\mathbf{k})) \\
&\qquad -f(t,x,c)f(t,x,c')\Big)\, \mathrm{dH}_{n-1}(\mathbf{k})\, \mathrm{d}c'\, \mathrm{d}c\ .
\end{aligned}
$$

Proof. This follows from the symmetry properties of the integral in (7.6). □

With $\zeta = -\ln(bef)$ one obtains the

7.7 *H*-Theorem. Assume $\mathrm{div}_c \mathbf{f} = 0$, that is (7.9). If $a, b \in \mathrm{IR}$ and $a > 0$ then

$$
\begin{aligned}
\eta(t,x) &:= -\int a \ln(bf(t,x,c))\, f(t,x,c)\, \mathrm{d}c, \\
\psi_i(t,x) &:= -\int a\, c_i \ln(bf(t,x,c))\, f(t,x,c)\, \mathrm{d}c
\end{aligned}
$$

satisfies

$$
\partial_t \eta(t,x) + \mathrm{div}\psi(t,x) = -\int a \ln(bf(t,x,c))\, r(t,x,c)\, \mathrm{d}c \geq 0\ ,
$$

provided r is given as in (7.6).

Proof. Since the test function is a function of f only, we compute $\frac{\mathrm{d}}{\mathrm{d}f}(\ln(bf)\cdot f) = \ln(bf) + 1 = \ln(bef) = -\zeta$ and therefore

$$
\begin{aligned}
&\partial_t\Big(-\int \log(bf)\, f\, \mathrm{d}c\Big) + \sum_i \partial_{x_i}\Big(-\int c_i \log(bf)\, f\, \mathrm{d}c\Big) \\
&= -\int \partial_t(\ln(bf)\, f)\, \mathrm{d}c - \sum_i \int c_i \partial_{x_i}(\ln(bf)\, f)\, \mathrm{d}c = \int \zeta(\partial_t f + \sum_i c_i \partial_{x_i} f)\, \mathrm{d}c \\
&= \int (\zeta r - \sum_i \zeta \mathbf{f}_i \partial_{c_i} f)\, \mathrm{d}c = \int \zeta r\, \mathrm{d}c + \sum_i \int \partial_{c_i}(\mathbf{f}_i \log(bf)\, f)\, \mathrm{d}c = \int \zeta r\, \mathrm{d}c.
\end{aligned}
$$

That this is nonnegative, is a consequence of 7.6, in fact,

$$
\begin{aligned}
&-\int \ln(bf(t,x,c))\, r(t,x,c)\, \mathrm{d}c \\
&= \int_{\mathrm{IR}^n}\int_{\mathrm{IR}^n}\int_{\partial \mathrm{B}_1(0)} \frac{1}{4}\ln\Big(\frac{f(t,x,c-q(c-c',\mathbf{k}))f(t,x,c'+q(c-c',\mathbf{k}))}{f(t,x,c)f(t,x,c')}\Big) \\
&\qquad \cdot\Big(f(t,x,c-q(c-c',\mathbf{k}))f(t,x,c'+q(c-c',\mathbf{k})) \\
&\qquad -f(t,x,c)f(t,x,c')\Big)\cdot w(c+c',\mathbf{k})\, \mathrm{dH}_{n-1}(\mathbf{k})\, \mathrm{d}c'\, \mathrm{d}c
\end{aligned}
$$

is nonnegative. □

Hence in this case the H-Theorem plays the role of the entropy principle, and it is do to the special form of the collision term, namely the "Stosszahl Ansatz". Finally, we note that the Boltzmann equation satifies the following

7.8 Remark. If $\mathrm{div}_c \mathbf{f} = 0$ as in 7.7 and 7.5 then Boltzmann's equation read

$$\partial_t f + \mathrm{div}_x(fc) + \mathrm{div}_c(f\mathbf{f}) = r,$$

which is of divergence type.

As third example we consider a particle system

$$m_\alpha \ddot{x}_\alpha(t) = \mathbf{f}_\alpha(t) - F_\alpha(t) \quad \text{for } \alpha \text{ in a locally finite set.} \tag{7.10}$$

Here $m_\alpha > 0$ as mass of the particle is a positive number, $\mathbf{f}_\alpha$ is an "external force", and F_α has the structure

$$F_\alpha(t) = \textstyle\sum_\beta F_{\alpha,\beta}(t) \quad \text{with} \quad F_{\beta,\alpha}(t) = -F_{\alpha,\beta}(t)\ , \tag{7.11}$$

where later in 7.11 we will assume that for $\beta \neq \alpha$

$$F_{\alpha,\beta}(t) = \nabla V_{\alpha,\beta}(x_\alpha(t) - x_\beta(t)) \quad \text{with } V_{\beta,\alpha}(z) = V_{\alpha,\beta}(-z) \tag{7.12}$$

and $F_{\alpha,\alpha} = 0$, where $V_{\alpha,\beta}$ are given "potentials". This includes the Lenard-Jones potential. It is assumed that the points x_α dont meet initially. Then, since the potential $z \mapsto V_{\alpha,\beta}(z)$ goes to $+\infty$ at $z \to 0$, the points $x_\alpha(t)$ cannot hit each other.

7.9 Objectivity. Let $\mathcal{P}_3$ consist of local solutions of (7.10), where x_α behaves under a change of observers by

$$x_\alpha(t) = X(t^*, x^*_\alpha(t^*)) \quad \text{for } t = T(t^*). \tag{7.13}$$

Moreover, let $m_\alpha = m^*_\alpha$ and

$$F_\alpha(t) = Q(t^*)F^*_\alpha(t^*) \quad \text{if } t = T(t^*). \tag{7.14}$$

Then $\mathcal{P}_3$ is objetive, if $\mathbf{f}_\alpha$ transforms as

$$\mathbf{f}_\alpha(t) = m^*_\alpha \dot{X}(t^*, x^*_\alpha) + 2m^*_\alpha \dot{Q}(t^*)\dot{x}^*_\alpha(t^*) + Q(t^*)\mathbf{f}^*_\alpha(t^*) \quad \text{if } t = T(t^*).$$

We want to write this as distributional balance laws. For a curve $t \to x(t)$ let δ_x be the time-space distribution, which for every time t is the Dirac measure in $x(t)$, and if $t \to y(t)$ is another curve, let $\delta_{x,y}$ be the time-space distributions, which for every time t is the 1-dimensional probability measure on the straight line between $x(t)$ and $y(t)$, therefore $\delta_{x,y} = \delta_{y,x}$. The definitions are

$$\begin{aligned}\langle\, \zeta\,,\, \delta_x \,\rangle \quad &:= \int_{\mathbb{R}} \zeta(t, x(t))\,\mathrm{d}t, \\ \langle\, \zeta\,,\, \delta_{x,y} \,\rangle \quad &:= \int_{\mathbb{R}} \int_0^1 \zeta(t, (1-s)x(t) + sy(t))\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

7.10 Conservation laws. Solutions of (7.10) satisfy the mass and momentum equation

$$\partial_t \left(\sum_\alpha m_\alpha \delta_{x_\alpha} \right) + \operatorname{div} \left(\sum_\alpha m_\alpha \dot{x}_\alpha \delta_{x_\alpha} \right) = 0$$

$$\partial_t \left(\sum_\alpha m_\alpha \dot{x}_\alpha \delta_{x_\alpha} \right) + \operatorname{div} \left(\sum_\alpha m_\alpha \dot{x}_\alpha \otimes \dot{x}_\alpha \delta_{x_\alpha} - \frac{1}{2} \sum_{\alpha,\beta} F_{\alpha,\beta} \otimes (x_\alpha - x_\beta) \delta_{x_\alpha, x_\beta} \right) = \sum_\alpha \mathbf{f}_\alpha \delta_{x_\alpha}$$

Proof. We compute for test functions $\zeta \in C_0^\infty(\mathrm{I\!R} \times \mathrm{I\!R}^n; \mathrm{I\!R})$ and all α

$$\begin{aligned}
&\langle \zeta \,,\, \partial_t \left(m_\alpha \delta_{x_\alpha} \right) \rangle + \langle \zeta \,,\, \operatorname{div} \left(m_\alpha \dot{x}_\alpha \delta_{x_\alpha} \right) \rangle \\
&= -m_\alpha \int_{\mathrm{I\!R}} \left((\partial_t \zeta)(t, x_\alpha(t)) + (\nabla \zeta)(t, x_\alpha(t)) \bullet \dot{x}_\alpha(t) \right) \mathrm{d}t \\
&= -m_\alpha \int_{\mathrm{I\!R}} \frac{\mathrm{d}}{\mathrm{d}t} (\zeta(t, x_\alpha(t))) \, \mathrm{d}t = 0 \; .
\end{aligned}$$

This implies the conservation of total mass, and is also the conservation of mass for each particle. Then for all test functions $\zeta \in C_0^\infty(\mathrm{I\!R} \times \mathrm{I\!R}^n; \mathrm{I\!R}^n)$

$$\begin{aligned}
&\langle \zeta \,,\, \partial_t \left(m_\alpha \dot{x}_\alpha \delta_{x_\alpha} \right) \rangle = - \int_{\mathrm{I\!R}} (\partial_t \zeta)(t, x_\alpha(t)) \bullet (m_\alpha \dot{x}_\alpha(t)) \, \mathrm{d}t \\
&= + \sum_i \int_{\mathrm{I\!R}} \dot{x}_{\alpha,i}(t) (\partial_i \zeta)(t, x_\alpha(t)) \bullet (m_\alpha \dot{x}_\alpha(t)) \, \mathrm{d}t + \int_{\mathrm{I\!R}} \zeta(t, x_\alpha(t)) \bullet \frac{\mathrm{d}}{\mathrm{d}t} (m_\alpha \dot{x}_\alpha(t)) \, \mathrm{d}t \; .
\end{aligned}$$

This gives

$$\begin{aligned}
&\langle \zeta \,,\, \partial_t \left(m_\alpha \dot{x}_\alpha \delta_{x_\alpha} \right) + \operatorname{div} (m_\alpha \dot{x}_\alpha \otimes \dot{x}_\alpha \delta_{x_\alpha}) \rangle = \int_{\mathrm{I\!R}} \zeta(t, x_\alpha(t)) \bullet \frac{\mathrm{d}}{\mathrm{d}t} (m_\alpha \dot{x}_\alpha(t)) \, \mathrm{d}t \\
&= \int_{\mathrm{I\!R}} \zeta(t, x_\alpha(t)) \bullet (\mathbf{f}_\alpha(t) - F_\alpha(t)) \, \mathrm{d}t = \langle \zeta \,,\, \mathbf{f}_\alpha \delta_{x_\alpha} \rangle - \langle \zeta \,,\, F_\alpha \delta_{x_\alpha} \rangle \; .
\end{aligned}$$

Thus we have to show that

$$\sum_\alpha \langle \zeta \,,\, F_\alpha \delta_{x_\alpha} \rangle = \frac{1}{2} \sum_{\alpha,\beta} \left\langle \mathrm{D}\zeta \,,\, F_{\alpha,\beta} \otimes (x_\alpha - x_\beta) \delta_{x_\alpha, x_\beta} \right\rangle \; .$$

To prove this, we use assumption (7.11) and obtain

$$\begin{aligned}
&\sum_\alpha \langle \zeta \,,\, F_\alpha \delta_{x_\alpha} \rangle = \sum_{\alpha,\beta} \langle \zeta \,,\, F_{\alpha,\beta} \delta_{x_\alpha} \rangle = \sum_{\alpha,\beta} \int_{\mathrm{I\!R}} \zeta(t, x_\alpha(t)) \bullet F_{\alpha,\beta}(t) \, \mathrm{d}t \\
&= \sum_{\alpha,\beta} \int_{\mathrm{I\!R}} \frac{1}{2} (\zeta(t, x_\alpha(t)) - \zeta(t, x_\beta(t))) \bullet F_{\alpha,\beta}(t) \, \mathrm{d}t \\
&= \frac{1}{2} \sum_{\alpha,\beta} \int_{\mathrm{I\!R}} \int_0^1 \mathrm{D}\zeta(t, (1-s) x_\beta(t) + s x_\alpha(t)) \bullet \left(F_{\alpha,\beta}(t) \otimes (x_\alpha(t) - x_\beta(t)) \right) \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2} \sum_{\alpha,\beta} \left\langle \mathrm{D}\zeta \,,\, F_{\alpha,\beta} \otimes (x_\alpha - x_\beta) \delta_{x_\alpha, x_\beta} \right\rangle \; .
\end{aligned}$$

□

The equation for mass and momentum are equivalent to the ODE in (7.10). There is another identity following from (7.10), it is the identity for the free energy.

7.11 Energy identity. Define the free energy f and its flux φ by

$$
\begin{aligned}
f &:= \sum_\alpha f_\alpha \delta_{x_\alpha} \quad \text{with} \quad f_\alpha := \frac{m_\alpha}{2}|\dot{x}_\alpha|^2 + \sum_{\beta:\beta\neq\alpha} \frac{1}{2} V_{\alpha,\beta}(x_\alpha - x_\beta) \ , \\
\varphi &:= \sum_\alpha f_\alpha \dot{x}_\alpha \delta_{x_\alpha} - \sum_{\alpha,\beta} \frac{1}{4} \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) \bullet (\dot{x}_\alpha + \dot{x}_\beta)(x_\alpha - x_\beta) \delta_{x_\alpha,x_\beta} \ .
\end{aligned}
$$

Then in the sense of distributions

$$
\partial_t f + \operatorname{div} \varphi = \sum_\alpha \dot{x}_\alpha \bullet \mathbf{f}_\alpha \delta_{x_\alpha}
$$

Proof. For the kinetic part of the free energy for any α

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_\alpha}{2}|\dot{x}_\alpha|^2\right) = m_\alpha \dot{x}_\alpha \bullet \ddot{x}_\alpha = \dot{x}_\alpha \bullet \mathbf{f}_\alpha - \dot{x}_\alpha \bullet F_\alpha \ ,
$$

and for the internal part using (7.11)

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_\beta \frac{1}{2} V_{\alpha,\beta}(x_\alpha - x_\beta)\right) = \sum_\beta \frac{1}{2} \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) \bullet (\dot{x}_\alpha - \dot{x}_\beta) \\
&= \dot{x}_\alpha \bullet \sum_\beta \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) - \sum_\beta \frac{1}{2} \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) \bullet (\dot{x}_\alpha + \dot{x}_\beta) \\
&= \dot{x}_\alpha \bullet F_\alpha - \sum_\beta \frac{1}{2} \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) \bullet (\dot{x}_\alpha + \dot{x}_\beta) \ ,
\end{aligned}
$$

which gives

$$
\frac{\mathrm{d}}{\mathrm{d}t} f_\alpha = \dot{x}_\alpha \bullet \mathbf{f}_\alpha - \sum_\beta \frac{1}{2} \nabla V_{\alpha,\beta}(x_\alpha - x_\beta) \bullet (\dot{x}_\alpha + \dot{x}_\beta) \ . \tag{7.15}
$$

We use this to compute for $\zeta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$

$$\begin{aligned}
&\langle \zeta \,,\, \partial_t(f_\alpha \delta_{x_\alpha}) + \operatorname{div}(f_\alpha \dot{x}_\alpha \delta_{x_\alpha}) \rangle \\
&= -\int_{\mathbb{R}} (\partial_t \zeta(t, x_\alpha(t)) + \nabla\zeta(t, x_\alpha(t)) \bullet \dot{x}_\alpha(t))\, f_\alpha(t)\, \mathrm{d}t \\
&= -\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t} (\zeta(t, x_\alpha(t)))\, f_\alpha(t)\, \mathrm{d}t = \int_{\mathbb{R}} \zeta(t, x_\alpha(t)) \frac{\mathrm{d}}{\mathrm{d}t} f_\alpha(t)\, \mathrm{d}t \\
&= \int_{\mathbb{R}} \zeta(t, x_\alpha(t)) \left(\dot{x}_\alpha(t) \bullet \mathbf{f}_\alpha(t) - \sum_\beta \frac{1}{2} \nabla V_{\alpha,\beta}(x_\alpha(t) - x_\beta(t)) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t)) \right) \mathrm{d}t \\
&= \langle \zeta \,,\, \dot{x}_\alpha \bullet \mathbf{f}_\alpha \delta_{x_\alpha} \rangle - \sum_\beta \frac{1}{2} \int_{\mathbb{R}} \zeta(t, x_\alpha(t)) \nabla V_{\alpha,\beta}(x_\alpha(t) - x_\beta(t)) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}t \\
&= \langle \zeta \,,\, \dot{x}_\alpha \bullet \mathbf{f}_\alpha \delta_{x_\alpha} \rangle - \sum_\beta \frac{1}{2} \int_{\mathbb{R}} \zeta(t, x_\alpha(t)) F_{\alpha,\beta}(t) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}t \;,
\end{aligned} \tag{7.16}$$

since $F_{\alpha,\beta} = \nabla V_{\alpha,\beta}(x_\alpha - x_\beta)$. Now we sum over α, take (7.12) in consideration, and obtain

$$\begin{aligned}
&\sum_\alpha \sum_\beta \frac{1}{2} \int_{\mathbb{R}} \zeta(t, x_\alpha(t)) F_{\alpha,\beta}(t) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}t \\
&= \sum_{\alpha,\beta} \frac{1}{4} \int_{\mathbb{R}} (\zeta(t, x_\alpha(t)) F_{\alpha,\beta}(t) + \zeta(t, x_\beta(t)) F_{\beta,\alpha}(t)) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}t \\
&= \sum_{\alpha,\beta} \frac{1}{4} \int_{\mathbb{R}} (\zeta(t, x_\alpha(t)) - \zeta(t, x_\beta(t)))\, F_{\alpha,\beta}(t) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}t \\
&= \sum_{\alpha,\beta} \frac{1}{4} \int_{\mathbb{R}} \int_0^1 \nabla\ \zeta(t, (1-s) x_\beta(t) + s x_\alpha(t)) \bullet (x_\alpha(t) - x_\beta(t)) \\
&\qquad\qquad \cdot F_{\alpha,\beta}(t) \bullet (\dot{x}_\alpha(t) + \dot{x}_\beta(t))\, \mathrm{d}s\, \mathrm{d}t \\
&= \left\langle \nabla\zeta \,,\, \sum_{\alpha,\beta} \frac{1}{4} F_{\alpha,\beta} \bullet (\dot{x}_\alpha + \dot{x}_\beta)(x_\alpha - x_\beta) \delta_{x_\alpha, x_\beta} \right\rangle
\end{aligned} \tag{7.17}$$

□

8 Mass-momentum system

As example for objectivity of balance laws we consider a "mass-momentum" vector $(\varrho, \mathbf{m})$ with $\mathbf{m} = (\mathbf{m}_k)_{k=1,\dots,n}$ on a moving interface Γ (see definition 2.1). The general system of balance laws for these quantities is of the form

$$\partial_t(\varrho \mu_\Gamma) + \sum_{i=1}^n \partial_{x_i}(\widetilde{J}_i \mu_\Gamma) = \tau \mu_\Gamma,$$

$$\partial_t(\mathbf{m}_k \mu_\Gamma) + \sum_{i=1}^n \partial_{x_i}(\widetilde{\Pi}_{ki} \mu_\Gamma) = \mathbf{f}_k \mu_\Gamma \quad \text{for } k = 1, \dots, n$$

in the space of distributions. Here

$$\widetilde{J} = \left(\widetilde{J}_i\right)_{i=1,\dots,n} \qquad \text{is the “mass flux” on } \Gamma,$$

τ the “mass production” on Γ,

$$\widetilde{\Pi} = \left(\widetilde{\Pi}_{ki}\right)_{k,i=1,\dots,n} \qquad \text{the “momentum flux” on } \Gamma,$$

$$\mathbf{f} = (\mathbf{f}_k)_{k=1,\dots,n} \qquad \text{the “momentum production” on } \Gamma.$$

If Γ is an open set then we have the usual situation of a fluid or a solid.

8.1 Differential system. The vector notation of the system is

$$\begin{aligned} \partial_t(\varrho\mu_\Gamma) + \operatorname{div}(\widetilde{J}\mu_\Gamma) &= \tau\mu_\Gamma, \\ \partial_t(\mathbf{m}\mu_\Gamma) + \operatorname{div}(\widetilde{\Pi}\mu_\Gamma) &= \mathbf{f}\mu_\Gamma. \end{aligned} \tag{8.1}$$

Note: We use the convention, that the divergence of a tensor acts on its last index.

This is also of the form as in (5.1) by writing it as

$$\partial_t\left(\begin{bmatrix}\varrho \\ \mathbf{m}\end{bmatrix}\mu_\Gamma\right) + \sum_{i=1}^n \partial_{x_i}\left(\begin{bmatrix}\widetilde{J}_i \\ \widetilde{\Pi}_i\end{bmatrix}\mu_\Gamma\right) = \begin{bmatrix}\tau \\ \mathbf{f}\end{bmatrix}\mu_\Gamma,$$

where $\widetilde{\Pi}_i := \left(\widetilde{\Pi}_{ki}\right)_{k=1,\dots,n}$. The type of this “mass-momentum system” is defined by a transformation rule (5.3) for test functions, which is of the form

$$\zeta^* = Z^{\mathrm{T}}\zeta\circ Y \quad \left(\text{that is } \zeta\circ Y = Z^{-\mathrm{T}}\zeta^*\right)$$

with matrix

$$Z := \mathrm{D}Y = \begin{bmatrix}1 & 0 \\ \dot{X} & Q\end{bmatrix} \quad \text{satisfying} \quad Z_{\prime 0} = \begin{bmatrix}0 & 0 \\ \ddot{X} & \dot{Q}\end{bmatrix},\ Z_{\prime j} = \begin{bmatrix}0 & 0 \\ \dot{X}_{\prime j} & 0\end{bmatrix}. \tag{8.2}$$

It follows from (5.4) that the mass-momentum system is objective, if

$$\begin{aligned} &\begin{bmatrix}\varrho \\ \mathbf{m}\end{bmatrix}\circ Y = \begin{bmatrix}1 & 0 \\ \dot{X} & Q\end{bmatrix}\begin{bmatrix}\varrho^* \\ \mathbf{m}^*\end{bmatrix}, \\ &\begin{bmatrix}\widetilde{J}_i \\ \widetilde{\Pi}_i\end{bmatrix}\circ Y = \begin{bmatrix}1 & 0 \\ \dot{X} & Q\end{bmatrix}\left(\dot{X}_i\begin{bmatrix}\varrho^* \\ \mathbf{m}^*\end{bmatrix} + \sum_{j=1}^n Q_{ij}\begin{bmatrix}\widetilde{J}^*_j \\ \widetilde{\Pi}^*_j\end{bmatrix}\right) \\ &\begin{bmatrix}\tau \\ \mathbf{f}\end{bmatrix}\circ Y = \begin{bmatrix}0 & 0 \\ \ddot{X} & \dot{Q}\end{bmatrix}\begin{bmatrix}\varrho^* \\ \mathbf{m}^*\end{bmatrix} + \sum_{j=1}^n \begin{bmatrix}0 & 0 \\ \dot{X}_{\prime j} & 0\end{bmatrix}\begin{bmatrix}\widetilde{J}^*_j \\ \widetilde{\Pi}^*_j\end{bmatrix} + \begin{bmatrix}1 & 0 \\ \dot{X} & Q\end{bmatrix}\begin{bmatrix}\tau^* \\ \mathbf{f}^*\end{bmatrix}. \end{aligned}$$

The first line of these rules are

$$\varrho\circ Y = \varrho^*, \quad \widetilde{J}\circ Y = \varrho^*\dot{X} + Q\widetilde{J}^*, \quad \tau\circ Y = \tau^*. \tag{8.3}$$

(We remark, that these properties one also derives for the mass equation in (8.1) applying the transformation rule (5.3) with $Z = 1$.) The second line of the rules are

$$\begin{aligned} \mathbf{m}\circ Y &= \varrho^*\dot{X} + Q\mathbf{m}^*, \\ \widetilde{\Pi}\circ Y &= \varrho^*\dot{X}\otimes\dot{X} + (Q\mathbf{m}^*)\otimes\dot{X} + \dot{X}\otimes(Q\widetilde{J}^*) + Q\widetilde{\Pi}^*Q^{\mathrm{T}}, \\ \mathbf{f}\circ Y &= \varrho^*\ddot{X} + \dot{Q}(\mathbf{m}^* + \widetilde{J}^*) + \tau^*\dot{X} + Q\mathbf{f}^*. \end{aligned} \tag{8.4}$$

In particular, ϱ and τ are objective scalars, and $\mathbf{m}$ and $\widetilde{J}$ have the same transformation rule. Altogether, we have derived the following lemma.

8.2 Lemma. The mass-momentum equation (8.1) is objective, if ϱ, $\widetilde{J}$, τ satisfy (8.3) and $\mathbf{m}$, $\widetilde{\Pi}$, $\mathbf{f}$ satisfy (8.4).

By 2.4 the mass-momentum equation is equivalent to differential equations with algebraic side conditions. These are

$$\begin{aligned} &(\widetilde{J} - \varrho\mathbf{v}_\Gamma)\bullet\mathbf{n} = 0 \quad \text{for } \mathbf{n}(t,x) \in T_x(\Gamma_t)^\perp, \\ &\partial_t^\Gamma\varrho + \mathrm{div}^\Gamma\widetilde{J} = \tau \quad \text{on } \Gamma, \\ &(\widetilde{\Pi} - \mathbf{m}\otimes\mathbf{v}_\Gamma)\mathbf{n} = 0 \quad \text{for } \mathbf{n}(t,x) \in T_x(\Gamma_t)^\perp, \\ &\partial_t^\Gamma\mathbf{m} + \mathrm{div}^\Gamma\widetilde{\Pi} = \mathbf{f} \quad \text{on } \Gamma, \end{aligned} \tag{8.5}$$

where the divergence operator div^Γ acts on the second index of $\widetilde{\Pi}$. The algebraic equations are relevant only if Γ is a proper surface. Indeed, if $\Gamma = \Omega$ (that is $d = n$ in definition 2.1) with an open set Ω, then $T_x(\Omega_t)^\perp = \{0\}$.

We now consider this case $\Gamma = \Omega$ and that the momentum is given by

$$\mathbf{m} = \varrho v, \tag{8.6}$$

where $\varrho > 0$ is assumed. It follows from the above rules, see (8.4), that

$$v\circ Y = \dot{X} + Qv^*. \tag{8.7}$$

This is the transformation rule for a “velocity” (see (3.5)). Defining the “diffusive mass flux” J and the “pressure tensor” Π by

$$\widetilde{J} = \varrho v + J, \quad \widetilde{\Pi} = \varrho\, v\otimes v + v\otimes J + \Pi. \tag{8.8}$$

Then

$$\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}J^* + \tau^*\dot{X} + Q\mathbf{f}^* \tag{8.9}$$

and it follows that the following is true.

8.3 Lemma. Under the above assumptions

ϱ and τ are objective scalars, v is a velocity,

J is an objective vector, Π is an objective tensor, and $\mathbf{f}$ transforms like a "force" in (8.9). Moreover, the equation of mass density and momentum density

$$\begin{aligned}\partial_t\varrho + \operatorname{div}(\varrho v + J) &= \tau,\\ \partial_t(\varrho v) + \operatorname{div}(\varrho v\otimes v + v\otimes J + \Pi) &= \mathbf{f}\end{aligned} \tag{8.10}$$

are satisfied.

Proof. The differential equations follow from (8.1) (or (8.5) by making use of 2.4 again). The derivation of the transformation rules is as follows. The rule for $\mathbf{m} = \varrho v$ is the same as for $\widetilde{J}$. Taking the difference one obtains $J\circ Y = QJ^*$. Moreover, the transformation rules for ϱ and v imply

$$\begin{aligned}(\varrho\, v\otimes v)\circ Y &= \varrho^*(\dot{X} + Qv^*)\otimes(\dot{X} + Qv^*)\\ &= \varrho^*\dot{X}\otimes\dot{X} + (Q\mathbf{m}^*)\otimes\dot{X} + \dot{X}\otimes(Q(\varrho^* v^*)) + Q(\varrho^*\; v^*\otimes v^*)\,Q^{\mathrm{T}}.\end{aligned}$$

Taking the difference with the transformation rule for $\widetilde{\Pi}$ one obtains

$$(\widetilde{\Pi} - \varrho\, v\otimes v)\circ Y = \dot{X}\otimes(Q(\widetilde{J}^* - \varrho^* v^*)) + Q(\widetilde{\Pi}^* - \varrho^*\; v^*\otimes v^*)\,Q^{\mathrm{T}},$$

where $\widetilde{J}^* - \varrho^* v^* = J^*$. Moreover, for $v\otimes J$ one computes

$$(v\otimes J)\circ Y = (\dot{X} + Qv^*)\otimes(QJ^*) = \dot{X}\otimes(QJ^*) + Q(v^*\otimes J^*)\,Q^{\mathrm{T}}$$

Subtraction of both rules gives $\Pi\circ Y = Q\Pi^* Q^{\mathrm{T}}$. □

9 Force

This section contains a discussion about the objectivity of momentum balance. The reason for this is the following transformation rule for the force. For simplicity let us consider the rule (which is (8.9) in the special case $\tau = 0$ and $J = 0$)

$$\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*\ . \tag{9.1}$$

This transformation rule does not allow a vanishing force for all observers. The ϱ^*-term is zero only for Galilean transformations, which are the linear Newtonian transformations. It is nonzero for example for Coriolis forces, which require a nonlinear Newtonian transformation. Therefore it is questionable, to which class $\mathbf{f}$ belongs. Let us restrict to the case of a single fluid, for which the conservation of mass ϱ and momentum $\mathbf{m} = \varrho v$ reads, see 8.3,

$$\begin{aligned}\partial_t\varrho + \operatorname{div}(\varrho v) &= 0,\\ \partial_t(\varrho v) + \operatorname{div}(\varrho v\otimes v + \Pi) &= \mathbf{f}\ .\end{aligned} \tag{9.2}$$

Whereas the class of Π is, for example, given as in section 11 or section 12, we shall discuss the class of forces $\mathbf{f}$ now.

The definition of objectivity says, that the description of a class of physical processes has to be independent of the observer. This principle, applied to the mass-momentum system (9.2), requires that this is of the same form for all observers. By the above considerations this is satisfied, if we define the class $\mathcal{P}$ of processes by all local solutions

ϱ, v, and $\mathbf{f}$ of the differential equations in (9.2), where ϱ is an objective scalar, v a velocity satisfying (8.7), and if $\mathbf{f}$ is an arbitrary vector field with the transformation rule (9.1).

It is obvious, that if we allow arbitrary vector fields $\mathbf{f}$ as forces, system (9.2) is a perfect objective description with transformation rule (9.1) for $\mathbf{f}$. However, such a class may not correspond to a real physical situation. Therefore, let us intent to describe a class of processes, where $\mathbf{f}$ is restricted to a certain subclass of vector fields and, of course, the rule (9.1) still should be valid.

First we deal with the v^*-term in (9.1). Let $f_{(1)} := \mathbf{f} - 2\varrho\mathbf{A}v$ with a matrix quantity $\mathbf{A}$. Then (9.1) is equivalent to

$$f_{(1)}\circ Y = \varrho^*(\ddot{X} - 2\mathbf{A}\circ Y\dot{X}) + 2\varrho^*(\dot{Q} - \mathbf{A}\circ YQ + Q\mathbf{A}^*)v^* + Qf^*_{(1)}\ .$$

The v^*-term vanishes, if we impose the following transformation rule

$$\mathbf{A}\circ Y = \dot{Q}Q^{\mathrm{T}} + Q\mathbf{A}^*Q^{\mathrm{T}} \tag{9.3}$$

for $\mathbf{A}$ (the same rule as for a velocity gradient in 10.4). Moreover, inserting the time derivatives of $X = Qx^* + \mathrm{b}$ we obtain that

$$f_{(1)}\circ Y = \varrho^*(\ddot{\mathrm{b}} - 2\mathbf{A}\circ Y\dot{\mathrm{b}}) + \varrho^*(\ddot{Q} - 2\mathbf{A}\circ Y\dot{Q})x^* + Qf^*_{(1)}.$$

Next we deal with the x^*-term. Let $f_{(2)} := f_{(1)} - \varrho\mathbf{C}x$ with a matrix quantity $\mathbf{C}$. Then the rule for $f_{(1)}$ is equivalent to

$$f_{(2)}\circ Y = \varrho^*(\ddot{\mathrm{b}} - 2\mathbf{A}\circ Y\dot{\mathrm{b}} - \mathbf{C}\circ Y\mathrm{b}) + \varrho^*(\ddot{Q} - 2\mathbf{A}\circ Y\dot{Q} - \mathbf{C}\circ YQ + Q\mathbf{C}^*)x^* + Qf^*_{(2)}.$$

The x^*-term vanishes, if we impose the following transformation rule

$$\mathbf{C}\circ Y = \ddot{Q}Q^{\mathrm{T}} - 2\mathbf{A}\circ Y\dot{Q}Q^{\mathrm{T}} + Q\mathbf{C}^*Q^{\mathrm{T}}$$

for $\mathbf{C}$. Using the identity $\dot{Q}Q^{\mathrm{T}} = -Q\dot{Q}^{\mathrm{T}}$ and inserting (9.3) this rule becomes

$$\mathbf{C}\circ Y = \ddot{Q}Q^{\mathrm{T}} + 2\dot{Q}\dot{Q}^{\mathrm{T}} + 2Q\mathbf{A}^*Q^{\mathrm{T}} + Q\mathbf{C}^*Q^{\mathrm{T}} \tag{9.4}$$

and we have

$$f_{(2)}\circ Y = \varrho^*(\ddot{\mathrm{b}} - 2\mathbf{A}\circ Y\dot{\mathrm{b}} - \mathbf{C}\circ Y\mathrm{b}) + Qf^*_{(2)}.$$

Setting $\mathbf{f}_0 := f_{(2)} - \varrho\mathbf{a}$ with a vector quantity $\mathbf{a}$ this is equivalent to the fact that $\mathbf{f}_0$ is an objective vector, if $\mathbf{a}$ satisfies

$$\mathbf{a}\circ Y = \ddot{\mathrm{b}} - 2\mathbf{A}\circ Y\dot{\mathrm{b}} - \mathbf{C}\circ Y\mathrm{b} + Q\mathbf{a}^*.$$

Inserting (9.3) and (9.4) this becomes

$$\mathbf{a}\circ Y = \ddot{\mathrm{b}} - 2(\dot{Q} + Q\mathbf{A}^*)(Q^{\mathrm{T}}\mathrm{b})^{\boldsymbol{\cdot}} - (\ddot{Q} + Q\mathbf{C}^*)Q^{\mathrm{T}}\mathrm{b} + Q\mathbf{a}^*. \tag{9.5}$$

Thus we obtain

9.1 Proposition. Let

$$\mathbf{f} = \varrho(2\mathbf{A}v + \mathbf{C}x + \mathbf{a}) + \mathbf{f}_0 \tag{9.6}$$

with an objective vector $\mathbf{f}_0$ and transformation rules (9.3), (9.4), and (9.5) for $(\mathbf{A}, \mathbf{C}, \mathbf{a})$. Then $\mathbf{f}$ is a force with transformation rule (9.1), and the identity (9.6) is objective, that is the same for all observers. We call $\mathbf{A}$ "Coriolis matrix", $\mathbf{C}$ "Euler-centrifugal matrix", and $\mathbf{a}$ "acceleration vector" (see also the terms in 9.2).

So far we have not introduced any constraint to the force term, since for a fixed observer the vector field $\mathbf{a}$ can be any function in time and space. We now introduce constraints for $\mathbf{f}$.

9.2 Proposition. Consider a class of forces $\mathbf{f}$ as in (9.6) with

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} = 0, \quad \nabla\mathbf{A} = 0, \quad \mathbf{C} = \partial_t\mathbf{A} - \mathbf{A}^2, \quad \nabla\mathbf{a} = 0.$$

Then this line of constraints is objective. In particular, the quantities $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{a}$ depend only on time. Moreover, there exist "inertial frames", that is frames with $\mathbf{A} = 0$, $\mathbf{C} = 0$, and $\mathbf{a} = 0$.

Meaning: The contributions in $\mathbf{f}$ have the following meaning (see [14, (2.50)]):

$\mathbf{A}v$	Coriolis acceleration,
$\partial_t\mathbf{A}\ x$	Euler acceleration,
$-\mathbf{A}^2 x$	centrifugal acceleration,
$\mathbf{a}$	acceleration of translation.

Proof. If $\mathbf{A}^*$ is antisymmetric, then transformation rule (9.3) gives that also $\mathbf{A}$ is antisymmetric. Hence $\mathbf{A}+\mathbf{A}^{\mathrm{T}} = 0$ is an objective equation. Moreover, all coefficients in (9.3), (9.4), and (9.5) depend only on time. This implies that $\nabla\mathbf{A} = 0$ is an objective equation, that the two equations $\nabla\mathbf{A} = 0$, $\nabla\mathbf{C} = 0$ are objective, and that the three equations $\nabla\mathbf{A} = 0$, $\nabla\mathbf{C} = 0$, $\nabla\mathbf{a} = 0$ also are objective.

Now assume the constraint $\nabla\mathbf{A} = 0$ for the quantity $\mathbf{A}$. It follows from (9.3), that $B := \partial_t\mathbf{A} - \mathbf{A}^2$ has the transformation rule

$$B \circ Y = \ddot{Q}Q^{\mathrm{T}} + 2\dot{Q}\dot{Q}^{\mathrm{T}} + 2Q\mathbf{A}^*Q^{\mathrm{T}} + QB^*Q^{\mathrm{T}}.$$

Together with (9.3) this gives, that $\mathbf{C} - B$ is an objective tensor. Hence the two equations $\nabla\mathbf{A} = 0$, $\mathbf{C} - B = 0$ are objective.

If $(\mathbf{A}, \mathbf{a})$ with these properties are given, we want to determine all observer transformations $t = t^*$, $x = Q(t^*)x^* + \mathrm{b}(t^*)$, for which $(\mathbf{A}^*, \mathbf{a}^*) = 0$. Now, $\mathbf{A}^* = 0$ in (9.3) if and only if Q satisfies $\dot{Q} = \mathbf{A}Q$. Since $\mathbf{A}$ is antisymmetric, this can be solved in the class of orthogonal transformations. Then, since $\mathbf{C}^* = 0$, (9.5) with $\mathbf{a}^* = 0$ is equivalent to

$$\ddot{\mathrm{b}} = \mathbf{a} + 2\dot{Q}\left(Q^{\mathrm{T}}\mathrm{b}\right)^{\bullet} + \ddot{Q}Q^{\mathrm{T}}\mathrm{b},$$

which gives b in terms of $\mathbf{a}$ and Q. □

Finally let us mention the following: In real physical situation it should be possible to check the validity of the structure of $\mathbf{f}$ in 9.1 with the constraints in 9.2. Since $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{a}$ are functions of time only, enough measurements at different space points or of different observers are needed.

However, this requires a full knowledge of the "internal force" $\mathbf{f}_0$. In every laboratory frame, solar frame, or galactic frame the influence of gravitational forces is present. The objective vector $\mathbf{f}_0$ is then given by a constitutive relation $\mathbf{f}_0 = -\varrho\nabla\phi$, where ϕ is the gravitational potential. It might be, that its effect is below the error of measurements.

10 Mass-momentum-energy system

The general system of balance laws for mass ϱ, momentum $\mathbf{m} = (\mathbf{m}_k)_{k=1,\dots,n}$, and "(total) energy" e on a moving interface Γ is of the form

$$\begin{aligned} \partial_t(\varrho\mu_\Gamma) + \textstyle\sum_{i=1}^n \partial_{x_i}(\widetilde{J}_i\mu_\Gamma) &= r\mu_\Gamma, \\ \partial_t(\mathbf{m}_k\mu_\Gamma) + \textstyle\sum_{i=1}^n \partial_{x_i}(\widetilde{\Pi}_{ki}\mu_\Gamma) &= \mathbf{f}_k\mu_\Gamma \quad \text{for } k = 1,\dots,n \\ \partial_t(e\mu_\Gamma) + \textstyle\sum_{i=1}^n \partial_{x_i}(\widetilde{q}_i\mu_\Gamma) &= g\mu_\Gamma \ . \end{aligned}$$

This holds in the space of distributions. In the case, that Γ is an open set (that is $d = n$ in definition 2.1), we are in the classical case. In addition to the quantities explained in section 8 here $\widetilde{q} = (\widetilde{q}_i)_{i=1,\dots,n}$ is the "energy flux" and g the "energy production".

10.1 Differential system. The vector notation of this system is

$$\begin{aligned} \partial_t(\varrho\mu_\Gamma) + \operatorname{div}(\widetilde{J}\mu_\Gamma) &= r\mu_\Gamma, \\ \partial_t(\mathbf{m}\mu_\Gamma) + \operatorname{div}(\widetilde{\Pi}\mu_\Gamma) &= \mathbf{f}\mu_\Gamma, \\ \partial_t(e\mu_\Gamma) + \operatorname{div}(\widetilde{q}\mu_\Gamma) &= g\mu_\Gamma \end{aligned}$$

with the convention, that the divergence of a tensor acts on its last index.

The fact, that the equations of mass and momentum are complemented by one equation, the equation for the energy, is called "first law of thermodynamics". Compare this with other models, like Grad's 13-moment theory, or the 84-moments theory, which are treated in [15, Chapter 9,10]. The mass-momentum-energy system 10.1 is of the form as in (5.1) by writing it as

$$\partial_t\left(\begin{bmatrix}\varrho\\ \mathbf{m}\\ e\end{bmatrix}\mu_\Gamma\right) + \textstyle\sum_{i=1}^n \partial_{x_i}\left(\begin{bmatrix}\widetilde{J}_i\\ \widetilde{\Pi}_i\\ \widetilde{q}_i\end{bmatrix}\mu_\Gamma\right) = \begin{bmatrix}r\\ \mathbf{f}\\ g\end{bmatrix}\mu_\Gamma,$$

which $\widetilde{\Pi}_i := \left(\widetilde{\Pi}_{ki}\right)_{k=1,\dots,n}$ as in section 8. The type of this system is defined by transformation rule (5.3) for test functions, where the matrix is

$$Z := \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & Q & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathrm{T}}Q & 1 \end{bmatrix} \tag{10.1}$$

with

$$Z'_0 = \begin{bmatrix} 0 & 0 & 0 \\ \ddot{X} & \dot{Q} & 0 \\ \ddot{X}\bullet\dot{X} & \ddot{X}^{\mathrm{T}}Q + \dot{X}^{\mathrm{T}}\dot{Q} & 0 \end{bmatrix}, \quad Z'_j = \begin{bmatrix} 0 & 0 & 0 \\ \dot{X}'_j & 0 & 0 \\ \dot{X}\bullet\dot{X}'_j & \dot{X}'_j{}^{\mathrm{T}}Q & 0 \end{bmatrix}$$

for $j = 1, \ldots, n$. Then 10.1 is objective, if (5.4) is satisfied, which here reads

$$\begin{aligned}
\begin{bmatrix} \varrho \\ \mathbf{m} \\ e \end{bmatrix}\circ Y &= Z \begin{bmatrix} \varrho^* \\ \mathbf{m}^* \\ e^* \end{bmatrix}, \\
\begin{bmatrix} \widetilde{J}_i \\ \widetilde{\Pi}_i \\ \widetilde{q}_i \end{bmatrix}\circ Y &= \dot{X}_i Z \begin{bmatrix} \varrho^* \\ \mathbf{m}^* \\ e^* \end{bmatrix} + \textstyle\sum_{j=1}^n Q_{ij} Z \begin{bmatrix} \widetilde{J}_j^* \\ \widetilde{\Pi}_j^* \\ \widetilde{q}_j^* \end{bmatrix} \qquad \text{for } i = 1, \ldots, n, \\
\begin{bmatrix} r \\ \mathbf{f} \\ g \end{bmatrix}\circ Y &= Z'_0 \begin{bmatrix} \varrho^* \\ \mathbf{m}^* \\ e^* \end{bmatrix} + \textstyle\sum_{j=1}^n Z'_j \begin{bmatrix} \widetilde{J}_j^* \\ \widetilde{\Pi}_j^* \\ \widetilde{q}_j^* \end{bmatrix} + Z \begin{bmatrix} r^* \\ \mathbf{f}^* \\ g^* \end{bmatrix}.
\end{aligned}$$

Due to the structure of the matrix Z, the properties of the quantities in the mass and momentum part follow as in section 8. That is (8.3) and (8.4) is satisfied and also the results about the the structure of the force in section 9 can be applied here. Therefore, we have to evaluate the energy part of this transformation rule. The first identity gives

$$e\circ Y = \tfrac{1}{2}|\dot{X}|^2\varrho^* + \dot{X}\bullet(Q\mathbf{m}^*) + e^*. \tag{10.2}$$

The energy part of the second identity is

$$\begin{aligned}
\widetilde{q}_i\circ Y &= \dot{X}_i \left(\tfrac{1}{2}|\dot{X}|^2\varrho^* + \dot{X}^{\mathrm{T}}Q\mathbf{m}^* + e^*\right) \\
&\quad + \textstyle\sum_{j=1}^n Q_{ij}\left(\tfrac{1}{2}|\dot{X}|^2\widetilde{J}_j^* + \dot{X}^{\mathrm{T}}Q\widetilde{\Pi}_j^* + \widetilde{q}_j^*\right),
\end{aligned}$$

which in vector notation is

$$\begin{aligned}
\widetilde{q}\circ Y &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \dot{X}^{\mathrm{T}}Q\mathbf{m}^* + e^*)\dot{X} \\
&\quad + \tfrac{1}{2}|\dot{X}|^2 Q\widetilde{J}^* + (Q\widetilde{\Pi}^* Q^{\mathrm{T}})^{\mathrm{T}}\dot{X} + Q\widetilde{q}^*.
\end{aligned} \tag{10.3}$$

The energy part of the third identity is

$$\begin{aligned}
g\circ Y &= \ddot{X}\bullet\dot{X}\varrho^* + (\ddot{X}^{\mathrm{T}}Q + \dot{X}^{\mathrm{T}}\dot{Q})\mathbf{m}^* \\
&\quad + \textstyle\sum_{j=1}^n \left(\dot{X}\bullet\dot{X}'_j\widetilde{J}_j^* + \dot{X}'_j{}^{\mathrm{T}}Q\widetilde{\Pi}_j^*\right) \\
&\quad + \tfrac{1}{2}|\dot{X}|^2 r^* + \dot{X}^{\mathrm{T}}Q\mathbf{f}^* + g^*,
\end{aligned}$$

that is

$$\begin{aligned}
g\circ Y &= \ddot{X}\bullet(\varrho^*\dot{X} + Q\mathbf{m}^*) + (\dot{Q}^{\mathrm{T}}\dot{X})\bullet(\mathbf{m}^* + \widetilde{J}^*) + (Q^{\mathrm{T}}\dot{Q})\bullet\widetilde{\Pi}^* \\
&\quad + \tfrac{1}{2}|\dot{X}|^2 r^* + \dot{X}\bullet Q\mathbf{f}^* + g^*.
\end{aligned} \tag{10.4}$$

Altogether, we have derived the following lemma.

10.2 Lemma. The mass-momentum-energy system 10.1 is objective, if ϱ, $\widetilde{J}$, r satisfy (8.3), if $\mathbf{m}$, $\widetilde{\Pi}$, $\mathbf{f}$ satisfy (8.4), and if e, $\widetilde{q}$, and g satisfy (10.2), (10.3), (10.4).

By 2.4 the system 10.1 is equivalent to the following differential equations and algebraic side conditions

$$\begin{aligned}
&(\widetilde{J} - \varrho \mathbf{v}_\Gamma) \bullet \mathbf{n} = 0 \quad \text{ for } \mathbf{n}(t,x) \in T_x(\Gamma_t)^\perp, \\
&\partial_t^\Gamma \varrho + \operatorname{div}^\Gamma \widetilde{J} = r \quad \text{ on } \Gamma, \\
&(\widetilde{\Pi} - \mathbf{m} \otimes \mathbf{v}_\Gamma)\mathbf{n} = 0 \quad \text{ for } \mathbf{n}(t,x) \in T_x(\Gamma_t)^\perp, \\
&\partial_t^\Gamma \mathbf{m} + \operatorname{div}^\Gamma \widetilde{\Pi} = \mathbf{f} \quad \text{ on } \Gamma, \\
&(\widetilde{q} - e \mathbf{v}_\Gamma) \bullet \mathbf{n} = 0 \quad \text{ for } \mathbf{n}(t,x) \in T_x(\Gamma_t)^\perp, \\
&\partial_t^\Gamma e + \operatorname{div}^\Gamma \widetilde{q} = g \quad \text{ on } \Gamma.
\end{aligned} \tag{10.5}$$

Here the algebraic equations are relevant only if Γ is a proper surface.

We now consider, as in section 8, the case that $\Gamma = \Omega$ with an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$. We assume that $\varrho > 0$ and consider a momentum of the form

$$\mathbf{m} = \varrho v \tag{10.6}$$

with a velocity v, that is satisfying (8.7). Define J, Π as in (8.8) and the "internal energy" ε by

$$\begin{aligned}
&\widetilde{J} = \varrho v + J, \quad \widetilde{\Pi} = \varrho\, v \otimes v + v \otimes J + \Pi, \\
&e = \varepsilon + \tfrac{\varrho}{2}|v|^2, \quad \widetilde{q} = ev + \tfrac{1}{2}|v|^2 J + \Pi^{\mathrm{T}} v + q,
\end{aligned} \tag{10.7}$$

where q is the "heat flux", a definition which indicates the role of q, however in general q may contain also other terms. Further, let $\widetilde{\mathbf{f}}$ and $\widetilde{g}$ be defined by

$$\widetilde{\mathbf{f}} = \mathbf{f} - rv \ , \quad \widetilde{g} = g + \tfrac{r}{2}|v|^2 - v \bullet \mathbf{f} \ . \tag{10.8}$$

Then we obtain

10.3 Lemma. With these definitions the mass-momentum-energy system 10.1 on a space-time domain reads

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho v + J) = r, \\
&\partial_t(\varrho v) + \operatorname{div}(v \otimes (\varrho v + J) + \Pi) = \mathbf{f}, \\
&\partial_t(\varepsilon + \tfrac{\varrho}{2}|v|^2) + \operatorname{div}(\varepsilon v + \tfrac{1}{2}|v|^2(\varrho v + J) + \Pi^{\mathrm{T}} v + q) = g.
\end{aligned} \tag{10.9}$$

Here

$$\begin{aligned}
&\varrho,\ \varepsilon,\ r \text{ are objective scalars, } v \text{ is a velocity (satisfying (8.7))}, \\
&J,\ q \text{ are objective vectors, } \Pi \text{ is an objective tensor}, \\
&\widetilde{\mathbf{f}} \circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}J^* + Q\widetilde{\mathbf{f}}^* \ , \\
&\widetilde{g} \circ Y = (Q^{\mathrm{T}}\dot{Q}) \bullet \Pi^{*\,\mathrm{A}} + \widetilde{g}^* \ ,
\end{aligned} \tag{10.10}$$

where $\widetilde{\mathbf{f}}$ and $\widetilde{g}$ are defined as in (10.8).

If the pressure tensor Π is assumed to be symmetric, the last identity means, that $\widetilde{g}$ is an objective scalar. This symmetry is connected to the moment of momentum.

Proof. For the properties of ϱ, v, J, r, Π, and $\mathbf{f}$ see section 8. The transformation rule for e in (10.2) now is

$$e \circ Y = \tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet (Qv^*) + e^*. \tag{10.11}$$

For the kinetic energy $\frac{\varrho}{2}|v|^2$ one computes

$$\begin{aligned}(\tfrac{\varrho}{2}|v|^2) \circ Y &= \tfrac{\varrho^*}{2}|\dot{X} + Qv^*|^2 \\ &= \tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet (Qv^*) + \tfrac{\varrho^*}{2}|v^*|^2,\end{aligned}$$

Taking the difference with (10.11) one obtains $\varepsilon \circ Y = \varepsilon^*$.

Next, insert $\mathbf{m}^*$, $\widetilde{J}^*$, and $\widetilde{\Pi}^*$ in rule (10.3). Since

$$\begin{aligned}(Q\widetilde{\Pi}^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X} &= \varrho^* Q(v^* \otimes v^*) Q^{\mathrm{T}} \dot{X} + Q(J^* \otimes v^*) Q^{\mathrm{T}} \dot{X} + (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X} \\ &= \dot{X} \bullet Qv^* (\varrho^* Qv^* + QJ^*) + (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X},\end{aligned}$$

this gives

$$\begin{aligned}\widetilde{q} \circ Y &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet Qv^* + e^*)\dot{X} \\ &\quad + (\tfrac{1}{2}|\dot{X}|^2 + \dot{X} \bullet Qv^*)(\varrho^* Qv^* + QJ^*) \\ &\quad + (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X} + Q\widetilde{q}^*. \\ &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet Qv^*)(\dot{X} + Qv^*) + e^* \dot{X} \\ &\quad + (\tfrac{1}{2}|\dot{X}|^2 + \dot{X} \bullet Qv^*) QJ^* + (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X} + Q\widetilde{q}^*.\end{aligned} \tag{10.12}$$

From known rules one computes

$$\begin{aligned}(\Pi^{\mathrm{T}} v) \circ Y &= (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} (\dot{X} + Qv^*) \\ &= (Q\Pi^* Q^{\mathrm{T}})^{\mathrm{T}} \dot{X} + Q(\Pi^{*\mathrm{T}} v^*),\end{aligned} \tag{10.13}$$

$$\begin{aligned}(\tfrac{1}{2}|v|^2 J) \circ Y &= (\tfrac{1}{2}|\dot{X}|^2 + \dot{X} \bullet Qv^* + \tfrac{1}{2}|v^*|^2) QJ^* \\ &= (\tfrac{1}{2}|\dot{X}|^2 + \dot{X} \bullet Qv^*) QJ^* + Q(\tfrac{1}{2}|v^*|^2 J^*),\end{aligned} \tag{10.14}$$

$$\begin{aligned}(ev) \circ Y &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet (Qv^*) + e^*)(\dot{X} + Qv^*) \\ &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^* \dot{X} \bullet (Qv^*))(\dot{X} + Qv^*) + e^* \dot{X} + Q(e^* v^*).\end{aligned} \tag{10.15}$$

Subtraction of (10.13), (10.14), (10.15) from (10.12) gives that $q \circ Y = Qq^*$.

Finally, inserting $\mathbf{m}^* = \varrho^* v^*$ in rule (10.4), this rule becomes

$$\begin{aligned}g \circ Y &= \varrho^* \ddot{X} \bullet (\dot{X} + Qv^*) + \dot{X} \bullet \dot{Q}(\varrho^* v^* + \widetilde{J}^*) + (Q^{\mathrm{T}} \dot{Q}) \bullet \widetilde{\Pi}^* \\ &\quad + \tfrac{r^*}{2}|\dot{X}|^2 + \dot{X} \bullet Q\mathbf{f}^* + g^*.\end{aligned} \tag{10.16}$$

Now

$$(\tfrac{r}{2}|v|^2) \circ Y = \tfrac{r^*}{2}|\dot{X}|^2 + r^* \dot{X} \bullet (Qv^*) + \tfrac{r^*}{2}|v^*|^2, \tag{10.17}$$

and from (8.7), (8.9)

$$\begin{aligned}(v\bullet\mathbf{f})\circ Y &= (\dot X + Qv^*)\bullet(\varrho^*(\ddot X + 2\dot Q v^*) + \dot Q J^* + r^*\dot X + Q\mathbf{f}^*)\\ &= (\dot X + Qv^*)\bullet(\varrho^*(\ddot X + 2\dot Q v^*) + \dot Q J^* + r^*\dot X) + \dot X\bullet Q\mathbf{f}^* + v^*\bullet\mathbf{f}^*.\end{aligned} \tag{10.18}$$

Combining (10.16), (10.17), (10.18) one obtains

$$\begin{aligned}\widetilde g\circ Y &= \dot X\bullet\dot Q(\varrho^* v^* + \widetilde J^*) - (\dot X + Qv^*)\bullet(2\varrho^*\dot Q v^* + \dot Q J^*)\\ &\quad +(Q^{\mathrm T}\dot Q)\bullet\widetilde\Pi^* + \widetilde g^*\\ &= \dot X\bullet\dot Q(\widetilde J^* - \varrho^* v^* - J^*)\\ &\quad +(Q^{\mathrm T}\dot Q)\bullet(\widetilde\Pi^* - 2\varrho^* v^*\otimes v^* - v^*\otimes J^*) + \widetilde g^*.\end{aligned}$$

Inserting the identities for $\widetilde J^*$, $\widetilde\Pi^*$ (see (10.7)) one obtains

$$\begin{aligned}\widetilde g\circ Y &= (Q^{\mathrm T}\dot Q)\bullet(\Pi^* - \varrho^* v^*\otimes v^*) + \widetilde g^*\\ &= (Q^{\mathrm T}\dot Q)\bullet(\Pi^* - \varrho^* v^*\otimes v^*)^{\mathrm A} + \widetilde g^* = (Q^{\mathrm T}\dot Q)\bullet\Pi^{*\,\mathrm A} + \widetilde g^*.\end{aligned}$$

Finally, subtracting $(rv)\circ Y = r\dot X + Q(r^* v^*)$ from (8.9) one obtains

$$\widetilde{\mathbf f}\circ Y = \varrho^*(\ddot X + 2\dot Q v^*) + \dot Q J^* + Q\widetilde{\mathbf f}^*.$$

□

Concerning the transformation rule for $\widetilde g$ we apply the following statement.

10.4 Proposition. If v is a velocity, then the velocity gradient $\mathrm D v = (\partial_j v_i)_{i,j=1,\dots,n}$ satisfies

$$(\mathrm D v)\circ Y = \dot Q\, Q^{\mathrm T} + Q \mathrm D v^*\, Q^{\mathrm T}.$$

It follows, that

$$(\mathrm D v)^{\mathrm S}\circ Y = Q\,(\mathrm D v^*)^{\mathrm S} Q^{\mathrm T}, \quad (\mathrm D v)^{\mathrm A}\circ Y = \dot Q\, Q^{\mathrm T} + Q\,(\mathrm D v^*)^{\mathrm A} Q^{\mathrm T},$$

that is $(\mathrm D v)^{\mathrm S}$, the symmetric part of $\mathrm D v$, is an objective tensor.

Proof. The transformation rule for velocities is

$$v_i\circ Y = \dot X_i + \textstyle\sum_{j=1}^n Q_{ij} v_j^*$$

for $i = 1,\dots,n$. Computing space derivatives of this identity for $l = 1,\dots,n$ one obtains

$$\textstyle\sum_{k=1}^n v_{i\,\prime k}\circ Y Q_{kl} = \dot Q_{il} + \sum_{j=1}^n Q_{ij} v^*_{j\,\prime l},$$

hence

$$v_{i\,\prime k}\circ Y = \textstyle\sum_{l=1}^n \dot Q_{il} Q_{kl} + \sum_{j,l=1}^n Q_{ij} Q_{kl} v^*_{j\,\prime l},$$

which is the required transformation rule. The antisymmetry of $\dot{Q}Q^{\mathrm{T}}$ implies the assertion for $(\mathrm{D}v)^{\mathrm{S}}$. $\square$

It follows, that

$$
\begin{aligned}
(\mathrm{D}v\bullet\Pi^{\mathrm{A}})\circ Y &= (\dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^* Q^{\mathrm{T}})\bullet(Q\Pi^* Q^{\mathrm{T}})^{\mathrm{A}} \\
&= (\dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^* Q^{\mathrm{T}})\bullet(Q\Pi^{*\,\mathrm{A}}Q^{\mathrm{T}}) \\
&= (Q^{\mathrm{T}}\dot{Q})\bullet\Pi^{*\,\mathrm{A}} + \mathrm{D}v^*\bullet\Pi^{*\,\mathrm{A}}.
\end{aligned}
$$

Therefore, $\mathrm{D}v\bullet\Pi^{\mathrm{A}}$ obeys the same transformation rule as $\widetilde{g}$, which implies that

$$
\widetilde{g} - \mathrm{D}v\bullet\Pi^{\mathrm{A}} = g + \tfrac{r}{2}|v|^2 - v\bullet\mathbf{f} - \mathrm{D}v\bullet\Pi^{\mathrm{A}} \quad \text{is an objective scalar.} \tag{10.19}
$$

11 Viscous fluids

Here we give a short report on the entropy principle for a single viscous fluid. We consider conservation of mass, momentum, and energy as in (10.9), that is

$$
\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v) = 0, \\
&\partial_t(\varrho v) + \operatorname{div}(\varrho\, v\otimes v + \Pi) = \mathbf{f}, \\
&\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}}v + q) = g.
\end{aligned} \tag{11.1}
$$

Here we have assumed that $J = 0$ and $r = 0$, and we have the constraint $\varrho > 0$, which is invariant under observer transformations. We consider $\mathbf{f}$ as an external force, containing for example Coriolis forces, so that section 9 can be applied. We also assume that Π is symmetric,

$$
\Pi = \Pi^{\mathrm{T}}, \tag{11.2}
$$

a property, which implies the conservation of moment of momentum. With the "inner energy" ε, given by

$$
e = \varepsilon + \frac{\varrho}{2}|v|^2, \tag{11.3}
$$

system (11.1) is equivalent to

$$
\begin{aligned}
&(\partial_t + v\bullet\nabla)\varrho + \varrho\operatorname{div}v = 0, \\
&\varrho(\partial_t + v\bullet\nabla)v + \operatorname{div}\Pi = \mathbf{f}, \\
&(\partial_t + v\bullet\nabla)\varepsilon + \operatorname{div}q + \mathrm{D}v\bullet(\varepsilon\,\mathrm{Id} + \Pi) = g - v\bullet\mathbf{f}.
\end{aligned} \tag{11.4}
$$

By section 10, objectivity of (11.1) means the following properties:

$$
\begin{aligned}
&\varrho,\ \varepsilon \text{ are objective scalars, } v \text{ is a velocity (as in (8.7)),} \\
&q \text{ is an objective vector, } \Pi \text{ a symmetric objective tensor,} \\
&\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*, \quad (g - v\bullet\mathbf{f})\circ Y = g^* - v^*\bullet\mathbf{f}^*.
\end{aligned} \tag{11.5}
$$

We now state the entropy principle.

11.1 Entropy principle. We assume that there is an entropy η and an entropy flux ψ, such that

$$h := \partial_t \eta + \operatorname{div}\psi \geq 0$$

in the domain of the process. In addition there are constitutive relations for (η, ψ), where we consider a dependence of η and ψ on the quantities in the the balance laws.

As a postulate η is an objective scalar, thus with $h = \partial_t\eta + \operatorname{div}\psi$ we derive (from (5.3) with $Z = 1$ for this single equation)

$$\eta,\ h \text{ are objective scalars,} \qquad \psi - \eta v \text{ is an objective vector.} \tag{11.6}$$

As constitutive relation for η we assume

11.2 Proposition. Let η depend on $(\varrho, v, \varepsilon)$, then objectivity implies, that

$$\eta = \widehat{\eta}(\varrho, \varepsilon).$$

Proof. If $\eta = \widehat{\eta}(\varrho, v, \varepsilon)$, then objectivity of $\widehat{\eta}$ implies $\eta^* = \widehat{\eta}(\varrho^*, v^*, \varepsilon^*)$. By (11.5) and (11.6) this gives

$$\widehat{\eta}(\varrho^*, \dot{X} + Qv^*, \varepsilon^*) = \widehat{\eta}(\varrho^*, v^*, \varepsilon^*).$$

This implies, that $\widehat{\eta}$ is independent of the velocity. □

If η depends on (ϱ, ε) as in 11.2, then one computes

$$\partial_t \eta = \eta'_\varrho \partial_t \varrho + \eta'_\varepsilon \partial_t \varepsilon,$$

and a similar expression one obtains for any first order derivative. Inserting this in the differential equations (11.4) one derives

$$\begin{aligned}
h &= \partial_t\eta + \operatorname{div}\psi \\
&= (\partial_t + v\bullet\nabla)\eta + \eta\operatorname{div}v + \operatorname{div}(\psi - \eta v) \\
&= \eta'_\varrho(\partial_t + v\bullet\nabla)\varrho + \eta'_\varepsilon(\partial_t + v\bullet\nabla)\varepsilon + \eta\operatorname{div}v + \operatorname{div}(\psi - \eta v) \\
&= \ \eta'_\varepsilon(g - v\bullet\mathbf{f}) \\
&\quad + \mathrm{D}v\bullet\left((\eta - \eta'_\varrho\varrho - \eta'_\varepsilon\varepsilon)\operatorname{Id} - \eta'_\varepsilon\Pi\right) \\
&\quad - \eta'_\varepsilon\operatorname{div}q + \operatorname{div}(\psi - \eta v).
\end{aligned}$$

Now we split Π into a diagonal part and a rest (here for a moment without any meaning),

$$\Pi = p\operatorname{Id} - S, \tag{11.7}$$

we use the standard computation involving the heat flux,

$$\eta'_\varepsilon\operatorname{div}q = \operatorname{div}(\eta'_\varepsilon q) - \nabla\eta'_\varepsilon\bullet q,$$

and we introduce a rest term in the entropy flux,

$$\psi = \eta v + \eta'_\varepsilon q + \psi_0.$$

Inserting this, we obtain for the entropy production

$$
\begin{aligned}
h &= \eta'_\varepsilon (g - v\bullet\mathbf{f}) \\
&\quad + \underbrace{\mathrm{div} v \cdot (\eta - \eta'_\varrho \varrho - \eta'_\varepsilon(\varepsilon + p))}_{\mathrm{div}v\text{-term}} \\
&\quad + \underbrace{\eta'_\varepsilon \mathrm{D}v\bullet S + (\nabla\eta'_\varepsilon)\bullet q}_{\text{remainder}} + \underbrace{\mathrm{div}\psi_0}_{\text{flux term}} \quad .
\end{aligned}
\tag{11.8}
$$

Now we assume that $\psi_0 = 0$. We will later justify this identity by the computations in 11.7. Thus setting $\psi_0 = 0$ we obtain

11.3 Theorem. Assume a constitutive relation for η as in 11.2 and for the entropy flux set

$$\psi = \eta v + \eta'_\varepsilon q.$$

Assume that $g = v\bullet\mathbf{f}$ and $\Pi = p\,\mathrm{Id} - S$. Then the entropy inequality is satisfied, if

(1) ***Gibbs relation.*** $\eta = \varrho\eta'_\varrho + (\varepsilon + p)\eta'_\varepsilon$,

(2) ***Dissipation.*** $\eta'_\varepsilon \mathrm{D}v\bullet S + \nabla\eta'_\varepsilon\bullet q \geq 0$.

This follows from (11.8). Moreover, the entropy identity reads

$$h = \partial_t\eta + \mathrm{div}\psi = \eta'_\varepsilon \mathrm{D}v\bullet S + \nabla\eta'_\varepsilon\bullet q \geq 0. \tag{11.9}$$

In concrete cases both terms on the right-hand side are nonnegative. That is, the splitting in (11.7) is so, that the part S is driven by the first derivatives $\mathrm{D}v$ of v to assure that $\eta'_\varepsilon \mathrm{D}v\bullet S$ is nonnegative, and the rest, which must be a multiple of the identity, satisfies Gibbs relation.

In theorem 11.3 we have shown that the entropy is fulfilled, if certain assumptions, among them 11.3(1) and 11.3(2), are satisfied. Now we want to take the entropy principle as assumption and draw conclusions from it, among them 11.3(1) and 11.3(2). The corresponding theorem will be 11.7.

For a Navier-Stokes-Fourier fluid, that is a heat conducting viscous fluid, see [14, Chapter 6], the class of physical processes $\mathcal{P}$ consists of local smooth solutions $(\varrho, v, \varepsilon, \mathbf{f})$ of (11.1), where Π and q are given by constitutive relations. Therefore one considers the density ϱ, the velocity v, and the internal energy ε as independent variables. We note that at this stage the temperature is not yet defined. Moreover, we consider $\mathbf{f}$ as an external quantity. And one specifies for Π and q a pointwise dependence on the values and first space derivatives of ϱ, v, ε:

11.4 Assumption (Solution property). We assume constitutive equations

$$
\begin{aligned}
\Pi &= \widehat{\Pi}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon), \\
q &= \widehat{q}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon).
\end{aligned}
\tag{11.10}
$$

and that $\widehat{\Pi}$ and $\widehat{q}$ are C^1-functions. For the entropy and entropy flux we assume similar equations

$$\eta = \widehat{\eta}(\varrho, v, \varepsilon), \quad \psi = \widehat{\psi}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon). \tag{11.11}$$

For an open and dense set of values $(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon)$ the following holds: If there is a polynomial $(\varrho, v, \varepsilon, \mathbf{f})$, which satisfies (11.1) at a point, then there is a solution of (11.1) in a neighbourhood of this point, which coincides with the polynomial at this point in all terms which occur in (11.1).

In this assumption the notion of polynomials is a trick to say that a set of values $(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon)$ together with $(\partial_t\varrho, \partial_t v, \partial_t\varepsilon, \mathrm{D}^2\varrho, \mathrm{D}^2 v, \mathrm{D}^2\varepsilon)$ exist, which solve the differentiated version (see (15.3)) of the system (11.1) at a point.

The following holds.

11.5 Proposition. If Π and q are given by constitutive functions as in (11.10), and if they are (affine) linear in the variables representing derivatives, then objectivity implies, for $n \geq 3$, that Π and q are of the form

$$\begin{aligned}
&\Pi = p\,\mathrm{Id} - S, \quad p = \widehat{p}(\varrho, \varepsilon),\\
&S = \widehat{a}(\varrho, \varepsilon)\,(\mathrm{D}v)^{\mathrm{S}} + \widehat{b}(\varrho, \varepsilon)\mathrm{div}(v)\,\mathrm{Id},\\
&q = \widehat{c}(\varrho, \varepsilon)\nabla\varepsilon + \widehat{d}(\varrho, \varepsilon)\nabla\varrho
\end{aligned}$$

with objective scalars p, a, b, c, and d.

We mention that for the identity for S the symmetry of Π (see (11.2)) has been used. Also we mention that in the following proof the assumption on objectivity is meant for solutions $(\varrho, v, \varepsilon, \mathbf{f})$ in $\mathcal{P}$. Then for an open and dense set of values $(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon)$ by assumption 11.4 we have a solution in $\mathcal{P}$, hence we can argue as in the following proof, which gives the conclusion. Since $\widehat{\Pi}$ and $\widehat{q}$ are continously differentiable, the conclusion holds for all values.

Proof for q. We have the identity

$$q = \widehat{q}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon).$$

Then, since the function $\widehat{q}$ is objective, also for another observer

$$q^* = \widehat{q}(\varrho^*, v^*, \varepsilon^*, \nabla\varrho^*, \mathrm{D}v^*, \nabla\varepsilon^*).$$

Since $q \circ Y = Qq^*$, we obtain, applying above transformation rules, which follow from (11.5) and which imply, for example, $(\nabla\varrho)\circ Y = Q\nabla\varrho^*$ and the same for $\nabla\varepsilon$, that

$$\begin{aligned}
&\widehat{q}(\varrho^*, \dot{X} + Qv^*, \varepsilon^*, Q\nabla\varrho^*, \dot{Q}\,Q^{\mathrm{T}} + Q\mathrm{D}v^*\,Q^{\mathrm{T}}, Q\nabla\varepsilon^*)\\
&= Q\widehat{q}(\varrho^*, v^*, \varepsilon^*, \nabla\varrho^*, \mathrm{D}v^*, \nabla\varepsilon^*).
\end{aligned}$$

For a given (t^*, x^*) we can choose an observer transformation with $Q(t^*) = \mathrm{Id}$, and such that $\dot{X}(t^*)$ is a given vector and $\dot{Q}(t^*)\,Q^{\mathrm{T}}(t^*)$ a given antisymmetric matrix. This implies (see the above explanation), that $\widehat{q}$ is independent of the v-variables and the antisymmetric part of the $\mathrm{D}v$-variable. Thus with a new constitutive function

$$q = \widehat{q}(\varrho, \varepsilon, \nabla\varrho, (\mathrm{D}v)^{\mathrm{S}}, \nabla\varepsilon)$$

and since $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor (see 10.4), the above identity becomes

$$\begin{aligned}&\widehat{q}(\varrho^*, \varepsilon^*, Q\nabla\varrho^*, Q\,(\mathrm{D}v^*)^{\mathrm{S}}Q^{\mathrm{T}}, Q\nabla\varepsilon^*)\\ &= Q\widehat{q}(\varrho^*, \varepsilon^*, \nabla\varrho^*, (\mathrm{D}v^*)^{\mathrm{S}}, \nabla\varepsilon^*).\end{aligned}$$

For constant solutions $(\varrho^*, v^*, \varepsilon^*)$ this gives

$$\widehat{q}(\varrho^*, \varepsilon^*, 0, 0, 0) = Q\widehat{q}(\varrho^*, \varepsilon^*, 0, 0, 0).$$

Since $Q(t^*)$ can be any orthogonal matrix, this implies

$$\widehat{q}(\varrho^*, \varepsilon^*, 0, 0, 0) = 0. \tag{11.12}$$

We reduce further objectivity properties to constant objective tensors. Using (11.12) and since $\widehat{q}$ is (affine) linear in the variables $\partial_j\varrho$, $\partial_j\varepsilon$, and $\partial_j v$, we have a representation

$$\begin{aligned}q_i = &\sum_{j=1}^{n} \widehat{a}_{ij}(\varrho, \varepsilon)\partial_j\varepsilon + \sum_{j=1}^{n} \widehat{b}_{ij}(\varrho, \varepsilon)\partial_j\varrho\\ &+ \sum_{k,l=1}^{n} \widehat{c}_{ikl}(\varrho, \varepsilon)\frac{\partial_k v_l + \partial_l v_k}{2}\end{aligned}$$

with coefficients a_{ij}, b_{ij}, and c_{ijk}, where we can assume that $c_{ikl} = c_{ilk}$ for all $i, k, l = 1, \ldots, n$. Using that ϱ, ε are objective scalars, that q, $\nabla\varrho$, $\nabla\varepsilon$ are objective vectors, and that $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor, the identity $q \circ Y = Qq^*$, that is

$$q_i \circ Y = \sum_{\bar{i}=1}^{n} Q_{i\bar{i}} q^*_{\bar{i}},$$

we obtain

$$\begin{aligned}&\sum_{j,\bar{j}=1}^{n} \widehat{a}_{ij}(\varrho^*, \varepsilon^*)Q_{j\bar{j}}\partial_{\bar{j}}\varepsilon^* + \sum_{j,\bar{j}=1}^{n} \widehat{b}_{ij}(\varrho^*, \varepsilon^*)Q_{j\bar{j}}\partial_{\bar{j}}\varrho^*\\ &\quad + \sum_{k,l,\bar{k},\bar{l}=1}^{n} \widehat{c}_{ikl}(\varrho^*, \varepsilon^*)Q_{k\bar{k}}Q_{l\bar{l}}\frac{\partial_{\bar{k}} v^*_{\bar{l}} + \partial_{\bar{l}} v^*_{\bar{k}}}{2}\\ &= \sum_{\bar{i},\bar{j}=1}^{n} Q_{i\bar{i}}\widehat{a}_{\bar{i}\bar{j}}(\varrho^*, \varepsilon^*)\partial_{\bar{j}}\varepsilon^* + \sum_{\bar{i},\bar{j}=1}^{n} Q_{i\bar{i}}\widehat{b}_{\bar{i}\bar{j}}(\varrho^*, \varepsilon^*)\partial_{\bar{j}}\varrho^*\\ &\quad + \sum_{\bar{i},\bar{k},\bar{l}=1}^{n} Q_{i\bar{i}}\widehat{c}_{\bar{i}\bar{k}\bar{l}}(\varrho^*, \varepsilon^*)\frac{\partial_{\bar{k}} v^*_{\bar{l}} + \partial_{\bar{l}} v^*_{\bar{k}}}{2}.\end{aligned}$$

Now fix (t^*, x^*). By the solution property 11.4 (see the above explanation) there is a process $(\varrho^*, v^*, \varepsilon^*)$ with given values and space derivatives at (t^*, x^*). Thus fixing $\varrho^*(t^*, x^*)$ and $\varepsilon^*(t^*, x^*)$, varying over all space derivatives at (t^*_0, x^*_0), we see that the following identities have to be satisfied at (t^*_0, x^*_0):

$$\begin{aligned}
&\textstyle\sum_{j=1}^n \widehat{a}_{ij}(\varrho^*,\varepsilon^*)Q_{j\bar{j}} = \sum_{\bar{i}=1}^n Q_{i\bar{i}}\widehat{a}_{\bar{i}\bar{j}}(\varrho^*,\varepsilon^*) && \text{for all } i,\bar{j},\\
&\textstyle\sum_{j=1}^n \widehat{b}_{ij}(\varrho^*,\varepsilon^*)Q_{j\bar{j}} = \sum_{\bar{i}=1}^n Q_{i\bar{i}}\widehat{b}_{\bar{i}\bar{j}}(\varrho^*,\varepsilon^*) && \text{for all } i,\bar{j},\\
&\textstyle\sum_{k,l=1}^n \widehat{c}_{ikl}(\varrho^*,\varepsilon^*)Q_{k\bar{k}}Q_{l\bar{l}} = \sum_{\bar{i}=1}^n Q_{i\bar{i}}\widehat{c}_{\bar{i}\bar{k}\bar{l}}(\varrho^*,\varepsilon^*) && \text{for all } i,\bar{k},\bar{l}.
\end{aligned}$$

Note, that for the last identity we have used the symmetry of c_{ikl} in k and l. The first identity is equivalent to

$$\widehat{a}_{ij}(\varrho^*,\varepsilon^*) = \textstyle\sum_{\bar{i},\bar{j}=1}^n Q_{i\bar{i}}Q_{j\bar{j}}\widehat{a}_{\bar{i}\bar{j}}(\varrho^*,\varepsilon^*) \quad \text{for all } i,j$$

and all orthogonal matrices Q with positive determinant. This says, that (for fixed values of $\varrho^*(t^*,x^*)$ and $\varepsilon^*(t^*,x^*)$) the tensor $(\widehat{a}_{ij}(\varrho^*,\varepsilon^*))_{i,j=1,\dots,n}$ behaves like a constant objective tensor, which implies that it is a multiple of the identity. The same follows for the b-term. The third identity is equivalent to

$$\widehat{c}_{ikl}(\varrho^*,\varepsilon^*) = \textstyle\sum_{\bar{i},\bar{k},\bar{l}=1}^n Q_{i\bar{i}}Q_{k\bar{k}}Q_{l\bar{l}}\widehat{c}_{\bar{i}\bar{k}\bar{l}}(\varrho^*,\varepsilon^*) \quad \text{for all } i,k,l,$$

and all orthogonal matrices Q with positive determinant. This says, that (for fixed values of $\varrho^*(t^*,x^*)$ and $\varepsilon^*(t^*,x^*)$) the 3-tensor $(\widehat{c}_{ikl}(\varrho^*,\varepsilon^*))_{i,k,l=1,\dots,n}$ behaves like a constant objective 3-tensor, which is symmetric in the last two indices. This implies that it has to vanish. Thus the assertion for q is proved. □

Proof for Π. For $\widehat{\Pi}$ one obtains independence of v and the antisymmetric part of $\mathrm{D}v$ in the same manner as for q. Then

$$\widehat{\Pi}(\varrho^*,\varepsilon^*,0,0,0) = Q\widehat{\Pi}(\varrho^*,\varepsilon^*,0,0,0)\,Q^{\mathrm{T}}$$

for all orthogonal matrices Q. This implies that $\widehat{\Pi}(\varrho^*,\varepsilon^*,0,0,0)$ (for fixed values of ϱ^* and ε^*) is a constant objective tensor, and therefore, for $n \geq 3$, is a multiple of the identity, that is,

$$\widehat{\Pi}(\varrho^*,\varepsilon^*,0,0,0) = \widehat{p}(\varrho^*,\varepsilon^*)\,\mathrm{Id}\,.$$

Then $\Pi = p\,\mathrm{Id} - S$ and S has a representation

$$\begin{aligned}
S_{ij} &= \sum_{k=1}^n \widehat{a}_{ijk}(\varrho,\varepsilon)\partial_k\varepsilon + \sum_{k=1}^n \widehat{b}_{ijk}(\varrho,\varepsilon)\partial_k\varrho\\
&\quad + \sum_{k,l=1}^n \widehat{c}_{ijkl}(\varrho,\varepsilon)\frac{\partial_k v_l + \partial_l v_k}{2},
\end{aligned}$$

where we can assume that $c_{ijkl} = c_{ijlk}$ for all $i,j,k,l = 1,\dots,n$. Now Π and then also S is an objective tensor, that is

$$S_{ij}\circ Y = \sum_{\bar{i},\bar{j}=1}^n Q_{i\bar{i}}Q_{j\bar{j}}S^*_{\bar{i},\bar{j}}.$$

This leads, with the above notation, to the identities

$$\begin{aligned}
&\textstyle\sum_{k=1}^{n} a_{ijk} Q_{k\bar{k}} = \sum_{\bar{i},\bar{j}=1}^{n} Q_{i\bar{i}} Q_{j\bar{j}} a_{\overline{ijk}} && \text{for all } i, j, \overline{k}, \\
&\textstyle\sum_{k=1}^{n} b_{ijk} Q_{k\bar{k}} = \sum_{\bar{i},\bar{j}=1}^{n} Q_{i\bar{i}} Q_{j\bar{j}} b_{\overline{ijk}} && \text{for all } i, j, \overline{k}, \\
&\textstyle\sum_{k,l=1}^{n} c_{ijkl} Q_{k\bar{k}} Q_{l\bar{l}} = \sum_{\bar{i},\bar{j}=1}^{n} Q_{i\bar{i}} Q_{j\bar{j}} c_{\overline{ijkl}} && \text{for all } i, j, \overline{k}, \overline{l}.
\end{aligned}$$

Again we rewrite this so that we have Q-terms only on the right-hand side. This gives, that $(a_{ijk})_{i,j,k=1,\ldots,n}$ behaves like a constant objective 3-tensor. This implies that it is antisymmetric in each pair of indices (for $n = 3$, for $n \geq 4$ it follows that $a_{ijk} = 0$), hence $a_{ijk} + a_{jik} = 0$. Therefore this term gives no contribution to the symmetric part of S. The same follows for the b-term. The third identity gives that $(c_{ijkl})_{i,j,k,l=1,\ldots,n}$ is a constant objective 4-tensor, which is symmetric in the last two indices. Since S, by (11.2), is symmetric, this implies that the symmetric part with respect to the first two indices is of the form

$$c_{ijkl} = \frac{a}{2}(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b\delta_{k,l}\delta_{i,j}$$

with two scalars a, b (see section 16). □

Further we obtain the following

11.6 Lemma. Let (η, ψ) be as in 11.4. Then objectivity implies that

$$\begin{aligned}
&\eta = \widehat{\eta}(\varrho, \varepsilon), \quad \psi = \eta v + \eta'_{\varepsilon} q + \psi_0, \\
&\psi_0 = \widehat{\psi}_0(\varrho, \varepsilon, \nabla\varrho, (\mathrm{D}v)^{\mathrm{S}}, \nabla\varepsilon)
\end{aligned}$$

and $\widehat{\psi}_0(\varrho, \varepsilon, 0, 0, 0) = 0$.

Proof. Apply similar arguments as above. □

Now we draw conclusion from the entropy principle and we are ready to formulate the "necessity theorem".

11.7 Theorem. Let 11.4 be satisfied, in particular

$$\eta = \widehat{\eta}(\varrho, v, \varepsilon),$$

and assume 11.5 and let $g = v \bullet \mathbf{f}$. Assume that the entropy principle holds. Then

$$\psi = \eta v + \eta'_{\varepsilon} q + \psi_0,$$

where $\operatorname{div}\psi_0 = 0$ (ψ_0 is of the form (11.17)), and

(1) ***Gibbs relation.*** $\eta = \varrho\eta'_{\varrho} + (\varepsilon + p)\eta'_{\varepsilon}$,

(2) ***Dissipation.*** $\eta'_{\varepsilon}\mathrm{D}v \bullet S \geq 0$, and $\nabla\eta'_{\varepsilon} \bullet q \geq 0$.

For the equation of the entropy flux ψ see also theorem 11.9. As a result in theorem 11.7 we have the identity

$$\begin{aligned}
0 \leq h &= \partial_t\eta + \operatorname{div}(\eta v + \eta'_{\varepsilon} q) \\
&= \eta'_{\varepsilon}\mathrm{D}v \bullet S + \nabla\eta'_{\varepsilon} \bullet q.
\end{aligned} \tag{11.13}$$

Proof. By the objectivity of η we know $\eta = \widehat{\eta}(\varrho, \varepsilon)$ (see 11.6). Then we get from 11.5 that for the entropy production, see (11.8),

$$\begin{aligned} 0 \le h &= \partial_t \eta + \mathrm{div}\,(\eta v + \eta_{'\varepsilon} q + \psi_0) \\ &= \mathrm{div} v \cdot (\eta - \eta_{'\varrho}\varrho - \eta_{'\varepsilon}(\varepsilon + p)) \\ &\quad + \eta_{'\varepsilon} \mathrm{D}v \bullet S + (\nabla \eta_{'\varepsilon}) \bullet q + \mathrm{div}\psi_0. \end{aligned}$$

Making use of $\psi_0 = \widehat{\psi}_0(\varrho, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon)$ and that ψ_0 depends only on the symmetric part of $\mathrm{D}v$, that is $\psi_{0\,'\partial_j v_k} = \psi_{0\,'\partial_k v_j}$, the formula becomes

$$\begin{aligned} 0 \le h \;=\; & \mathrm{div} v \cdot (\eta - \eta_{'\varrho}\varrho - \eta_{'\varepsilon}(\varepsilon + p)) + \eta_{'\varepsilon} \mathrm{D}v \bullet \widehat{S}(\varrho, \varepsilon, \mathrm{D}v^{\mathrm{S}}) \\ & + \textstyle\sum_j (\eta_{'\varepsilon\varepsilon} q_j + \psi_{0j\,'\varepsilon}) \partial_j \varepsilon + \sum_j (\eta_{'\varepsilon\varrho} q_j + \psi_{0j\,'\varrho}) \partial_j \varrho \\ & + \textstyle\sum_{ij} \psi_{0i\,'\partial_j\varepsilon} \partial_i \partial_j \varepsilon + \sum_{ij} \psi_{0i\,'\partial_j\varrho} \partial_i \partial_j \varrho + \sum_{ijk} \psi_{0i\,'\partial_j v_k} \partial_i \partial_j v_k. \end{aligned} \tag{11.14}$$

We know, that this is nonnegative for all physical processes. That is $h \ge 0$ for functions $(\varrho, v, \varepsilon, \mathbf{f})$, which satisfy the equations (11.1) with $g = v \bullet \mathbf{f}$. We want to conclude that an algebraic version of this is satisfied. Now, by assumption 11.4 there exists a solution of system (11.1) in a neighbourhood of some point (t, x) with given values

$$(\varrho, v, \varepsilon, \partial_t\varrho, \nabla\varrho, \partial_t v, \mathrm{D}v, \partial_t\varepsilon, \nabla\varepsilon, \mathrm{D}^2\varrho, \mathrm{D}^2 v, \mathrm{D}^2\varepsilon), \tag{11.15}$$

provided the values at this point belong to a dense set and satisfy the algebraic version of system (11.1) (which is the differentiated version obtained in a similar way as the function h in (11.14) above). Since for all processes the entropy principle is satisfied, the inequality $h \ge 0$ holds in this neighbourhood and therefore (11.14) in the above point. Since the involved functions are C^1, we obtain that this conclusion is true for all arguments.

Therefore the algebraic version (if we write the arguments as in (11.15)) of this conclusion is satisfied: That is, the term in (11.14) is nonnegative, if the differentiated version of system (11.1) is fulfilled. Since each equation of the algebraic form of (11.1) has an explicit time derivative of $(\partial_t\varrho, \partial_t v, \partial_t\varepsilon)$, whereas the form (11.14) contains no time derivative, we conclude that (11.14) has to be satisfied for all arguments.

We now draw conclusions from this algebraic version. Let us first assume that $\mathrm{div}\,\psi_0 = 0$. Then the last two lines in (11.14) vanish, hence the entropy production (11.14) has a term quadratic in $(\mathrm{D}v)^{\mathrm{S}}$, the term containing $\widehat{S}$, a term quadratic in $(\nabla\varrho, \nabla\varepsilon)$, the term containing $\widehat{q}$, and a term linear in $\mathrm{div}\, v$. Then it follows that the coefficient of the linear term has to vanish, which is Gibb's relation, and that the two quadratic terms have to be nonnegative, which are the two dissipative terms in the theorem.

Therefore in order to show the theorem, we have to show that $\mathrm{div}\,\psi_0 = 0$. The expression (11.14) is explicit linear in the arguments $(\mathrm{D}^2\varrho, \mathrm{D}^2 v, \mathrm{D}^2\varepsilon)$. Dividing these arguments by a small positive δ, multiplying the inequality with δ, and letting $\delta \to 0$, we conclude, that the coefficients (taking the symmetry into account) have to vanish, that is

$$\psi_{0i\,'\partial_j\varepsilon} + \psi_{0j\,'\partial_i\varepsilon} = 0, \quad \psi_{0i\,'\partial_j\varrho} + \psi_{0j\,'\partial_i\varrho} = 0, \quad \psi_{0i\,'\partial_j v_k} + \psi_{0j\,'\partial_i v_k} = 0. \tag{11.16}$$

Denote by

$$C = \left(\psi_{0i\,'\partial_{j_1}\varepsilon\partial_{j_2}\varepsilon}\right)_{i,j_1,j_2=1,\dots,n}$$

the 3-matrix C, which is antisymmetric in (i, j_1) and symmetric in (i, j_2). It follows that $C = 0$ by 11.8, hence

$$\psi_{0i\,'\partial_{j_1}\varepsilon\partial_{j_2}\varepsilon} = 0.$$

In the same way $\psi_{0i\,'\partial_{j_1}\varrho\partial_{j_2}\varrho} = 0$ and $\psi_{0i\,'\partial_{j_1}v_k\partial_{j_1}v_k} = 0$. Now denote by

$$C = \left(\psi_{0i\,'\partial_j v_k}\right)_{i,j,k=1,\dots,n}$$

the 3-matrix C, which is antisymmetric in (i, j) and symmetric in (j, k). Consequently $C = 0$ by 11.8, which means that ψ_0 is independent of $\mathrm{D}v$.

All this implies, since $\widehat{\psi}_0(\varrho, \varepsilon, 0, 0, 0) = 0$, that

$$\psi_{0l} = \sum_j c^1_{lj}\partial_j\varrho + \sum_j c^2_{lj}\partial_j\varepsilon + \sum_{ij} d_{lij}\partial_i\varrho\partial_j\varepsilon$$

with constitutive relations $c^1_{lj} = \widehat{c^1}_{lj}(\varrho, \varepsilon)$ and the same for c^2_{lj} and d_{lij}. Since ψ_0 is an objective vector, the matrix $C = \left(c^m_{lj}\right)_{lj}$ satisfies $C = QCQ^{\mathrm{T}}$ for all rotations Q satisfying $\det Q = 1$. Moreover, by the above identity it is antisymmetric, that is $C + C^{\mathrm{T}} = 0$. Since $n \geq 3$, it follows that $C = 0$. Hence it remains the quadratic term

$$\begin{aligned} &\psi_{0l} = \textstyle\sum_{ij} d_{lij}\partial_i\varrho\partial_j\varepsilon, \\ &d_{lij} = \widehat{d}_{lij}(\varrho, \varepsilon) \text{ antisymmetric in } (l, i) \text{ and } (l, j). \end{aligned} \tag{11.17}$$

From this it follows that $\operatorname{div}\psi_0 = 0$. □

11.8 Lemma. Suppose $C = (C_{ijk})_{i,j,k=1,\dots,N}$ is a 3-matrix, which is antisymmetric in the first two indices, and symmetric in the last two indices. Then $C = 0$.

Proof. It is

$$\begin{aligned} C_{lkj} &= -C_{klj} = -C_{kjl} = C_{jkl}, \\ C_{lkj} &= C_{ljk} = -C_{jlk} = -C_{jkl}. \end{aligned}$$

Hence $C_{jkl} = 0$. □

In the above proof we used the so called “algebraic entropy principle”, which is a general procedure, and therefore can be applied under quite general circumstances. Now, under an additional assumption, we can show that $\psi_0 = 0$.

11.9 Theorem. If in addition to theorem 11.7

$$\begin{aligned} q_{i\,'\partial_j\varrho} = q_{j\,'\partial_i\varrho}, \quad q_{i\,'\partial_j\varepsilon} = q_{j\,'\partial_i\varepsilon}, \\ \psi_{i\,'\partial_j\varrho} = \psi_{j\,'\partial_i\varrho}, \quad \psi_{i\,'\partial_j\varepsilon} = \psi_{j\,'\partial_i\varepsilon}, \end{aligned}$$

then $\psi = \eta v + \eta_{'\varepsilon} q$, that is $\psi_0 = 0$.

Proof. We have to show that (11.17) implies that $\psi_0 = 0$. It follows, that the assumptions are also valid for ψ_0,

$$\psi_{0i\,'\partial_j\varrho} = \psi_{0j\,'\partial_i\varrho}, \quad \psi_{0i\,'\partial_j\varepsilon} = \psi_{0j\,'\partial_i\varepsilon}.$$

Then (11.16) implies that $\psi_{0i\,'\partial_j\varrho} = 0$ and $\psi_{0i\,'\partial_j\varepsilon} = 0$, so that $\psi_0 = 0$ by (11.17). □

12 Elastic solids

The physical formulation of every material starts in the physical domain. Therefore, for a single body, the basis for the formulation is the conservation of mass, momentum, and energy (as in (10.9) with $J = 0$, see also (11.1)), that is

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v) = r, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v \otimes v + \Pi) = \mathbf{f}, \\ &\partial_t e + \operatorname{div}(e v + \Pi^{\mathrm{T}} v + q) = g \end{aligned} \tag{12.1}$$

in a time-space domain Ω, see (12.3). Objectivity of the system (12.1) requires by section 10.3 the following:

$$\begin{aligned} &\varrho,\ \varepsilon,\ r \text{ are objective scalars, } e = \varepsilon + \tfrac{\varrho}{2}|v|^2, \\ &v \text{ is a velocity (see (8.7)),} \\ &q \text{ is an objective vector, } \Pi \text{ an objective tensor,} \end{aligned} \tag{12.2}$$

and $\mathbf{f}$ is an external force and g an energy source (satisfying the identities in (10.10)). In contrast to fluids the velocity v is not an independent variable, is it given by a map

$$\begin{aligned} &\phi :]t_1, t_2[\times \mathcal{B} \to \mathrm{I\!R}^n, \quad (t, \underline{x}) \mapsto (t, x) = \tau(t, \underline{x}) := (t, \phi(t, \underline{x})), \\ &\Omega_t = \{x \in \mathrm{I\!R}^n\,;\ (t, x) \in \Omega\} = \{\phi(t, \underline{x})\,;\ \underline{x} \in \mathcal{B}\}, \end{aligned} \tag{12.3}$$

via the formula

$$v(t, x) = \partial_t \phi(t, \underline{x}) \quad \text{for} \quad x = \phi(t, \underline{x}). \tag{12.4}$$

Here $\mathcal{B}$ is the "unperturbed" body, so that v can be considered as the velocity of a "single particle". Is is assumed that $\underline{x} \mapsto \phi(t, \underline{x})$ is an isomorphism with

$$\det \mathrm{D}_{\underline{x}} \phi > 0.$$

Formula (12.4) is consistent with the definition of a velocity, since the transformation formula of ϕ between observers is

$$\phi(t, \underline{x}) = X(t^*, \phi^*(t^*, \underline{x})) \quad \text{for } t = T(t^*). \tag{12.5}$$

Here T denotes the time transformation for Newtonian observer transformation. We remark that $\phi(T(t^*), \underline{x}) = X(t^*, \phi^*(t^*, \underline{x}))$ implies, since $T(t^*)$ is only a shift in time,

$$\partial_t \phi(T(t^*), \underline{x}) = \dot{X}(t^*, \phi^*(t^*, \underline{x})) + Q(t^*) \partial_{t^*} \phi^*(t^*, \underline{x}).$$

This is consistent with the transformation rule for a velocity.

The inverse of ϕ is denoted by

$$\underline{x} = \xi(t, x) \quad \text{for} \quad x = \phi(t, \underline{x}). \tag{12.6}$$

Then equation (12.4) can be considered as a constitutive equation for v, which in terms of ξ instead of ϕ is formulated in the following lemma.

12.1 Lemma. The coordinates ξ_i, $i = 1, \dots, n$, are objective scalars and we have the constitutive equation $v = -\left(\mathrm{D}_x\xi\right)^{-1}\partial_t\xi$ for the velocity.

Proof. The fact, that ξ_i is an objective scalar, that is $\xi_i \circ Y = \xi_i^*$ for $i = 1, \dots, n$, or

$$\underline{x} = \xi(t,x) = \xi^*(t^*,x^*) \quad \text{for } (t,x) = Y(t^*,x^*), \tag{12.7}$$

is equivalent to (12.5). The identity $\underline{x} = \xi(t, \phi(t,\underline{x}))$ implies

$$0 = \partial_t\xi(t,\phi(t,\underline{x})) + \mathrm{D}_x\xi(t,\phi(t,\underline{x}))\partial_t\phi(t,\underline{x}),$$

so that (12.4) is equivalent to $0 = \partial_t\xi + (\mathrm{D}_x\xi)v$. □

We obtain the following theorem, where we refer to [13] for readers, who want to have a comparison with existing literature.

12.2 Theorem. With the transformation in (12.3) we define

$V := \partial_t\phi = \left(\partial_t\phi_i\right)_i$ the velocity,
$F := \mathrm{D}\phi = \left(\partial_{x_j}\phi_i\right)_{ij}$ the deformation gradient,
$J := \det F > 0$ the determinant,
$P := J \cdot (-\Pi\circ\tau)\, F^{-\mathrm{T}}$ the (first) Piola-Kirchhoff stress tensor,
$\underline{\varrho} := J \cdot (\varrho\circ\tau)$ the reference density,
$\underline{e} := J \cdot (e\circ\tau)$ the reference energy,
$\underline{q} := J \cdot F^{-1}(q\circ\tau)$ the reference heat flux.

Then the differential equations (12.1) in reference coordinates read

$$\begin{aligned} &\partial_t\underline{\varrho} = J \cdot (r\circ\tau), \\ &\partial_t(\underline{\varrho}V) - \mathrm{div}P = J \cdot (\mathbf{f}\circ\tau), \\ &\partial_t\underline{e} + \mathrm{div}(-P^{\mathrm{T}}V + \underline{q}) = J \cdot (g\circ\tau). \end{aligned} \tag{12.8}$$

Proof. The first equation in (12.1) in its weak form reads

$$\int_\Omega \left(\partial_t\eta \cdot \varrho + \nabla\eta\bullet(\varrho v) + \eta \cdot r\right)\, \mathrm{dL}_{n+1} = 0$$

for $\eta \in C_0^\infty(\Omega;\mathrm{I\!R})$. We transform this into an integral over $]t_1,t_2[\times\mathcal{B} = \tau^{-1}(\Omega)$. Defining

$$\widetilde{\eta}(t,\underline{x}) = \eta(t,\phi(t,\underline{x})), \quad \tau(t,\underline{x}) = (t,\phi(t,\underline{x})),$$

we obtain

$$\partial_t\widetilde{\eta} = (\partial_t\eta)\circ\tau + (\nabla\eta)\circ\tau\bullet\partial_t\phi = (\partial_t\eta + v\bullet\nabla\eta)\circ\tau,$$

and the weak equation becomes, since $J = \det \mathrm{D}_{\underline{x}}\phi = \left|\det \mathrm{D}_{(t,\underline{x})}\tau\right|$,

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \left(\partial_t\widetilde{\eta} \cdot J\varrho\circ\tau + \widetilde{\eta} \cdot Jr\circ\tau\right)\, \mathrm{dL}_n\, \mathrm{dL}_1 = 0.$$

In its strong version this is

$$\partial_t(J\varrho\circ\tau) = Jr\circ\tau.$$

The second equation in (12.1) is for $\zeta \in C_0^\infty(\Omega; \mathbb{R}^n)$

$$\int_\Omega \left(\partial_t\zeta \bullet (\varrho v) + \sum_{i,j=1}^n \partial_j\zeta_i \cdot (\varrho v_i v_j + \Pi_{ij}) + \zeta \bullet \mathbf{f} \right) \mathrm{dL}_{n+1} = 0.$$

Transforming this via $\widetilde{\zeta}(t, \underline{x}) = \zeta(t, \phi(t, \underline{x}))$, so that

$$\partial_t \widetilde{\zeta} = (\partial_t\zeta + v \bullet \nabla\zeta)\circ\tau, \quad \mathrm{D}_{\underline{x}}\widetilde{\zeta} = (\mathrm{D}_x\zeta)\circ\tau \mathrm{D}_{\underline{x}}\phi,$$

we see that the above integral equals

$$\begin{aligned}
&= \int_\Omega \left((\partial_t\zeta + \sum_{j=1}^n v_j\partial_j\zeta)\bullet(\varrho v) + (\mathrm{D}_x\zeta)\bullet\Pi + \zeta\bullet\mathbf{f} \right) \mathrm{dL}_{n+1} \\
&= \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\partial_t\widetilde{\zeta}\bullet(J(\varrho v)\circ\tau) + J \cdot (\mathrm{D}_{\underline{x}}\widetilde{\zeta}\,(\mathrm{D}_{\underline{x}}\phi)^{-1})\bullet(\Pi\circ\tau) + \widetilde{\zeta}\bullet(J\mathbf{f}\circ\tau) \right) \mathrm{dL}_n\,\mathrm{dL}_1 \\
&= \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\partial_t\widetilde{\zeta}\bullet(J(\varrho\circ\tau)V) + (\mathrm{D}_{\underline{x}}\widetilde{\zeta})\bullet(J(\Pi\circ\tau)\,(\mathrm{D}_{\underline{x}}\phi)^{-\mathrm{T}}) + \widetilde{\zeta}\bullet(J\mathbf{f}\circ\tau) \right) \mathrm{dL}_n\,\mathrm{dL}_1.
\end{aligned}$$

In its strong version this is

$$\partial_t(J(\varrho\circ\tau)V) + \mathrm{div}_{\underline{x}}(J(\Pi\circ\tau)\,(\mathrm{D}_{\underline{x}}\phi)^{-\mathrm{T}}) = J\mathbf{f}\circ\tau.$$

The energy equation in (12.1) is for $\eta \in C_0^\infty(\Omega; \mathbb{R})$

$$\int_\Omega \left(\partial_t\eta \cdot e + \nabla\eta\bullet(ev + \Pi^{\mathrm{T}}v + q) + \eta \cdot g \right) \mathrm{dL}_{n+1} = 0.$$

For $\widetilde{\eta}$, defined as above, this becomes

$$\int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\partial_t\widetilde{\eta} \cdot J(e\circ\tau) + J \cdot ((\mathrm{D}\phi)^{-\mathrm{T}}\nabla\widetilde{\eta})\bullet((\Pi\circ\tau)^{\mathrm{T}}V + q\circ\tau) + \widetilde{\eta} \cdot (Jg\circ\tau) \right) \mathrm{dL}_n\,\mathrm{dL}_1 = 0.$$

Since

$$\begin{aligned}
& J \cdot \left((\mathrm{D}\phi)^{-\mathrm{T}}\nabla\widetilde{\eta} \right) \bullet \left((\Pi\circ\tau)^{\mathrm{T}}V + q\circ\tau \right) \\
&= J \cdot \nabla\widetilde{\eta}\bullet \left((\mathrm{D}\phi)^{-1}(\Pi\circ\tau)^{\mathrm{T}}V + (\mathrm{D}\phi)^{-1}(q\circ\tau) \right) \\
&= J \cdot \nabla\widetilde{\eta}\bullet \left(\left((\Pi\circ\tau)\,(\mathrm{D}\phi)^{-\mathrm{T}} \right)^{\mathrm{T}} V + (\mathrm{D}\phi)^{-1}(q\circ\tau) \right) \\
&= \nabla\widetilde{\eta}\bullet \left(-P^{\mathrm{T}}V + \underline{q} \right) ,
\end{aligned}$$

the result follows. □

Equivalently, system (12.1) can be written as (see (11.4))

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}(\varrho v) = r, \\ &\varrho(\partial_t v + v \bullet \nabla v) + \mathrm{div}\Pi = \widetilde{\mathbf{f}}, \\ &\partial_t \varepsilon + \mathrm{div}(\varepsilon v + q) + \mathrm{D}v \bullet \Pi = \widetilde{g}, \end{aligned} \tag{12.9}$$

where $\widetilde{\mathbf{f}}$ and $\widetilde{g}$ are defined in (10.8). The last differential equation we compute in reference coordinates, where

$$\underline{r} = J \cdot (r \circ \tau), \quad \underline{\mathbf{f}} = J \cdot (\mathbf{f} \circ \tau), \quad \underline{g} = J \cdot (\widetilde{g} \circ \tau)$$

12.3 Theorem. System (12.8) is equivalent to

$$\begin{aligned} &\partial_t \underline{\varrho} = \underline{r}, \\ &\partial_t (\underline{\varrho} V) - \mathrm{div} P = \underline{\mathbf{f}}, \\ &\partial_t \underline{e} + \mathrm{div}\underline{q} = (\mathrm{D}V) \bullet P + \underline{g}. \end{aligned} \tag{12.10}$$

Here

$$\underline{e} = \underline{\varepsilon} + \frac{\underline{\varrho}}{2}|V|^2, \quad \underline{\varepsilon} = J \cdot (\varepsilon \circ \tau).$$

Remark: Introducing

$$\begin{aligned} &S := F^{-1}P \quad \text{the second Piola-Kirchhoff stress tensor,} \\ &D := \frac{1}{2}\left(F^{\mathrm{T}}\mathrm{D}V + (\mathrm{D}V)^{\mathrm{T}}F\right) \quad \text{the material rate of deformation tensor,} \end{aligned}$$

then, if Π is symmetric, $(\mathrm{D}V) \bullet P = D \bullet S$.

The first two equations of (12.8), resp. (12.9), are equivalent to the mass and momentum equation. If $r = 0$, then the first equation of (12.10) reduces to a given function $\underline{\varrho} = \widehat{\underline{\varrho}}(\underline{x})$. Consequently the second and third equation are equations for (V, ε).

Proof. Similar as in the previous proof we obtain from the last equation in (12.9)

$$\partial_t \underline{\varepsilon} + \mathrm{div}_{\underline{x}} \underline{q} = J(\mathrm{D}_x v \bullet (-\Pi)) \circ \tau + J(g \circ \tau).$$

For the first term on the right hand side we compute

$$J(\mathrm{D}_x v \bullet (-\Pi)) \circ \tau = ((\mathrm{D}_x v) \circ \tau) \bullet (-J(\Pi \circ \tau)) = ((\mathrm{D}_x v) \circ \tau \mathrm{D}_{\underline{x}} \phi) \bullet (-J(\Pi \circ \tau)\,(\mathrm{D}_{\underline{x}} \phi)^{-\mathrm{T}}),$$

so that we can use the definition of the first Piola-Kirchhoff stress tensor, moreover,

$$((\mathrm{D}_x v) \circ \tau \mathrm{D}_{\underline{x}} \phi)(t, \underline{x}) = (\mathrm{D}_x v)(t, \phi(t, \underline{x})) \mathrm{D}_{\underline{x}} \phi(t, \underline{x}) = \mathrm{D}_{\underline{x}}\left(v(t, \phi(t, \underline{x}))\right) = \mathrm{D}_{\underline{x}} V(t, \underline{x}).$$

□

Proof of Remark. The identity $(\mathrm{D}V)\bullet P = (\mathrm{D}V)\bullet(FS) = (F^{\mathrm{T}}\mathrm{D}V)\bullet S$ holds in any case. If Π is symmetric, then also $S = JF^{-1}(-\Pi\circ\tau)F^{-\mathrm{T}}$ is symmetric. It follows that $(F^{\mathrm{T}}\mathrm{D}V)\bullet S = D\bullet S$. □

We now state the entropy principle.

12.4 Entropy principle. We assume that there is an entropy η and an entropy flux ψ, such that

$$h := \partial_t \eta + \operatorname{div}\psi \geq 0$$

for all solutions of the problem with given constitutive relations. In reference coordinates

$$\underline{\eta} := J\cdot\eta\circ\tau, \quad \underline{\psi} := J\cdot F^{-1}(\psi - \eta v)\circ\tau,$$

this is equivalent to

$$\underline{h} := \partial_t\underline{\eta} + \operatorname{div}\underline{\psi} \geq 0.$$

As in the above proofs it follows that $\underline{h} = J\cdot h\circ\tau$. It is assumed, that the entropy equation transforms like an objective scalar, that is η is an objective scalar and $\psi - \eta v$ an objective vector. To specify the entropy principle, one has to prescribe the physical processes one has in mind. We consider here the most elementary case of a constitutive dependence on $(\xi, \varepsilon, \mathrm{D}\xi)$, or equivalently, on $(\underline{x}, \underline{\varepsilon}, \mathrm{D}\phi)$ in reference coordinates.

12.5 Lemma. Let us assume a constitutive relation $\eta = \widehat{\eta}(\xi, \varepsilon, \mathrm{D}\xi)$. This is equivalent to $\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F)$. Then

$$\underline{\eta}_{,\underline{\varepsilon}} = \eta_{,\varepsilon}\circ\tau, \quad \eta_{,\varepsilon} = \underline{\eta}_{,\underline{\varepsilon}}\circ\tau^{-1},$$

$$\underline{\eta}_{,F} = J\cdot\Big((\eta - \varepsilon\eta_{,\varepsilon})\,\mathrm{Id} - (\mathrm{D}\xi)^{\mathrm{T}}\eta_{,\mathrm{D}\xi}\Big)\circ\tau \;\; F^{-\mathrm{T}}\ ,$$

$$\eta_{,\mathrm{D}\xi} = \det \mathrm{D}\xi\cdot\Big((\underline{\eta} - \underline{\varepsilon}\underline{\eta}_{,\underline{\varepsilon}})\,\mathrm{Id} - F^{\mathrm{T}}\underline{\eta}_{,F}\Big)\circ\tau^{-1} \;\; (\mathrm{D}\xi)^{-\mathrm{T}}\ .$$

Proof. Since $\xi\circ\tau(t,\underline{x}) = \underline{x}$ one gets $(\mathrm{D}\xi)\circ\tau = F^{-1}$, and since $J = \det F$ one obtains

$$\underline{\eta} = J\cdot\eta\circ\tau = J\cdot\widehat{\eta}\left(\underline{x}, \tfrac{1}{J}\underline{\varepsilon}, F^{-1}\right),$$

hence

$$\widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F) := \det F\cdot\widehat{\eta}\left(\underline{x}, \tfrac{1}{\det F}\underline{\varepsilon}, F^{-1}\right). \tag{12.11}$$

On the other hand, if $\widehat{\underline{\eta}}$ is given, define

$$\widehat{\eta}(\xi, \varepsilon, M) := \det M\cdot\widehat{\underline{\eta}}\left(\xi, \tfrac{1}{\det M}\varepsilon, M^{-1}\right).$$

From these representations one derives the equations for the derivatives. □

With this constitutive relation for the entropy we have the following

12.6 Theorem. Assume that

(1) $\eta = \widehat{\eta}(\xi, \varepsilon, \mathrm{D}\xi)$ and $\psi = \eta v + \eta_{,\varepsilon} q + \psi_0$. Then for solutions of the differential equations

$$\begin{aligned} h = &\ \partial_t \eta + \operatorname{div} \psi \\ = &\ \eta_{,\varepsilon} \widetilde{g} + \operatorname{div} \psi_0 + (\nabla \eta_{,\varepsilon}) \bullet q \\ &+ \Big((\eta - \varepsilon \eta_{,\varepsilon}) \,\mathrm{Id} - (\mathrm{D}\xi)^{\mathrm{T}} \eta_{,\mathrm{D}\xi} - \eta_{,\varepsilon} \Pi \Big) \bullet \mathrm{D}v. \end{aligned}$$

(2) $\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F)$ and $\underline{\psi} = \underline{\eta}_{,\underline{\varepsilon}} \underline{q} + \widetilde{\psi}_0$. Then for solutions of the differential equations

$$\begin{aligned} \underline{h} = &\ \partial_t \underline{\eta} + \operatorname{div} \underline{\psi} \\ = &\ \underline{\eta}_{,\underline{\varepsilon}} J \cdot (\widetilde{g} \circ \tau) + \operatorname{div} \widetilde{\psi}_0 + (\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q} \\ &+ (\underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F}) \bullet \mathrm{D}V \ . \end{aligned}$$

The two statements in this theorem, the formulation 12.6(2) in reference coordinates and 12.6(1) in the physical domain, are equivalent statements. This can be shown by transforming the corresponding terms, and therefore only one of the following proofs is necessary.

Proof of (1). Since $\eta = \widehat{\eta}(\xi, \varepsilon, \mathrm{D}\xi)$ we compute

$$\begin{aligned} h = &\ \partial_t \eta + \operatorname{div} \psi = \partial_t \eta + \operatorname{div}(\eta v + \eta_{,\varepsilon} q + \psi_0) \\ = &\ (\partial_t + v \bullet \nabla) \eta + \eta \operatorname{div} v + \operatorname{div}(\eta_{,\varepsilon} q + \psi_0) \\ = &\ \eta_{,\xi} (\partial_t + v \bullet \nabla) \xi + \eta_{,\varepsilon} (\partial_t + v \bullet \nabla) \varepsilon + \eta_{,\mathrm{D}\xi} \bullet \left((\partial_t + v \bullet \nabla) \mathrm{D}\xi \right) \\ &+ \eta \operatorname{div} v + \operatorname{div}(\eta_{,\varepsilon} q + \psi_0) \,. \end{aligned}$$

We use the last equation in (12.9) to replace

$$(\partial_t + v \bullet \nabla) \varepsilon = \widetilde{g} - \operatorname{div} q - \varepsilon \operatorname{div} v - \Pi \bullet \mathrm{D}v.$$

Besides this there are two identities in connection with the reference coordinates

$$\begin{aligned} 0 &= \partial_t \xi + v \bullet \nabla \xi, \\ 0 &= \partial_t \mathrm{D}\xi + v \bullet \nabla \mathrm{D}\xi + \mathrm{D}\xi \mathrm{D}v. \end{aligned}$$

Using this the entropy production becomes

$$\begin{aligned} h = &\ \eta_{,\varepsilon} (\widetilde{g} - \operatorname{div} q - \varepsilon \operatorname{div} v - \Pi \bullet \mathrm{D}v) \\ &- \eta_{,\mathrm{D}\xi} \bullet (\mathrm{D}\xi \mathrm{D}v) \\ &+ \eta \operatorname{div} v + \operatorname{div}(\eta_{,\varepsilon} q + \psi_0) \\ = &\ \eta_{,\varepsilon} \widetilde{g} + \underbrace{\operatorname{div} \psi_0}_{\text{flux term}} + \underbrace{(\nabla \eta_{,\varepsilon}) \bullet q}_{\text{remainder}} \\ &+ \underbrace{((\eta - \varepsilon \eta_{,\varepsilon}) \,\mathrm{Id} - (\mathrm{D}\xi)^{\mathrm{T}} \eta_{,\mathrm{D}\xi} - \eta_{,\varepsilon} \Pi) \bullet \mathrm{D}v}_{\mathrm{D}v\text{-term}}, \end{aligned}$$

where we have used that $\eta_{,\mathrm{D}\xi} \bullet (\mathrm{D}\xi \mathrm{D}v) = ((\mathrm{D}\xi)^{\mathrm{T}} \eta_{,\mathrm{D}\xi}) \bullet \mathrm{D}v$. □

Proof of (2). Since $\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F)$ we obtain

$$\begin{aligned} \underline{h} = \quad & \partial_t \underline{\eta} + \operatorname{div} \underline{\psi} = \partial_t \underline{\eta} + \operatorname{div}\Big(\underline{\eta}_{,\underline{\varepsilon}} \underline{q} + \widetilde{\psi}_0\Big) \\ = \quad & \underline{\eta}_{,\underline{\varepsilon}} \partial_t \underline{\varepsilon} + \underline{\eta}_{,F} \bullet \partial_t F + \operatorname{div}\Big(\underline{\eta}_{,\underline{\varepsilon}} \underline{q} + \widetilde{\psi}_0\Big) . \end{aligned}$$

We use the last equation in (12.10) to replace

$$\partial_t \underline{\varepsilon} = J \cdot (\widetilde{g} \circ \tau) - \operatorname{div} \underline{q} + P \bullet \mathrm{D}V \ .$$

Using that $\partial_t F = \mathrm{D}V$ the formula for $\underline{h}$ becomes

$$\begin{aligned} \underline{h} = \quad & \underline{\eta}_{,\underline{\varepsilon}} J \cdot (\widetilde{g} \circ \tau) + \underbrace{\operatorname{div} \widetilde{\psi}_0}_{\text{flux term}} + \underbrace{(\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q}}_{\text{remainder}} \\ & + \underbrace{(\underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F}) \bullet \mathrm{D}V}_{\text{linear in } \mathrm{D}V} . \end{aligned}$$

□

The following statements will be formulated only in reference coordinates. Similar as in 12.6 an equivalent formulation in physical coordinates is always possible. We begin with a sufficient condition for the entropy principle.

12.7 Theorem. Let $\underline{r} = 0$ and $\underline{g} = 0$ (that is $r = 0$ and $g = v \bullet \mathbf{f}$), and assume that

$$\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F), \quad \underline{\psi} = \underline{\eta}_{,\underline{\varepsilon}} \underline{q}.$$

Then the entropy principle is satisfied, if

$$\begin{aligned} & \underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F} = 0 \ , \\ & \underline{h} = (\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q} \geq 0 \ . \end{aligned} \tag{12.12}$$

Proof. From the assumptions on the rates and on $\underline{\psi}$ we obtain that the entropy production in reference coordinates, by the previous theorem, is

$$\underline{h} = (\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q} + (\underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F}) \bullet \mathrm{D}V \ .$$

The second term on the right-hand side vanishes by assumption. Then the entropy principle reduces to $\underline{h} = (\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q} \geq 0$. □

In this theorem we have shown that the entropy principle is fulfilled, if certain assumptions, among them (12.12), are satisfied. Now we want to assume the entropy principle and draw conclusions from it, among them (12.12). For this we need the following assumptions.

12.8 Assumption (Solution property). We assume constitutive relations

$$\begin{aligned} P &= \widehat{P}(\underline{x}, \underline{\varepsilon}, F, \nabla \underline{\varepsilon}, \mathrm{D}F) \ , \\ \underline{q} &= \widehat{\underline{q}}(\underline{x}, \underline{\varepsilon}, F, \nabla \underline{\varepsilon}, \mathrm{D}F) \ , \end{aligned} \tag{12.13}$$

where $\widehat{P}$ and $\widehat{\underline{q}}$ are C^1-functions, and similar for the entropy and entropy flux

$$\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F) \ , \quad \underline{\psi} = \widehat{\underline{\psi}}(\underline{x}, \underline{\varepsilon}, F, \nabla \underline{\varepsilon}, \mathrm{D}F) \ . \tag{12.14}$$

For an open and dense set of values $(\underline{x}, \underline{\varepsilon}, F, \nabla \underline{\varepsilon}, \mathrm{D}F)$ the following holds: If there is a polynomial $(\xi, \underline{\varepsilon}, \mathbf{f})$, which solves (12.1) at a point and coincides with the given values, then there is a solution of (12.1) in a neighbourhood of this point, which coincides with the polynomial at this point in all terms which occur in (12.1).

Since $\partial_k F_{ij} = \partial_{kj}\phi_i = \partial_j F_{ik}$ for all $i, j, k = 1, \ldots, n$, we assume that functions, which depend on $\mathrm{D}F_{ij} = (\partial_{kj}\phi_i)_{k=1,\ldots,n}$, are symmetric in (k, j). For example, we have the identity

$$\underline{\psi}_{\prime \partial_k F_{ij}} = \underline{\psi}_{\prime \partial_j F_{ik}}. \tag{12.15}$$

With this assumption we can formulate our theorem. For the heat flux and the entropy flux we assume in addition the Onsager's type assumption

$$\begin{aligned} q_{l \prime \partial_k F_{ij}} &= q_{k \prime \partial_l F_{ij}}, \qquad & q_{l \prime \partial_k \underline{\varepsilon}} &= q_{k \prime \partial_l \underline{\varepsilon}}, \\ \widetilde{\psi}_{l \prime \partial_k F_{ij}} &= \widetilde{\psi}_{k \prime \partial_l F_{ij}}, \qquad & \widetilde{\psi}_{l \prime \partial_k \underline{\varepsilon}} &= \widetilde{\psi}_{k \prime \partial_l \underline{\varepsilon}}. \end{aligned} \tag{12.16}$$

12.9 Theorem. Let us assume that $r = 0$ and $g = v \bullet \mathbf{f}$. Further, let us assume the constitutive relations in (12.13) and (12.14), in particular,

$$\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F) \ ,$$

and $\widehat{\underline{\psi}}(\underline{x}, \underline{\varepsilon}, F, 0, 0) = 0$, and the assumption in 12.8. Then, if (12.16) is satisfied and if the entropy prinicple holds, we conclude

$$\begin{aligned} \underline{\eta}_{\prime \underline{\varepsilon}} P &= -\underline{\eta}_{\prime F} \ , \\ (\nabla \underline{\eta}_{\prime \underline{\varepsilon}}) \bullet \underline{q} &\geq 0 \ , \end{aligned}$$

and $\widehat{\underline{q}}(\underline{x}, \underline{\varepsilon}, F, 0, 0) = 0$. In addition, the entropy flux is given by

$$\underline{\psi} = \underline{\eta}_{\prime \underline{\varepsilon}} \underline{q}.$$

The residual entropy inequality is

$$0 \leq h = (\nabla \underline{\eta}_{\prime \underline{\varepsilon}}) \bullet \underline{q}$$

The theorem can also be formulated in the physical domain, where Π, q, and ψ depend on $(\xi, \varepsilon, \partial_t \xi, \mathrm{D}\xi, \nabla \varepsilon, \mathrm{D}^2 \xi)$, and the entropy η depends only on $(\xi, \varepsilon, \mathrm{D}\xi)$. Of course, in addition η is an objective scalar, q and $\psi - \eta v$ are objective vectors, and Π is an objective tensor. What are the consequences for the quantities in reference coordinates? The connection between q and $\underline{q}$ is $q \circ \tau = J^{-1} F \underline{q}$, and for another observer $q^* \circ \tau^* = J^{*\,-1} F^* \underline{q}^*$. Then the identity $q \circ Y = Q q^*$, the transformation rule for the heat flux, implies, since $\tau(t, \underline{x}) = Y \circ \tau^*(t^*, \underline{x})$ for $t = T(t^*)$,

$$\begin{aligned} \frac{1}{J(t,\underline{x})} F(t,\underline{x}) \underline{q}(t,\underline{x}) &= q \circ \tau(t,\underline{x}) = Q(t^*) q^* \circ \tau^*(t^*,\underline{x}) \\ &= \frac{1}{J^*(t^*,\underline{x})} Q(t^*) F^*(t^*,\underline{x}) \underline{q}^*(t^*,\underline{x}) \ . \end{aligned}$$

Since $\tau(t,\underline{x}) = Y \circ \tau^*(t^*,\underline{x})$ implies $F(t,\underline{x}) = Q(t^*) F^*(t^*,\underline{x})$ and $J(t,\underline{x}) = J^*(t^*,\underline{x})$, one obtains

$$\underline{q}(t,\underline{x}) = \underline{q}^*(t^*,\underline{x}),$$

In the same way $\underline{\psi}(t,\underline{x}) = \underline{\psi}^*(t^*,\underline{x})$, and similar $P(t,\underline{x}) = Q(t^*) P^*(t^*,\underline{x})$. The objectivity of $\widehat{\underline{q}}$, for example, then implies

$$\widehat{\underline{q}}(\underline{x}, \underline{\varepsilon}^*, QF^*, \nabla\underline{\varepsilon}^*, Q\mathrm{D}F^*) = \widehat{\underline{q}}(\underline{x}, \underline{\varepsilon}^*, F^*, \nabla\underline{\varepsilon}^*, \mathrm{D}F^*)$$

We do not use these identities in the following proof.

Proof. With $\widetilde{\psi}_0 := \underline{\psi} - \underline{\eta}_{,\underline{\varepsilon}} \underline{q}$ we obtain from 12.6(2) for all solutions of the differential equations, since $\underline{\eta}$ depends only on $(\underline{x}, \underline{\varepsilon}, F)$,

$$\begin{aligned} 0 \le \underline{h} = &\ \operatorname{div} \widetilde{\psi}_0 + (\nabla \underline{\eta}_{,\underline{\varepsilon}}) \bullet \underline{q} + (\underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F}) \bullet \mathrm{D}V \\ = &\ \textstyle\sum_i \left(\underline{\eta}_{,\underline{\varepsilon} \underline{x}_i} \underline{q} + \widetilde{\psi}_{0\,,\underline{x}_i} \right) \bullet e_i + \left(\underline{\eta}_{,\underline{\varepsilon}\underline{\varepsilon}} \underline{q} + \widetilde{\psi}_{0\,,\underline{\varepsilon}} \right) \bullet \nabla \underline{\varepsilon} \\ &+ \textstyle\sum_{ij} \left(\underline{\eta}_{,\underline{\varepsilon} F_{ij}} \underline{q} + \widetilde{\psi}_{0\,,F_{ij}} \right) \bullet \nabla F_{ij} + (\underline{\eta}_{,\underline{\varepsilon}} P + \underline{\eta}_{,F}) \bullet \mathrm{D}V \\ &+ \textstyle\sum_k \widetilde{\psi}_{0\,,\partial_k \underline{\varepsilon}} \bullet \nabla \partial_k \underline{\varepsilon} + \sum_{ijk} \widetilde{\psi}_{0\,,\partial_k F_{ij}} \bullet \nabla \partial_k F_{ij} \ . \end{aligned} \tag{12.17}$$

We want to conclude that an algebraic version of this inequality is satisfied. Now q, P and $\widetilde{\psi}_0$ depend on $(\underline{x}, \underline{\varepsilon}, F, \nabla\underline{\varepsilon}, \mathrm{D}F)$ and η on $(\underline{x}, \underline{\varepsilon}, F)$. For a dense set of arguments, there exists a solution of the differential equations (12.8), or equivalently (12.10). For this solution the entropy inequality is satisfied, that is, (12.17) holds. At a point (t, x), the arguments

$$(\underline{\varepsilon}, V, F, \partial_t \underline{\varepsilon}, \nabla\underline{\varepsilon}, \partial_t V, \mathrm{D}F, \mathrm{D}^2 \underline{\varepsilon}, \mathrm{D}^2 F) \tag{12.18}$$

of this solution satisfy the algebraic version of (12.10) (which is the differentiated version obtained in a similar way as the function $\underline{h}$ in (12.17) above). Therefore at this point (12.17) is satisfied. Since the involved functions are C^1, we obtain the conclusion for all arguments.

Now each equation of the algebraic form of (12.10) has an explicit time derivative of

$$(\partial_t \underline{\varrho}, \partial_t V, \partial_t \underline{\varepsilon}) = (\partial_t \underline{\varrho}, \partial_t^2 \phi, \partial_t \underline{\varepsilon}),$$

whereas the form (12.17) does not contain these time derivatives. We conclude that (12.17) has to be satisfied for all arguments.

We now draw conclusions from this algebraic version. The expression (12.17) is explicit linear in the arguments

$$(\mathrm{D}V, \mathrm{D}^2\underline{\varepsilon}, \mathrm{D}^2F) = (\mathrm{D}\partial_t\phi, \mathrm{D}^2\underline{\varepsilon}, \mathrm{D}^3\phi). \tag{12.19}$$

Dividing the arguments in (12.19) by a small positive δ, multiplying (12.17) by δ, and letting $\delta \to 0$, we conclude, that the coefficients have to vanish. This gives

$$\underline{\eta}_{\prime\underline{\varepsilon}}P + \underline{\eta}_{\prime F} = 0, \tag{12.20}$$

and (taking the symmetry, see (12.15), into account)

$$\begin{gathered}\widetilde{\psi}_{0l\,\prime\partial_k F_{ij}} + \widetilde{\psi}_{0k\,\prime\partial_l F_{ij}} + \widetilde{\psi}_{0j\,\prime\partial_l F_{ik}} = 0,\\ \widetilde{\psi}_{0l\,\prime\partial_k\underline{\varepsilon}} + \widetilde{\psi}_{0k\,\prime\partial_l\underline{\varepsilon}} = 0.\end{gathered} \tag{12.21}$$

Using (12.16) we derive that

$$\widetilde{\psi}_{0l\,\prime\partial_k F_{ij}} = 0\ , \quad \widetilde{\psi}_{0l\,\prime\partial_k\underline{\varepsilon}} = 0$$

and therefore $\widetilde{\psi}_0$ is independent of $\partial_k F_{ij}$ and $\partial_k\underline{\varepsilon}$. (See the arguments in 12.10 for a general $\widetilde{\psi}_0$.) Then it follows from the assumption $\widehat{\underline{\psi}}(\underline{x}, \underline{\varepsilon}, F, 0, 0) = 0$ that $\widetilde{\psi}_0 = 0$. Thus the entropy inequality reduces to $0 \leq h = (\nabla\underline{\eta}_{\prime\underline{\varepsilon}})\bullet\underline{q}$. $\square$

12.10 Lemma. Without the assumption (12.16) one has for the remainder $\widetilde{\psi}_0 := \underline{\psi} - \underline{\eta}_{\prime\underline{\varepsilon}}\underline{q}$ the following statements.

(1) $\widetilde{\psi}_0 = \widehat{\psi}_0(\underline{x}, \underline{\varepsilon}, F, \nabla\underline{\varepsilon}, \mathrm{D}F)$ is a polynomial in $(\nabla\underline{\varepsilon}, \nabla F_{i,j})$.

(2) An example for $\widetilde{\psi}_0$ is

$$c \cdot \left(\textstyle\sum_i \partial_{2k}\varphi_i\partial_k\varphi_i\ e_1 - \sum_i \partial_{1k}\varphi_i\partial_k\varphi_i\ e_2\right),$$

with the property $\mathrm{div}\widetilde{\psi}_0 = 0$, if $c = \text{const}$.

Proof (1). That $\widetilde{\psi}_0$ is linear in $\nabla\underline{\varepsilon}$ follows from the second line in (12.21) (see the arguments following (11.16)). Using the first line in (12.21) one gets

$$\begin{aligned}0 &= \widetilde{\psi}_{0j\,\prime\partial_j F_{ij}\partial_{l_1}F_{il_2}} = (\widetilde{\psi}_{0j\,\prime\partial_{l_1}F_{il_2}})_{\prime\partial_j F_{ij}}\\ &= -(\widetilde{\psi}_{0l_1\,\prime\partial_{l_2}F_{ij}} + \widetilde{\psi}_{0l_2\,\prime\partial_{l_1}F_{ij}})_{\prime\partial_j F_{ij}}\\ &= -(\widetilde{\psi}_{0l_1\,\prime\partial_j F_{ij}})_{\prime\partial_{l_2}F_{ij}} - (\widetilde{\psi}_{0l_2\,\prime\partial_j F_{ij}})_{\prime\partial_{l_1}F_{ij}}\\ &= 2(\widetilde{\psi}_{0j\,\prime\partial_{l_1}F_{ij}})_{\prime\partial_{l_2}F_{ij}} + 2(\widetilde{\psi}_{0j\,\prime\partial_{l_2}F_{ij}})_{\prime\partial_{l_1}F_{ij}}\\ &= 4\widetilde{\psi}_{0j\,\prime\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}},\end{aligned}$$

and therefore

$$
\begin{aligned}
0 &= (\widetilde{\psi}_{0j}{}'_{\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}})'_{\partial_{k_1}F_{ik_2}} \\
&= (\widetilde{\psi}_{0j}{}'_{\partial_{k_1}F_{ik_2}})'_{\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}} \\
&= -(\widetilde{\psi}_{0k_1}{}'_{\partial_{k_2}F_{ij}} + \widetilde{\psi}_{0k_2}{}'_{\partial_{k_1}F_{ij}})'_{\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}} \\
&= -\widetilde{\psi}_{0k_1}{}'_{\partial_{k_2}F_{ij}\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}} + \widetilde{\psi}_{0k_2}{}'_{\partial_{k_1}F_{ij}\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}}.
\end{aligned}
$$

Then $C = (C_{k_1k_2l_2l_1})_{k_1,k_2,l_2}$ with

$$
C_{k_1k_2l_2l_1} := \widetilde{\psi}_{0k_1}{}'_{\partial_{k_2}F_{ij}\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}} \tag{12.22}
$$

is antisymmetric in (k_1, k_2) while symmetric in (k_2, l_2), and therefore $C = 0$ by lemma 11.8, that is

$$
\widetilde{\psi}_{0k}{}'_{\partial_{l_3}F_{ij}\partial_{l_2}F_{ij}\partial_{l_1}F_{ij}} = 0
$$

or in words, $\widetilde{\psi}_0$ is quadratic in each vector variable ∇F_{ij} for $i, j = 1, \ldots, n$, and multi-quadratic as a function of all these variables. □

13 Navier-Stokes-Korteweg equation

The evaluation of the entropy principle depends on the number of arguments, on which the quantities of the equations depend. Therefore it also depends on the arguments of the entropy. Too many and too few arguments may lead to unsatisfactory results. We will give here an example (see [7]). In contrast to section 11, where the entropy $\eta = \widehat{\eta}(\varrho, \varepsilon)$ has been a function of ϱ and ε, here the entropy depends in addition on $\nabla\varrho$, that is $\eta = \widehat{\eta}(\varrho, \varepsilon, \nabla\varrho)$, see 13.1.

Again we consider the equations for mass, momentum, and energy as in (11.1), that is

$$
\begin{aligned}
&\partial_t\varrho + \mathrm{div}(\varrho v) = 0, \\
&\partial_t(\varrho v) + \mathrm{div}(\varrho\, v\otimes v + \Pi) = \mathbf{f}, \\
&\partial_t e + \mathrm{div}(ev + \Pi^{\mathrm{T}}v + q) = v\bullet\mathbf{f}, \quad e = \varepsilon + \tfrac{\varrho}{2}|v|^2\ ,
\end{aligned} \tag{13.1}
$$

where ε is the internal energy. Further, we assume that

$$
\Pi^{\mathrm{T}} = \Pi, \tag{13.2}
$$

which implies the conservation of moment of momentum. Then, as shown in section 10, objectivity of (13.1) means

$$
\begin{aligned}
&\varrho,\ \varepsilon \text{ are objective scalars}, \quad v \text{ is a velocity}, \\
&q \text{ is an objective vector}, \quad \Pi \text{ a symmetric objective tensor}, \\
&\mathbf{f} \text{ is a force, } g \text{ is an energy production.}
\end{aligned} \tag{13.3}
$$

As shown in section 11, system (13.1) is equivalent to

$$
\begin{aligned}
&(\partial_t + v\bullet\nabla)\varrho + \varrho\mathrm{div}v = 0, \\
&\varrho(\partial_t + v\bullet\nabla)v + \mathrm{div}\Pi = \mathbf{f}, \\
&(\partial_t + v\bullet\nabla)\varepsilon + \mathrm{div}q + \mathrm{D}v\bullet(\varepsilon\,\mathrm{Id} + \Pi) = 0.
\end{aligned} \tag{13.4}
$$

In contrast to section 11 the entropy is allowed to depend on $\nabla\varrho$.

13.1 Entropy principle. As processes we consider the solutions of (13.1) with certain constitutive relations. The entropy principle associates to each process an entropy η and an entropy flux ψ satisfying

$$\partial_t\eta + \operatorname{div}\psi \geq 0 \tag{13.5}$$

in the domain of the process. Here we postulate, that

$$\eta = \widehat{\eta}(\varrho, v, \varepsilon, \nabla\varrho), \quad \psi = \widehat{\psi}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon). \tag{13.6}$$

where η is an objective scalar.

A consequent application of Truesdell's principle of equipresence would mean, that one takes an entropy η of the form $\eta = \widehat{\eta}(\varrho, v, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon)$. We will not discuss this here, and therefore stay with (13.6).

13.2 Lemma. Objectivity implies with a new function $\widehat{\eta}$, that

$$\eta = \widehat{\eta}(\varrho, \varepsilon, \frac{1}{2}|\nabla\varrho|^2).$$

Remark: The third variable we denote by s, so $(\varrho, \varepsilon, s) \mapsto \widehat{\eta}(\varrho, \varepsilon, s)$. A statement about the dependence on $(\varrho, \varepsilon, \nabla\varrho)$ with the property, that η depends only on the modulus of $\nabla\varrho$, would also be adequate.

Proof. Let $\widehat{\eta}$ be the function in (13.6), which is the same for all observers. Since the entropy is an objective scalar, we derive

$$\widehat{\eta}(\varrho^*, \dot{X} + Qv^*, \varepsilon^*, Q^{\mathrm{T}}\nabla\varrho^*) = \eta\circ Y = \eta^* = \widehat{\eta}(\varrho^*, v^*, \varepsilon^*, \nabla\varrho^*).$$

Since $\dot{X}$ can be chosen independent of Q, the function $\widehat{\eta}$ can not depend on v^*. Then $\widehat{\eta}$ has the same value for $\nabla\varrho^*$ and $Q^{\mathrm{T}}\nabla\varrho^*$. Since Q can be chosen arbitrarily as an orthogonal transformation with $\det Q = 1$, the function $\widehat{\eta}$ can only depend on the modulus $|\nabla\varrho^*|$. Thus with a new function $\eta^* = \widehat{\eta}(\varrho^*, \varepsilon^*, \frac{1}{2}|\nabla\varrho^*|^2)$. □

Assume for the entropy the dependence in 13.2. We compute the first derivatives

$$\begin{aligned}\partial_t\eta &= \eta'_\varrho\partial_t\varrho + \sum_i \eta'_s\partial_i\varrho\partial_t\partial_i\varrho + \eta'_\varepsilon\partial_t\varepsilon,\\ \partial_j\eta &= \eta'_\varrho\partial_j\varrho + \sum_i \eta'_s\partial_i\varrho\partial_j\partial_i\varrho + \eta'_\varepsilon\partial_j\varepsilon,\end{aligned}$$

and therefore, with $\dot{g} := \partial_t g + v\bullet\nabla g$ for any function g,

$$\dot{\eta} = \eta'_\varrho\dot{\varrho} + \sum_i \eta'_s\partial_i\varrho(\dot{\partial_i\varrho}) + \eta'_\varepsilon\dot{\varepsilon}.$$

Since

$$(\dot{\partial_i \varrho}) = \partial_i \dot\varrho + v\bullet\nabla\partial_i\varrho - \partial_i(v\bullet\nabla\varrho) = \partial_i\dot\varrho - (\partial_i v)\bullet\nabla\varrho, \tag{13.7}$$

we get

$$\begin{aligned}\sum_i \eta'_s \partial_i\varrho(\dot{\partial_i\varrho}) &= \sum_i \eta'_s\partial_i\varrho\partial_i\dot\varrho - \sum_{i,j}\eta'_s\partial_i\varrho\partial_i v_j\partial_j\varrho \\ &= -\sum_i \partial_i(\eta'_s\partial_i\varrho)\cdot\dot\varrho + \sum_i \partial_i(\dot\varrho\eta'_s\partial_i\varrho) - \sum_{i,j}\eta'_s\partial_i\varrho\partial_i v_j\partial_j\varrho.\end{aligned}$$

Defining the first variation of η with respect to ϱ by

$$\frac{\delta\eta}{\delta\varrho} := \eta'_\varrho - \operatorname{div}(\eta'_s\nabla\varrho)\,, \tag{13.8}$$

we obtain

$$\dot\eta = \frac{\delta\eta}{\delta\varrho}\dot\varrho + \eta'_\varepsilon\dot\varepsilon + \operatorname{div}(\dot\varrho\eta'_s\nabla\varrho) - (\nabla\varrho\otimes(\eta'_s\nabla\varrho))\bullet\mathrm{D}v.$$

Now we compute the entropy production $h = \partial_t\eta + \operatorname{div}\psi$, which for any functions ϱ, v, ε can be written as in [14, section 1.3.1.2], that is as a linear combination of the underlying equations. However, due to the fact that the entropy depends on the gradient of the mass density, there is an additional term $\dot\varrho\eta'_s\nabla\varrho$ in the entropy flux. A similar term also arises in connection with phase transition problems (see [3]). Thus we obtain for the entropy production using the above computations

$$\begin{aligned} h :=\ & \partial_t\eta + \operatorname{div}\psi = \dot\eta + \eta\operatorname{div} v + \operatorname{div}(\psi - \eta v) \\ =\ & \frac{\delta\eta}{\delta\varrho}\dot\varrho + \eta'_\varepsilon\dot\varepsilon + \operatorname{div}(\psi - \eta v + \dot\varrho\eta'_s\nabla\varrho) + (\eta\,\mathrm{Id} - \nabla\varrho\otimes(\eta'_s\nabla\varrho))\bullet\mathrm{D}v \\ =\ & \eta'_\varepsilon(g - v\bullet\mathbf{f}) + \underbrace{\operatorname{div}(\psi - \eta v - \eta'_\varepsilon q + \dot\varrho\eta'_s\nabla\varrho)}_{\text{flux term}} + \underbrace{\nabla\eta'_\varepsilon\bullet q}_{\text{heat term}} \\ & \underbrace{+\,\mathrm{D}v\bullet\left((\eta - \varrho\frac{\delta\eta}{\delta\varrho} - \varepsilon\eta'_\varepsilon)\,\mathrm{Id} - \eta'_\varepsilon\Pi - \nabla\varrho\otimes(\eta'_s\nabla\varrho)\right)}_{\text{at least linear in } \mathrm{D}v}.\end{aligned}$$

As sufficient condition for the entropy principle 13.1 we obtain the

13.3 Theorem. Assume that η has the assumption in 13.2, hence

$$\eta = \widehat\eta(\varrho, \varepsilon, \tfrac{1}{2}|\nabla\varrho|^2)$$

and that the entropy flux is

$$\psi = \eta v + \eta'_\varepsilon q - \dot\varrho\eta'_s\nabla\varrho, \quad \dot\varrho = (\partial_t + v\bullet\nabla)\varrho.$$

Then the entropy inequality is satisfied, if $g = v\bullet\mathbf{f}$, $\eta'_\varepsilon \neq 0$ and if p and S satisfy

(1) ***Pressure tensor.*** $\Pi = p\,\mathrm{Id} - \frac{1}{\eta'_\varepsilon}\nabla\varrho\otimes(\eta'_s\nabla\varrho) - S$,

(2) *Gibbs relation.* $\eta = \varrho\frac{\delta\eta}{\delta\varrho} + (\varepsilon + p)\eta'_\varepsilon$,

(3) *Dissipation.* $\eta'_\varepsilon \mathrm{D}v \bullet S + \nabla\eta'_\varepsilon \bullet q \geq 0$.

Proof. This follows from the above formula, where one chooses $\widetilde{S}$ so, that

$$\widetilde{S} = (\eta - \varrho\tfrac{\delta\eta}{\delta\varrho} - \varepsilon\eta'_\varepsilon)\,\mathrm{Id} - \eta'_\varepsilon\Pi - \nabla\varrho\otimes(\eta'_s\nabla\varrho). \tag{13.9}$$

With this and the assumptions in the theorem

$$h = \nabla\eta'_\varepsilon \bullet q + \mathrm{D}v \bullet \widetilde{S}.$$

We split (13.9) into two equations by defining an scalar $\widetilde{p}$ by

$$\widetilde{p} := \eta - \varrho\frac{\delta\eta}{\delta\varrho} - \varepsilon\eta'_\varepsilon,$$

then the formula (13.9) reads

$$\eta'_\varepsilon\Pi = \widetilde{p}\,\mathrm{Id} - \widetilde{S} - \nabla\varrho\otimes(\eta'_s\nabla\varrho).$$

Now p and S are so that $\widetilde{p}\,\mathrm{Id} - \widetilde{S} = \eta'_\varepsilon(p\,\mathrm{Id} - S)$. □

A necessary condition for the entropy principle 13.1 can also be obtained, but will not be presented in this paper. For the form of p and Π see also the computation following 14.4, where constitutive relations are written in terms of the free energy instead of the entropy. One obtains (without arguments)

$$\begin{aligned} p &= \varrho \cdot \left(f_0{}'_\varrho - \theta\mathrm{div}(\tfrac{1}{\theta} f_0{}'_{\nabla\varrho})\right) - f_0, \quad \theta = \tfrac{1}{\eta'_\varepsilon} \\ \Pi &= p\,\mathrm{Id} + \nabla\varrho \times f_0{}'_{\nabla\varrho} - S \end{aligned} \tag{13.10}$$

Often the Navier-Stokes-Korteweg equation, in the isothermal situation, is written down in the form

$$\begin{aligned} &\partial_t\varrho + \mathrm{div}(\varrho v) = 0, \\ &\partial_t(\varrho v) + \mathrm{div}(\varrho\, v\otimes v + p_1\,\mathrm{Id} - S) = \delta\varrho\nabla\Delta\varrho. \end{aligned}$$

A proof is contained in section 14, where we introduce the free energy as a function of absolute temperature.

14 Absolute temperature

One of the basic computations in thermodynamics is the definition of temperature and easy properties about it. Often the properties look more familiar, if we introduce as independent variable the absolute temperature θ instead of the internal energy ε. A quantity containing the temperature is the "free energy", defined by

$$f = e - \theta\eta. \tag{14.1}$$

The temperature, by postulate, is an objective scalar, and therefore the free energy transforms like the (total) energy. The above formula is the case for fluids in section 11 and 13 and for solids in section 12.

We consider the general case and the following conclusions are part of classical equilibrium thermodynamics. In details we have the following definitions, where we first consider the case of a fluid.

14.1 Definition (Fluid). Suppose that $\eta'_\varepsilon > 0$ in section 11. We assume that the variables (ϱ, ε) can be transformed into variables (ϱ, θ) via

$$\theta = \widehat{\theta}(\varrho, \varepsilon) := \frac{1}{\widehat{\eta}'_\varepsilon(\varrho, \varepsilon)}.$$

Moreover, the "free energy" f is defined by $f = \frac{\varrho}{2}|v|^2 + f_0$, where the "internal free energy" is

$$f_0 = \widehat{f}_0(\varrho, \theta) := \varepsilon - \theta\, \widehat{\eta}(\varrho, \varepsilon) \quad \text{for } \theta = \widehat{\theta}(\varrho, \varepsilon).$$

Note, that f is a function of θ. The property, that θ may attain any positive number, is an assumption on η, and a similar property applies to the other examples below. We compute (omitting the arguments)

$$f'_\theta = -\eta, \quad f'_\varrho = -\theta\eta'_\varrho,$$

and Gibbs relation 11.7(1) is equivalent to (omitting arguments)

$$p = \varrho f'_\varrho - f.$$

For solids we have the following definitions.

14.2 Definition (Solid). Suppose that $\underline{\eta}'_{\underline{\varepsilon}} > 0$ (or equivalently $\eta'_\varepsilon > 0$) in section 12. We assume that the variables $(\underline{x}, \underline{\varepsilon}, F)$ can be transformed into variables $(\underline{x}, \underline{\theta}, F)$ via

$$\underline{\theta} = \widehat{\underline{\theta}}(\underline{x}, \underline{\varepsilon}, F) := \frac{1}{\widehat{\underline{\eta}}'_{\underline{\varepsilon}}(\underline{x}, \underline{\varepsilon}, F)}.$$

We define the "free energy" in reference coordinates by

$$\underline{f} = \frac{\varrho}{2}|V|^2 + \underline{f}_0,$$

where the "internal free energy" in reference coordinates is defined by

$$\underline{f}_0 = \widehat{\underline{f}}_0(\underline{x}, \underline{\theta}, F) := \underline{\varepsilon} - \underline{\theta}\, \widehat{\underline{\eta}}(\underline{x}, \underline{\varepsilon}, F) \quad \text{for } \underline{\theta} = \widehat{\underline{\theta}}(\underline{x}, \underline{\varepsilon}, F).$$

Note, that $\underline{f}$ is a function of $\underline{\theta}$. With the above definitions one has the following.

14.3 Lemma. $P = -\underline{\theta}\underline{\eta}'_F = \underline{f}'_F$ under previous assumptions.

Proof. It is

$$\underline{f}_0 = \widehat{\underline{f}}_0(\underline{x}, \underline{\theta}, F) = \widehat{\underline{\varepsilon}}(\underline{x}, \underline{\theta}, F) - \underline{\theta} \cdot \widehat{\underline{\eta}}(\underline{x}, \widehat{\underline{\varepsilon}}(\underline{x}, \underline{\theta}, F), F) \ .$$

Since $\underline{\theta} \cdot \widehat{\underline{\eta}}_{'\underline{\varepsilon}}(\underline{x}, \widehat{\underline{\varepsilon}}(\underline{x}, \underline{\theta}, F), F) = 1$, we obtain $\underline{f}_{0'\underline{x}} = -\underline{\theta}\underline{\eta}_{'\underline{x}}$ and $\underline{f}_{0'F} = -\underline{\theta}\underline{\eta}_{'F}$. It also follows that $\underline{f}_{0'\underline{\theta}} = -\underline{\eta}$. Note that $\underline{f}_{'F} = \underline{f}_{0'F}$. □

One can introduce the free energy f by $\underline{f} = J \cdot f \circ \tau$ et cetera, or define the quantities in physical coordinates. In any case, one obtains (omitting the arguments)

$$\begin{aligned} \theta &= \underline{\theta}, \\ f_{'\theta} = -\eta, &\quad f_{'\varrho} = -\theta\eta_{'\varrho}, \\ \underline{f}_{'\underline{\theta}} = -\underline{\eta}, &\quad \underline{f}_{'F} = -\underline{\theta}\underline{\eta}_{'F} \end{aligned}$$

and other things.

The following is an example in nonclassical thermodynamics.

14.4 Definition (Gradient fluid). Suppose that $\eta_{'\varepsilon} > 0$ in section 13. We assume that the variables $(\varrho, \varepsilon, \nabla\varrho)$ can be transformed into variables $(\varrho, \theta, \nabla\varrho)$ via

$$\theta = \widehat{\theta}(\varrho, \varepsilon, \nabla\varrho) := \frac{1}{\widehat{\eta}_{'\varepsilon}(\varrho, \varepsilon, \nabla\varrho)}.$$

Moreover, the "free energy" f is defined by $f = \frac{\varrho}{2}|v|^2 + f_0$, where the "internal free energy"

$$f_0 = \widehat{f}_0(\varrho, \theta, \nabla\varrho) := \varepsilon - \theta\widehat{\eta}(\varrho, \varepsilon, \nabla\varrho) \quad \text{for } \theta = \widehat{\theta}(\varrho, \varepsilon, \nabla\varrho).$$

Then (omitting the arguments)

$$f_{0'\theta} = -\eta, \quad f_{0'\varrho} = -\theta\eta_{'\varrho}, \quad f_{0'\nabla\varrho} = -\theta\eta_{'\nabla\varrho}.$$

Let us consider a quadratic dependence on $\nabla\varrho$

$$f_0 = \widehat{f}_0(\varrho, \theta, \nabla\varrho) = \widehat{f}_1(\varrho, \theta) + \frac{\widehat{\delta}(\varrho, \theta)}{2}|\nabla\varrho|^2. \tag{14.2}$$

Then (omitting the arguments)

$$\begin{aligned} \eta &= -f_{0'\theta} = -f_{1'\theta} - \frac{\delta_{'\theta}}{2}|\nabla\varrho|^2, \\ \varepsilon &= f_0 + \theta\eta = f_1 - \theta f_{1'\theta} + \frac{\delta - \theta\delta_{'\theta}}{2}|\nabla\varrho|^2, \end{aligned}$$

so that in general the entropy η and the internal energy ε contains a gradient term, it is

$$\eta_{'\nabla\varrho} = -\frac{\delta}{2}\nabla\varrho, \quad \theta_{'\nabla\varrho} = \theta_{'\varepsilon} \cdot (\theta\delta_{'\theta} - \delta)\nabla\varrho.$$

From 13.3(1) we have

$$\Pi = p\,\mathrm{Id} - S - \theta\nabla\varrho \otimes \eta_{'\nabla\varrho} = p\,\mathrm{Id} - S + \nabla\varrho \otimes f_{0'\nabla\varrho},$$

and from 13.3(2)

$$\begin{aligned} p &= \theta\eta - \varepsilon - \theta\varrho\frac{\delta\eta}{\delta\varrho} = \theta\eta - \varepsilon - \theta\varrho\left(\eta'_\varrho - \operatorname{div}\eta'_{\nabla\varrho}\right) \\ &= -f_0 + \varrho\left(f_0{}'_\varrho - \theta\operatorname{div}\left(\frac{1}{\theta}f_0{}'_{\nabla\varrho}\right)\right) \\ &= \varrho f_1{}'_\varrho - f_1 + + \frac{\varrho\delta'_\varrho - \delta}{2}|\nabla\varrho|^2 - \theta\varrho\operatorname{div}\left(\frac{\delta}{\theta}\nabla\varrho\right). \end{aligned}$$

Hence the momentum equation becomes

$$\partial_t(\varrho v) + \operatorname{div}(\varrho\, v\otimes v + \Pi_1 + \Pi_2) = \mathbf{f},$$

where $\Pi = \Pi_1 + \Pi_2$ has the form

$$\begin{aligned} \Pi_1 = &\quad (\varrho f_1{}'_\varrho - f_1)\operatorname{Id} - S, \\ \Pi_2 = &\quad \left(\frac{\varrho\delta'_\varrho - \delta}{2}|\nabla\varrho|^2 - \theta\varrho\operatorname{div}\left(\frac{\delta}{\theta}\nabla\varrho\right)\right)\operatorname{Id} + \delta\nabla\varrho\otimes\nabla\varrho. \end{aligned}$$

14.5 Lemma. For $\delta = \text{const}$ in the isothermal case $\operatorname{div}\Pi_2 = -\delta\varrho\nabla\Delta\varrho$.

Proof. In the isothermal case $\theta = \text{const}$ and for $\delta = \text{const}$ one has the form

$$\Pi_2 = -\delta\left(\frac{1}{2}|\nabla\varrho|^2 + \varrho\Delta\varrho\right)\operatorname{Id} + \delta\nabla\varrho\otimes\nabla\varrho,$$

which implies the assertion. □

15 Solution property

We prove in this section the "solution property" for fluids and solids. These conditions are formulated in 11.4 and 12.8.

The proofs are here in a common section, since as a main tool we use the Cauchy-Kowalevski theorem (for a detailed presentation of the theorem see [11]). This theorem for our purpose requires some assumptions, which are formulated in the main theorems 15.1 and 15.2. The results of this section are not connected to parabolic existence results. They are a tool to show that enough local solutions exist, and therefore useful to prove that the entropy principle is satisfied.

In section 11 we introduced (11.4) as equation for a single fluid, which for $g = v\bullet\mathbf{f}$ reads

$$\begin{aligned} &(\partial_t + v\bullet\nabla)\varrho + \varrho\operatorname{div}v = 0, \\ &\varrho(\partial_t + v\bullet\nabla)v + \operatorname{div}\Pi = \mathbf{f}, \\ &(\partial_t + v\bullet\nabla)\varepsilon + \operatorname{div}q + \mathrm{D}v\bullet(\varepsilon\operatorname{Id} + \Pi) = 0. \end{aligned} \tag{15.1}$$

We assumed that physical processes are defined by independent variables ϱ, v, ε and constitutive relations

$$\begin{aligned} \Pi &= \widehat{\Pi}(\varrho, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon), \\ q &= \widehat{q}(\varrho, \varepsilon, \nabla\varrho, \mathrm{D}v, \nabla\varepsilon). \end{aligned} \tag{15.2}$$

(In the following we denote by ϱ, ε, $\partial_j\varrho$, $\partial_j v_k$, and $\partial_j\varepsilon$ also the variables of these functions.) We mention, that objectivity implies that Π and q do not depend on v, and that $\Pi_{'\partial_j v_k} = \Pi_{'\partial_k v_j}$ and $q_{'\partial_j v_k} = q_{'\partial_k v_j}$. With the constitutive functions $\widehat{\Pi}$ and $\widehat{q}$ in (15.2) one can write the system of differential equations (15.1) in the "explicit form"

$$\begin{aligned}
&\partial_t\varrho + v\bullet\nabla\varrho + \varrho\mathrm{div}v = 0,\\
&\varrho(\partial_t v_k + v\bullet\nabla v_k) + \textstyle\sum_{ij}\Pi_{ki'\partial_j\varrho}\partial_{ij}\varrho + \sum_{ijl}\Pi_{ki'\partial_j v_l}\partial_{ij}v_l + \sum_{ij}\Pi_{ki'\partial_j\varepsilon}\partial_{ij}\varepsilon\\
&\qquad + \textstyle\sum_i\Pi_{ki'\varrho}\partial_i\varrho + \sum_i\Pi_{ki'\varepsilon}\partial_i\varepsilon = \mathbf{f}_k \quad \text{for } k = 1,\dots,n,\\
&\partial_t\varepsilon + v\bullet\nabla\varepsilon + \textstyle\sum_{ij}q_{i'\partial_j\varrho}\partial_{ij}\varrho + \sum_{ijl}q_{i'\partial_j v_l}\partial_{ij}v_l + \sum_{ij}q_{i'\partial_j\varepsilon}\partial_{ij}\varepsilon\\
&\qquad + \textstyle\sum_i q_{i'\varrho}\partial_i\varrho + \sum_i q_{i'\varepsilon}\partial_i\varepsilon. + \mathrm{D}v\bullet(\varepsilon\,\mathrm{Id} + \Pi) = 0
\end{aligned} \tag{15.3}$$

The main result is the following statement.

15.1 Theorem. Assume that $\widehat{\Pi}$ and $\widehat{q}$ are continuous differentiable, and for an open and dense set of arguments these functions are real analytic. Also the function $\mathbf{f}$ is continuous and on an open and dense set real analytic. Moreover, assume that $\widehat{\Pi}$ is independent of $\nabla\varrho$ and $\nabla\varepsilon$ and that $\widehat{q}$ is independent of $\mathrm{D}v$. Let the matrix

$$\left(\textstyle\sum_{i,j} e_i e_j \Pi_{ki'\partial_j v_l}\right)_{k,l} \text{ be invertible, and } \textstyle\sum_{i,j} e_i e_j q_{i'\partial_j\varepsilon} \neq 0$$

for some unit vector $e = (e_i)_{i=1,\dots,n}$. Then for an open and dense set of arguments

$$(\varrho, v, \varepsilon, (\partial_j\varrho)_j, (\partial_j v_k)_{jk}, (\partial_j\varepsilon)_j) \tag{15.4}$$

it holds: If this together with $(\partial_t\varrho, (\partial_t v_k)_k, \partial_t\varepsilon, (\partial_{ij}\varrho)_{ij}, (\partial_{ij}v_k)_{ijk}, (\partial_{ij}\varepsilon)_{ij})$ defines a solution of (15.3) at a single point, then there exists a local solution of (15.1) (and of (15.3)) in a neighbourhood of this point, which coincides with the data at this single point.

The assumptions on Π and q are as in section 11. We mention, that as an example $\Pi = p\,\mathrm{Id} - S$, see 11.5, satisfies the assumption of the theorem with $p = \widehat{p}(\varrho,\varepsilon)$ and

$$\begin{aligned}
S_{ki} &= \frac{a}{2}(\partial_i v_k + \partial_k v_i) + b\ \mathrm{div}v\ \delta_{ki},\\
q &= c\nabla\eta_{'\varepsilon} = c\eta_{'\varepsilon\varrho}\nabla\varrho + c\eta_{'\varepsilon\varepsilon}\nabla\varepsilon,
\end{aligned}$$

where a, b and c are functions of (ϱ,ε) like p, and where we assume differentiability and analyticity as in the theorem. Then, with e being the n-th unit vector,

$$S_{kn'\partial_n v_l} = \frac{a}{2}(\delta_{kl} + \delta_{kn}\delta_{ln}) + b\ \delta_{kn}\delta_{ln},$$

so that the assumption in the theorem is fulfilled if $a \neq 0$, $b + a \neq 0$, and $c\eta_{'\varepsilon\varepsilon} \neq 0$. (The more restrictive assumption for the entropy principle in section 11 is $a > 0$, $b + \frac{a}{n} > 0$, and $c\eta_{'\varepsilon\varepsilon} > 0$.)

Proof. We assume that e is the n-th unit vector. We write system (15.1) in the form

$$\begin{aligned} &\textstyle\sum_i \partial_i(\varrho v_i) = -\partial_t \varrho, \\ &\textstyle\sum_i \partial_i \Pi_{ki} = -\varrho(\partial_t + v \bullet \nabla) v_k + \mathbf{f}_k \text{ for } k = 1, \dots, n, \\ &\textstyle\sum_i \partial_i q_i = -(\partial_t + v \bullet \nabla)\varepsilon - \mathrm{D}v \bullet (\varepsilon \,\mathrm{Id} + \Pi). \end{aligned} \tag{15.5}$$

We compute the n-th derivative of the first equation, and obtain

$$\begin{aligned} \partial_n \left(\textstyle\sum_i \partial_i(\varrho v_i)\right) = &\textstyle\sum_{i \le n}(v_i \partial_i \partial_n \varrho + \varrho \partial_i \partial_n v_i)) \\ &+ \textstyle\sum_{i \le n}(\partial_i \varrho \partial_n v_i + \partial_n \varrho \partial_i v_i). \end{aligned}$$

Thus (15.5) becomes, see (15.3),

$$\begin{aligned} &v_n \partial_{nn} \varrho + \varrho \partial_{nn} v_n = -\textstyle\sum_{i<n}(v_i \partial_{ni} \varrho + \varrho \partial_{ni} v_i) \\ &\qquad - \textstyle\sum_{i \le n}(\partial_i \varrho \partial_n v_i + \partial_n \varrho \partial_i v_i) - \partial_{nt} \varrho, \\ &\textstyle\sum_l \Pi_{kn\,'\partial_n v_l} \partial_{nn} v_l = -\textstyle\sum_{i+j<2n} \sum_l \Pi_{ki\,'\partial_j v_l} \partial_{ij} v_l \\ &\qquad - \textstyle\sum_{i \le n} \Pi_{ki\,'\varrho} \partial_i \varrho - \sum_{i \le n} \Pi_{ki\,'\varepsilon} \partial_i \varepsilon \\ &\qquad -\varrho \partial_t v_k - \varrho v \bullet \nabla v_k + \mathbf{f}_k \quad \text{ for } k = 1, \dots, n, \\ &q_{n\,'\partial_n \varrho} \partial_{nn} \varrho + q_{n\,'\partial_n \varepsilon} \partial_{nn} \varepsilon \\ &\qquad = -\textstyle\sum_{i+j<2n} \left(q_{i\,'\partial_j \varrho} \partial_{ij} \varrho + q_{i\,'\partial_j \varepsilon} \partial_{ij} \varepsilon\right) \\ &\qquad - \textstyle\sum_{i \le n} q_{i\,'\varrho} \partial_i \varrho - \sum_{\le n} q_{i\,'\varepsilon} \partial_i \varepsilon \\ &\qquad -\partial_t \varepsilon - \mathrm{D}v \bullet (\varepsilon \,\mathrm{Id} + \Pi). \end{aligned}$$

These differential equations are solvable by the Cauchy-Kowalevski theorem, if we introduce $R_j := \partial_j \varrho$, $V_{kj} := \partial_j v_k$, $E_j := \partial_j \varepsilon$ for $j = 1, \dots, n$. Then let us define

$$(\widetilde{i}, \widetilde{j}) := \begin{cases} (i, j) & \text{if } i < n, \\ (j, i) & \text{if } j < n, \end{cases} \tag{15.6}$$

an assignment where one has the choice for $i < n$ and $j < n$, and consider

$$\begin{aligned} &\Pi_{ki} = \widehat{\Pi}_{ki}(\varrho, \varepsilon, (R_j)_j, (V_{kj})_{kj}, (E_j)_j), \\ &\Pi_{ki\,'\partial_j \varrho} = \widehat{\Pi}_{ki\,'\partial_j \varrho}(\varrho, \varepsilon, (R_j)_j, (V_{kj})_{kj}, (E_j)_j), \end{aligned}$$

and the same for other constitutive functions and derivatives. With this the equations become

$$
\begin{aligned}
& v_n \partial_n R_n + \varrho \partial_n V_{nn} = -\sum\nolimits_{i<n} (v_i \partial_i R_n + \varrho \partial_i V_{in}) \\
& \qquad - \sum\nolimits_{i \le n} (R_i V_{in} + R_n V_{ii}) - \partial_t R_n, \\
& \sum\nolimits_l \Pi_{kn\,'\partial_n v_l} \partial_n V_{ln} = -\sum\nolimits_{i+j<2n} \sum\nolimits_l \Pi_{ki\,'\partial_j v_l} \partial_{\tilde{i}} V_{l\tilde{j}} \\
& \qquad - \sum\nolimits_{i \le n} \left(\Pi_{ki\,'\varrho} R_i + \Pi_{ki\,'\varepsilon} E_i \right) \\
& \qquad - \varrho (\partial_t v_k + \sum\nolimits_{i \le n} v_i V_{ki}) + \mathbf{f}_k \quad \text{ for } k = 1, \ldots, n, \\
& q_{n\,'\partial_n \varrho} \partial_n R_n + q_{n\,'\partial_n \varepsilon} \partial_n E_n \\
& \qquad = -\sum\nolimits_{i+j<2n} \left(q_{i\,'\partial_j \varrho} \partial_{\tilde{i}} R_{\tilde{j}} + q_{i\,'\partial_j \varepsilon} \partial_{\tilde{i}} E_{\tilde{j}} \right) \\
& \qquad - \sum\nolimits_{i \le n} \left(q_{i\,'\varrho} R_i + q_{i\,'\varepsilon} E_i \right) \\
& \qquad - \partial_t \varepsilon - \sum\nolimits_{k,i \le n} V_{ki} (\varepsilon \,\mathrm{Id} + \Pi) \ .
\end{aligned}
\tag{15.7}
$$

This equations have to be complemented by

$$
\begin{aligned}
& \partial_n \varrho = R_n, \quad \partial_n v_k = V_{kn}, \quad \partial_n \varepsilon = E_n, \\
& \partial_n R_i = \partial_i R_n, \quad \partial_n V_{ki} = \partial_i V_{kn}, \quad \partial_n E_i = \partial_i E_n \quad \text{ for } i < n, \\
& R_i = \partial_i \varrho, \quad V_{ki} = \partial_i v_k, \quad E_i = \partial_i \varepsilon \quad \text{ on } \{ x_n = \text{const} \} \text{ for } i < n,
\end{aligned}
\tag{15.8}
$$

and, of course by data, which ensure the first differential equation at least initially. This is

$$
v_n R_n + \varrho V_{nn} = -\sum\nolimits_{i<n} (v_i \partial_i \varrho + \varrho \partial_i v_i) - \partial_t \varrho \quad \text{ on } \{(t,x) \,;\, x_n = \text{const}\} \ , \tag{15.9}
$$

which can be satisfied by choosing R_n or V_{nn} appropriately. The application of the Cauchy-Kowalevski theorem is possible, since the matrix

$$
\begin{bmatrix}
v_n & 0 & \cdots & 0 & \varrho & 0 \\
0 & \Pi_{1n\,'\partial_n v_1} & \cdots & \Pi_{1n\,'\partial_n v_{n-1}} & \Pi_{1n\,'\partial_n v_n} & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & \Pi_{nn\,'\partial_n v_1} & \cdots & \Pi_{nn\,'\partial_n v_{n-1}} & \Pi_{nn\,'\partial_n v_n} & 0 \\
q_{n\,'\partial_n \varrho} & 0 & \cdots & 0 & 0 & q_{n\,'\partial_n \varepsilon}
\end{bmatrix}
$$

is invertible for an open and dense set of v_n values. Thus the system is admissible for the set of unknowns $(\varrho, (v_k)_k, \varepsilon, (R_j)_j, (V_{kj})_{kj}, (E_j)_j)$, and the Cauchy-Kowalevski theorem can be applied to the non-characteristic surface $\{(t,x) \,;\, x_n = \text{const}\}$. Thus a local solution exists, which then is a solution of the three differential equations (15.5) with given data in a point. □

We now come to our second application. In section 12 we had as equation for a single body (12.9) with $r = 0$ and $g = v \bullet \mathbf{f}$. In reference coordinates this is (12.10), where $\underline{\varrho} = \underline{\varrho}(\underline{x})$ is the mass distribution and, with $\underline{\mathbf{f}} := J \cdot (\mathbf{f} \circ \tau)$,

$$
\begin{aligned}
& \partial_t (\underline{\varrho} V) - \mathrm{div} P = \underline{\mathbf{f}}, \\
& \partial_t \underline{\varepsilon} + \mathrm{div} \underline{q} = (\mathrm{D} V) \bullet P.
\end{aligned}
\tag{15.10}
$$

We assume that physical processes are defined by a function $\underline{x} \mapsto \underline{\varrho}(\underline{x})$, by independent variables ϕ, $\underline{\varepsilon}$, and constitutive relations

$$\begin{aligned} &V = \partial_t \phi, \quad F = \mathrm{D}\phi, \\ &P = \widehat{P}(\underline{x}, \underline{\varepsilon}, F), \\ &\underline{q} = \widehat{\underline{q}}(\underline{x}, \underline{\varepsilon}, F, \nabla\underline{\varepsilon}, \mathrm{D}F). \end{aligned} \tag{15.11}$$

(In the following we denote by $\underline{x}$, $\underline{\varepsilon}$, F_{ml}, $\partial_j \underline{\varepsilon}$, and $\partial_j F_{ml}$ also the variables of these functions.) Since

$$\partial_j F_{ml} = \partial_{jl}\phi_m = \partial_{lj}\phi_m = \partial_l F_{mj} \tag{15.12}$$

we can assume that $\underline{q}_{\prime \partial_j F_{ml}} = \underline{q}_{\prime \partial_l F_{mj}}$. The constitutive functions $\widehat{P}$ and $\widehat{\underline{q}}$ depend on different variables, that is why the principle of equipresence, as formulated by Truesdell, is not satisfied. Using this definition of $\widehat{P}$ and $\widehat{\underline{q}}$ one can write the system of differential equations (15.10) in the "explicit form"

$$\begin{aligned} &\underline{\varrho}\partial_t V_k - \textstyle\sum_i P_{ki\prime\underline{\varepsilon}} \partial_i \underline{\varepsilon} - \sum_{iml} P_{ki\prime F_{ml}} \partial_i F_{ml} - \sum_i P_{ki\prime \underline{x}_i} \\ &\qquad = \underline{\mathbf{f}}_k \quad \text{for } k = 1, \ldots, n, \\ &\partial_t \underline{\varepsilon} + \textstyle\sum_{ij} \underline{q}_{i\prime \partial_j \underline{\varepsilon}} \partial_{ij} \underline{\varepsilon} + \sum_{ijml} \underline{q}_{i\prime \partial_j F_{ml}} \partial_{ij} F_{ml} \\ &\qquad + \textstyle\sum_i \underline{q}_{i\prime \underline{\varepsilon}} \partial_i \underline{\varepsilon} + \sum_{iml} \underline{q}_{i\prime F_{ml}} \partial_i F_{ml} + \sum_i \underline{q}_{i\prime z_i} = (\mathrm{D}V)\bullet P. \end{aligned} \tag{15.13}$$

The result is the following statement.

15.2 Theorem. Assume that $\widehat{P}$ and $\widehat{\underline{q}}$ are continuous differentiable, and for an open and dense set of arguments $(\underline{x}, \underline{\varepsilon}, F, \nabla\underline{\varepsilon}, \mathrm{D}F)$ these functions are real analytic. Also the function $\underline{\mathbf{f}}$ is continuous and on an open and dense set real analytic. Let the matrix

$$\begin{bmatrix} \sum_i e_i P_{1i\prime\underline{\varepsilon}} & \sum_{ij} e_i e_j P_{1i\prime F_{1j}} & \cdots & \sum_{ij} e_i e_j P_{1i\prime F_{nj}} \\ \vdots & \vdots & & \vdots \\ \sum_i e_i P_{ni\prime\underline{\varepsilon}} & \sum_{ij} e_i e_j P_{ni\prime F_{1j}} & \cdots & \sum_{ij} e_i e_j P_{ni\prime F_{nj}} \\ \sum_{ij} e_i e_j \underline{q}_{i\prime\partial_j\underline{\varepsilon}} & \sum_{ij} e_i e_j \underline{q}_{i\prime\partial_j F_{1j}} & \cdots & \sum_{ijl} e_i e_j e_l \underline{q}_{i\prime\partial_j F_{nj}} \end{bmatrix}$$

be invertible for some unit vector $e = (e_i)_{i=1,\ldots,n}$. Then for an open and dense set of arguments

$$(\underline{x}, \underline{\varepsilon}, F, (\partial_j \underline{\varepsilon})_j, (\partial_j F)_j) \tag{15.14}$$

it holds: If this together with $(\partial_t \phi, \partial_t^2 \phi, \partial_t \underline{\varepsilon}, (\partial_{ijk}\phi)_{ijk}, (\partial_{ij}\underline{\varepsilon})_{ij})$ defines a solution of (15.13) at a single point, then there exists a local solution of (15.10) (and of (15.13)) in a neighbourhood of this point, which coincides with the data at this single point.

If e is the n-th unit vector, the matrix has the form

$$\begin{bmatrix} P_{1n\prime\underline{\varepsilon}} & P_{1n\prime F_{1n}} & \cdots & P_{1n\prime F_{nn}} \\ \vdots & \vdots & & \vdots \\ P_{nn\prime\underline{\varepsilon}} & P_{nn\prime F_{1n}} & \cdots & P_{nn\prime F_{nn}} \\ \underline{q}_{n\prime\partial_n\underline{\varepsilon}} & \underline{q}_{n\prime\partial_n F_{1n}} & \cdots & \underline{q}_{n\prime\partial_n F_{nn}} \end{bmatrix} \tag{15.15}$$

As an example one takes P and q as in (12.12)

$$P = \frac{1}{\underline{\eta}_{'\underline{\varepsilon}}}\underline{\eta}_{'F}, \quad \underline{q} = c\nabla\underline{\eta}_{'\underline{\varepsilon}}.$$

where c is a function of $(\underline{x}, \underline{\varepsilon}, F)$, and where we assume differentiability and analyticity as in the theorem. Then $\underline{q} = c\left(\underline{\eta}_{'\underline{\varepsilon}\underline{\varepsilon}}\nabla\underline{\varepsilon} + \sum_{ml}\underline{\eta}_{'\underline{\varepsilon}F_{ml}}\nabla F_{ml} + \underline{\eta}_{'\underline{\varepsilon}z}\right)$ and, with e being the n-th unit vector, the matrix has the entries

$$P_{kn'\underline{\varepsilon}} = -\frac{\underline{\eta}_{'F_{kn}}}{\underline{\eta}^2_{\underline{\varepsilon}}}\underline{\eta}_{'\underline{\varepsilon}\underline{\varepsilon}} + \frac{1}{\underline{\eta}_{'\underline{\varepsilon}}}\underline{\eta}_{'F_{kn}\underline{\varepsilon}}, \qquad P_{kn'F_{ln}} = -\frac{\underline{\eta}_{'F_{kn}}}{\underline{\eta}^2_{\underline{\varepsilon}}}\underline{\eta}_{'\underline{\varepsilon}F_{ln}} + \frac{1}{\underline{\eta}_{'\underline{\varepsilon}}}\underline{\eta}_{'F_{kn}F_{ln}}$$

$$\underline{q}_{n'\partial_n\underline{\varepsilon}} = c\underline{\eta}_{'\underline{\varepsilon}\underline{\varepsilon}}, \qquad \underline{q}_{n'\partial_n F_{ln}} = c\underline{\eta}_{'\underline{\varepsilon}F_{ln}},$$

so that the assumption in the theorem is fulfilled if $c \neq 0$ and $\mathrm{D}^2_{(\underline{\varepsilon},F)}\underline{\eta} \neq 0$. (The more restrictive assumption for the entropy principle in section 12, see (12.12), is $c > 0$, $\underline{\eta}_{'\underline{\varepsilon}} > 0$. And the concavity of $(\underline{\varepsilon}, F) \mapsto \underline{\eta}(\underline{x}, \underline{\varepsilon}, F)$ in the form that $\mathrm{D}^2_{(\underline{\varepsilon},F)}\underline{\eta} < 0$ is the usual assumption.)

Proof. We assume that e is the n-th unit vector. We compute the n-th derivative of the momentum equation, the first equation in (15.13), and obtain

$$\begin{aligned}
&\textstyle\sum_i P_{ki'\underline{\varepsilon}}\partial_{ni}\underline{\varepsilon} + \sum_{iml} P_{ki'F_{ml}}\partial_{ni}F_{ml} \\
&\quad = \underline{\varrho}\partial_{tt}F_{kn} + \partial_n\underline{\varrho}\cdot\partial_t^2\varphi_k - \partial_n\underline{\mathbf{f}}_k \\
&\quad \textstyle - \sum_i \partial_n(P_{ki'\underline{\varepsilon}})\partial_i\underline{\varepsilon} - \sum_{iml}\partial_n(P_{ki'F_{ml}})\partial_i F_{ml} - \sum_i \partial_n(P_{ki'\underline{x}_i}) \\
&\quad \text{for } k = 1,\dots,n, \\
&\textstyle\sum_{ij}\underline{q}_{i'\partial_j\underline{\varepsilon}}\partial_{ij}\underline{\varepsilon} + \sum_{ijml}\underline{q}_{i'\partial_j F_{ml}}\partial_{ij}F_{ml} \\
&\quad = (\mathrm{D}V)\bullet P - \partial_t\underline{\varepsilon} \\
&\quad \textstyle - \sum_i \underline{q}_{i'\underline{\varepsilon}}\partial_i\underline{\varepsilon} - \sum_{iml}\underline{q}_{i'F_{ml}}\partial_i F_{ml} - \sum_i \underline{q}_{i'\underline{x}_i}
\end{aligned}$$

where the second equation is the second in (15.13). We consider these equations in a neighbourhood of $\{x_n = \text{const}\}$ with (15.16), to ensure the validity of the first equation in (15.13).

To apply the Cauchy-Kowalevski theorem, we use the variables $\underline{\varepsilon}$, φ_m for $m = 1,\dots,n$, and introduce the variables E_i and F_{mi} for $i = 0,\dots,n$, and G_{mij} for $i,j = 0,\dots,n$. Besides the auxiliary equations

$$\begin{aligned}
&\partial_n\underline{\varepsilon} = E_n, \quad \partial_n\varphi_m = F_{mn}, \\
&\partial_n F_{mj} = G_{mjn} \quad \text{for } i = 0,\dots,n, \\
&\partial_n G_{mij} = \begin{cases} \partial_i G_{mnj} \text{ if } 0 \le i < n \\ \partial_j G_{min} \text{ if } 0 \le j < n \end{cases} \qquad \text{for } 0 \le i+j < 2n,
\end{aligned}$$

and the auxiliary Cauchy data

$$\begin{aligned}
&E_i = \partial_i\underline{\varepsilon}, \quad F_{mi} = \partial_i\varphi_m \quad \text{for } i = 0,\dots,n-1, \\
&G_{mij} = \begin{cases} \partial_i F_{mj} \text{ if } 0 \le i < n \\ \partial_j F_{mi} \text{ if } 0 \le j < n \end{cases} \qquad \text{for } 0 \le i+j < 2n,
\end{aligned}$$

we have the equations for E_n and for G_{mnn} for $m = 1, \dots, n$, where $(\widetilde{i}, \widetilde{j})$ is chosen exactly as in (15.6),

$$
\begin{aligned}
&P_{kn'\underline{\varepsilon}} \partial_n E_n + \textstyle\sum_m P_{kn'F_{mn}} \partial_n G_{mnn} \\
&\quad = - \textstyle\sum_{1 \le i < n} P_{ki'\underline{\varepsilon}} \partial_i E_n - \sum_{2 \le i+j < 2n} \sum_m P_{ki'F_{mj}} \partial_{\widetilde{i}} G_{m\widetilde{j}n} \\
&\quad + \underline{\varrho} \partial_0 G_{kn0} + \partial_n \underline{\varrho} \cdot G_{k00} - \partial_n \underline{\mathbf{f}}_k \\
&\quad - \textstyle\sum_{1 \le i \le n} P_{ki'\underline{\varepsilon}\underline{\varepsilon}} E_n E_i - \sum_{1 \le i,j \le n} \sum_m P_{ki'\underline{\varepsilon}F_{mj}} (G_{mjn} E_i + E_n G_{mji}) \\
&\quad - \textstyle\sum_{1 \le i,j,\underline{j} \le n} \sum_{m\underline{m}} P_{ki'F_{mj}F_{\underline{m}\underline{j}}} G_{\underline{m}\underline{j}n} G_{mji} \\
&\quad - \textstyle\sum_{1 \le i \le n} P_{ki'\underline{x}_i\underline{\varepsilon}} E_n - \sum_{1 \le i,j \le n} \sum_m P_{ki'\underline{x}_i F_{mj}} G_{mjn} \\
&\quad - \textstyle\sum_{1 \le i \le n} P_{ki'\underline{\varepsilon}\underline{x}_n} E_i - \sum_{1 \le i,j \le n} \sum_m P_{ki'F_{mj}\underline{x}_n} G_{mji} - \sum_{1 \le i \le n} P_{ki'\underline{x}_i\underline{x}_n}
\end{aligned}
$$

for $k = 1, \dots, n$,

$$
\begin{aligned}
&\underline{q}_{n'\partial_n\underline{\varepsilon}} \partial_n E_n + \textstyle\sum_m \underline{q}_{n'\partial_n F_{mn}} \partial_n G_{mnn} \\
&\quad = - \textstyle\sum_{2 \le i+j < 2n} \underline{q}_{i'\partial_j\underline{\varepsilon}} \partial_{\widetilde{i}} E_{\widetilde{j}} - \sum_{3 \le i+j+l < 3n} \sum_m \underline{q}_{i'\partial_j F_{ml}} \begin{cases} \partial_i G_{mlj} & \text{if } i < n \\ \partial_j G_{mli} & \text{if } j < n \\ \partial_l G_{mij} & \text{if } l < n \end{cases} \\
&\quad + \textstyle\sum_{1 \le j \le n} \sum_m G_{mj0} P_{mj} - E_0 \\
&\quad - \textstyle\sum_{1 \le i \le n} \underline{q}_{i'\underline{\varepsilon}} E_i - \sum_{2 \le i+l \le 2n} \sum_m \underline{q}_{i'F_{ml}} G_{mli} - \sum_{1 \le i \le n} \underline{q}_{i'\underline{x}_i}.
\end{aligned}
$$

In addition we have to satisfy Cauchy data, which ensure the first differential equation of (15.13), on $\{x_n = \text{const}\}$,

$$
\begin{aligned}
&P_{kn'\underline{\varepsilon}} E_n + \textstyle\sum_m P_{kn'F_{mn}} G_{mnn} \\
&\quad = - \textstyle\sum_{1 \le i < n} P_{ki'\underline{\varepsilon}} E_i - \sum_{2 \le i+j < 2n} \sum_m P_{ki'F_{mj}} G_{mji} \\
&\quad + \underline{\varrho} \cdot G_{k00} - \underline{\mathbf{f}}_k - \textstyle\sum_{1 \le i \le n} P_{ki'\underline{x}_i},
\end{aligned}
\tag{15.16}
$$

which corresponds to a choice of Cauchy data for E_n and G_{mnn}. They exist since the determinant of the matrix in (15.15) is nonzero.

The terms with the second n-th derivative on the left side have a nonzero determinant, by assumption on the matrix (15.15). Therefore we can apply the Cauchy-Kowalevski theorem similar as in the first statement. □

16 Appendix (Constant objective tensors)

In this section we proof necessary structures for objective quantities. We consider a constant objective m-tensor $C = (c_{i_1,\dots,i_m})_{i_1,\dots,i_m=1,\dots,n}$. This means, that for all orthogonal matrices Q in $\mathbb{R}^n$ with $\det Q = 1$ the following identity is satisfied:

$$
c_{i_1,\dots,i_m} = \sum_{\bar{i}_1,\dots,\bar{i}_m=1}^{n} Q_{i_1\bar{i}_1} \cdot \ldots \cdot Q_{i_m\bar{i}_m} c_{\bar{i}_1,\dots,\bar{i}_m}. \tag{16.1}
$$

We assume $n \geq 2$ (for $n = 1$ there is only $Q = \mathrm{Id}$). Setting $Q = \exp(sA)$ with an antisymmetric matrix A, and taking the derivative with respect to s in (16.1) at $s = 0$ one obtains

$$\begin{aligned} 0 = &\sum_{\bar{i}_1=1}^{n} A_{i_1\bar{i}_1} c_{\bar{i}_1,i_2,\ldots,i_m} + \sum_{\bar{i}_2=1}^{n} A_{i_2\bar{i}_2} c_{i_1,\bar{i}_2,i_3,\ldots,i_m} \\ &+ \ldots + \sum_{\bar{i}_m=1}^{n} A_{i_m\bar{i}_m} c_{i_1,\ldots,i_{m-1},\bar{i}_m}. \end{aligned} \tag{16.2}$$

One can show

16.1 Theorem. For a constant objective m-tensor $C = (c_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1,\ldots,n}$ properties (16.2) and (16.1) are equivalent.

Proof. Assume (16.2) holds. Denote the right-hand side of (16.1) by

$$F_i(Q) := \sum_{\bar{i}_1,\ldots,\bar{i}_m=1}^{n} Q_{i_1\bar{i}_1} \cdot \ldots \cdot Q_{i_m\bar{i}_m} c_{\bar{i}_1,\ldots,\bar{i}_m} \quad \text{for } i = (i_1, \ldots, i_m).$$

Consider a smooth curve $s \mapsto Q(s)$ with $Q(0) = \mathrm{Id}$. Then with $A(s) := \dot{Q}(s)Q^{\mathrm{T}}(s)$ one computes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} F_i(Q(s)) &= \sum_{k=1}^{n} A_{i_1k} F_{k,i_2,\ldots,i_m}(Q(s)) + \sum_{k=1}^{n} A_{i_2k} F_{i_1,k,i_3,\ldots,i_m}(Q(s)) \\ &+ \ldots + \sum_{k=1}^{n} A_{i_mk} F_{i_1,,\ldots,i_{m-1},k}(Q(s)). \end{aligned}$$

Using (16.2), we see that the same differential equation holds for the function $s \mapsto F_i(Q(s)) - c_i$. Since $F_i(Q(0)) - c_i = 0$ for all i, we obtain $F_i(Q(s)) - c_i = 0$ for all s and i.

Since the set of orthogonal matrices with positive determinant is a connected manifold, we can reach any such matrix with a curve starting at the identity. □

Varying over all antisymmetric matrices one sees that (16.2) is equivalent to the fact, that

$$\begin{aligned} &\delta_{i_1,r} c_{s,i_2,\ldots,i_m} + \delta_{i_2,r} c_{i_1,s,i_3,\ldots,i_m} + \ldots + \delta_{i_m,r} c_{i_1,\ldots,i_{m-1},s} \\ &\text{is symmetric in } r, s \in \{1, \ldots, n\} \end{aligned} \tag{16.3}$$

for all $i_1, \ldots, i_m = 1, \ldots, n$. We consider property (16.3) in the subsequent arguments and show consequences of it. We do not claim that this is the most efficient way to derive these conclusions, but at least there is a unified background.

- ***Case*** $m = 1$.

Then (16.3) reads

$$\delta_{i,r} c_s = \delta_{i,s} c_r \quad \text{for all } i \text{ and } r \neq s.$$

Setting $i = r$ we get $c_s = 0$, and this for all s, hence

16.2 Lemma. If $m = 1$ and C satisfies (16.1), then $C = 0$.

- ***Case*** $m = 2$**.**

Then (16.3) reads

$$\delta_{i,r}c_{s,j} + \delta_{j,r}c_{i,s} = \delta_{i,s}c_{r,j} + \delta_{j,s}c_{i,r} \quad \text{for all } i, j \text{ and } r \neq s.$$

Setting $i = r$, $j = s$ we get

$$c_{s,s} = c_{r,r} \quad \text{for all } r \neq s \ ,$$

hence for some number a

$$c_{i,i} = a \quad \text{for all } i \ .$$

If $n \geq 3$ set $i = r$ and let $j \neq r, s$. This gives

$$c_{s,j} = 0 \quad \text{for all } j \neq s \ .$$

Thus

$$C = a\,\mathrm{Id}\,.$$

For $n = 2$ set $i = j = r$ and obtain

$$c_{s,r} + c_{r,s} = 0 \quad \text{for } s \neq r \ ,$$

hence for some number b

$$C = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{16.4}$$

16.3 Lemma. If $m = 2$ and C satisfies (16.1), then if $n = 3$ the matrix C is a multiple of the identity. If $n = 2$ then C has the representation (16.4).

- ***Case*** $m = 3$**.**

Then (16.3) reads

$$\begin{aligned} &\delta_{i,r}c_{s,j,k} + \delta_{j,r}c_{i,s,k} + \delta_{k,r}c_{i,j,s} \\ &= \delta_{i,s}c_{r,j,k} + \delta_{j,s}c_{i,r,k} + \delta_{k,s}c_{i,j,r} \quad \text{for all } i, j, k \text{ and } r \neq s. \end{aligned} \tag{16.5}$$

We consider the case $n \geq 3$. For $r = k = j$, and three different i, k, and s this gives

$$c_{i,s,k} + c_{i,k,s} = 0 \quad \text{for all } s, k \neq i \text{ with } s \neq k.$$

For $i = j$, $r = k$, and three different i, k, and s the identity gives

$$c_{i,i,s} = 0 \quad \text{for all } s \neq i,$$

and for $j = k$, $r = i$, and different i, k, and s this gives

$$c_{s,j,j} = 0 \quad \text{for all } s \neq j.$$

For $k = i \neq j$ and $s = i$, $r = j$ this gives $c_{i,i,i} = c_{j,j,i} + c_{i,j,j}$, which is 0 by the previous results, thus

$$c_{i,i,i} = 0 \quad \text{for all } i.$$

Therefore we have seen that

$$c_{i,j,k} \quad \text{is antisymmetric in } j, k.$$

Interchanging two indices leads to a tensor, which again is objective, hence the above antisymmetry applies to the new tensor. It follows that C is antisymmetric in every pair of indices. Every such 3-tensor is objective.

16.4 Lemma. If $m = 3$ then C satisfies (16.1) if and only if C is antisymmetric in every pair of indices. If $n = 3$, then C satisfies (16.6), if $n \geq 4$, then $C = 0$.

For $n \geq 4$ obtain for empty set $\{r, s\} \cap \{i, j\}$

$$\delta_{k,r} c_{i,j,s} = \delta_{k,s} c_{i,j,r}$$

and then for $k = s \neq r$

$$c_{i,j,r} = 0 \quad \text{for all } r \neq i, j.$$

Together with the above antisymmetry it follows that $C = 0$. For $n = 3$ we have

$$a := c_{1,2,3} = -c_{1,3,2} = c_{3,1,2} = -c_{3,2,1} = c_{2,3,1} = -c_{2,1,3},$$

and all other components vanish. Hence for vectors $\xi \in \mathbb{R}^3$

$$C\xi := \left(\textstyle\sum_{k=1}^{3} c_{ijk}\xi_k\right)_{i,j=1,\ldots,n} = a \cdot \begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix}. \tag{16.6}$$

- ***Case*** $m = 4$**.**

We consider the case $n \geq 3$. Equation (16.3) reads

$$\begin{aligned} &\delta_{i,r} c_{s,j,k,l} + \delta_{j,r} c_{i,s,k,l} + \delta_{k,r} c_{i,j,s,l} + \delta_{l,r} c_{i,j,k,s} \\ &= \delta_{i,s} c_{r,j,k,l} + \delta_{j,s} c_{i,r,k,l} + \delta_{k,s} c_{i,j,r,l} + \delta_{l,s} c_{i,j,k,r} \quad \text{for all } i, j, k, l \text{ and } r \neq s. \end{aligned} \tag{16.7}$$

Let us assume, that C is symmetric in the last two indices, that is

$$c_{i,j,k,l} = c_{i,j,l,k} \quad \text{for all } i, j, k, l.$$

Set $i = j$ in (16.7). Then

$$\begin{aligned} &\delta_{i,r}(c_{s,i,k,l} + c_{i,s,k,l}) + \delta_{k,r} c_{i,i,s,l} + \delta_{l,r} c_{i,i,k,s} \\ &= \delta_{i,s}(c_{r,i,k,l} + c_{i,r,k,l}) + \delta_{k,s} c_{i,i,r,l} + \delta_{l,s} c_{i,i,k,r} \quad \text{for } r \neq s \text{ and all } i, k, l. \end{aligned} \tag{16.8}$$

For $k, l, r, s \neq i$ with $r \neq s$ one obtains

$$\delta_{k,r} c_{i,i,s,l} + \delta_{l,r} c_{i,i,k,s} = \delta_{k,s} c_{i,i,r,l} + \delta_{l,s} c_{i,i,k,r}.$$

This is the characterization of the objective 2-tensor $(c_{i,i,k,l})_{k,l=1,\dots,n}$ in $n-1$ dimensions. Since $n-1 \geq 2$ and symmetry is assumed, the above result 16.3 implies with a real number b_i

$$c_{i,i,k,l} = b_i \delta_{k,l} \quad \text{for all } k, l \neq i. \tag{16.9}$$

For $r, s \neq i$ with $r \neq s$ and $k = r$, $l = i$ one obtains

$$c_{i,i,s,i} = 0 \quad \text{for all } s \neq i. \tag{16.10}$$

Now, in (16.8), set $k = i$ and let $l, r, s \neq i$ with $r \neq s$. One obtains

$$\delta_{l,r} c_{i,i,i,s} = \delta_{l,s} c_{i,i,i,r} \quad \text{for all } i \text{ and } l, r, s \neq i \text{ with } r \neq s.$$

For $l = r$ this gives

$$c_{i,i,i,s} = 0 \quad \text{for all } s \neq i.$$

Together with (16.9) this gives

$$\begin{aligned} &c_{i,i,k,l} = 0 \quad \text{for all } k \neq l, \\ &c_{i,i,k,k} = b_i \quad \text{for all } k \neq i, \\ &c_{i,i,i,i} \quad \text{so far undetermined.} \end{aligned} \tag{16.11}$$

Now, in (16.8), set $r = i$ and let $k, l, s \neq i$ (then $r \neq s$). One obtains

$$c_{s,i,k,l} + c_{i,s,k,l} = \delta_{k,s} c_{i,i,i,l} + \delta_{l,s} c_{i,i,k,i}.$$

The right-hand side vanishes by the first identity in (16.11), hence

$$c_{s,i,k,l} + c_{i,s,k,l} = 0 \quad \text{for all } k, l, s \neq i,$$

or relabeled

$$c_{j,i,k,l} + c_{i,j,k,l} = 0 \quad \text{for all } i \neq j \text{ and all } k, l \neq i \text{ or } k, l \neq j.$$

Denoting the symmetrization with respect to the first two indices by

$$c'_{i,j,k,l} := \frac{1}{2}(c_{i,j,k,l} + c_{j,i,k,l}) \quad \text{for all } i, j, k, l \tag{16.12}$$

we obtain $c'_{i,j,k,l} = c'_{j,i,k,l} = 0$ for all $i \neq j$ and all $k, l \neq i$ or $k, l \neq j$, that is

$$c'_{i,j,k,l} = 0 \quad \text{for all } i \neq j \text{ and } k, l \text{ with } \{k, l\} \neq \{i, j\}. \tag{16.13}$$

For $\{k, l\} = \{i, j\}$ we get

$$a_{i,j} := c'_{i,j,i,j} = c'_{i,j,j,i} = c'_{j,i,i,j} = c'_{j,i,j,i} = a_{j,i}.$$

Now let $i \neq j$ in (16.7). Then for $r = i$ and $s \neq i$ this identity becomes

$$\begin{aligned} & c_{s,j,k,l} + \delta_{k,i} c_{i,j,s,l} + \delta_{l,i} c_{i,j,k,s} \\ & = \delta_{j,s} c_{i,i,k,l} + \delta_{k,s} c_{i,j,i,l} + \delta_{l,s} c_{i,j,k,i} \quad \text{for } i \neq j, \text{ and } s \neq i, \text{ and all } k, l. \end{aligned} \tag{16.14}$$

As first case in (16.14) let $s = j$. Then

$$\begin{aligned} & c_{j,j,k,l} + \delta_{k,i} c_{i,j,j,l} + \delta_{l,i} c_{i,j,k,j} \\ & = c_{i,i,k,l} + \delta_{k,j} c_{i,j,i,l} + \delta_{l,j} c_{i,j,k,i} \quad \text{for all } k, l. \end{aligned}$$

For $k = l \neq i, j$ this gives

$$c_{j,j,k,k} = c_{i,i,k,k} \quad \text{for all } i \neq j \text{ and } k \neq i, j \ ,$$

thus in the second identity in (16.11) for some number b

$$b_i = b \quad \text{for all } i.$$

For $k = l = i$ we obtain

$$c_{j,j,i,i} + c_{i,j,j,i} + c_{i,j,i,j} = c_{i,i,i,i},$$

which by definition of b becomes

$$c_{i,i,i,i} = b + c_{i,j,j,i} + c_{i,j,i,j},$$

and for $k = l = j$ we obtain

$$c_{j,j,j,j} = c_{i,i,j,j} + c_{i,j,i,j} + c_{i,j,j,i} = b + c_{i,j,i,j} + c_{i,j,j,i},$$

and interchanging i, j

$$c_{i,i,i,i} = b + c_{j,i,j,i} + c_{j,i,i,j}.$$

Adding up both equations for $c_{i,i,i,i}$ we arrive at

$$c_{i,i,i,i} = b + c'_{i,j,j,i} + c'_{i,j,i,j} = b + 2a_{i,j} \tag{16.15}$$

by definition of $a_{i,j}$. As second case in (16.14) let $s \neq i, j$. Then

$$\begin{aligned} & c_{s,j,k,l} + \delta_{k,i} c_{i,j,s,l} + \delta_{l,i} c_{i,j,k,s} \\ & = \delta_{k,s} c_{i,j,i,l} + \delta_{l,s} c_{i,j,k,i} \quad \text{for } i \neq j \text{ and } s \neq i, j, \text{ and all } k, l. \end{aligned}$$

For $k = s$ and $l = j$ this gives

$$c_{s,j,s,j} = c_{i,j,i,j} \quad \text{for all } i \neq j \text{ and } s \neq i, j.$$

From now on let us consider only $C' := \left(c'_{i,j,k,l}\right)_{i,j,k,l=1,\ldots,n}$ given by (16.12), that is the symmetric part of C with respect to the first two indices. Since also C' is a constant objective 4-tensor (the same for the corresponding antisymmetric part), we can apply all results also to C'. In particular, the last identity becomes

$$c'_{s,j,s,j} = c'_{i,j,i,j} \quad \text{for all } i \neq j \text{ and } s \neq i, j,$$

which by symmetry means that $c'_{\widetilde{i},\widetilde{j},\widetilde{i},\widetilde{j}} = c'_{i,j,i,j}$ for all $i \neq j$ and $\widetilde{i} \neq \widetilde{j}$. Hence for some number a

$$a_{i,j} = a \quad \text{for all } i \neq j.$$

Therefore $c'_{i,i,i,i} = b + 2a$ from (16.15). Summing up, we have shown that C' has the following structure:

$$\begin{aligned} &c'_{i,j,k,l} = 0 \quad \text{except that} \\ &c'_{i,i,k,k} = b \quad \text{for all } k \neq i, \\ &c'_{i,i,i,i} = b + 2a \quad \text{for all } i, \\ &c'_{i,j,i,j} = c'_{i,j,j,i} = c'_{j,i,i,j} = c'_{j,i,j,i} = a \quad \text{for all } j \neq i. \end{aligned} \tag{16.16}$$

This means that

$$c'_{i,j,k,l} = a(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b\delta_{k,l}\delta_{i,j} \quad \text{for all } i,j,k,l. \tag{16.17}$$

Or equivalently, for all matrices $M = (m_{i,j})_{i,j=1,\dots,n}$

$$\textstyle\sum_{k,l=1}^{n} c'_{i,j,k,l} m_{k,l} = a \cdot (m_{i,j} + m_{j,i}) + b \cdot \operatorname{trace}(M) \cdot \delta_{i,j},$$

that is,

$$C'(M) := \Big(\textstyle\sum_{k,l=1}^{n} c'_{i,j,k,l} m_{k,l}\Big)_{i,j=1,\dots,n} = a \cdot (M + M^{\mathrm{T}}) + b \cdot \operatorname{trace}(M) \cdot \mathrm{Id}\,. \tag{16.18}$$

16.5 Lemma. If $m = 4$, $n \geq 3$, and C' satisfies (16.1), then if C' is symmetric in the last two arguments and C' is symmetric in the first two arguments it has the form in (16.18).

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