Technical University Munich 2019

Lectures on

Mathematical Continuum Mechanics

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This is the english version of the script, so far it is only partly translated. The script will be further developed parallel to the lecture. This version is preliminary, it is subject to corrections.

To my parents

Contents

Introduction

Die Naturwissenschaft beschreibt und erklärt
die Natur nicht einfach, sie ist Teil des Wechselspiels
zwischen der Natur und uns selbst.
Werner Heisenberg (1901-1976)

Die mathematische Modellierung physikalischer Phänomene führt zu Erhaltungsgleichungen, die von allen Beobachtern gleich formuliert werden müssen. Daher stellen wir in der Vorlesung folgende Prinzipien auf:

- die Formulierung mit Erhaltungssätzen,
- die Objektivität bei Beobachtertransformationen,
- das Entropieprinzip bzw. die freie Energieungleichung,

wobei das Entropieprinzip ausdrückt, dass wir es mit irreversiblen Prozessen zu tun haben. Diese Prinzipien haben Auswirkungen auf die Behandlung physikalischer Effekte, sie haben Konsequenzen was die mathematische Existenztheorie betrifft, als auch für die Entwicklung von numerischen Algorithmen. Es soll in dieser Vorlesung dargestellt werden, wie diese Prinzipien in Standardsituationen aussehen und welche Konsequenzen zu ziehen sind. Die abstrakten Formulierungen als partielle Differentialgleichung werden so in Zusammenhang mit alltäglichen Gleichungen gebracht. Die Idee zu dieser Vorlesung ist aus meiner Veröffentlichung [19] entstanden und ich hoffe sehr, dass dieses Skript dazu beiträgt zu verstehen, wie die physikalische Theorie auf ein einfaches System von Axiomen aufgebaut ist.

Es sei bemerkt, dass die allgemeinen Prinzipien in einem strengen Sinne zu verstehen sind, obwohl das im Text nicht immer so zum Ausdruck kommt. Das gilt in Standardsituationen als auch bei speziellen Theorien, sie sind allgemeine physikalische Prinzipien. Dies bestimmt im wesentlichen den Aufbau des Skriptes. Im ersten Abschnitt werden Erhaltungssätze vorgestellt, und zwar geben wir diese in der üblichen Differentialschreibweise an. Eine Formulierung mit Hilfe von Testvolumina wird als Einführung in das Kapitel I angegeben. Da viele physikalische Vorgänge nichtklassische Lösungen beinhalten, wird danach, also möglichst früh, der Begriff der Distribution eingeführt. Nichtklassische Lösungen sind etwa bei der Selbstgravitation und bei der Temperaturmessung der Standardfall. Es werden in dieser Vorlesung jedoch nur solche Beweise über Distributionslösungen gebracht, bei denen keine Größen auf der Fläche auftreten, obwohl dies häufig der Fall wäre. Das heißt, der Gauß'sche Satz im Raum ist hinreichend für die Beweise, bei denen die Flächen von der Zeit nicht abhängen. Das Kapitel I enthält auch die Darstellung der Erhaltungssätze in Lagrange Koordinaten. Dazu wird eine allgemeine Transformationsformel bewiesen, die auch später bei der Beobachterunabhängigkeit sowohl im klassischen Newton'schen Fall, als auch bei den Lorentztransformationen benutzt wird. Damit sind in diesem Kapitel I alle mathematischen Hilfsmittel zusammengestellt.

Das Kapitel II enthält alle Aussagen über die Objektivität, wobei bei diesem Begriff gemeint ist, dass physikalische Aussagen unabhängig vom Beobachter getroffen werden müssen. Dies ist notwendig, da sonst eine Kommunikation zwischen beteiligten Wissenschaftlern unnötig verkompliziert wird, bzw. eine phsikalische Beschreibung in Büchern bzw. elektronisch unmöglich wird. Große Teile dieses Skripts basieren auf klassischen Newton Transformationen, die in Abschnitt II.1 behandelt werden. Um die Abhängigkeit der Theorie von den Transformationen zu verdeutlichen, geben wir in diesem Kapitel auch Lorentz Transformationen an, die allerdings erst im Kapitel VI benötigt werden.

Das nächste Kapitel III handelt von der Energie und Entropie. Es ist eines der herausragenden Ergebnisse des 19. und 20. Jahrhunderts, die Irreversibilität von Prozessen mit einem Anstieg der Entropiedichte und des Entropieflusses in Verbindung zu setzen. Dabei wird hier der Standpunkt vertreten, dass diese Größen an sich von vornherein unbekannt sind. Erst durch die Anwendung des Prinzips wird deutlich, welche Bedingungen das Entropieprinzip an die konstitutiven Funktionen stellt. Die Aufgabe besteht also darin, das Entropieprinzip mit zu berücksichtigen und so zu einem tragfähigen Modell zu kommen.

Das ist nun Aufgabe des Kapitels IV, in dem aus den verschiedensten Bereichen Modellgleichungen dargestellt werden, und zwar unter Benutzung des Entropieprinzips bzw. der Energieungleichung. Es wird klar, dass alle in den Beispielgleichungen gemachten Ungleichungen auf dieses Prinzip zurückzuführen sind.

Hinweise für die Lehrenden

Die Anwendung der Distributionstheorie ist wesentlich für dieses Skript, und wird gleich im zweiten Paragraphen eingeführt, wobei es zur Darstellung der Punktmechanik gebraucht wird. In den weiteren Kapiteln werden sie auf eindimensionalen Kurven und zweidimensionalen Flächen im $\mathbb{R}^n$ angewandt. Es wird, bei gleicher Definition, auch zwischen Distributionen in $\mathcal{D}'(\mathbb{R}^n)$ und Distributionen in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ unterschieden, in dieser Vorlesung ist jede Dimension vertreten.

Die erste Vorlesung wurde ein Semester im Umfang von 4Std/Woche gehalten (im Wintersemester 2011). Dies umfasste die grundlegenden Kapitel I-III, und insgesamt fünf Abschnitte aus Kapitel IV. Die Wahl der Abschnitte kann nach der besonderen Situation der Universität oder nach den speziellen Wünschen des Lehrenden gewählt werden.

Im Skript wurden oft mehrere Beweise gegeben, obwohl in der Vorlesung jeweils nur ein Beweis dargestellt wurde. Zum Teil sind auch Beweise aufgeschrieben, die in der Vorlesung garnicht gebracht wurden. Dies ist bei der Auswahl des Stoffes zu berücksichtigen.

Der Text ist z.Z. noch im Entwicklungsstadium und wird ständig verbessert und erweitert. Die vorhandenen Paragraphen werden aber mit Sicherheit bleiben.

I Mass and momentum

The equations of continuum physics are based on systems of conservation laws. In this chapter we focus on the simplest such system, namely the conservation of mass and momentum. Mathematically, we will introduce conservation laws and distributions. These are the main tools of this chapter.

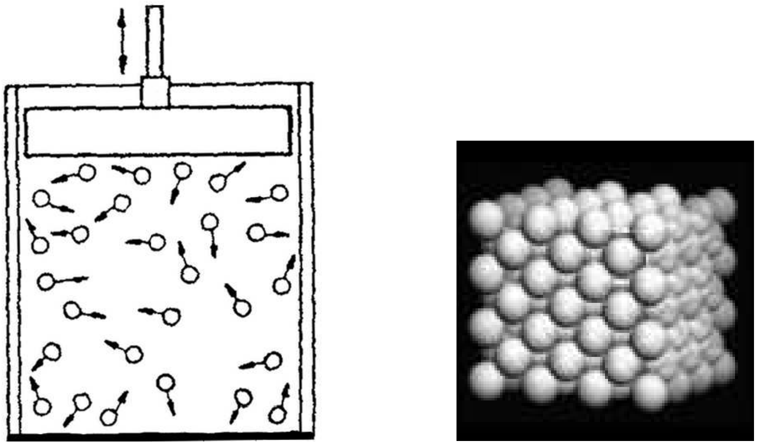

Fig. 1: Gas and solid

For engineers, the conservation laws are introduced with the help of test volumes $V \subset \mathbb{R}^n$, where $n \leq 3$ in the physical case. One writes the change of a physical quantity, whose density is u, as [1]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V u(t,x)\,\mathrm{d}x = -\int_{\partial V} q(t,x)\bullet\nu_V(x)\,\mathrm{dH}^{n-1} + \int_V \mathbf{r}(t,x)\,\mathrm{d}x\,.$$

Here q is the flux across the boundary of the test volume V, which is independent of time and where ν_V is the outer normal, and $\mathbf{r}$ is the rate at which the quantity u in the volume is changed. The fact that no other terms occur is the characteristic of continuum physics. Another equivalent formulation of conservation laws is the version as differential equation for C^1-functions

$$\partial_t u + \mathrm{div} q = \mathbf{r}\,. \tag{I0.1}$$

[1] Wir verwenden die Bezeichnung H^m für das m-dimensionale Hausdorffmaß in jedem $\mathbb{R}^n$ mit $n \geq m$ und L^n für das n-dimensionale Lebesguemaß im $\mathbb{R}^n$.

This follows from the formulation for test volumes using the Gauss's theorem, as one can see from the following calculation:

$$\begin{aligned}\int_V \partial_t u(t,x)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V u(t,x)\,\mathrm{d}x \\ &= -\int_{\partial V} q(t,x)\bullet\nu_V(x)\,\mathrm{dH}^{n-1}(x) + \int_V \mathbf{r}(t,x)\,\mathrm{d}x \\ &= \int_V (-\operatorname{div}q(t,x) + \mathbf{r}(t,x))\,\mathrm{d}x\,,\end{aligned}$$

consequently,

$$\int_V (\partial_t u(t,x) + \operatorname{div}q(t,x) - \mathbf{r}(t,x))\,\mathrm{d}x = 0\,.$$

Since the test domain V is arbitrary, we obtain the differential equation (I0.1). It should be mentioned that the formulation with test volumes follows from the strong differential equation just by reversing the above conclusions.

Es hat seinen besonderen Grund, dass in der Kontinuumsphysik die Formulierung mit Differentialgleichungen gewählt wird, und es berührt überhaupt nicht die Struktur der Materie im Kleinen. So ist in Fig. 1 auf der linken Seite dargestellt, wie sich die Atome irregulär bewegen, so dass man nicht mehr weiß, ob und wie die Atome im Moment zuvor angeordnet waren. Währenddessen ist auf der rechten Seite die Situation in einem Festkörper dargestellt. Hier bewegen sich die Atome nach denselben Gesetzen, aber sie bleiben fast immer in derselben Anordnung. Das liegt daran, dass die auf die Atome wirkenden Kräfte ihr Vorzeichen ändern, bevor sie selbst ihre Ordnung zu verlieren drohen. Also kommen wir zu dem folgenden Schluss: Wir müssen (t,x) als einen "Punkt" interpretieren, der viele Atome mitsamt ihren lokalen Gesetzen beinhaltet, und die makroskopischen Erhaltungsgleichungen sind zu verstehen als eine Methode, diese lokalen Gesetze von Ort zu Ort zu "vermitteln". Nichtsdestotrotz geben diese makroskopischen Gleichungen das Verhalten der Materie in der Natur wieder, wir werden dies bei der Massen- und Impulserhaltung im einzelnen sehen. Die später eingeführte Temperatur ist dann wie eine "Verschlüsselung" der lokalen Bewegung der Atome.

However, many important functions are not classical solutions of the differential equation, for example, the Earth's gravitational field at the Earth's surface. In this case, the formulation with test volumes becomes more complex. Therefore, we use test functions instead of test volumes. On the space of test functions

$$\mathscr{D}(\mathcal{U}) := \{\zeta \in C^\infty(\mathcal{U})\,;\ \zeta \text{ has compact support in } \mathcal{U}\}\,,$$

L. Schwartz's theory of distributions had two important effects in mathematical analysis. First of all, it provided a rigorous justification for a number of formal manipulations that had become quite common in the technical literature. The second and more important effect was that it opened up a new area of mathematical research, which in turn provided an impetus in the development of a number of mathematical disciplines, such as ordinary and partial differential equations, operational calculus, transformation theory, and functional analysis. However, the subject has remained pretty much in the realm of advanced mathematics, and only a few aspects of it have found their way into the technical literature.

To be sure, a certain type of distribution (in particular, the delta function and its derivatives) had been used in the physical and engineering sciences for quite some time before the advent of distribution theory. Indeed, the delta function dates back to the nineteenth century. A summary of its history is given by Van der Pol and Bremmer (see Van der Pol and Bremmer [1], pp. 62–66, in the bibliography, Appendix D). On the other hand, distribution theory appears to have first been formulated in 1936 by S. L. Soboleff (see Soboleff [1]) and then developed in a symmetric and thorough way by L. Schwartz (see Schwartz [1]), whose books appeared in 1950 and 1951. A somewhat different version of this theory was proposed by S. Bochner around 1927 (see Bochner [1], chap. VI), who used it to generalize the Fourier transformation for functions $f(t)$ that grow as some power of t as $|t|$ approaches infinity.

Fig. 2: Relevance of distributions (from [79])

where $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ is an open set, we consider linear forms U, Q, and R in the dual space $\mathscr{D}'(\mathcal{U})$ (see the section 2) so that

$$\partial_t U + \mathrm{div} Q = R \text{ in } \mathscr{D}'(\mathcal{U}) \,, \tag{I0.2}$$

where $\mathscr{D}'(\mathcal{U})$ is called the space of distributions. This formulation instead of (I0.1), see (I2.3), has the advantage that it is more general and much easier. This becomes particularly clear when one goes to descriptions of conservation laws on surfaces. Both representations are very common in literature, the representation of conservation laws using test volumes can be found usually in physics books. Both formulations are equivalent as one can see if one replaces the characteristic functions $\mathcal{X}_V$ (in the formulation with test volumes) by smooth functions ζ (in the formulation with test functions), which can be made rigorous by an approximation argument, that is, by a convolution of the characteristic function.

1 Conservation laws

We consider scalar conservation laws of the following form:

$$\begin{array}{|c|}\hline \textbf{\textit{Conservation law}}: \\ \partial_t u + \operatorname{div} q = \mathbf{r} \\ \hline u \text{ physical quantity,} \\ q \text{ associated flux,} \\ \mathbf{r} \text{ source term.} \\ \hline \end{array} \tag{I1.1}$$

So we have real-valued functions u, $\mathbf{r}$, and q_i for $i = 1, \ldots, n$. Here n is the space dimension. In physical reality this is 3, but it may also be 1 and 2, when the quantities do not depend of the other space filling coordinates. Mathematically, n can be arbitrary. The functions depend on the time $t \in \mathbb{R}$ and from the location $x \in \mathbb{R}^n$, that is, $(t, x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ and $\mathcal{U}$ is the considered region. This definition of a conservation law is only defined, if u and q are differentiable and r is continuous. For a continuously differentiable vector field $q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ we write

$$q = (q_i)_{i=1,\ldots,n} = (q_1, \ldots, q_n) = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix},$$ [2]

so we identify vectors with column matrices.

1.1 Remark. Die Erhaltungsgleichung $\partial_t u + \operatorname{div} q = \mathbf{r}$ in t und x kann auch aufgefasst werden als Divergenzgleichung $\underline{\operatorname{div}}(u, q) = \mathbf{r}$ in (t, x), wobei $\underline{\operatorname{div}} := (\partial_t, \operatorname{div})$.

For derivatives we have the following definitions.[3]

1.2 Definition of derivatives. For a function $g : \mathbb{R}^N \to \mathbb{R}^M$ and a vector $e \in \mathbb{R}^N$ the ***directional derivative*** in direction e is given by

$$\partial_e g(y) = \lim_{h \to 0} \frac{1}{h}(g(y + he) - g(y)) \in \mathbb{R}^M .$$

Important: The same definition holds if e is replaced by a map $y \mapsto e(y) \in \mathbb{R}^N$, that is, the directional derivative depends on the variable.
All other derivatives are based on this definition.

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[2] Es ist $\mathbb{R}^N$ die Menge der N-Vektoren (mit runden Klammern) und $\mathbb{R}^{N \times M}$ die Menge der $N \times M$-Matrizen (mit eckigen Klammern). Die lineare Abbildung $I : \mathbb{R}^N \to \mathbb{R}^{N \times 1}$ wird hier nicht geschrieben, d.h. $\mathbb{R}^N$ und $\mathbb{R}^{N \times 1}$ werden "identifiziert".

[3] *Note:* We do not always specify the exact mathematical assumptions, e.g. the difference between differentiability and partial differentiability.

(1) If $\mathbb{R}^N = \mathbb{R} \times \mathbb{R}^n$, hence $N = n+1$, we write for the variables $y = (t,x)$ and for $e = (0,\widetilde{e}) \in \mathbb{R} \times \mathbb{R}^n$

$$\begin{aligned}\partial_e g(t,x) &= \partial_{(0,\widetilde{e})} g(t,x) \text{ (as mapping on } \mathbb{R}^N = \mathbb{R}\times\mathbb{R}^n\text{)}\\ &= \partial_{\widetilde{e}} g(t,x) \text{ (as mapping } g(t,\bullet):\mathbb{R}^n \to \mathbb{R}\text{)} \ .\end{aligned}$$

(2) On $\mathbb{R}^n$ we define for $i = 1,\dots,n$

$$\mathbf{e}_i := (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^n \text{ with a 1 on the } i\text{-th position.} \tag{I1.2}$$

Then $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is the ***standard orthonormal basis*** of $\mathbb{R}^n$.

(3) The following formulas hold for $g:\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$:

$$\partial_i g(t,x) := \partial_{x_i} g(t,x) := \begin{cases} \partial_{(0,\mathbf{e}_i)} g(t,x) \text{ as mapping on } \mathbb{R}\times\mathbb{R}^n,\\ \partial_{\mathbf{e}_i} g(t,x) \text{ as mapping } g(t,\bullet):\mathbb{R}^n \to \mathbb{R},\\ \lim\limits_{h\to 0} \frac{1}{h}(g(t,x+h\mathbf{e}_i) - g(t,x))\,, \end{cases}$$

$$\partial_t g(t,x) := \partial_{(1,0)} g(t,x) = \lim_{h\to 0}\frac{1}{h}(g(t+h,x) - g(t,x))\,.$$

(4) For a mapping $g:\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ the ***gradient*** of g is given by

$$\nabla g = (\partial_{x_i} g)_{i=1,\dots,n} = \begin{bmatrix}\partial_1 g\\ \vdots\\ \partial_n g\end{bmatrix},$$

that is, $(t,x) \mapsto \nabla g(t,x) \in \mathbb{R}^n$ is a vector field. *Important:* The notation ∇ as well as the following notation involves only the space variables.

(5) The ***(space) derivative*** of a vector field $q = (q_1,\dots,q_n)$ is

$$\mathrm{D}q = (\partial_{x_i} q_k)_{k,i=1,\dots,n} = \begin{bmatrix}\partial_1 q_1 & \cdots & \partial_n q_1\\ \vdots & & \vdots\\ \partial_1 q_n & \cdots & \partial_n q_n\end{bmatrix}.$$

Remark: In literature sometimes the gradient ∇q of the vector field q is used, and we define it as [4]

$$\nabla q = (\partial_{x_i} q_k)_{i,k=1,\dots,n} = (\mathrm{D}q)^{\mathrm{T}} = \begin{bmatrix}\partial_1 q_1 & \cdots & \partial_1 q_n\\ \vdots & & \vdots\\ \partial_n q_1 & \cdots & \partial_n q_n\end{bmatrix}.$$

For $n = 1$ this is in accordance with the gradient of a function.

[4] Throughout this book we use the following notation for matrices M: The transposed matrix is M^{T}, the symmetric part is $M^{\mathrm{S}} = \frac{1}{2}(M + M^{\mathrm{T}})$, and the antisymmetric part or skew symmetric part is $M^{\mathrm{A}} = \frac{1}{2}(M - M^{\mathrm{T}})$.

(6) The ***divergence*** of q is given by the trace of $\mathrm{D}q$

$$\operatorname{div} q := \sum_{i=1}^{n} \partial_{x_i} q_i = \operatorname{trace} \mathrm{D}q\,.$$

(7) Für ein Vektorfeld q und eine Richtung $e:\mathbb{R}^n \to \mathbb{R}^n$ gilt [5]

$$\partial_e q = (e\bullet\nabla)q = \mathrm{D}q\, e \quad \text{für alle } e \in \mathbb{R}^n.$$

Remark: It is $\operatorname{div} q = \nabla\bullet q$ where $\nabla\bullet := \sum_j \mathbf{e}_j\bullet\partial_j$ in the world of the "nabla operator".

Please, keep these definitions in mind, we use them systematically in this script. Certain identities for derivatives can be found in exercise 7.2.

1.3 Representation of the divergence operator. For a differentiable vector field q and orthonormal bases $\{e_1(t,x),\dots,e_n(t,x)\}$ of the Euclidean space $\mathbb{R}^n$ it holds

$$\operatorname{div} q = \sum_{i=1}^{n} \partial_{x_i} q_i = \sum_{i=1}^{n} e_i\bullet\partial_{e_i} q\,. \tag{I1.3}$$

Here the basis vectors can depend arbitrarily on (t,x).

It should be noted that in general

$$\operatorname{div} q \neq \sum_{i=1}^{n} \partial_{e_i}(e_i\bullet q) = \underbrace{\sum_{i=1}^{n} e_i\bullet\partial_{e_i} q}_{=\operatorname{div} q} + \Big(\underbrace{\sum_{i=1}^{n} \partial_{e_i} e_i}_{\text{general } \neq 0}\Big)\bullet q\,,$$

if e_i are variable vectors. Remember that for the divergence operator property (I1.3) is true. This fact includes the isotropy of the empty space.

Proof. The orthonormality of $\{e_1,\dots,e_n\}$ means that

$$e_i\bullet e_j = \delta_{ij} \text{ für } i,j = 1,\dots,n.$$

With [6]

$$e_i = (e_{ik})_{k=1,\dots,n} \text{ therefore } e_{ik} = e_i\bullet\mathbf{e}_k$$

the orthonormality is

$$\sum_{k=1}^{n} e_{ik}e_{jk} = \delta_{ij}$$

[5] The ***Euclidean scalar product*** is given by $x\bullet y := \sum_{i=1}^{n} x_i y_i$ for $x,y \in \mathbb{R}^n$. In analogy we define the ***scalar product for matrices*** by $R\,\mathbf{:}\,S := R\bullet S := \sum_{i,j=1}^{n} R_{ij}S_{ij}$ for $R,S \in \mathbb{R}^{n\times n}$. Here $\mathbb{R}^{n\times n}$ stands for the set of real $n\times n$-matrices.

[6] for $\mathbf{e}_k$ see (I1.2)

or

$$E\,E^{\mathrm{T}} = \mathrm{Id} \text{ if } E = (e_{ik})_{i,k=1,\dots,n} \ . \tag{I1.4}$$

Das besagt, dass E^{T} die Rechtsinverse von E ist, was aber gleich der Linksinversen ist, eine Aussage für endliche Matrizen, denn

$$(E^{\mathrm{T}}\,E - \mathrm{Id})\,E^{\mathrm{T}} = E^{\mathrm{T}}\,(E\,E^{\mathrm{T}} - \mathrm{Id}) = 0\,,$$

und da E^{T} injektiv ist (folgt aus (I1.4)), somit surjektiv ist, schließen wir $E^{\mathrm{T}}\,E - \mathrm{Id} = 0$, also

$$E^{\mathrm{T}}\,E = \mathrm{Id}$$

und damit

$$\delta_{kl} = (E^{\mathrm{T}}\,E)_{kl} = \sum_{i=1}^{n} e_{ik}e_{il}\,.$$

Dann ist wegen 1.2(7)

$$\begin{aligned}
&\sum_{i=1}^{n} e_i\bullet\partial_{e_i} q = \sum_{i=1}^{n} e_i\bullet(\mathrm{D}q)e_i \\
&= \sum_{k,l=1}^{n} \underbrace{\mathbf{e}_k\bullet(\mathrm{D}q)\mathbf{e}_l}_{=\,\partial_l q_k} \sum_{i=1}^{n} (\underbrace{e_i\bullet\mathbf{e}_k}_{=\,e_{ik}})(\underbrace{e_i\bullet\mathbf{e}_l}_{=\,e_{il}}) \\
&= \sum_{k,l=1}^{n} \partial_l q_k \sum_{i=1}^{n} e_{ik}e_{il} = \sum_{k,l=1}^{n} \partial_l q_k \delta_{kl} = \sum_{k=1}^{n} \partial_k q_k = \mathrm{div}\, q\ .
\end{aligned}$$

□

Now we give some examples for q in order to calculate $\mathrm{div} q$.

1.4 Example. Let a matrix $(t,r) \mapsto A(t,r) \in \mathbb{R}^{n\times n}$ be given and

$$q(t,x) := A(t,|x|)x.$$

(1) If A depends only on time t, then

$$\mathrm{div}\, q = \mathrm{trace}\, A.$$

(2) For continously differentiable A we compute for $x \in \mathbb{R}^n \setminus \{0\}$

$$\mathrm{div}\, q(t,x) = \mathrm{trace}\, A(t,|x|) + \frac{1}{|x|}\big(x\bullet\partial_r A(t,|x|)\big)\,x.$$

(3) Let $n = 2$ and a is continuous differentiable. If

$$A(t,r) = a(t,r)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then

$$q(t,x) = a(t,|x|)\,\mathrm{i}x \quad \text{with} \quad \mathrm{div} q = 0.$$

Proof. Siehe die Übung 7.6. □

1.5 Plane polar coordinates. Let $n = 2$ and for $x \in \mathbb{R}^2 \setminus \{0\}$ let[7]

$$\mathbf{e}_r = \widehat{\mathbf{e}}_r(x) := \frac{x}{|x|}\,, \quad \mathbf{e}_\theta = \widehat{\mathbf{e}}_\theta(x) := \frac{\mathrm{i}x}{|x|} = \frac{(-x_2, x_1)}{|x|}\,. \tag{I1.5}$$

Then $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is an orthonormal system of $\mathbb{R}^2$ and

$$\begin{aligned} \partial_{\mathbf{e}_r}\widehat{\mathbf{e}}_r(x) = 0\,, &\quad \partial_{\mathbf{e}_\theta}\widehat{\mathbf{e}}_r(x) = \frac{1}{|x|}\widehat{\mathbf{e}}_\theta(x)\,, \\ \partial_{\mathbf{e}_r}\widehat{\mathbf{e}}_\theta(x) = 0\,, &\quad \partial_{\mathbf{e}_\theta}\widehat{\mathbf{e}}_\theta(x) = -\frac{1}{|x|}\widehat{\mathbf{e}}_r(x)\,. \end{aligned} \tag{I1.6}$$

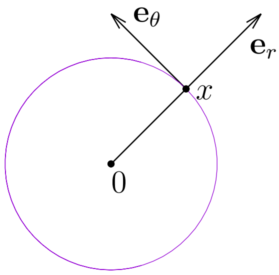


Fig. 3: The orthonormal system $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ for $x \in \mathbb{R}^2 \setminus \{0\}$

Proof. On $\mathbb{R}^2$ ***polar coordinates*** are given by

$$x = \tau(r, \theta) = r\mathrm{e}^{\mathrm{i}\theta}$$

and then

$$\widehat{\mathbf{e}}_r \circ \tau = \mathrm{e}^{\mathrm{i}\theta}, \quad \widehat{\mathbf{e}}_\theta \circ \tau = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,.$$

It holds for functions g

$$(\partial_{\mathbf{e}_r} g)\circ\tau = \partial_r(g\circ\tau)\,, \quad (\partial_{\mathbf{e}_\theta} g)\circ\tau = \frac{1}{r}\partial_\theta(g\circ\tau)\,,$$

due to

$$\begin{aligned} ((\partial_{\mathbf{e}_r} g)\circ\tau)(r,\theta) = &\lim_{h\to 0}\frac{1}{h}(g(r\mathrm{e}^{\mathrm{i}\theta} + h\mathrm{e}^{\mathrm{i}\theta}) - g(r\mathrm{e}^{\mathrm{i}\theta})) \\ = &\lim_{h\to 0}\frac{1}{h}(g((r+h)\mathrm{e}^{\mathrm{i}\theta}) - g(r\mathrm{e}^{\mathrm{i}\theta})) = \partial_r(g\circ\tau)\,, \end{aligned}$$

[7] Here $\mathrm{i}x$ is the complex multiplication of two numbers $\mathrm{i} \in \mathbb{C}$ and $x = x_1 + \mathrm{i}x_2 \in \mathbb{C}$. The complex numbers $\mathbb{C}$ are identified with $\mathbb{R}^2$, hence $\mathrm{i} = (0, 1)$. We recomment engineering students the section [9, 6.1 General Principles], and mathematics students are refered to the introductory lectures in mathematics.

$$\begin{aligned}((\partial_{\mathbf{e}_\theta} g)\circ\tau)(r,\theta) &= \lim_{h\to 0}\frac{1}{h}(g(r\mathrm{e}^{\mathrm{i}\theta}+h\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})-g(r\mathrm{e}^{\mathrm{i}\theta}))\\ &= \lim_{h\to 0}\frac{1}{h}(g(r\underbrace{(1+\frac{h}{r}\mathrm{i})}_{=\mathrm{e}^{\mathrm{i}\frac{h}{r}}+\mathcal{O}(h^2)}\mathrm{e}^{\mathrm{i}\theta})-g(r\mathrm{e}^{\mathrm{i}\theta}))\\ &= \lim_{\widetilde{h}\to 0}\frac{1}{r\widetilde{h}}(g(r\mathrm{e}^{\mathrm{i}(\theta+\widetilde{h})}+\mathcal{O}(\widetilde{h}^2))-g(r\mathrm{e}^{\mathrm{i}\theta}))=\frac{1}{r}\partial_\theta(g\circ\tau)\,.\end{aligned}$$

Then

$$\begin{aligned}(\partial_{\mathbf{e}_r}\widehat{\mathbf{e}}_r)\circ\tau &= \partial_r(\widehat{\mathbf{e}}_r\circ\tau)=\partial_r(\mathrm{e}^{\mathrm{i}\theta})=0\,,\\ (\partial_{\mathbf{e}_\theta}\widehat{\mathbf{e}}_r)\circ\tau &= \frac{1}{r}\partial_\theta(\widehat{\mathbf{e}}_r\circ\tau)=\frac{1}{r}\partial_\theta(\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{r}\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}=\frac{1}{r}\widehat{\mathbf{e}}_\theta\circ\tau\,,\\ (\partial_{\mathbf{e}_r}\widehat{\mathbf{e}}_\theta)\circ\tau &= \partial_r(\widehat{\mathbf{e}}_\theta\circ\tau)=\partial_r(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})=0\,,\\ (\partial_{\mathbf{e}_\theta}\widehat{\mathbf{e}}_\theta)\circ\tau &= \frac{1}{r}\partial_\theta(\widehat{\mathbf{e}}_\theta\circ\tau)=\frac{1}{r}\partial_\theta(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})=-\frac{1}{r}\mathrm{e}^{\mathrm{i}\theta}=-\frac{1}{r}\widehat{\mathbf{e}}_r\circ\tau\,.\end{aligned}$$

□

We use this in order to calculate the divergence of a vector field $q:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$.

1.6 Plain divergence. Each vector field $q:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ has a unique representation in $\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})$:

$$q=s_1\mathbf{e}_r+s_2\mathbf{e}_\theta\,,\quad \text{where}\quad s_1=q\bullet\mathbf{e}_r\,,\quad s_2=q\bullet\mathbf{e}_\theta\,,$$

with $s_1,s_2:\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})\to\mathbb{R}$. Here the orthonormal system is chosen as in 1.5. Then in $\mathbb{R}\times(\mathbb{R}^2\setminus\{0\})$

$$\operatorname{div} q(t,x)=\partial_{\mathbf{e}_r}s_1(t,x)+\frac{s_1(t,x)}{|x|}+\partial_{\mathbf{e}_\theta}s_2(t,x)\,.$$

If the vector field q is directed outward seen from the origin, then $s_1\geq 0$ and $s_2=0$. If the vector field q rotates around the origin, then $s_1=0$ and s_2 is arbitrary.

Convention: We write $\mathbf{e}_r$ instead of $\widehat{\mathbf{e}}_r$ etc. because so the formulas become more handsome.

Proof 1. Version. Es ist nach 1.3

$$\begin{aligned}&\operatorname{div} q=\mathbf{e}_r\bullet\partial_{\mathbf{e}_r}q+\mathbf{e}_\theta\bullet\partial_{\mathbf{e}_\theta}q\\ &=\mathbf{e}_r\bullet\partial_{\mathbf{e}_r}(s_1\mathbf{e}_r+s_2\mathbf{e}_\theta)+\mathbf{e}_\theta\bullet\partial_{\mathbf{e}_\theta}(s_1\mathbf{e}_r+s_2\mathbf{e}_\theta)\\ &=\partial_{\mathbf{e}_r}s_1+\partial_{\mathbf{e}_\theta}s_2\\ &+s_1(\mathbf{e}_r\bullet\partial_{\mathbf{e}_r}\mathbf{e}_r+\mathbf{e}_\theta\bullet\partial_{\mathbf{e}_\theta}\mathbf{e}_r)+s_2(\mathbf{e}_r\bullet\partial_{\mathbf{e}_r}\mathbf{e}_\theta+\mathbf{e}_\theta\bullet\partial_{\mathbf{e}_\theta}\mathbf{e}_\theta)\,.\end{aligned}$$

Unter Benutzung der Regeln (I1.6) folgt mir $r=|x|$, dass dies

$$=\partial_{\mathbf{e}_r}s_1+\partial_{\mathbf{e}_\theta}s_2+\frac{1}{r}s_1\,,$$

also folgt die Behauptung. □

Proof 2. Version. Es ist nach 1.3

$$\begin{aligned}\operatorname{div} q &= \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} q + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} q \\ &= \partial_{\mathbf{e}_r}(\mathbf{e}_r \bullet q) + \partial_{\mathbf{e}_\theta}(\mathbf{e}_\theta \bullet q) - (\partial_{\mathbf{e}_r}\mathbf{e}_r + \partial_{\mathbf{e}_\theta}\mathbf{e}_\theta)\bullet q \\ &= \partial_{\mathbf{e}_r} s_1 + \partial_{\mathbf{e}_\theta} s_2 - (\partial_{\mathbf{e}_r}\mathbf{e}_r + \partial_{\mathbf{e}_\theta}\mathbf{e}_\theta)\bullet q\,.\end{aligned}$$

Unter Benutzung der Regeln (I1.6) folgt, dass dies

$$\begin{aligned}&= \partial_{\mathbf{e}_r} s_1 + \partial_{\mathbf{e}_\theta} s_2 + \frac{1}{r}\mathbf{e}_r \bullet q \\ &= \partial_{\mathbf{e}_r} s_1 + \partial_{\mathbf{e}_\theta} s_2 + \frac{1}{r} s_1\,,\end{aligned}$$

also folgt die Behauptung. □

The most famous example of a conservation law is the mass conservation, that is, we write $u = \varrho$, where $\varrho > 0$ is the "mass density", which is the mass per volume

$$\boxed{\varrho = \frac{\text{mass}\,[kg]}{\text{volume}\,[m^3]}}$$

And we set $q = \varrho v + \mathbf{J}$, where v denotes the "velocity" of the mass and $\mathbf{J}$ the "mass diffusion" (we will derive this equation in detail in II.3.4). Hence we get the [8]

General mass conservation:

$$\partial_t \varrho + \operatorname{div}_x(\varrho v + \mathbf{J}) = \mathbf{r} \tag{I1.7}$$

$\varrho \geq 0$ mass density,

$q = \varrho v + \mathbf{J}$ mass flux,

$v = (v_i)_{i=1,\dots,n}$ velocity,

$\mathbf{J} = (\mathbf{J}_i)_{i=1,\dots,n}$ mass diffusion,

$\mathbf{r}$ source term of the mass,

what we can also write as

$$\partial_t \varrho + \underbrace{\operatorname{div}(\varrho v)}_{\text{transport}} = \underbrace{\mathbf{r} - \operatorname{div}\mathbf{J}}_{\text{change of mass}}\,.$$

The $\mathbf{J}$-term has a twofold meaning. It can be written as $-\operatorname{div}\mathbf{J}$ on the right-hand side of the equation, then it is an "external" term, or it can be written as $\mathbf{J}$ as part of the flux, then it is an "internal" term (for more information on $\mathbf{J}$ specially in systems see section IV.13).

[8] We write "div_x" instead of "div" in order to focus on the variables (t, x).

Components in dry air		Volume ratio = Molar ratio compared to dry air		Molar mass	Molar mass in air		Atmospheric boiling point		
Name	Formula	[mol/mol$_{air}$]	[vol%]	[g/mol], [kg/kmol]	[g/mol$_{air}$], [kg/kmol$_{air}$]	[wt%]	[K]	[°C]	[°F]
Nitrogen	N_2	0.78084	78.084	28.013	21.873983	75.52	77.4	-195.8	-320.4
Oxygen	O_2	0.20946	20.946	31.999	6.702469	23.14	90.2	-183.0	-297.3
Argon	Ar	0.00934	0.934	39.948	0.373114	1.29	87.3	-185.8	-302.5
Carbon dioxide	CO_2	0.00033	0.033	44.010	0.014677	0.051	194.7	-78.5	-109.2
Neon	Ne	0.00001818	0.001818	20.180	0.000367	0.0013	27.2	-246.0	-410.7
Helium	He	0.00000524	0.000524	4.003	0.000021	0.00007	4.2	-269.0	-452.1
Methane	CH_4	0.00000179	0.000179	16.042	0.000029	0.00010	111.7	-161.5	-258.7
Krypton	Kr	0.0000010	0.0001	83.798	0.000084	0.00029	119.8	-153.4	-244.0
Hydrogen	H_2	0.0000005	0.00005	2.016	0.000001	0.000003	20.3	-252.9	-423.1
Xenon	Xe	0.00000009	0.000009	131.293	0.000012	0.00004	165.1	-108.1	-162.5
Average molar mass of air					28.9647				

Fig. 4: Atmosphere of Earth: "Components in Dry Air" from [127]. See also [Wikipedia: Atmosphere of Earth] [129] "Water vapor H_2O strongly varies locally $0.001\% - 5\%$". Here the mass of components in air are relevant.

If one considers a system of Gases, that is, if one is confronted with a total mass, which is the mixture of several constituents, an example is given in Fig. 4, one has

$$\varrho = \sum_{k=1}^{k_0} \varrho_k\,, \tag{I1.8}$$

where ϱ_k are the k_0 single masses and ϱ is the total mass.

1.7 Theorem. Let masses ϱ_k as in (I1.8) be given satifying the general partial mass equation

$$\partial_t \varrho_k + \mathrm{div}_x(\varrho_k v + \mathbf{J}_k) = \mathbf{r}_k \text{ für } k = 1, \ldots, k_0\,. \tag{I1.9}$$

We introduce the ***concentration*** of the component k by

$$c_k := \frac{\varrho_k}{\varrho}\,, \quad \text{so} \quad \varrho_k = c_k \varrho \quad \text{and} \quad \varrho > 0\,.$$

Then if

$$\mathbf{J} := \sum_{k=1}^{k_0} \mathbf{J}_k = 0\,, \quad \mathbf{r} := \sum_{k=1}^{k_0} \mathbf{r}_k = 0$$

the system (I1.9) is equivalent to

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}_x(\varrho v) = 0\,,\\ &\varrho\big(\partial_t c_k + v \bullet \nabla c_k\big) + \mathrm{div}_x \mathbf{J}_k = \mathbf{r}_k \text{ for } k = 1, \ldots, k_0\,. \end{aligned} \tag{I1.10}$$

Attention: Since $c_1 + c_2 + \cdots + c_{k_0} = 1$ the last k_0 equations in (I1.10) are linearly dependent.

Proof. Taking the sum of (I1.9) we get

$$\partial_t\varrho + \mathrm{div}_x\big(\varrho v + \underbrace{\sum_k \mathbf{J}_k}_{=0}\big) = \underbrace{\sum_k \mathbf{r}_k}_{=0} \;,$$

hence $\partial_t\varrho + \mathrm{div}_x(\varrho v) = 0$. The single equations of (I1.9) then become

$$\begin{aligned}
&\mathbf{r}_k - \mathrm{div}_x\mathbf{J}_k = \partial_t\varrho_k + \mathrm{div}_x(\varrho_k v) = \partial_t(c_k\varrho) + \mathrm{div}_x(c_k\varrho v) \\
&= c_k(\partial_t\varrho + \mathrm{div}_x(\varrho v)) + \varrho\big(\partial_t c_k + \sum_{i=1}^n v_i\partial_{x_i}c_k\big) \\
&= \varrho\big(\partial_t c_k + v\bullet\nabla c_k\big)\,.
\end{aligned}$$

□

The mass conservation of the total mass is usually valid without the **J** and **r** terms. Thus we assume that $\mathbf{r} = 0$ and $\mathbf{J} = 0$. Then the often used equation reads

Conservation of mass:

$$\partial_t\varrho + \mathrm{div}_x(\varrho v) = 0 \tag{I1.11}$$

$\varrho > 0$ mass density,

$q = \varrho v$ mass flux,

$v = (v_i)_{i=1,\dots,n}$ velocity.

We consider now this differential equation.

1.8 Relativity of velocity. Assume (ϱ, v) satisfies the mass conservation (I1.11). We move the mass density with a constant velocity $v_0 \in \mathbb{R}^n$, that is, we define

$$\varrho^*(t,x) := \varrho(t, x + tv_0)\,.$$

Is there a v^* such that for (ϱ^*, v^*) the equation (I1.11) is satisfied? Yes, for

$$v^*(t,x) = v(t, x + tv_0) - v_0\,.$$

Remark: This is the Doppler effect for constant v_0. We will study this phenomenon in detail in section II.3.

So (ϱ^*, v^*) and (ϱ, v) fulfill the same equation, thus, solutions of (I1.11) correspond to each other. The proof shows that this follows from a change of coordinates.

Proof. We ask, what the property

$$\partial_t \varrho^* + \operatorname{div}_x(\varrho^* v^*) = 0 \tag{I1.12}$$

for v^* means. We consider the transformation

$$Y\left(\begin{bmatrix} t \\ x \end{bmatrix}\right) = \begin{bmatrix} t \\ x + tv_0 \end{bmatrix}.$$

Then $\varrho \circ Y = \varrho^*$ and for any vector field q

$$\operatorname{div}_x(q \circ Y) = (\operatorname{div}_x q) \circ Y, \tag{I1.13}$$

hence we compute

$$\begin{aligned}
&\partial_t \varrho^* = \partial_t(\varrho \circ Y) = (\partial_t \varrho) \circ Y + v_0 \bullet (\nabla \varrho) \circ Y \\
&= -\big(\operatorname{div}_x(\varrho v)\big) \circ Y + v_0 \bullet (\nabla \varrho) \circ Y \ \ (\text{since } \partial_t \varrho + \operatorname{div}_x(\varrho v) = 0) \\
&= -\big(\operatorname{div}_x(\varrho v)\big) \circ Y + \big(\operatorname{div}_x(\varrho v_0)\big) \circ Y \ \ (\text{since } v_0 \text{ is constant}) \\
&= -\big(\operatorname{div}_x\big(\varrho(v - v_0)\big)\big) \circ Y = -\operatorname{div}_x\big((\varrho(v - v_0)) \circ Y\big) \ \ (\text{nach (I1.13)}) \\
&= -\operatorname{div}_x\big(\varrho^*((v - v_0) \circ Y)\big).
\end{aligned}$$

Therefore the differential equation (I1.12) is satisfied, if

$$v^* = (v - v_0) \circ Y = v \circ Y - v_0,$$

which was the guess in the assertion. □

In the following we consider a particle without mass in a fluid, or we think about a flag which is assigned to a moving mass point.

1.9 Particle in a fluid. Let a fluid be modelled by a mass density ϱ satisfying

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0. \tag{I1.14}$$

We are moving with the fluid, i.e., at time t we are somewhere, say, at the point $\xi(t) \in \mathbb{R}^n$, and we drift with the velocity v, i.e., ξ is given by the differential equation

$$\dot{\xi}(t) = v(t, \xi(t)).$$

Here is $\dot{\xi}$ the time derivative of $t \mapsto \xi(t)$. Define $\bar{\varrho}(t) := \varrho(t, \xi(t))$ the mass density at the position we are at time t. Then

$$\dot{\bar{\varrho}}(t) + a(t)\bar{\varrho}(t) = 0, \quad a(t) := (\operatorname{div} v)(t, \xi(t)).$$

This means that the rate at which the mass density at our position changes is $-a(t)$. Therefore one writes (I1.14) as [9]

$$\overset{\circ}{\varrho} + \varrho \operatorname{div} v = 0, \quad \overset{\circ}{\varrho} := \partial_t \varrho + v \bullet \nabla \varrho. \tag{I1.15}$$

[9] It is $\overset{\circ}{h} := \partial_t h + v \bullet \nabla h$ as "material derivative" for each function h defined and in literature this is usually written as $\overset{\circ}{h} = \dot{h}$.

Proof. It is

$$\partial_t \varrho + \operatorname{div}(\varrho v) = (\partial_t \varrho + v \bullet \nabla \varrho) + \varrho \operatorname{div} v\,,$$

which implies (I1.15). We then compute

$$\begin{aligned}
\dot{\bar{\varrho}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\varrho(t,\xi(t))) = (\partial_t \varrho)(t,\xi(t)) + \sum_{i=1}^{n} (\partial_{x_i}\varrho)(t,\xi(t))\,\dot{\xi}_i(t) \\
&= (\partial_t \varrho + v \bullet (\nabla \varrho))(t,\xi(t)) \quad \big(\text{da } \dot{\xi}_i(t) = v_i(t,\xi(t))\big) \\
&= -(\varrho \operatorname{div} v)(t,\xi(t)) = -(\operatorname{div} v)(t,\xi(t))\bar{\varrho}(t)\,,
\end{aligned}$$

which is the statement. □

If one considers polar coordinates (r,θ) for $n=2$, it can be understood as the case that the space functions do not depend on x_3. We now consider the case $n=3$ and describe cylindrical coordinates (r,θ,z) with $x_3 = z$, and we allow functions to depend on all variables.

1.10 Cylinder coordinates. In $\mathbb{R} \times \mathbb{R}^3$ we consider the transformation

$$(t,x) = (t,x_1,x_2,x_3) = \tau(t,r,\theta,z)$$

given by

$$\begin{aligned}
t &= \tau_0(t,r,\theta,z) := t\,, \\
x_1 &= \tau_1(t,r,\theta,z) := r\cos\theta\,, \\
x_2 &= \tau_2(t,r,\theta,z) := r\sin\theta\,, \\
x_3 &= \tau_3(t,r,\theta,z) := z\,.
\end{aligned}$$

We want to write the conservation law (I1.1)

$$\partial_t u + \operatorname{div} q = \mathbf{r}$$

in cylindrical coordinates. To this we decompose the flux vector q with respect to the cylindrical coordinates as

$$q = q_r \mathbf{e}_r + q_\theta \mathbf{e}_\theta + q_z \mathbf{e}_z\,, \tag{I1.16}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z \in \mathbb{R}^3$ (compare (I1.6) where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are considered in the plane) with $\mathbf{e}_r = \widehat{\mathbf{e}}_r(x)$, $\mathbf{e}_\theta = \widehat{\mathbf{e}}_\theta(x)$, $\mathbf{e}_z = \widehat{\mathbf{e}}_z(x)$ are given by [10]

$$\begin{aligned}
&\widehat{\mathbf{e}}_r(x) := (x_1^2 + x_2^2)^{-\frac{1}{2}}(x_1,x_2,0)\,, \qquad \tau'_r = (0,\cos\theta,\sin\theta,0) = (0,\widehat{\mathbf{e}}_r\circ\tau)\,, \\
&\widehat{\mathbf{e}}_\theta(x) := (x_1^2 + x_2^2)^{-\frac{1}{2}}(-x_2,x_1,0)\,, \quad \frac{1}{r}\tau'_\theta = (0,-\sin\theta,\cos\theta,0) = (0,\widehat{\mathbf{e}}_\theta\circ\tau)\,, \\
&\widehat{\mathbf{e}}_z(x) := (0,0,1)\,, \qquad \tau'_z = (0,0,0,1) = (0,\widehat{\mathbf{e}}_z\circ\tau)\,, \\
&\{\widehat{\mathbf{e}}_r(x), \widehat{\mathbf{e}}_\theta(x), \widehat{\mathbf{e}}_z(x)\} \text{ for } x \neq 0 \text{ is an orthonormal basis of } \mathbb{R}^3.
\end{aligned}$$

[10] **Notation for partial derivative:** We denote partial derivatives also as "readjusted derivative", e.g. in the formula $\tau'_r(t,r,\theta,z) := \partial_r \tau(t,r,\theta,z)$. We will use this mainly for coefficient functions, this way the presentation of differential equations becomes better readable.

Further, if we define $\underline{u} = u \circ \tau$, $\underline{\mathbf{r}} = \mathbf{r} \circ \tau$, $\underline{q} = q \circ \tau$ (and therefore $\underline{q_r} = q_r \circ \tau$, $\underline{q_\theta} = q_\theta \circ \tau$, $\underline{q_z} = q_z \circ \tau$) it follows that

$$\partial_t \underline{u} + \partial_z \underline{q_z} + \underbrace{\partial_r \underline{q_r} + \frac{1}{r}\underline{q_r}}_{= \frac{1}{r}\partial_r (r\underline{q_r})} + \frac{1}{r}\partial_\theta \underline{q_\theta} = \underline{\mathbf{r}} \,. \tag{I1.17}$$

Multiplying the equation by r, we obtain

$$\partial_t (r \cdot \underline{u}) + \partial_z (r \cdot \underline{q_z}) + \partial_r (r \cdot \underline{q_r}) + \partial_\theta \underline{q_\theta} = r \cdot \underline{\mathbf{r}} \,, \tag{I1.18}$$

which is an equation also of divergence structure (compare the result in 5.4).

Convention: We write $\mathbf{e}_r$ instead of $\widehat{\mathbf{e}}_r$ etc. because so the formulas become more handsome.

Proof (1.Version). We compute using lemma 1.3

$$\begin{aligned} \mathrm{div} q = \; & \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} q + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} q + \mathbf{e}_z \bullet \partial_{\mathbf{e}_z} q \\ = \; & \partial_{\mathbf{e}_r} (\mathbf{e}_r \bullet q) + \partial_{\mathbf{e}_\theta} (\mathbf{e}_\theta \bullet q) + \partial_{\mathbf{e}_z} (\mathbf{e}_z \bullet q) \\ & - (\partial_{\mathbf{e}_r} \mathbf{e}_r + \partial_{\mathbf{e}_\theta} \mathbf{e}_\theta + \partial_{\mathbf{e}_z} \mathbf{e}_z) \bullet q \\ = \; & \partial_{\mathbf{e}_r} q_r + \partial_{\mathbf{e}_\theta} q_\theta + \partial_{\mathbf{e}_z} q_z + \frac{1}{r} q_r \end{aligned}$$

where $r = \sqrt{x_1^2 + x_2^2}$, since

$$\partial_{\mathbf{e}_r} \mathbf{e}_r = 0, \;\; \partial_{\mathbf{e}_\theta} \mathbf{e}_\theta = -\frac{1}{r}\mathbf{e}_r, \;\; \partial_{\mathbf{e}_z} \mathbf{e}_z = 0 \text{ (see (I1.6))}.$$

Then, since for any function g

$$\begin{aligned} (\partial_{\mathbf{e}_r} g) \circ \tau &= \partial_r (g \circ \tau) \,, \\ (\partial_{\mathbf{e}_\theta} g) \circ \tau &= \frac{1}{r}\partial_\theta (g \circ \tau) \,, \\ (\partial_{\mathbf{e}_z} g) \circ \tau &= \partial_z (g \circ \tau) \,, \end{aligned}$$

we obtain

$$(\mathrm{div} q) \circ \tau = \; \partial_r (q_r \circ \tau) + \frac{1}{r}\partial_\theta (q_\theta \circ \tau) + \partial_z (q_z \circ \tau) + \frac{1}{r} q_r \circ \tau,$$

the assertion. □

Proof (2. Version). We compute, since $\mathbf{e}_z$ is constant, using lemma 1.3

$$\begin{aligned}\operatorname{div} q = {} & \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} q + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} q + \mathbf{e}_z \bullet \partial_{\mathbf{e}_z} q \\ = {} & \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} (q_r \mathbf{e}_r + q_\theta \mathbf{e}_\theta + q_z \mathbf{e}_z) \\ & + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} (q_r \mathbf{e}_r + q_\theta \mathbf{e}_\theta + q_z \mathbf{e}_z) \\ & + \mathbf{e}_z \bullet \partial_{\mathbf{e}_z} (q_r \mathbf{e}_r + q_\theta \mathbf{e}_\theta + q_z \mathbf{e}_z) \\ = {} & \partial_{\mathbf{e}_r} q_r + \partial_{\mathbf{e}_\theta} q_\theta + \partial_{\mathbf{e}_z} q_z \\ & + q_r (\mathbf{e}_r \bullet \partial_{\mathbf{e}_r} \mathbf{e}_r + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} \mathbf{e}_r + \mathbf{e}_z \bullet \partial_{\mathbf{e}_z} \mathbf{e}_r) \\ & + q_\theta (\mathbf{e}_r \bullet \partial_{\mathbf{e}_r} \mathbf{e}_\theta + \mathbf{e}_\theta \bullet \partial_{\mathbf{e}_\theta} \mathbf{e}_\theta + \mathbf{e}_z \bullet \partial_{\mathbf{e}_z} \mathbf{e}_\theta) \\ = {} & \partial_{\mathbf{e}_r} q_r + \partial_{\mathbf{e}_\theta} q_\theta + \partial_{\mathbf{e}_z} q_z + \frac{1}{r} q_r \,,\end{aligned}$$

since

$$\begin{aligned}&\partial_{\mathbf{e}_r} \mathbf{e}_r = 0,\ \partial_{\mathbf{e}_\theta} \mathbf{e}_r = \frac{1}{r} \mathbf{e}_\theta,\ \partial_{\mathbf{e}_z} \mathbf{e}_r = 0, \\ &\partial_{\mathbf{e}_r} \mathbf{e}_\theta = 0,\ \partial_{\mathbf{e}_\theta} \mathbf{e}_\theta = -\frac{1}{r} \mathbf{e}_r,\ \partial_{\mathbf{e}_z} \mathbf{e}_\theta = 0.\end{aligned} \qquad \text{(see (I1.6))}.$$

Then, since for any function g

$$\begin{aligned}(\partial_{\mathbf{e}_r} g) \circ \tau &= \partial_r (g \circ \tau)\,, \\ (\partial_{\mathbf{e}_\theta} g) \circ \tau &= \frac{1}{r} \partial_\theta (g \circ \tau)\,, \\ (\partial_{\mathbf{e}_z} g) \circ \tau &= \partial_z (g \circ \tau)\,,\end{aligned}$$

we obtain

$$(\operatorname{div} q) \circ \tau = \partial_r (q_r \circ \tau) + \frac{1}{r} \partial_\theta (q_\theta \circ \tau) + \partial_z (q_z \circ \tau) + \frac{1}{r} q_r \circ \tau,$$

the assertion. □

2 Distributions

We multiply the scalar conservation law (I1.1) for C^1-functions u, q_i, $\mathbf{r}$

$$\partial_t u + \operatorname{div} q = \mathbf{r} \quad \text{in } \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \tag{I2.1}$$

with a ***test function*** $\zeta \in C_0^\infty(\mathcal{U})$ and obtain after integration by parts

$$\begin{aligned} 0 &= \int_{\mathcal{U}} \zeta(-\partial_t u - \operatorname{div} q + \mathbf{r}) \, \mathrm{dL}^{n+1} \\ &= \int_{\mathcal{U}} (\partial_t \zeta \cdot u + \nabla \zeta \bullet q + \zeta \cdot \mathbf{r}) \, \mathrm{dL}^{n+1} \end{aligned}$$

where the last integral exists, if the functions u, q_i und r are in $L^1_{\mathrm{loc}}(\mathcal{U})$. Therefore the conservation law contains the following three contributions

$$\begin{aligned} \zeta &\mapsto \int_{\mathcal{U}} \partial_t \zeta \cdot u \, \mathrm{dL}^{n+1} \,, \\ \zeta &\mapsto \int_{\mathcal{U}} \nabla \zeta \bullet q \, \mathrm{dL}^{n+1} \,, \\ \zeta &\mapsto \int_{\mathcal{U}} \zeta \cdot \mathbf{r} \, \mathrm{dL}^{n+1} \,, \end{aligned} \tag{I2.2}$$

which are all linear in the test function ζ. These linear functions are, as we shall see, distributions with $N = n + 1$.

Definition of Distributions

We start with the essential property of distributions.

2.1 Distributions. Let $\mathcal{U} \subset \mathbb{R}^N$ be an open set. We denote by

$$\mathscr{D}(\mathcal{U}) := C_0^\infty(\mathcal{U})$$

the space of ***test functions***. We consider mappings

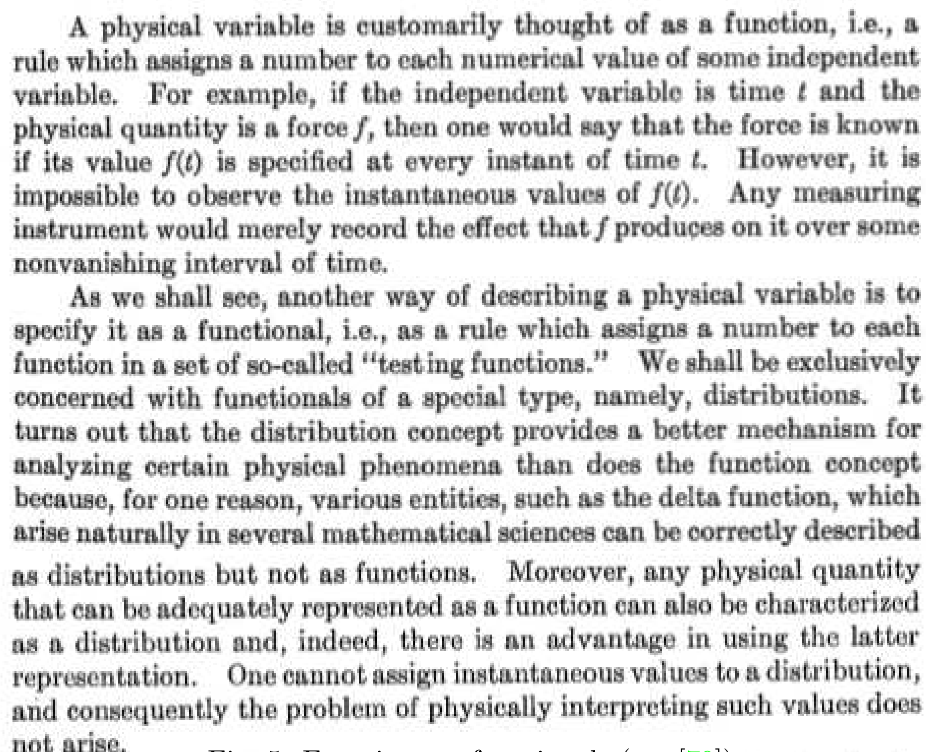
$$T : \mathscr{D}(\mathcal{U}) \to \mathbb{R} \text{ linear}$$

and call them ***distributions*** with the notation $T \in \mathscr{D}'(\mathcal{U})$, if they satisfy the estimate 2.4(1). We introduce the notation

$$\langle \zeta \,, T \rangle_{\mathscr{D}(\mathcal{U})} := T(\zeta) \,,$$

which is motivated by the integral in (I2.2). Often we simply write $\langle \zeta \,, T \rangle = \langle \zeta \,, T \rangle_{\mathscr{D}(\mathcal{U})}$ if the domain $\mathcal{U}$ is fixed.

There are two things which are important for a distribution, taking the derivative and multiplying with a function.


A physical variable is customarily thought of as a function, i.e., a rule which assigns a number to each numerical value of some independent variable. For example, if the independent variable is time t and the physical quantity is a force f, then one would say that the force is known if its value $f(t)$ is specified at every instant of time t. However, it is impossible to observe the instantaneous values of $f(t)$. Any measuring instrument would merely record the effect that f produces on it over some nonvanishing interval of time.

As we shall see, another way of describing a physical variable is to specify it as a functional, i.e., as a rule which assigns a number to each function in a set of so-called "testing functions." We shall be exclusively concerned with functionals of a special type, namely, distributions. It turns out that the distribution concept provides a better mechanism for analyzing certain physical phenomena than does the function concept because, for one reason, various entities, such as the delta function, which arise naturally in several mathematical sciences can be correctly described as distributions but not as functions. Moreover, any physical quantity that can be adequately represented as a function can also be characterized as a distribution and, indeed, there is an advantage in using the latter representation. One cannot assign instantaneous values to a distribution, and consequently the problem of physically interpreting such values does not arise.

Fig. 5: Functions as functionals (see [79])

2.2 Operations on distributions.

(1) ***Derivative.*** For $j \in \{1, \dots, N\}$ a linear map $\partial_j T : \mathscr{D}(\mathcal{U}) \to \mathbb{R}$ is defined by

$$\langle \zeta \, , \, \partial_j T \rangle_{\mathscr{D}(\mathcal{U})} := \langle -\partial_j \zeta \, , \, T \rangle_{\mathscr{D}(\mathcal{U})} \; .$$

General: For higher derivatives see 2.6(1).

Definition in spacetime: Let $N = n+1$ with $n \geq 1$. Then $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ and j runs from 0 to n. We then have $\partial_0 = \partial_t$ and $\partial_i = \partial_{x_i}$ for $i = 1, \dots, n$.

(2) ***Multiplication.*** For $a \in C^\infty_{\mathrm{loc}}(\mathcal{U})$ a linear map $aT : \mathscr{D}(\mathcal{U}) \to \mathbb{R}$ is defined by

$$\langle \zeta \, , \, aT \rangle_{\mathscr{D}(\mathcal{U})} := \langle a\zeta \, , \, T \rangle_{\mathscr{D}(\mathcal{U})} \; .$$

Both, $\partial_j T$ and aT are again distributions, since they still satisfy 2.4(1).

These are all definitions for distributions we need, and for our three terms (I2.2) in the conservation law we have to define

2.3 Functions as distribution. Let us consider special mappings $T = [g]$ where $g \in L^1_{\mathrm{loc}}(\mathcal{U})$, defined for test functions $\zeta \in \mathscr{D}(\mathcal{U})$ by

$$\langle \zeta \, , \, [g] \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta \cdot g \, \mathrm{dL}^N \, .$$

Remark: The Lebesgue-measurable function g can be reconstructed from its distribution $[g]$ (see exercise 7.9). *Hint:* See also the definition in 2.5.

The remark says that g can be recovered from its distribution almost everywhere (see also the text in Fig. 5). Similarly this follows for the derivative $\partial_i[g]$, provided this distribution is represented by a function. For example, if g is a Lipschitz continuous function, it is $\partial_i[g] = [g_i]$ with a bounded measurable function g_i (see the definition in 2.6(2))

References: Zur Geschichte der Distributionen siehe [80]. Mathematische Einführungen werden für $N = 1$ in [79], für beliebiges N in [72, in Abschnitt 3], [74, Kapitel I-II], [76, Kapitel 1-9], [78, Kapitel 1-2] gegeben. Ich habe auch ein eigenes Skript [71] dazu angefertigt. Siehe auch die sehr gute Darstellung in [Wikipedia: Distribution (Mathematik)].

We have yet to specify the full definition of distributions.

2.4 Estimate satified by distributions. Let $\mathcal{U} \subset \mathbb{R}^N$ be an open set and consider the space $\mathscr{D}(\mathcal{U}) = C_0^\infty(\mathcal{U})$. A ***distribution***, that is an element in $\mathscr{D}'(\mathcal{U})$, satifies by definition one of the following equivalent properties:

(1) A map $T \in \mathscr{D}'(\mathcal{U})$ is a linear mapping $T : \mathscr{D}(\mathcal{U}) \to \mathbb{R}$ which satisfies [11]

$$\forall\, U \subset\subset \mathcal{U} :\ \exists\, k_U \in \mathbb{N} \cup \{0\} \text{ and } C_U \geq 0 :$$
$$\forall\, \zeta \in \mathscr{D}(\mathcal{U}) \text{ with } \operatorname{supp} \zeta \subset U :\ \left| \langle \zeta\,,\, T \rangle_{\mathscr{D}(\mathcal{U})} \right| \leq C_U \|\zeta\|_{C^{k_U}(\overline{U})}\,.$$

(2) $\mathscr{D}'(\mathcal{U})$ is the set of ***linear continuous mappings***, in fact the dual space of $\mathscr{D}(\mathcal{U})$, if we assign $\mathscr{D}(\mathcal{U})$ with the following topology $\mathcal{T}$:

$$\mathcal{T} := \{ V \subset C_c^\infty(\mathcal{U})\,;\ \forall\, \zeta \in V :\ \exists\, \boldsymbol{\varepsilon} :\ \zeta + V_{\boldsymbol{\varepsilon}} \subset V \}\,.$$

Thereby $\boldsymbol{\varepsilon} = (\varepsilon_j)_{j\in\mathbb{N}}$ and

$$V_{\boldsymbol{\varepsilon}} := \operatorname{conv}\Big(\bigcup_{j\in\mathbb{N}} \{ \zeta \in C_c^\infty(\mathcal{U})\,;\ \operatorname{supp}(\zeta) \subset U_j \text{ and } p(\zeta) < \varepsilon_j \} \Big)\,,$$
$$(U_j)_{j\in\mathbb{N}} \text{ is an open covering of } \mathcal{U} \text{ with } \overline{U_j} \subset \mathcal{U} \text{ compact},$$
$$p(\zeta) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|\zeta\|_{C^k(\overline{U})}}{1 + \|\zeta\|_{C^k(\overline{U})}} \text{ for } \operatorname{supp}(\zeta) \subset U\,,\ \overline{U} \text{ compact in } \mathcal{U}\,.$$

Result: Hence $\mathscr{D}(\mathcal{U})$ becomes a locally convex topoplogical vector space, see [72, 3.19] and [71, section 6] where also the completeness of the dual space $\mathscr{D}'(\mathcal{U})$ is discussed.

Proof of equivalence: The statements (1) and (2) are equivalent. For example see [72, 3.21 Der Dualraum von $\mathscr{D}(\mathcal{U})$], but you can visit any book involving distributions.

Es ist effektiv mit der Eigenschaft 2.4(1) zu arbeiten, so wie das in [71] dargestellt wird. Setzen wir in der Abschätzung 2.4(1) $k_U = 0$ für alle $U \subset\subset \mathcal{U}$, so erhalten wir

2.5 Radon-measures as distribution. A Radon measure μ on $\mathcal{U}$ is a linear map $\mu : C_c^0(\mathcal{U}) \to \mathbb{R}$, we write $\zeta \mapsto \langle \zeta\,,\, \mu \rangle_{C_c^0(\mathcal{U})}$, such that

$$\forall\, U \subset\subset \mathcal{U} :\ \exists\, C_U \geq 0 :$$
$$\forall\, \zeta \in C^0(\mathcal{U}) \text{ with } \operatorname{supp} \zeta \subset U :\ \left| \langle \zeta\,,\, \mu \rangle_{C_c^0(\mathcal{U})} \right| \leq C_U \|\zeta\|_{C^0(\overline{U})}\,.$$

[11] $U \subset\subset \mathcal{U}$ means that $U \subset \mathcal{U}$ and $\overline{U}$ is compact in $\mathcal{U}$, in words: U is relative compact in $\mathcal{U}$.

If $g \in L^1_{\mathrm{loc}}(\mathcal{U})$ then obviously $\mu := [g]$ is a Radon measure, see 2.3. *Remark:* For a measure theoretical definition of Radon measures on the Borel sets of an open set $\mathcal{U} \subset \mathbb{R}^N$ we identify

$$\mu(E) = \int_{\mathcal{U}} \mathcal{X}_E \,\mathrm{d}\mu = \langle\, \mathcal{X}_E \,,\, \mu \,\rangle_{\mathscr{D}'(\mathcal{U})} \text{ for Borel sets } E \subset\subset \mathcal{U}\,.$$

Reference: This definition you find in Tartar [77, Definition 4.3]. For a measure theoretical definition see [75, Definition 2.2] and [Wikipedia: Radon measure].

The mathematical definition of distributions essentially shows that functions as distributions are dense in the set of distributions (siehe [71, End of section 2]). However, we will not use this estimate (except in 2.10). Here some of the important properties which distributions have.

2.6 Some properties of distributions.

(1) ***Higher derivatives.*** For all multi-indices s the ***distributional derivative*** $\partial^s T$ is the linear map $\partial^s T : \mathscr{D}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\langle\, \zeta \,,\, \partial^s T \,\rangle_{\mathscr{D}(\mathcal{U})} = (-1)^{|\,s\,|} \langle\, \partial^s \zeta \,,\, T \,\rangle_{\mathscr{D}(\mathcal{U})} \qquad \text{for } \zeta \in \mathscr{D}(\mathcal{U}).$$

Es gilt

$$\partial^s T = \partial^{r_1}(\partial^{r_2} T) \text{ for all } r_1, r_2 \text{ with } r_1 + r_2 = s\,.$$

(2) ***Partial derivative.*** Für $g \in C^1(\mathcal{U})$ gilt $\partial_j[g] = [\partial_j g]$ wegen der Regel der partiellen Integration. Man definiert daher in Analogie dazu

$$W^{1,p}_{\mathrm{loc}}(\mathcal{U}) := \{g \in L^p_{\mathrm{loc}}(\mathcal{U})\,;\ \forall\, i:\ \exists\, g_i \in L^p_{\mathrm{loc}}(\mathcal{U}):\ \partial_i[g] = [g_i]\}\,.$$

Hierbei ist $1 \le p \le \infty$. (Entsprechend ist $W^{k,p}_{\mathrm{loc}}(\mathcal{U})$ definiert.)

(3) ***Vector valued distributions.*** Analog ist die Definition von $[g]$ für vektorwertiges $g : \mathcal{U} \to \mathbb{R}^M$, es ist dann [12] für $\zeta \in \mathscr{D}(\mathcal{U}; \mathbb{R}^M) = C_0^\infty(\mathcal{U}; \mathbb{R}^M)$

$$\langle\, \zeta \,,\, [g] \,\rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta \bullet g \,\mathrm{dL}^N\,.$$

Wir schreiben dann $[g] \in \mathscr{D}'(\mathcal{U}; \mathbb{R}^M)$ (siehe auch [71, 5.4]).

(4) ***Order of a distribution.*** A distribution T is of order k, if 2.4(1) is satisfied always with the same $k_U = k$. It holds: If T is a distribution of order k, then $\partial^s T$ is a distribution of order $k + |\,s\,|$.

(5) ***Extended distributions.*** Ist T eine Distribution der Ordnung k, so kann T eindeutig fortgesetzt werden zu einer linearen Abbildung auf $C_0^k(\mathcal{U})$. Es ist also $\langle\, \zeta \,,\, T \,\rangle := T(\zeta)$ für $\zeta \in C_0^k(\mathcal{U})$ als Fortsetzung definiert. Es folgt, dass aT als Distribution definiert ist für $a \in C^k(\mathcal{U})$, es ist (siehe auch [71, 4.1]).

$$\langle\, \zeta \,,\, aT \,\rangle_{\mathscr{D}(\mathcal{U})} := \langle\, a\zeta \,,\, T \,\rangle_{C_0^k(\mathcal{U})} \text{ für } \zeta \in \mathscr{D}(\mathcal{U}).$$

What does it mean for our conservation law?

Back to the conservation law

We will now write conservation laws in the context of distributions, where we set $N = n + 1$, i.e. it is $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ and the distributions, we consider, live in spacetime:

[12] With "$\bullet$" we denote the scalar product of the Euclidic space.

2.7 Distributions in spacetime. Let $N = n+1$ with $n \geq 1$ and $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$. Then for $g \in L^1_{\mathrm{loc}}(\mathcal{U})$ the distribution $[g] \in \mathscr{D}(\mathcal{U})$ satisfies

$$\langle\, \zeta\,,\, [g]\,\rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta \cdot g \,\mathrm{dL}^{n+1} = \int_{\mathbb{R}} \int_{\mathcal{U}_t} \zeta(t,x) g(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,$$

where $\mathcal{U}_t := \{x \in \mathbb{R}^n\,;\ (t,x) \in \mathcal{U}\}$.

With these definitions we obtain for the law $\partial_t u + \operatorname{div} q = \mathbf{r}$

$$\begin{aligned}
0 &= \int_{\mathcal{U}} \zeta(-\partial_t u - \operatorname{div} q + \mathbf{r})\,\mathrm{dL}^{n+1} \\
&= \int_{\mathcal{U}} (\partial_t \zeta \cdot u + \nabla\zeta \bullet q + \zeta \cdot \mathbf{r})\,\mathrm{dL}^{n+1} \\
&= \int_{\mathcal{U}} \partial_t \zeta \cdot u \,\mathrm{dL}^{n+1} + \int_{\mathcal{U}} \nabla\zeta \bullet q \,\mathrm{dL}^{n+1} + \int_{\mathcal{U}} \zeta \cdot \mathbf{r} \,\mathrm{dL}^{n+1} \\
&= \langle\, \partial_t\zeta\,,\, [u]\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \nabla\zeta\,,\, [q]\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \zeta\,,\, [\mathbf{r}]\,\rangle_{\mathscr{D}(\mathcal{U})} \\
&= \langle\, \zeta\,,\, -\partial_t[u] - \operatorname{div}[q] + [\mathbf{r}]\,\rangle_{\mathscr{D}(\mathcal{U})}\ ,
\end{aligned}$$

where $[u]$, $[\mathbf{r}]$, $[q_j]$ $(j = 1,\dots,n)$ are defined as in 2.3. Consequently the conservation law (I2.1) now is for functions $u, \mathbf{r}, q_j \in L^1_{\mathrm{loc}}(\mathcal{U})$

$$\partial_t[u] + \operatorname{div}[q] = [\mathbf{r}] \text{ in } \mathscr{D}'(\mathcal{U})\,, \tag{I2.3}$$

and for general distributions $U, Q_j, R : \mathscr{D}(\mathcal{U}) \to \mathbb{R}$ the equation becomes

Distributional conservation law:

$$\partial_t U + \operatorname{div} Q = R \text{ in } \mathscr{D}'(\mathcal{U}), \tag{I2.4}$$

$$U, Q_j, R \in \mathscr{D}'(\mathcal{U}) \text{ for } j = 1,\dots,n.$$

This definition means that for $\zeta \in \mathscr{D}(\mathcal{U})$

$$\begin{aligned}
0 &= \langle\, \zeta\,,\, -\partial_t U - \operatorname{div} Q + R\,\rangle_{\mathscr{D}(\mathcal{U})} \\
&= \langle\, \partial_t\zeta\,,\, U\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \nabla\zeta\,,\, Q\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \zeta\,,\, R\,\rangle_{\mathscr{D}(\mathcal{U})} \\
&= \langle\, \partial_t\zeta\,,\, U\,\rangle_{\mathscr{D}(\mathcal{U})} + \sum_j \langle\, \partial_j\zeta\,,\, Q_j\,\rangle_{\mathscr{D}(\mathcal{U})} + \langle\, \zeta\,,\, R\,\rangle_{\mathscr{D}(\mathcal{U})}\ .
\end{aligned}$$

We now apply it to several examples.

Mass points

As first example we consider a trajectory $t \mapsto \xi(t) \in \mathbb{R}^n$.

2.8 Trajectory as Distribution. Let a trajectory in time and space

$$t \mapsto (t, \xi(t)) \in \mathbb{R} \times \mathbb{R}^n$$

be given. This trajectory defines a distribution $\boldsymbol{\mu}_\xi \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ by

$$\begin{aligned} \langle\, \zeta \,,\, \boldsymbol{\mu}_\xi \,\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} &:= \int_{\mathbb{R}} \zeta(t, \xi(t))\, \mathrm{d}t \\ &= \int_{\mathbb{R}} \langle\, \zeta(t, \bullet) \,,\, \boldsymbol{\delta}_{\xi(t)} \,\rangle_{\mathscr{D}(\mathbb{R}^n)} \,\mathrm{d}t \text{ for } \zeta \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)\,. \end{aligned} \tag{I2.5}$$

Es ist $\boldsymbol{\mu}_\xi$ ein Radon-Maß wie in Definition 2.5.

Dirac Distribution: Für $x_0 \in \mathbb{R}^n$ ist $\boldsymbol{\delta}_{x_0} \in \mathscr{D}'(\mathbb{R}^n)$ definiert durch

$$\langle\, \zeta \,,\, \boldsymbol{\delta}_{x_0} \,\rangle_{\mathscr{D}(\mathbb{R}^n)} = \zeta(x_0) \text{ für } \zeta \in \mathscr{D}(\mathbb{R}^n). \tag{I2.6}$$

Our example now shows that the motion of a mass point is a solution of the general distributional mass conservation. For this we take a variable mass $t \mapsto m(t) \in \mathbb{R}$ on the trajectory. If m is a bounded function then also $m\boldsymbol{\mu}_\xi$ is a Radon measure and the following holds.

2.9 Moving mass point. Let $\boldsymbol{\mu}_\xi$ be the distribution defined in 2.8. Let two differentiable maps $t \mapsto m(t) \in \mathbb{R}$, and continuous maps $t \mapsto \mathbf{r}(t) \in \mathbb{R}$ and $(t, x) \mapsto v(t, x) \in \mathbb{R}^n$ be given satisfying the distributional mass conservation law with mass production

$$\partial_t(m\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) = \mathbf{r}\boldsymbol{\mu}_\xi \tag{I2.7}$$

in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$. This is satisfied if and only if

$$\dot{m}(t) = \mathbf{r}(t) \quad \text{and} \quad v(t, \xi(t)) = \dot{\xi}(t)\,, \tag{I2.8}$$

as long as $m(t) > 0$.

Proof (I2.8)⇒(I2.7) *in the case* $\mathbf{r} = 0$. Im Falle $\dot{m} = \mathbf{r} = 0$ ist $m = \text{const}$, also hat der Körper eine konstante Masse. Wir haben zu zeigen, dass

$$\partial_t(m\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) = 0 \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)\,.$$

Now for test functions $\zeta \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$

$$\begin{aligned} &-\langle\, \zeta \,,\, \partial_t(m\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) \,\rangle \\ &= m \,\langle\, \partial_t\zeta \,,\, \boldsymbol{\mu}_\xi \,\rangle + m \sum_{i=1}^n \langle\, \partial_{x_i}\zeta \,,\, v_i\boldsymbol{\mu}_\xi \,\rangle \\ &= m \int_{\mathbb{R}} \Big((\partial_t\zeta)(t, \xi(t)) + \sum_{i=1}^n (\partial_{x_i}\zeta)(t, \xi(t))\, \underbrace{v_i(t, \xi(t))}_{= \dot{\xi}_i(t)} \Big)\, \mathrm{d}t \\ &= m \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta(t, \xi(t))\big)\, \mathrm{d}t = 0\,, \end{aligned}$$

because ζ has compact support. □

Proof (I2.8)⇒(I2.7) *in the general case.* With test functions $\zeta \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ we compute

$$\begin{aligned}
&\big\langle \zeta\,,\, -\partial_t(m\boldsymbol{\mu}_\xi) - \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) + \mathbf{r}\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \big\langle \partial_t\zeta\,,\, m\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} + \big\langle \nabla\zeta\,,\, mv\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} + \big\langle \zeta\,,\, \mathbf{r}\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \int_{\mathbb{R}} \Big(m(t)\partial_t\zeta(t,\xi(t)) + m(t)\nabla\zeta(t,\xi(t))\bullet \underbrace{v(t,\xi(t))}_{=\dot\xi(t)} + \zeta(t,\xi(t))\mathbf{r}(t)\Big)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \Big(m(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta(t,\xi(t))\big) + \zeta(t,\xi(t))\mathbf{r}(t)\Big)\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\big(m(t)\zeta(t,\xi(t))\big)\,\mathrm{d}t + \int_{\mathbb{R}} \zeta(t,\xi(t))\big(\underbrace{\mathbf{r}(t) - \dot m(t)}_{=0}\big)\,\mathrm{d}t = 0\,,
\end{aligned}$$

also $-\partial_t(m\boldsymbol{\mu}_\xi) - \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) + \mathbf{r}\boldsymbol{\mu}_\xi = 0$ in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$. □

Proof (I2.7)⇒(I2.8). Choose a test function $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n)$. Then

$$\begin{aligned}
0 &= \big\langle \zeta\,,\, -\partial_t(m\boldsymbol{\mu}_\xi) - \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) + \mathbf{r}\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \big\langle \partial_t\zeta\,,\, m\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} + \big\langle \nabla\zeta\,,\, mv\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} + \big\langle \zeta\,,\, \mathbf{r}\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \int_{\mathbb{R}} \Big(m(t)\partial_t\zeta(t,\xi(t)) + m(t)\nabla\zeta(t,\xi(t))\bullet v(t,\xi(t)) + \zeta(t,\xi(t))\mathbf{r}(t)\Big)\,\mathrm{d}t\,,
\end{aligned}$$

For the first term we get

$$\begin{aligned}
&m(t)\partial_t\zeta(t,\xi(t)) \\
&= m(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta(t,\xi(t))\big) - m(t)\dot\xi(t)\bullet\nabla\zeta(t,\xi(t)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(m(t)\zeta(t,\xi(t))\big) - \dot m(t)\zeta(t,\xi(t)) - m(t)\dot\xi(t)\bullet\nabla\zeta(t,\xi(t))\,,
\end{aligned}$$

and integrating this

$$\begin{aligned}
&\int_{\mathbb{R}} m(t)\partial_t\zeta(t,\xi(t))\,\mathrm{d}(t) \\
&= -\int_{\mathbb{R}} \Big(\dot m(t)\zeta(t,\xi(t)) + m(t)\dot\xi(t)\bullet\nabla\zeta(t,\xi(t))\Big)\,\mathrm{d}(t)\,.
\end{aligned}$$

Plugging this in the above identity we obtain

$$\begin{aligned}
0 &= \big\langle \zeta\,,\, -\partial_t(m\boldsymbol{\mu}_\xi) - \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) + \mathbf{r}\boldsymbol{\mu}_\xi \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \int_{\mathbb{R}} \Big(m(t)\nabla\zeta(t,\xi(t))\bullet\big(v(t,\xi(t)) - \dot\xi(t)\big) + \zeta(t,\xi(t))\big(\mathbf{r}(t) - \dot m(t)\big)\Big)\,\mathrm{d}t\,.
\end{aligned} \tag{I2.9}$$

This identity is true for all $\zeta \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. We now choose special test functions ζ. First we choose a function $\chi \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ with

$$\chi = 1,\ \nabla\chi = 0 \text{ on } \Gamma := \{(t,\xi(t))\,;\ t\in\mathbb{R}\}$$

and $\chi(t,x) = 0$ whenever the disteance of (t,x) from Γ is grater than 1. Then let $\zeta(t,x) = \eta(t)\chi(t,x)$ with a function $\eta \in C_c^\infty(\mathbb{R})$. Taking this test function in (I2.9), we obtain, since $\nabla\zeta(t,\xi(t)) = 0$ for all t,

$$0 = \int_{\mathbb{R}} \zeta(t,\xi(t))\big(\mathbf{r}(t) - \dot{m}(t)\big)\,\mathrm{d}t = \int_{\mathbb{R}} \eta(t)\big(\mathbf{r}(t) - \dot{m}(t)\big)\,\mathrm{d}t\,.$$

Since this is true for any function $\eta \in C_c^\infty(\mathbb{R})$ we conclude

$$\mathbf{r}(t) - \dot{m}(t) = 0\,.$$

Thus the first assertion is shown. With this the equation (I2.9) reduces to

$$0 = \int_{\mathbb{R}} m(t)\nabla\zeta(t,\xi(t)){\bullet}\big(v(t,\xi(t)) - \dot{\xi}(t)\big)\,\mathrm{d}t$$

for all $\zeta \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n)$. Now we choose $\zeta(t,x) = \chi(t,x)\,x{\bullet}w(t)$ with a vector function $w \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$. Then $\nabla\zeta(t,\xi(t)) = w(t)$ for all t and therefore

$$0 = \int_{\mathbb{R}} m(t)w(t){\bullet}\big(v(t,\xi(t)) - \dot{\xi}(t)\big)\,\mathrm{d}t$$

for all $w \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$. We conclude, if $m(t) > 0$,

$$v(t,\xi(t)) - \dot{\xi}(t) = 0\,,$$

that is the second assertion. □

It follows from the distributional mass conservation that if $\mathbf{r} = 0$ then the mass has to be constant. This proves that the distributional conservation law is the right thing to consider. In the context of momentum conservation in the next section 3 we come back to this example. In [21, Flug eines Asteroiden] we take this distributional problem and clarify in a "total mass balance", how the ejected material is distributed in the surrounding vacuum.

Gravitational law

As another example consider the gravity, the corresponding field equation has a distributional solution, so it is not a smooth function in the general case, because the characteristic function for the mass density has a jump. It turns out that in this case the solution is not a C^2-function. The field equation is

Newton's gravitation:

$$\mathrm{div}(-\nabla[\boldsymbol{\phi}]) = [\varrho]$$

in entire $\mathbb{R}\times\mathbb{R}^n$ (physically $n=3$), i.e. in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$

ϱ total mass density (as a function),

$\boldsymbol{\phi}$ gravitational field (is a function),

$\boldsymbol{\phi}(t,x)\to 0$ for $|x|\to\infty$ (if $n=3$),

$G = 6.67384\cdot 10^{-11}\frac{m^3}{kg\,s^2}$ gravitational constant,

$\phi_{literature} = -4\pi G\boldsymbol{\phi}$ potential in the literature ($n=3$). (I2.10)

One can imagine this equation also as conservation law

$$\partial_t 0 + \mathrm{div}(-\nabla[\boldsymbol{\phi}]) = [\varrho]\,,$$

thus is seems to be a general mass conservation without any mass. (But this is misleading, since the 0 arises if the speed of light goes to ∞.) In the literature the gravitational field is $\phi_{literature}$ and therefore the equation reads $\Delta\phi_{literature} = 4\pi G\varrho$. One can also write $\mathrm{div}\nabla[\boldsymbol{\phi}] = \Delta[\boldsymbol{\phi}]$, hence $-\Delta[\boldsymbol{\phi}] = [\varrho]$. In general the gravitational field $\boldsymbol{\phi}$ and the mass ϱ may be distributions Φ and R satisfying the equation

General gravitational law:

$$\mathrm{div}(-\nabla\Phi) = R \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$$

R the total mass as distribution,

Φ the gravity field as distribution. (I2.11)

In the law of gravitation the time t occurs only as a parameter. There is no explicit time derivative in the Newtonian physics considered here. Therefore, the general gravity law is related to the distributional Poisson equation, which is:

Distributional Poisson equation:

$$-\Delta\Phi = R \text{ in } \mathscr{D}'(\mathbb{R}^n)$$

R the source term as a distribution,

Φ the solution as a distribution.

Δ the Laplace operator in $\mathbb{R}^n$. (I2.12)

In the following we apply the Poisson equation where n is the space dimension and where the time is a parameter. We compare it with Newton's law in spacetime $\mathbb{R}\times\mathbb{R}^n$ with dimension $n+1$.

2.10 Remark. We assume that $U_t, R_t \in \mathscr{D}'(\mathbb{R}^n)$ for $t \in \mathbb{R}$ are distributions of order k, that is,

$$|\langle \zeta\,,\, U_t \rangle_{\mathscr{D}(\mathbb{R}^n)}| + |\langle \zeta\,,\, R_t \rangle_{\mathscr{D}(\mathbb{R}^n)}| \le C(t)\|\zeta\|_{C^k(\mathbb{R}^n)}$$

with an integrable function $C \in L^1(\mathbb{R})$. If they satisfy the Poisson equation

$$-\operatorname{div}\nabla U_t = R_t \text{ in } \mathscr{D}'(\mathbb{R}^n) \text{ for almost all } t,$$

then (under the assumption of measurability on $t \mapsto R_t$, $t \mapsto U_t$)

$$\langle \zeta\,,\, R \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} := \int_{\mathbb{R}} \langle \zeta(t,\bullet)\,,\, R_t \rangle_{\mathscr{D}(\mathbb{R}^n)}\,\mathrm{d}t\,,$$
$$\langle \zeta\,,\, U \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} := \int_{\mathbb{R}} \langle \zeta(t,\bullet)\,,\, U_t \rangle_{\mathscr{D}(\mathbb{R}^n)}\,\mathrm{d}t$$

define distributions $U, R \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ and they fulfill the general law of gravitation

$$-\operatorname{div}\nabla U = R \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n).$$

Attention: Not each distribution $R \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ can be represented as shown (see, e.g. Exercise 7.13, but keep 2.11 in mind).

Proof. Both U and R are distributions, and it is

$$\begin{aligned}&\langle \zeta\,,\, \Delta U + R \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} = \langle \Delta\zeta\,,\, U \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} + \langle \zeta\,,\, R \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)}\\ &= \int_{\mathbb{R}} \Big(\langle \Delta\zeta(t,\bullet)\,,\, U_t \rangle_{\mathscr{D}(\mathbb{R}^n)} + \langle \zeta(t,\bullet)\,,\, R_t \rangle_{\mathscr{D}(\mathbb{R}^n)} \Big)\,\mathrm{d}t\\ &= \int_{\mathbb{R}} \langle \zeta(t,\bullet)\,,\, \Delta U_t + R_t \rangle_{\mathscr{D}(\mathbb{R}^n)}\,\mathrm{d}t\,.\end{aligned}$$

□

As an example we choose a moving mass point.

2.11 Example. If $R \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ is a distribution that belongs to a mass point with the trajectory $\{(t,\xi(t))\,;\ t\in\mathbb{R}\}$, then the definition $R = m\boldsymbol{\mu}_\xi$ implies that

$$\langle \zeta\,,\, R \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} = m\int_{\mathbb{R}} \zeta(t,\xi(t))\,\mathrm{d}t = \int_{\mathbb{R}} \langle \zeta(t,\bullet)\,,\, R_t \rangle_{\mathscr{D}(\mathbb{R}^n)}\,\mathrm{d}t\,,$$

where $R_t = m\boldsymbol{\delta}_{\xi(t)}$, i.e.

$$\langle \eta\,,\, R_t \rangle_{\mathscr{D}(\mathbb{R}^n)} := m\cdot\eta(\xi(t)) \text{ for } \eta \in \mathscr{D}(\mathbb{R}^n)\,,$$

hence R_t is given by the Dirac distribution.

We are now focusing first on the Poisson equation. Here we can consider in (I2.12) as a special case $R = \boldsymbol{\delta}_{x_0}$ for $x_0 \in \mathbb{R}^n$, see 2.8 for the Dirac distribution. Then the solution $\Phi = [\phi]$ to $R = \boldsymbol{\delta}_0$ with $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is the fundamental solution for the negative Laplace operator:

2.12 Fundamental solution for the Laplace operator. Let $n \ge 3$. The solution $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ of the equation

$$-\Delta[\phi] = \boldsymbol{\delta}_0 \text{ in } \mathscr{D}'(\mathbb{R}^n)\,,$$

with the boundary condition $\phi(x) \to 0$ as $|x| \to \infty$, is given by

$$\phi(x) := \frac{1}{\sigma_n(n-2)}|x|^{2-n} \text{ for } |x| > 0\,. \tag{I2.13}$$

Remark: It is ϕ the fundamental solution for $-\Delta$, that is, the negative Laplace operator. *Definition:* It is $\sigma_n := \mathrm{H}^{n-1}(\partial\mathrm{B}_1(0)) = n\kappa_n$ the ***surface of the unit sphere*** in $\mathbb{R}^n$, and $\kappa_n := \mathrm{L}^n(\partial\mathrm{B}_1(0))$ the ***volume of the unit ball*** in $\mathbb{R}^n$.

n	1	2	3	arbitrary
κ_n	2	π	$\frac{4}{3}\pi$	$\mathrm{L}^n(\mathrm{B}_1(0))$
σ_n	2	2π	4π	$\mathrm{H}^{n-1}(\partial\mathrm{B}_1(0)) = n\kappa_n$

(I2.14)

Proof. It is for $\zeta \in \mathscr{D}(\mathbb{R}^n)$

$$\begin{aligned}
\langle \zeta\,,\, -\partial_i[\phi]\,\rangle &= \langle\, \partial_i\zeta\,,\,[\phi]\,\rangle = \frac{1}{\sigma_n(n-2)}\int_{\mathbb{R}^n}\partial_i\zeta(x)\frac{\mathrm{d}x}{|x|^{n-2}} \\
&= \frac{1}{\sigma_n(n-2)}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon(0)}\partial_{x_i}\zeta(x)\frac{\mathrm{d}x}{|x|^{n-2}} \\
&= -\lim_{\varepsilon\to 0}\frac{1}{\sigma_n(n-2)}\int_{\partial\mathrm{B}_\varepsilon(0)}\zeta(x)\mathbf{e}_i\bullet\nu_{\mathrm{B}_\varepsilon(0)}(x)\frac{1}{\varepsilon^{n-2}}\,\mathrm{dH}^{n-1}(x) \\
&\quad -\lim_{\varepsilon\to 0}\frac{1}{\sigma_n(n-2)}\int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon(0)}\zeta(x)\partial_{x_i}\frac{1}{|x|^{n-2}}\,\mathrm{d}x \\
&= \frac{1}{\sigma_n}\int_{\mathbb{R}^n}\zeta(x)\frac{x_i}{|x|^n}\,\mathrm{d}x \\
&= \langle\,\zeta\,,\,[F_i]\,\rangle
\end{aligned}$$

with

$$F(x) := \frac{1}{\sigma_n}\frac{x}{|x|^n}\,,$$

hence $-\nabla[\phi] = [F]$ in $\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n)$. Now

$$\begin{aligned}\langle \zeta\,,\ \mathrm{div}[F]\,\rangle &= \ \langle -\nabla\zeta\,,\,[F]\,\rangle = -\frac{1}{\sigma_n}\int_{\mathbb{R}^n}\nabla\zeta(x)\bullet\frac{x}{|x|^n}\,\mathrm{d}x \\ &= \ -\frac{1}{\sigma_n}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus \mathrm{B}_\varepsilon(0)}\nabla\zeta(x)\bullet\frac{x}{|x|^n}\,\mathrm{d}x \\ &= \ \frac{1}{\sigma_n}\lim_{\varepsilon\to 0}\int_{\partial \mathrm{B}_\varepsilon(0)}\zeta(x)\ \underbrace{\nu_{\mathrm{B}_\varepsilon(0)}(x)\bullet\frac{x}{|x|^n}}_{=\frac{1}{\varepsilon^{n-1}}}\ \mathrm{dH}^{n-1}(x) \\ &\qquad +\lim_{\varepsilon\to 0}\frac{1}{\sigma_n}\int_{\mathbb{R}^n\setminus \mathrm{B}_\varepsilon(0)}\zeta(x)\ \underbrace{\mathrm{div}\frac{x}{|x|^n}}_{=0}\ \mathrm{d}x \quad \text{(see 7.12)} \\ &= \ \zeta(0) = \langle\,\zeta\,,\,\boldsymbol{\delta}_0\,\rangle\ ,\end{aligned}$$

that is, F is the fundamental solution of the divergence operator. □

In the case $n = 1, 2$ there are also fundamental solutions of the Laplace operator, however they are physically only of interest in finite neighbourhoods of the singularity. They are

$$\phi(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| \text{ if } n = 2, \\ -\dfrac{1}{2}|x| \text{ if } n = 1. \end{cases}$$

It is

$$\left.\begin{aligned} -\nabla[\phi] = [F] \text{ in } \mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n) \\ F(x) := \frac{1}{\sigma_n}\frac{x}{|x|^n} \end{aligned}\right\} \text{ for all } n \geq 1.$$

In the case $n = 3$ the fundamental solution is related to the solution of the general gravity law for a mass point.

2.13 Gravitational potential of a point-shaped star. Let $n \geq 3$ (physically $n = 3$), $m > 0$ the mass and $R := m\boldsymbol{\mu}_\xi$ the "density of a mass point" which moves by $t \mapsto \xi(t)$. Then the solution, i.e. the distribution Φ, of the general law of gravitation

$$\mathrm{div}(-\nabla\Phi) = R := m\boldsymbol{\mu}_\xi \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$$

is given by $\Phi = [\boldsymbol{\phi}]$,

$$\boldsymbol{\phi}(t,x) := \frac{m}{\sigma_n(n-2)}|x-\xi(t)|^{2-n} \text{ if } |x-\xi(t)| > 0\,. \tag{I2.15}$$

The solution is uniquely determined by the condition that as $|x| \to \infty$ the potential $\boldsymbol{\phi}(t,x) \to 0$. In the physical case $n = 3$ the solution is given by

$$\boldsymbol{\phi}(t,x) := \frac{m}{4\pi|x-\xi(t)|} \text{ for } x \neq \xi(t)\,. \tag{I2.16}$$

Proof. This follows essentially in the same way as the proof of 2.12. The difference is that one deals with integrals over $\mathbb{R}\times\mathbb{R}^n$. The uniqueness is derived from the following. We mention that $\boldsymbol{\phi}(t,\bullet)\in L^1_{loc}(\mathbb{R}^n)$. Another way to prove this is to apply 2.17. □

2.14 Uniqueness. Let $n\geq 3$ and $R\in\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$. Then there exists at most one $\phi\in L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^n)$ with

$$\begin{aligned}&\mathrm{div}(-\nabla[\phi])=R \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n),\\ &\phi(t,x)\to 0 \text{ for } |x|\to\infty \text{ for almost all } t.\end{aligned}$$

Proof. Because ϕ_1 and ϕ_2 are solutions to R, it follows with $\phi:=\phi_1-\phi_2$ that

$$\begin{aligned}&-\Delta[\phi]=0 \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)\,,\\ &\phi(t,x)\to 0 \text{ für } |x|\to\infty \text{ for almost all } t.\end{aligned}$$

It follows for $\zeta(t,x)=\eta_0(t)\eta_1(x)$

$$0=\langle\,\zeta\,,\,\Delta[\phi]\,\rangle=\langle\,\Delta\zeta\,,\,[\phi]\,\rangle=\int_{\mathbb{R}}\eta_0(t)\int_{\mathbb{R}^n}\Delta\eta_1(x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

Since this holds for all η_0, it follows for almost all t

$$0=\int_{\mathbb{R}^n}\Delta\eta_1(x)\phi(t,x)\,\mathrm{d}x\,,$$

hence $\phi(t,\bullet)$ or better $[\phi(t,\bullet)]$ is a harmonic distribution, that is,

$$\Delta[\phi(t,\bullet)]=0 \text{ in } \mathscr{D}'(\mathbb{R}^n)\,,$$

and for such functions the mean value property of spheres applies (see [PDE]), i.e.

$$\phi(t,x_0)=\frac{1}{\sigma_n r^{n-1}}\int_{\partial\mathrm{B}_r(x_0)}\phi(t,x)\,\mathrm{dH}^{n-1}(x)\to 0 \text{ as } r\to\infty\,.$$

Consequently, it is $\phi=0$. □

We will now calculate the gravitational force of a planet. The solution is a distribution because it models the boundary between a solid body and vacuum, that is, the density makes a jump. (Hence it is $\Delta\phi\in L^\infty$, where we do not use brackets for distributions. The best of what could be shown by the regularity theory is that $\phi\in C^{1,1}$. This is because there is the sharp statement that $\Delta\phi\in L^p$ implies $\phi\in W^{2,p}$ for $p<\infty$.) This is konsistent with the following theorem.

2.15 Theorem. Let $\Gamma \subset D \subset \mathbb{R}^N$ be a C^1-surface. Further, let a decomposition $D = D_+ \cup \Gamma \cup D_-$ with disjoint open sets D_+ and D_- (so that Γ has no boundary in D) be given and define

$$\varrho = \begin{cases} \varrho_+ \text{ in } D_+, & \varrho_+ \in C^0(\overline{D_+}), \\ \varrho_- \text{ in } D_-, & \varrho_- \in C^0(\overline{D_-}), \end{cases} \qquad \phi = \begin{cases} \phi_+ \text{ in } D_+, & \phi_+ \in C^2(\overline{D_+}), \\ \phi_- \text{ in } D_-, & \phi_- \in C^2(\overline{D_-}). \end{cases}$$

Then

$$-\Delta\Phi = R \text{ in } \mathscr{D}'(D), \quad \Phi = [\phi], \quad R = [\varrho], \tag{I2.17}$$

is equivalent to

$$\begin{gathered} -\Delta\phi_+ = \varrho_+ \text{ in } D_+ \;, \quad -\Delta\phi_- = \varrho_- \text{ in } D_- \;, \\ \left.\begin{aligned} \phi_+ &= \phi_- \\ \partial_\nu\phi_+ &= \partial_\nu\phi_- \end{aligned}\right\} \text{ on } \Gamma. \end{gathered} \tag{I2.18}$$

Consequently $\phi \in C^1(D)$, where $\nu = \nu_{D_+} = -\nu_{D_-}$.

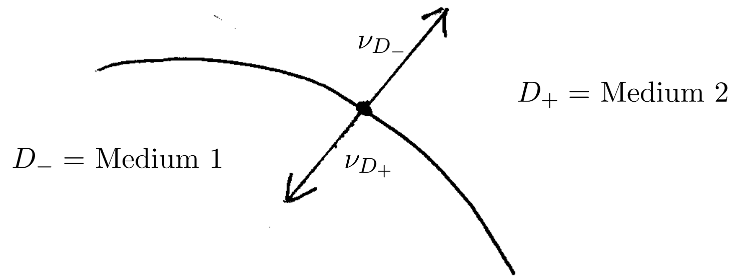


Fig. 6: Jump conditions at the interface

Note: For the result 2.15 the requirement $R = [\varrho]$ with $\varrho \in L^\infty(D)$ is essential. For example if we have [13]

$$R = \varrho_+ \mathrm{L}^N \llcorner D_+ + \varrho_- \mathrm{L}^N \llcorner D_- + \varrho_0 \mathrm{H}^{N-1} \llcorner \Gamma \,,$$

then ϕ is only continuous with

$$\partial_\nu\phi_+ = \partial_\nu\phi_- + \varrho_0$$

on Γ. The case $\varrho_0 \neq 0$ is treated in 2.18.

Proof (I2.17)⇒(I2.18). For test functions $\zeta \in \mathscr{D}(D)$ it holds

$$\langle\, \zeta \,,\, \Delta[\phi] \,\rangle = -\langle\, \zeta \,,\, [\varrho] \,\rangle \;.$$

The right-hand side is

$$-\langle\, \zeta \,,\, [\varrho] \,\rangle = -\int_D \zeta\varrho \,\mathrm{dL}^N = -\int_{D_+} \zeta\varrho_+ \,\mathrm{dL}^N - \int_{D_-} \zeta\varrho_- \,\mathrm{dL}^N.$$

[13] Ist μ ein Maß, so ist das Maß $\mu \llcorner S$ definiert durch $(\mu \llcorner S)(E) := \mu(S \cap E)$.

The left-hand side is

$$
\begin{aligned}
&\langle \zeta \,,\, \Delta[\phi] \,\rangle = \langle\, \Delta\zeta \,,\, [\phi] \,\rangle = \int_D \Delta\zeta \cdot \phi \, \mathrm{dL}^N \\
&= \int_{D_+} \Delta\zeta \cdot \phi_+ \, \mathrm{dL}^N + \int_{D_-} \Delta\zeta \cdot \phi_- \, \mathrm{dL}^N \\
&= -\int_{D_+} \nabla\zeta \bullet \nabla\phi_+ \, \mathrm{dL}^N - \int_{D_-} \nabla\zeta \bullet \nabla\phi_- \, \mathrm{dL}^N \\
&+ \int_\Gamma \nabla\zeta \bullet \left(\phi_+ \nu_{D_+} + \phi_- \nu_{D_-} \right) \, \mathrm{dH}^{N-1} \\
&= \int_{D_+} \zeta \Delta\phi_+ \, \mathrm{dL}^N + \int_{D_-} \zeta \Delta\phi_- \, \mathrm{dL}^N \\
&+ \int_\Gamma \left(\nabla\zeta \bullet \left(\phi_+ \nu_{D_+} + \phi_- \nu_{D_-} \right) - \zeta \left(\nabla\phi_+ \bullet \nu_{D_+} + \nabla\phi_- \bullet \nu_{D_-} \right) \right) \, \mathrm{dH}^{N-1} \,.
\end{aligned}
$$

We now choose ζ in $C_0^\infty(D_+)$ and conclude

$$\int_{D_+} \zeta \Delta\phi_+ \, \mathrm{dL}^N = -\int_{D_+} \zeta \varrho_+ \, \mathrm{dL}^N$$

for all such test functions, hence

$$\Delta\phi_+ = -\varrho_+ \text{ in } D_+ \,.$$

Accordingly in the same way, it means chosing ζ in $C_0^\infty(D_-)$, it follows

$$\Delta\phi_- = -\varrho_- \text{ in } D_- \,.$$

By plugging these identities into the above equation we obtain for arbitrary test functions

$$0 = \int_\Gamma \left(\nabla\zeta \bullet \left(\phi_+ \nu_{D_+} + \phi_- \nu_{D_-} \right) - \zeta \left(\nabla\phi_+ \bullet \nu_{D_+} + \nabla\phi_- \bullet \nu_{D_-} \right) \right) \, \mathrm{dH}^{N-1} \,.$$

Now we extend this argument from $\zeta \in C_0^\infty(D)$ to $\zeta \in C_0^1(D)$ by an approximation. Having done this we choose a function $\eta \in C^1(D)$ which vanishes on Γ and for which $\nabla\eta \neq 0$ on Γ (e.g. $\eta(x) = \mathrm{dist}\,(x, \Gamma)$ auf D_+ und $\eta(x) = -\mathrm{dist}\,(x, \Gamma)$ auf D_-). Set

$$\zeta = \zeta_0 \cdot \eta \in C_0^1(D) \,,$$

where $\zeta_0 \in C_0^\infty(D)$. Then on Γ

$$\zeta = 0 \,, \quad \nabla\zeta = \zeta_0 \nabla\eta$$

is satisfied. It follows

$$0 = \int_\Gamma \zeta_0 \nabla\eta \bullet \left(\phi_+ \nu_{D_+} + \phi_- \nu_{D_-} \right) \, \mathrm{dH}^{N-1}$$

and, since ζ_0 is arbitrary, $0 = \nabla\eta\bullet\left(\phi_+\nu_{D_+} + \phi_-\nu_{D_-}\right)$ on Γ. Since $\nabla\eta$ points in the direction of a normal ν we obtain

$$0 = \nu\bullet\left(\phi_+\nu_{D_+} + \phi_-\nu_{D_-}\right) = \nu\bullet\nu_{D_+}\left(\phi_+ - \phi_-\right)$$

and therefore

$$\phi_+ = \phi_- \text{ on } \Gamma\,,$$

i.e. ϕ is continuous across Γ. Thus the identity for arbitrary test functions is now

$$0 = \int_\Gamma \zeta\left(\nabla\phi_+\bullet\nu_{D_+} + \nabla\phi_-\bullet\nu_{D_-}\right)\,\mathrm{dH}^{N-1}\,.$$

It follows since ζ is an arbitrary test function

$$\nabla\phi_+\bullet\nu_{D_+} + \nabla\phi_-\bullet\nu_{D_-} = 0 \text{ on } \Gamma\,,$$

hence i.e. the differentiability of ϕ. □

Proof (I2.18)⇒(I2.17). For test functions $\zeta \in C_0^\infty(D)$

$$\begin{aligned}
0 &= \int_{D_+} \zeta(\Delta\phi_+ + \varrho_+)\,\mathrm{dL}^N + \int_{D_-} \zeta(\Delta\phi_- + \varrho_-)\,\mathrm{dL}^N \\
&= -\int_{D_+} \nabla\zeta\bullet\nabla\phi_+\,\mathrm{dL}^N - \int_{D_-} \nabla\zeta\bullet\nabla\phi_-\,\mathrm{dL}^N \\
&\quad - \int_\Gamma -\zeta\,\underbrace{\left(\nabla\phi_+\bullet\nu_{D_+} + \nabla\phi_-\bullet\nu_{D_-}\right)}_{=\,\partial_{\nu_{D_+}}\phi_+ - \partial_{\nu_{D_+}}\phi_- = 0}\,\mathrm{dH}^{N-1} + \int_D \zeta\varrho\,\mathrm{dL}^N \\
&= \int_{D_+} \Delta\zeta\cdot\phi_+\,\mathrm{dL}^N + \int_{D_-} \Delta\zeta\cdot\phi_-\,\mathrm{dL}^N \\
&\quad - \int_\Gamma \nabla\zeta\bullet\,\underbrace{\left(\phi_+\nu_{D_+} + \phi_-\nu_{D_-}\right)}_{=\,(\phi_+ - \phi_-)\nu_{D_+} = 0}\,\mathrm{dH}^{N-1} + \int_D \zeta\varrho\,\mathrm{dL}^N \\
&= \int_D \Delta\zeta\cdot\phi\,\mathrm{dL}^N + \int_D \zeta\varrho\,\mathrm{dL}^N \\
&= \langle\,\Delta\zeta\,,\,[\phi]\,\rangle + \langle\,\zeta\,,\,[\varrho]\,\rangle = \langle\,\zeta\,,\,\Delta[\phi] + [\varrho]\,\rangle
\end{aligned}$$

also $\Delta[\phi] + [\varrho] = 0$ in $\mathscr{D}'(D)$. □

Proof der Bemerkung. Wir betrachten nur den Fall $R = \varrho_0[\mathrm{H}^{N-1}\llcorner\Gamma]$. Der Beweis ist derselbe bis auf die Tatsache, dass nun

$$-\langle\,\zeta\,,\,R\,\rangle = -\int_\Gamma \zeta\varrho_0\,\mathrm{dH}^{N-1}$$

und der Term auf Γ

$$-\int_\Gamma \zeta\varrho_0 \,\mathrm{dH}^{N-1} = \int_\Gamma \Big(\nabla\zeta\bullet\left(\phi_+\nu_{D_+} + \phi_-\nu_{D_-}\right) \\ -\zeta\left(\nabla\phi_+\bullet\nu_{D_+} + \nabla\phi_-\bullet\nu_{D_-}\right)\Big)\,\mathrm{dH}^{N-1}$$

ist. Mit dergleichen Argumentation wie oben ist dann $\phi_- = \phi_+$ auf Γ und daher

$$0 = \int_\Gamma \zeta\left(\varrho_0 - \nabla\phi_+\bullet\nu_{D_+} - \nabla\phi_-\bullet\nu_{D_-}\right)\,\mathrm{dH}^{N-1}$$

für alle ζ, weswegen $\nabla\phi_+\bullet\nu_{D_+} + \nabla\phi_-\bullet\nu_{D_-} = \varrho_0$. □

The situation in 2.15 occurs for example for the gravity when the body has a smooth surface. This is true for the spherical case.

2.16 Gravitational potential of a globe. Let be $n \geq 3$ (physically $n = 3$), $m > 0$ the mass and $t \mapsto \xi(t)$ the motion of the center of the planet. Then

$$\varrho(t,x) = \frac{m}{\mathrm{L}^n(\mathrm{B}_R(\xi(t)))}\mathcal{X}_{\mathrm{B}_R(\xi(t))}(x)$$

is the mass distribution of the planet idealized as a homogeneous mass density on a sphere of radius R (see also 4.5). The total mass of the planet is

$$m = \int_{\mathbb{R}^n} \varrho(t,x)\,\mathrm{d}x\,.$$

We are seeking a solution $\boldsymbol{\phi}$ of the differential equation

$$\begin{gathered}\operatorname{div}(-[\nabla\boldsymbol{\phi}]) = [\varrho] \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n) \\ \text{with } \boldsymbol{\phi}(t,x) \to 0 \text{ if } |x| \to \infty,\end{gathered} \tag{I2.19}$$

which is of order C^1 in the space variables (see statement 2.15).
Assertion: The solution, which disappears at infinity (for $n \geq 3$), is [14]

$$\boldsymbol{\phi}(t,x) = \begin{cases} \dfrac{m}{2\kappa_n}\dfrac{1}{R^n}\left(\dfrac{R^2}{n-2} - \dfrac{|x-\xi(t)|^2}{n}\right) & \text{if } |x-\xi(t)| \leq R, \\ \dfrac{m}{n\kappa_n(n-2)}\dfrac{1}{|x-\xi(t)|^{n-2}} & \text{if } |x-\xi(t)| \geq R. \end{cases} \tag{I2.20}$$

For $n = 3$ the solution is given by

$$\boldsymbol{\phi}(t,x) = \begin{cases} \dfrac{3m}{8\pi}\dfrac{1}{R^3}\left(R^2 - \dfrac{|x-\xi(t)|^2}{3}\right) & \text{if } |x-\xi(t)| \leq R, \\ \dfrac{m}{4\pi}\dfrac{1}{|x-\xi(t)|} & \text{if } |x-\xi(t)| \geq R. \end{cases} \tag{I2.21}$$

[14] It is $\kappa_n := \mathrm{L}^n(\mathrm{B}_1(0))$, see (I2.14).

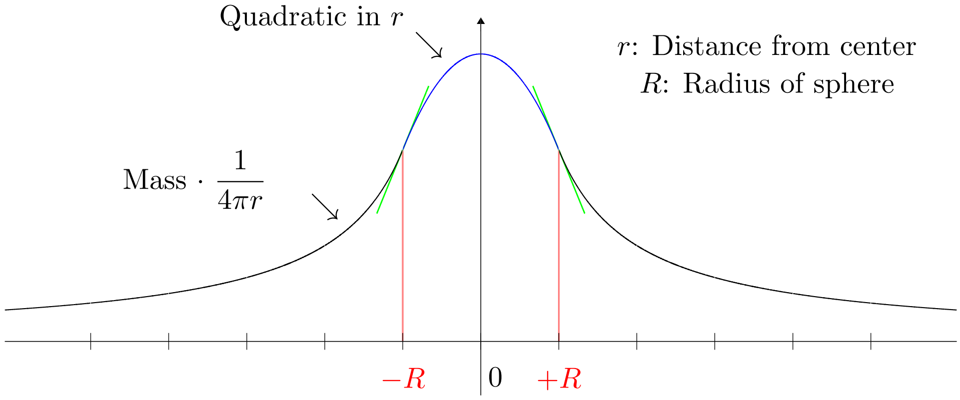


Fig. 7: Gravitational field of an incompressible ball ($n = 3$)

Proof. Without restrictions let $\xi(t) = 0$. Since $\kappa_n R^n = \mathrm{L}^n(\mathrm{B}_R(0))$ one computes

$$\varrho(t,x) = \frac{m}{\kappa_n R^n}\mathcal{X}_{\mathrm{B}_R(0)}(x) = m_R \mathcal{X}_{\mathrm{B}_R(0)}(x) \text{ if } m_R := \frac{m}{\kappa_n R^n}\,.$$

Further let

$$\boldsymbol{\phi}(t,x) := \begin{cases} \boldsymbol{\phi}_-(t,x) \text{ if } |x| < R, \\ \boldsymbol{\phi}_+(t,x) \text{ if } |x| > R, \end{cases}$$

where

$$-\Delta\boldsymbol{\phi}_-(t,x) = m_R \text{ if } |x| < R,$$
$$-\Delta\boldsymbol{\phi}_+(t,x) = 0 \text{ if } |x| > R\,.$$

This is satisfied if

$$\boldsymbol{\phi}_-(t,x) := c_0 - \frac{m_R}{2n}|x|^2,$$
$$\boldsymbol{\phi}_+(t,x) := c_\infty \frac{1}{|x|^{n-2}}\,.$$

Then $\boldsymbol{\phi}$ is continuous in space, if $\boldsymbol{\phi}_-(t,x) = \boldsymbol{\phi}_+(t,x)$ for $x \in \partial\mathrm{B}_R(0)$, i.e.

$$c_0 - \frac{m_R}{2n}R^2 = c_\infty R^{2-n}\,. \tag{I2.22}$$

Then $\boldsymbol{\phi}$ is continuous differentiable in space if and only if $\partial_\nu\boldsymbol{\phi}_-(t,x) = \partial_\nu\boldsymbol{\phi}_+(t,x)$ for $x \in \partial\mathrm{B}_R(0)$, i.e.

$$\frac{m_R}{n}R = (n-2)c_\infty R^{1-n}\,. \tag{I2.23}$$

From (I2.22) and (I2.23) it follows that $\boldsymbol{\phi}$ is continuously differentiable in spacetime, and it is

$$\begin{aligned} c_\infty &= \frac{m_R}{n(n-2)}R^n = \frac{m}{n\kappa_n(n-2)}\,, \\ c_0 &= m_R\left(\frac{1}{2n} + \frac{1}{n(n-2)}\right)R^2 = \frac{m}{2\kappa_n(n-2)}R^{2-n}\,. \end{aligned} \tag{I2.24}$$

The conditions in (I2.24) yield (I2.20). Hence (I2.17) is shown and therefore theorem 2.15 implies that (I2.18) holds. The theorem is proved. □

Usually it is only an approximation if we consider a planet to be a ball. The reason is that there are mountains on the surface, or ϱ inside the planet is not constant, see Fig. 8 and [GRACE_globe_animation.gif]. But whatever ϱ is, in any case (I2.19) says what the gravitation potential has to be. We now treat the case that the planet degenerates to a point of mass m.

2.17 Convergence to a mass point. Let $n \geq 3$. As fixed mass $m > 0$ and as $R \to 0$ the gravity solution of 2.16 converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^n)$ to a solution $\boldsymbol{\phi}$ of

$$-\operatorname{div}([\nabla \boldsymbol{\phi}]) = m\boldsymbol{\mu}_\xi \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)\,,$$
$$\boldsymbol{\phi}(t,x) \to 0 \text{ as } |x| \to \infty\,.$$

This solution is given by [15]

$$\boldsymbol{\phi}(t,x) := \frac{m}{\sigma_n(n-2)}|x-\xi(t)|^{2-n} \text{ if } |x-\xi(t)| > 0\,. \tag{I2.25}$$

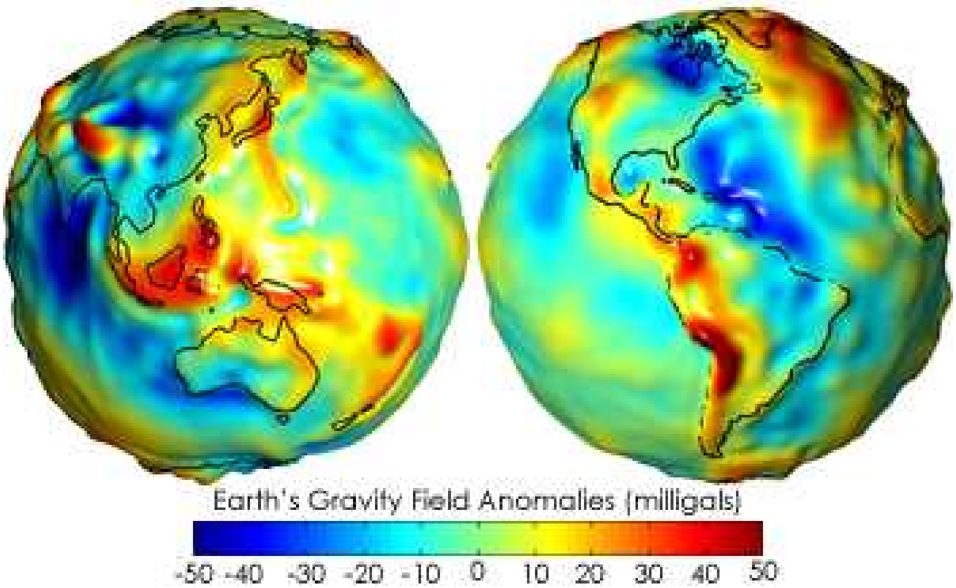


Fig. 8: "Earth's gravity measured by NASA GRACE mission, showing deviations from the theoretical gravity of an idealized smooth Earth, the so-called earth ellipsoid. Red shows the areas where gravity is stronger than the smooth, standard value, and blue reveals areas where gravity is weaker." [Wikipedia: Gravity of Earth].

[15] We define $\sigma_n := \mathrm{H}^{n-1}(\partial \mathrm{B}_1(0))$, $\partial \mathrm{B}_1(0) \subset \mathbb{R}^n$, so that $\sigma_n = n\kappa_n$.

Proof of convergence. The gravity solution $\boldsymbol{\phi}_R$ and ϱ_R of 2.16 fulfills

$$\mathrm{div}(-[\nabla\boldsymbol{\phi}_R]) = [\varrho_R] \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)\,,$$

or with test functions $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$

$$\int \zeta\varrho_R \,\mathrm{dL}^{n+1} = \int \nabla\zeta\bullet\nabla\boldsymbol{\phi}_R \,\mathrm{dL}^{n+1} = -\int \Delta\zeta\cdot\boldsymbol{\phi}_R \,\mathrm{dL}^{n+1}\,.$$

Now it holds $\boldsymbol{\phi}_R \to \boldsymbol{\phi}$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^n)$. This is because of Lebesgue's convergence theorem and the estimate

$$\begin{aligned}&\boldsymbol{\phi}_R(t,x) = \boldsymbol{\phi}(t,x) \text{ if } |x-\xi(t)| \geq R\,,\\ &0 \leq \boldsymbol{\phi}_R(t,x) \leq \boldsymbol{\phi}(t,x) \text{ if } |x-\xi(t)| \leq R\,.\end{aligned}$$

The first identity follows from the definition of $\boldsymbol{\phi}_R$. The second inequality reads $\boldsymbol{\phi}_R(t,x) \leq \boldsymbol{\phi}(t,x)$ for $0 \leq r = |x-\xi(t)| \leq R$ This holds if and only if

$$\begin{aligned}&\frac{m}{2\kappa_n}\frac{1}{R^n}\left(\frac{R^2}{n-2}-\frac{r^2}{n}\right) \leq \frac{m}{n\kappa_n(n-2)}r^{2-n}\\ \Longleftrightarrow\quad &\frac{R^2}{(n-2)R^n} \leq \frac{2}{n(n-2)}r^{2-n}+\frac{r^2}{nR^n}\\ \Longleftrightarrow\quad &\frac{s^{n-2}}{n-2} \leq \frac{2}{n(n-2)}+\frac{s^n}{n} \text{ for } s=\frac{r}{R}\leq 1\\ \Longleftrightarrow\quad &s^{n-2} \leq \frac{2}{n}+\frac{n-2}{n}s^n \text{ for } s=\frac{r}{R}\leq 1,\end{aligned}$$

which is true by Young's inequality. (For the L^1-convergence it is enough to show that $\boldsymbol{\phi}_R(t,x) \leq C|x-\xi(t)|^{2-n}$ for all R, where C is independent of R.) Also $\varrho_R\mathrm{L}^{n+1} \to m\boldsymbol{\mu}_\xi$ as $R\to 0$, which follows from

$$\begin{aligned}&\int \zeta\varrho_R \,\mathrm{dL}^{n+1} = \int_{\mathbb{R}}\int_{\mathrm{B}_R(\xi(t))} \zeta(t,x)\frac{m}{\kappa_n R^n}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{\mathbb{R}}\int_{\mathrm{B}_1(0)} \zeta(t,\xi(t)+Ry)\frac{m}{\kappa_n}\,\mathrm{d}y\,\mathrm{d}t \to \int_{\mathbb{R}} \zeta(t,\xi(t))m\,\mathrm{d}t = \left\langle \zeta\,,\, m\boldsymbol{\mu}_\xi \right\rangle\,.\end{aligned}$$

Therefore altogether

$$\left\langle \zeta\,,\, m\boldsymbol{\mu}_\xi \right\rangle = -\left\langle \Delta\zeta\,,\,\boldsymbol{\phi}\right\rangle\,,$$

qed. □

Therefore, the solution outside a "star" (if the star has a constant mass density) coincides with the solution obtained if one sets the "star as a point mass" with the same total mass. In the next section 3 we will consider the conservation of momentum and we will show that in the stationary incompressible case homogeneous stars produce a gravitational field like the one here (see 4.5). In the compressible case we refer to section IV.16, where radially symmetric mass distributions of stars are considered.

That the solution of the gravity equation is C^1, is not true if the mass density is supported on a surface. As it turns out the solution is only Lipschitz continuous. The following example is for a homogeneous mass distribution.

2.18 Hollow sphere. Let $n \geq 3$, $m > 0$ be constant and $t \mapsto \xi(t)$ the movement of the center of a shell. Its support is supposed to ly on $\partial \mathrm{B}_R(\xi(t))$. Then let $\boldsymbol{\mu} \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ be given by

$$\langle \zeta \,,\, \boldsymbol{\mu} \rangle := \int_{\mathbb{R}} \int_{\partial \mathrm{B}_R(\xi(t))} \zeta(t,x) \, \mathrm{dH}^{n-1}(x) \, \mathrm{d}t$$

for $\zeta \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n)$. Further, let

$$\varrho_s(t,x) := \frac{m}{\mathrm{H}^{n-1}(\partial \mathrm{B}_R(0))} \mathcal{X}_{\partial \mathrm{B}_R(\xi(t))}(x)$$

the constant mass density on $\partial \mathrm{B}_R(\xi(t))$. Then the solution $\boldsymbol{\phi}$ of equation

$$\begin{aligned} &\mathrm{div}(-\nabla[\boldsymbol{\phi}]) = \varrho_s \boldsymbol{\mu} \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)\,, \\ &\boldsymbol{\phi}(t,x) \to 0 \text{ as } |x| \to \infty, \end{aligned}$$

is given by

$$\boldsymbol{\phi}(t,x) = \begin{cases} \dfrac{m}{\sigma_n (n-2)} |x - \xi(t)|^{2-n} \text{ if } |x - \xi(t)| \geq R\,, \\ \dfrac{m R^{2-n}}{\sigma_n (n-2)} \text{ if } |x - \xi(t)| \leq R\,. \end{cases}$$

The solution $\boldsymbol{\phi}$ is thus only of class C^0.

Proof. If $\boldsymbol{\phi}$ is as in the formula, we get

$$\nabla \boldsymbol{\phi}(t,x) = \begin{cases} -\dfrac{m}{\sigma_n} \dfrac{x - \xi(t)}{|x - \xi(t)|^n} \text{ if } x \in \mathbb{R}^n \setminus \overline{\mathrm{B}_R(\xi(t))}\,, \\ 0 \text{ if } x \in \mathrm{B}_R(\xi(t)) \,, \end{cases}$$

where $\Delta \boldsymbol{\phi} = \mathrm{div} \nabla \boldsymbol{\phi} = 0$ in $\mathbb{R}^n \setminus \overline{\mathrm{B}_R(\xi(t))}$. Hence for $\zeta \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$

$$\begin{aligned} &\langle \zeta \,,\, \nabla[\boldsymbol{\phi}] \rangle = -\langle \, \mathrm{div}\zeta \,,\, [\boldsymbol{\phi}] \rangle = -\int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathrm{div}\zeta(t,x) \cdot \boldsymbol{\phi}(t,x) \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n \setminus \overline{\mathrm{B}_R(\xi(t))}} \zeta(t,x) \bullet \nabla \boldsymbol{\phi}(t,x) \, \mathrm{d}x \, \mathrm{d}t\,, \end{aligned}$$

because ϕ is continuous. Therefore it holds for $\eta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$

$$\begin{aligned}
&\langle\, \eta\,,\ \mathrm{div}(-\nabla[\boldsymbol{\phi}])\,\rangle = \langle\, \nabla\eta\,,\ \nabla[\boldsymbol{\phi}]\,\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n\setminus\overline{\mathrm{B}_R(\xi(t))}} \nabla\eta(t,x)\bullet\nabla\boldsymbol{\phi}(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}^n\setminus\overline{\mathrm{B}_R(\xi(t)))}} \eta(t,x)\,\underbrace{\Delta\boldsymbol{\phi}(t,x)}_{=0}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathbb{R}}\int_{\partial\mathrm{B}_R(\xi(t)))} \eta(t,x)\,\underbrace{\nabla\boldsymbol{\phi}(t,x)\bullet\nu_{\mathbb{R}^n\setminus\overline{\mathrm{B}_R(\xi(t))}}}_{=\dfrac{m}{\sigma_n|x-\xi(t)|^{n-1}}}\,\mathrm{dH}^{n-1}(x)\,\mathrm{d}t \\
&= \langle\, \eta\,,\ \varrho_s\boldsymbol{\mu}\,\rangle\ ,
\end{aligned}$$

where in the last integral $\nabla\boldsymbol{\phi}(t,x)$ is taken from outside, i.e.

$$\nabla\boldsymbol{\phi}(t,x) = \lim_{h\searrow 0}\nabla\boldsymbol{\phi}(t,x+h\nu_{\mathrm{B}_R(\xi(t))})$$

□

3 Conservation of momentum

The momentum conservation needs for its formulation a mass conservation, which results from the observer transformations in Section II.3. Hence, this system of mass-momentum balance reads

General mass-momentum equation:

$$\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}} \quad [16] \tag{I3.1}$$

where besides the quantities in (I1.7)

$\Pi = (\Pi_{ij})_{i,j=1,\dots,n}$ pressure tensor,

$\widetilde{\mathbf{f}} = \left(\widetilde{\mathbf{f}}_i\right)_{i=1,\dots,n}$ general force density.

Here at first $(\varrho, \mathbf{J}, \mathbf{r})$ and $(v, \Pi, \widetilde{\mathbf{f}})$ are arbitrary terms, so we have written down the general version of the conservation equations. The $\widetilde{\mathbf{f}}$-term includes both "external forces" and "internal forces" such as the self-gravity. Strictly speaking $\widetilde{\mathbf{f}}$ is a "general force density" and there is a correspondence between the flux and the production terms, that is, between

$$\begin{bmatrix} \mathbf{J} \\ v\,\mathbf{J}^{\mathrm{T}} + \Pi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ (\mathbf{r} + \mathbf{J}\bullet\nabla)v + \mathbf{f} \end{bmatrix}, \tag{I3.2}$$

where the "classical force density" $\mathbf{f} := \widetilde{\mathbf{f}} - (\mathbf{r} + \mathbf{J}\bullet\nabla)v$ is introduced in (II3.17). So, for example, there is a correspondence between the pressure term Π and the force term $\mathbf{f}$, as it was between $\mathbf{J}$ and $\mathbf{r}$ (see the remark following (I1.7)). Thus parts of the forces can be written under the divergence term, that is, as part of the pressure tensor. Such terms will be denoted as "internal force" (see as example for mass points 3.4). The v-terms in the fluxes and on the right side result from objectivity reasons (see section II.3). Below the time derivative we have ϱ as the ***mass density*** and ϱv as the ***moment density***. In general, the divergence is defined by the fact that it acts on the last index, as one can see in the following

Definition: If

$$M = (M_{ij})_{i,j=1,\dots,n} = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & & \vdots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}$$

is a matrix-valued function, the divergence of it is defined by

$$\operatorname{div} M := \left(\sum_{j=1}^{n} \partial_{x_j} M_{ij} \right)_{i=1,\dots,n}.$$

[16] While $(x,y) \mapsto x\bullet y = x^{\mathrm{T}} y$ denotes the scalar product, the ***tensor product*** is expressed by $(x,y) \mapsto x\, y^{\mathrm{T}} = x \otimes y$.

Further, in the above equation there is

$$v\,v^{\mathrm{T}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} [\,v_1 \quad \dots \quad v_n\,] = \begin{bmatrix} v_1 v_1 & \dots & v_1 v_n \\ \vdots & & \vdots \\ v_n v_1 & \dots & v_n v_n \end{bmatrix} = (v_i v_j)_{i,j=1,\dots,n}\,,$$

$$v\,\mathbf{J}^{\mathrm{T}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} [\,\mathbf{J}_1 \quad \dots \quad \mathbf{J}_n\,] = \begin{bmatrix} v_1 \mathbf{J}_1 & \dots & v_1 \mathbf{J}_n \\ \vdots & & \vdots \\ v_n \mathbf{J}_1 & \dots & v_n \mathbf{J}_n \end{bmatrix} = (v_i \mathbf{J}_j)_{i,j=1,\dots,n}\,.$$

Thus

$$\varrho v\,v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi = (\varrho v_i v_j + v_i \mathbf{J}_j + \Pi_{ij})_{i,j=1,\dots,n}$$

and therefore the system of differential equations can be written as a system of $n+1$ equations

$$\boxed{\begin{aligned} &\partial_t \varrho + \sum_{j=1}^{n} \partial_j (\varrho v_j + \mathbf{J}_j) = \mathbf{r}\,, \\ &\partial_t (\varrho v_k) + \sum_{j=1}^{n} \partial_j (\varrho v_k v_j + v_k \mathbf{J}_j + \Pi_{kj}) = \widetilde{\mathbf{f}}_k \text{ for } k = 1, \dots, n. \end{aligned}} \tag{I3.3}$$

Momentum of mass points

To begin with we consider the motion of a single mass point, the trajectory is again denoted by $t \mapsto \xi(t) \in \mathbb{R}^n$ (as in 2.8) and the mass-momentum conservation has a distributional formulation which we will present in 3.1. We show that these distributional differential equations are equivalent to ordinary differential equations of first order for m and of second order for ξ.

3.1 Mass point. We consider the mass point introduced in 2.9 which moves with $t \mapsto \xi(t) \in \mathbb{R}^n$ and whose total mass is given by $t \mapsto m(t) > 0$. Further consider maps $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{r} : \mathbb{R} \to \mathbb{R}$ and the general force $\widetilde{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^n$. Then the following is equivalent:

(1) The distributional equations

$$\begin{aligned} &\partial_t (m \boldsymbol{\mu}_\xi) + \mathrm{div}\,(m v \boldsymbol{\mu}_\xi) = \mathbf{r} \boldsymbol{\mu}_\xi\,, \\ &\partial_t (m v \boldsymbol{\mu}_\xi) + \mathrm{div}\,(m v\,v^{\mathrm{T}}\, \boldsymbol{\mu}_\xi) = \widetilde{\mathbf{f}} \boldsymbol{\mu}_\xi \end{aligned} \tag{I3.4}$$

are fulfilled. Here the distribution $\boldsymbol{\mu}_\xi$ is given by 2.8.

(2) It is $v(t, \xi(t)) = \dot{\xi}(t)$ the velocity, and the ordinary differential equations

$$\dot{m} = \mathbf{r}\,, \quad \left(m\dot{\xi} \right)^{\boldsymbol{\cdot}} = \widetilde{\mathbf{f}} \tag{I3.5}$$

are satisfied.

(3) It is $v(t,\xi(t)) = \dot\xi(t)$ the velocity and if the force is defined by the formula $\mathbf{f}(t) := \widetilde{\mathbf{f}}(t) - \mathbf{r}(t)v(t,\xi(t))$ then

$$\dot m = \mathbf{r}\,, \quad m\ddot\xi = \mathbf{f} \tag{I3.6}$$

Zusatz: If $\mathbf{r} = 0$ then $m = \text{const}$.

We mention that here $\widetilde{\mathbf{f}}\boldsymbol{\mu}_\xi$ is the "general force density" in the distributional momentum equation, whereas $\widetilde{\mathbf{f}}$, the right-hand side of the ODE (I3.5), is called "general force". (In II.3.8 the difference between $\widetilde{\mathbf{f}}$ and $\mathbf{f}$ becomes clear in the general case. We see here how this difference is introduced in the ODE case.)

Proof (1)⇒(2). The first differential equation in (I3.4), that is the mass conservation, is treated as in 2.9. This results in the equations $v(t,\xi(t)) = \dot\xi(t)$ and in the ordinary differential equation $\dot m = \mathbf{r}$. Further, for the second equation it applies for all $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ that

$$\begin{aligned}
&\Big\langle \zeta\,,\, -\partial_t(mv\boldsymbol{\mu}_\xi) - \operatorname{div}_x(mv\,v^{\mathrm{T}}\,\boldsymbol{\mu}_\xi) + \widetilde{\mathbf{f}}\boldsymbol{\mu}_\xi \Big\rangle \\
&= \sum_k \Big\langle \zeta_k\,,\, -\partial_t(mv_k\boldsymbol{\mu}_\xi) - \operatorname{div}_x(mv_k v\boldsymbol{\mu}_\xi) + \widetilde{\mathbf{f}}_k\boldsymbol{\mu}_\xi \Big\rangle \\
&= \sum_k \Big(\big\langle \partial_t\zeta_k\,,\, mv_k\boldsymbol{\mu}_\xi \big\rangle + \big\langle \nabla\zeta_k\,,\, mv_k v\boldsymbol{\mu}_\xi \big\rangle + \Big\langle \zeta_k\,,\, \widetilde{\mathbf{f}}_k\boldsymbol{\mu}_\xi \Big\rangle \Big) \\
&= \sum_k \int_{\mathbb{R}} m(t)\, \underbrace{v_k(t,\xi(t))}_{=\dot\xi_k(t)}\, \underbrace{(\partial_t\zeta_k + v\bullet\nabla\zeta_k)(t,\xi(t))}_{=\frac{\mathrm{d}}{\mathrm{d}t}\zeta_k(t,\xi(t))}\,\mathrm{d}t \\
&\quad + \sum_k \int_{\mathbb{R}} \zeta_k(t,\xi(t))\widetilde{\mathbf{f}}_k(t)\,\mathrm{d}t \\
&= \sum_k \int_{\mathbb{R}} \Big(-\frac{\mathrm{d}}{\mathrm{d}t}(m(t)\dot\xi_k(t)) + \widetilde{\mathbf{f}}_k(t) \Big)\, \zeta_k(t,\xi(t))\,\mathrm{d}t\,.
\end{aligned}$$

Since this is true for arbitrary test function we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m(t)\dot\xi(t)\big) = \widetilde{\mathbf{f}}(t)\,. \tag{I3.7}$$

This is the second ordinary differential equation. □

Proof (3). We see that

$$m\ddot\xi + \dot m\dot\xi = \Big(m\dot\xi\Big)^{\boldsymbol{\cdot}} = \widetilde{\mathbf{f}} = \mathbf{f} + \mathbf{r}v = \mathbf{f} + \dot m\dot\xi\,,$$

hence $m\ddot\xi = \mathbf{f}$. □

Die Gleichung (I3.5) besagt also, wie Newton in seinen Principia schreibt, siehe Newton [118, Axiomata sive Leges Motus: Lex.II] oder Newton [119, Axioms, or the Laws of Motion: Law 2][17],

Change in momentum = General force

und das ist bei sich ändernder Masse richtig. Es ist aber genauso richtig, dass Gleichung (I3.6) sagt

Mass × Acceleration = Force

was auch bei sich ändernder Masse richtig ist. Die Newton'sche Physik ist also in den distributionellen Masse-Impuls Gleichungen enthalten.

Die distributionelle Masse-Impuls Gleichung ist auch nützlich bei zwei Massenpunkten, die sich zu einer Zeit t_0 treffen. Bei diesem Zusammenstoß kann vieles passieren. Wir betrachten hier die Situation, dass nach dem Stoß wieder zwei Massepartikel vorhanden sind, zum Zeitpunkt t_0 sonst keine Auswirkung in der Masse-Impuls Gleichung zu bemerken ist. It can also happen that there are several particles after the collision (as in Fig. 9), which leads to corresponding formulas, or during the collision a light flash is emitted, which changes the formulas dramatically. We refer to III.6.5 where we also consider the energy balance and to III.6.6 where we present realistic situations such as elastic collision and plastic collisions.

3.2 Collision of mass points. Let be given two mass points, as in 2.9,

$$t \mapsto \xi^\alpha(t) \in \mathbb{R}^n \text{ continuous}, \qquad \alpha = 1, 2,$$

whose trajectories meet exactly in the spacetime point (t_0, x_0),

$$x_0 = \xi^1(t_0) = \xi^2(t_0)\,.$$

We denote the distributions $\boldsymbol{\mu}_{\xi^\alpha}$ as in 2.8. The masses are given by bounded continuous functions $t \mapsto m^\alpha(t) > 0$ for $t \neq t_0$. Thus, the distributional total mass of the system is given by

$$\sum_{\alpha=1,2} m^\alpha \boldsymbol{\mu}_{\xi^\alpha} \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)\,.$$

Assertion: Let the distributional equations

$$\begin{aligned} &\partial_t \Big(\sum_\alpha m^\alpha \boldsymbol{\mu}_{\xi^\alpha} \Big) + \operatorname{div} \Big(\sum_\alpha m^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha} \Big) = 0\,, \\ &\partial_t \Big(\sum_\alpha m^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha} \Big) + \operatorname{div} \Big(\sum_\alpha m^\alpha v^\alpha v^{\alpha\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha} \Big) = \sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\xi^\alpha} \end{aligned} \tag{I3.8}$$

[17] I. Bernard Cohen writes there in "A Guide to Newton's Principia": 'For example, in law 2, Newton writes that a "change in motion" is "proportional to the motive force". Here he means "change in the quantity of motion" or, in our terminology, change in momentum.'

be satisfied, where $v^\alpha(t,\xi(t)) := \dot\xi^\alpha(t)$ are the velocities, and and $\widetilde{\mathbf{f}}^\alpha = \mathbf{f}^\alpha$. Let the derivatives $\dot\xi^\alpha(t)$ for $t \neq t_0$ be piecewise continuous up to the point t_0, as well as the vector fields $\mathbf{f}^\alpha$. It follows

$$\left.\begin{aligned} &m^\alpha \ddot\xi^\alpha = \mathbf{f}^\alpha\,, \\ &m^\alpha \text{ locally constant in } t \end{aligned}\right\} \quad \begin{aligned} &\text{for } t \neq t_0 \\ &\text{and } \alpha = 1,2, \end{aligned}$$
$$m_-^1 + m_-^2 = m_+^1 + m_+^2 \text{ (mass conservation in } t_0),$$
$$m_-^1 v_-^1 + m_-^2 v_-^2 = m_+^1 v_+^1 + m_+^2 v_+^2 \text{ (momentum conservation in } t_0), \tag{I3.9}$$

where

$$m_-^\alpha := \lim_{t\nearrow t_0} m^\alpha(t)\,, \qquad m_+^\alpha := \lim_{t\searrow t_0} m^\alpha(t)$$
$$v_-^\alpha := \lim_{t\nearrow t_0} v^\alpha(t,\xi^\alpha(t))\,, \qquad v_+^\alpha := \lim_{t\searrow t_0} v^\alpha(t,\xi^\alpha(t))\,.$$

Thus, it is not described what happenes to the particles when colliding, but it is set up a total mass balance and a total momentum balance. This is done under the assumption that after the collision the only thing which is left are again two particles.

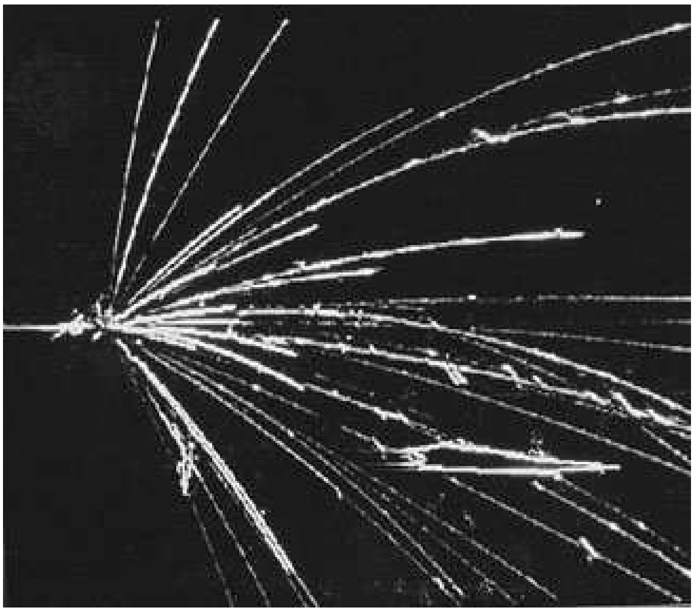


Fig. 9: "Particle tracks from the collision of an accelerated nucleus of a niobium atom with another niobium nucleus. The single line on the left is the track of the incoming projectile nucleus, and the other tracks are fragments from the collision." (Courtesy of the Department of Physics and Astronomy, Michigan State University)

Proof. Outside the point (t_0,x_0) the two trajectories ξ^α are apart from each other. Let $t \neq t_0$. In a neighbourhood of the point $(t,\xi^\alpha(t))$ we have to consider only the α-phase, this means that in this neighbourhood we have to consider

$$\partial_t \left(m^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) + \operatorname{div}\left(m^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) = 0\,,$$
$$\partial_t \left(m^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) + \operatorname{div}\left(m^\alpha v^\alpha v^{\alpha\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha}\right) = \mathbf{f}^\alpha \boldsymbol{\mu}_{\xi^\alpha}\,.$$

Due to 3.1 and 2.9, it follows that m^α is constant in this region, that $v^\alpha(t,\xi^\alpha(t)) = \dot\xi^\alpha(t)$, and that $m^\alpha \ddot\xi^\alpha(t) = \mathbf{f}^\alpha(t,\xi^\alpha(t))$ for $t \neq t_0$.
Therefore we have to compute the mass and momentum contribution near the point (t_0, x_0). We write the mass conservation and the components of the momentum conservation, see (I3.8), in one equation

$$\partial_t \left(\sum_\alpha g^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) + \operatorname{div}\left(\sum_\alpha g^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) = \sum_\alpha r^\alpha \boldsymbol{\mu}_{\xi^\alpha}\,,$$

where

$$g^\alpha := m^\alpha\,,\ r^\alpha := 0 \text{ for the mass conservation,}$$
$$g^\alpha := m^\alpha v_k^\alpha\,,\ r^\alpha := \mathbf{f}_k^\alpha,\ k = 1,\dots,n \text{ for the momentum conservation.}$$

We now choose test functions $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$ which have a support in a neighborhood of (t_0, x_0). We calculate

$$\begin{aligned}
0 = {} & \left\langle \zeta\,,\, -\partial_t\left(\sum_\alpha g^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) - \operatorname{div}\left(\sum_\alpha g^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right) + \sum_\alpha r^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right\rangle \\
= {} & \left\langle \partial_t\zeta\,,\, \sum_\alpha g^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right\rangle + \left\langle \nabla\zeta\,,\, \sum_\alpha g^\alpha v^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right\rangle + \left\langle \zeta\,,\, \sum_\alpha r^\alpha \boldsymbol{\mu}_{\xi^\alpha}\right\rangle \\
= {} & \sum_\alpha \Bigg(\int_{\mathbb{R}\setminus\{t_0\}} (\partial_t\zeta)(t,\xi^\alpha(t))\, g^\alpha(t,\xi^\alpha(t))\,\mathrm{d}t \\
& \quad + \int_{\mathbb{R}\setminus\{t_0\}} (\nabla\zeta)(t,\xi^\alpha(t))\bullet \underbrace{(g^\alpha v^\alpha)(t,\xi^\alpha(t))}_{= g^\alpha(t,\xi^\alpha(t))\dot\xi^\alpha(t)}\,\mathrm{d}t \\
& \quad + \int_{\mathbb{R}\setminus\{t_0\}} \zeta(t,\xi^\alpha(t))\, r^\alpha(t,\xi^\alpha(t))\,\mathrm{d}t \Bigg)
\end{aligned}$$

$$= \sum_\alpha \int_{\mathbb{R}\setminus\{t_0\}} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta(t,\xi^\alpha(t))\big) g^\alpha(t,\xi^\alpha(t)) + \zeta(t,\xi^\alpha(t)) r^\alpha(t,\xi^\alpha(t)) \Big)\,\mathrm{d}t$$

(now we integrate by parts)

$$= \sum_\alpha \int_{\mathbb{R}\setminus\{t_0\}} \zeta(t,\xi^\alpha(t)) \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\big(g^\alpha(t,\xi^\alpha(t))\big) + r^\alpha(t,\xi^\alpha(t)) \Big)\,\mathrm{d}t$$

$$+ \sum_\alpha \underbrace{\int_{\mathbb{R}\setminus\{t_0\}} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\zeta(t,\xi^\alpha(t)) g^\alpha(t,\xi^\alpha(t))\Big)\,\mathrm{d}t}_{\substack{= \zeta(t_0,\xi^\alpha(t_0))(g^\alpha_- - g^\alpha_+) \\ = \zeta(t_0,x_0)(g^\alpha_- - g^\alpha_+)}}$$

$$= \sum_\alpha \int_{\mathbb{R}\setminus\{t_0\}} \zeta(t,\xi^\alpha(t)) \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\big(g^\alpha(t,\xi^\alpha(t))\big) + r^\alpha(t,\xi^\alpha(t)) \Big)\,\mathrm{d}t$$

$$+ \zeta(t_0,x_0) \sum_\alpha (g^\alpha_- - g^\alpha_+)\,,$$

where

$$g^\alpha_- := \lim_{t \nearrow t_0} g^\alpha(t,\xi^\alpha(t))\,, \quad g^\alpha_+ := \lim_{t \searrow t_0} g^\alpha(t,\xi^\alpha(t))\,.$$

Since the test function ζ is arbitrarily, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(g^\alpha(t,\xi(t))\big) = r^\alpha(t,\xi(t)) \text{ for } t \neq t_0 \text{ and } \alpha = 1,2,$$

$$\sum_\alpha g^\alpha_- = \sum_\alpha g^\alpha_+\,.$$

This gives all the equations in (I3.9). □

Gravity applied to space objects

Wir betrachten nun das Newton'sche Gravitationsgesetz (I2.10)

$$\mathrm{div}(-\nabla[\boldsymbol{\phi}]) = [\varrho]$$

für die gesamte Massendichte ϱ. Wir stellen uns die Frage, wie $\boldsymbol{\phi}$ als Kraft auf die Impulserhaltung (I3.31)

$$\begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho v) &= 0\,, \\ \partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + \Pi) &= \mathbf{f} \end{aligned} \tag{I3.10}$$

wirkt. Es ist dies die Newton'sche Kraft(dichte), die für $\mathbf{f}$ bedeutet

Newton's force density:

$$\mathbf{f} = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} \tag{I3.11}$$

$\mathbf{f}$ force density,

wobei hier angenommen wird, dass es die alleinige Kraft ist, im Allgemeinen können noch andere Kräfte wirksam sein. Die zugehörige Beschleunigung ist

$$a = \mathfrak{g}\nabla\boldsymbol{\phi}\,. \tag{I3.12}$$

Bemerkung: Let $n = 3$. It is

$$\mathfrak{g} = 4\pi G \quad \text{with} \quad G = 6.67384 \cdot 10^{-11}\,\frac{m^3}{kg\,s^2} \tag{I3.13}$$

being the ***gravitational constant***. Here is a list of some dimensions:

$\boldsymbol{\phi}$	$\frac{kg}{m}$	$\operatorname{div}\nabla\boldsymbol{\phi}\,,\ \varrho$	$\frac{kg}{m^3}$
$\nabla\boldsymbol{\phi}$	$\frac{kg}{m^2}$	$\partial_t\varrho$	$\frac{kg}{m^3 s}$
$a\,,\ \mathfrak{g}\nabla\boldsymbol{\phi}$	$\frac{m}{s^2}$	$\varrho a\,,\ \mathbf{f}\,,\ \mathfrak{g}\varrho\nabla\boldsymbol{\phi}$	$\frac{kg}{m^2 s^2}$

Here a is an acceleration.

Wir denken uns nun die gesamte Massendichte ϱ aus disjunkten Teilmassen ϱ_α zusammengesetzt, etwa ein Teil des Himmels bestehend aus Sonnen und Planeten. Für das Gravitationspotential gilt dann wegen der Linearität des Gravitationsgesetzes

$$\varrho = \sum_\alpha \varrho_\alpha\,, \quad \boldsymbol{\phi} = \sum_\alpha \boldsymbol{\phi}_\alpha\,, \quad \operatorname{div}(-\nabla[\boldsymbol{\phi}_\alpha]) = [\varrho_\alpha]\,.$$

Da wir annehmen, dass die Teilmassen alle verschiedenen Träger haben, sagen wir disjunkte $D_\alpha \subset \mathbb{R} \times \mathbb{R}^n$ für den α-Träger, können wir definieren

$$v = v_\alpha + u_\alpha \quad \text{und} \quad \Pi = \Pi_\alpha \quad \text{in } D_\alpha,$$

wobei v_α die Bewegung des Himmelkörpers als Ganzes und u_α z.B. die lokale Rotationsbewegung ist. Es gilt damit nach (I3.10) für jedes α

$$\begin{aligned}
&\operatorname{div}(-\nabla[\boldsymbol{\phi}_\alpha]) = [\varrho_\alpha]\,,\\
&\partial_t[\varrho_\alpha] + \operatorname{div}[\varrho_\alpha v_\alpha + \varrho_\alpha u_\alpha] = 0\,,\\
&\partial_t[\varrho_\alpha v_\alpha + \varrho_\alpha u_\alpha] + \operatorname{div}[\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + \varrho_\alpha u_\alpha {v_\alpha}^{\mathrm{T}} + \varrho_\alpha v_\alpha {u_\alpha}^{\mathrm{T}}]\\
&\qquad\qquad = \mathfrak{g}\sum_\beta [\varrho_\alpha \nabla\boldsymbol{\phi}_\beta] - \operatorname{div}[\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}} + \Pi_\alpha]\,.
\end{aligned} \tag{I3.14}$$

Wir lassen nun die Teilkörper gegen Punktmassen konvergieren, also konvergiert für alle α

$$\begin{aligned}
&[\varrho_\alpha] \to m_\alpha \boldsymbol{\mu}_{\xi_\alpha} \text{ punktweise in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n),\\
&v_\alpha \text{ gleichmäßig in Raum und Zeit } \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

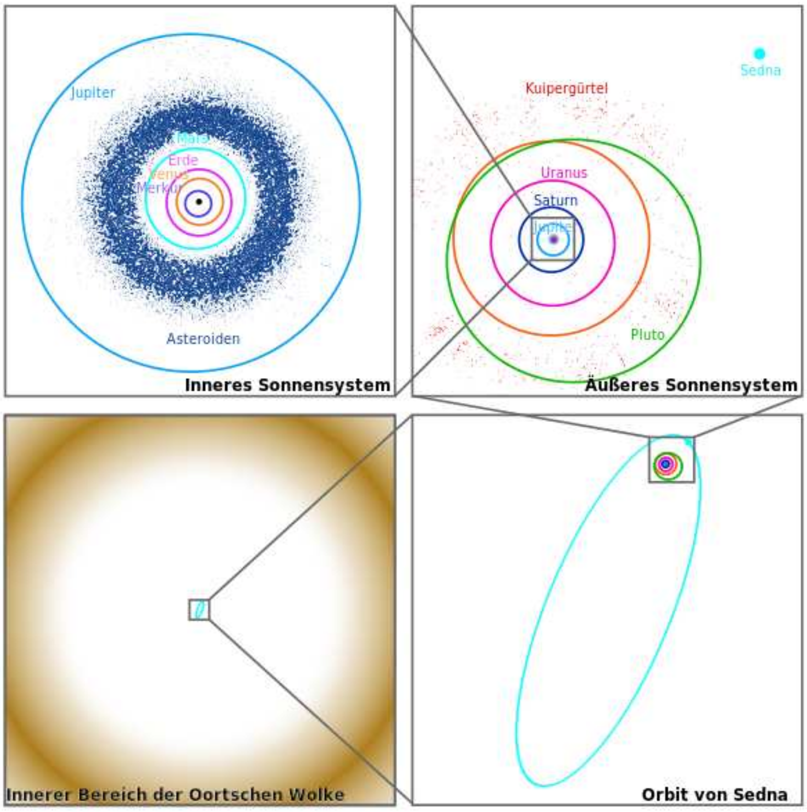


Fig. 10: "Die Umlaufbahnen der Objekte des Sonnensystems im Maßstab" aus [Wikipedia: Sonnensystem] (2-dimensionale Projektion)

Weiter konvergiert dann

$$[\varrho_\alpha u_\alpha] \to 0 \text{ punktweise in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n).$$

Aber es gibt Probleme mit dem Term $\varrho_\alpha \nabla\boldsymbol{\phi}_\alpha$, da hier beide Faktoren entarten, ϱ_α geht gegen einen Punkt und dort geht $\nabla\boldsymbol{\phi}_\alpha$ gegen unendlich, es gibt also keinen einfachen Limes. Jedoch gilt in der hier gegebenen Situation, dass

$$\mathfrak{g}[\varrho_\alpha \nabla\boldsymbol{\phi}_\alpha] - \operatorname{div}[\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}} + \Pi_\alpha] \to 0 \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n) \tag{I3.15}$$

punktweise, d.h. für jede Testfunktion in $\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)$. Siehe dazu im stationären Fall 4.5 für Kugeln, IV.16.5 für rotierende inkompressible Planeten, und IV.16.3 für allgemeine Planeten. Es sei auch auf [21, Apropos Newton] hingewiesen. Setzen wir nun diese Resultate in (I3.14) ein, so lauten die

Gleichungen im Limes

$$\begin{aligned}
&\operatorname{div}(-\nabla[\boldsymbol{\phi}_\alpha]) = m_\alpha \boldsymbol{\mu}_{\xi_\alpha}\,,\\
&\partial_t(m_\alpha \boldsymbol{\mu}_{\xi_\alpha}) + \operatorname{div}(m_\alpha v_\alpha \boldsymbol{\mu}_{\xi_\alpha}) = 0\,,\\
&\partial_t(m_\alpha v_\alpha \boldsymbol{\mu}_{\xi_\alpha}) + \operatorname{div}(m_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} \boldsymbol{\mu}_{\xi_\alpha}) = \mathfrak{g} m_\alpha \Big(\sum_{\beta:\beta\neq\alpha} \nabla\boldsymbol{\phi}_\beta\Big)\boldsymbol{\mu}_{\xi_\alpha}\,.
\end{aligned} \tag{I3.16}$$

Nach 3.1 sind diese Gleichungen äquivalent dazu, dass für alle α die Masse m_α konstant ist, dass $v_\alpha(t,\xi(t)) = \dot{\xi}(t)$ ist, und dass gilt

$$\begin{aligned}
&\operatorname{div}(-\nabla[\boldsymbol{\phi}_\alpha]) = m_\alpha \boldsymbol{\mu}_{\xi_\alpha}\,,\\
&\ddot{\xi}_\alpha(t) = \mathfrak{g} \sum_{\beta:\beta\neq\alpha} \nabla\boldsymbol{\phi}_\beta(t,\xi_\alpha(t))\,.
\end{aligned} \tag{I3.17}$$

wobei wir die letzte Gleichung noch durch m_α dividiert haben. In diesem Zusammenhang sei auf [21, N-body problem] verwiesen, wo der Einfluss von Planeten auf die Perihelbewegung mit der allgemeinen Formel (I3.17) numerisch gezeigt wird.

Momentum of a single planet

We consider now the Sun system and assume that we are in the center of gravity, hence

$$\sum_\alpha m_\alpha \xi_\alpha(t) = 0\,,$$

and we orientate ourselves on stars in the surroundings. This is the reason why we took only one **f**-term in (I3.11). Now the Sun takes about 99.86%

Fig. 11: "Fotomontage zum Größenvergleich zwischen Erde (links) und Sonne. Das Kerngebiet (Umbra) des großen Sonnenflecks hat etwa 5-fachen Erddurchmesser" aus [Wikipedia: Sonne]

of the mass of the whole Sun system (see Fig. 11), hence $m_\alpha << m_0$, where m_0 denotes the mass of the Sun and m_α, $\alpha \neq 0$, the mass of the planets. Since the major mass is the mass of the Sun, the individual masses of the planets are not considered (for a better model see [21, N-Körper Problem]), and therefore the differential equations in (I3.17), where now the movements of the planets become independent, are approximated by

$$\begin{aligned} &\operatorname{div}(-\nabla[\boldsymbol{\phi}_0]) = m_0 \boldsymbol{\mu}_0\,, \\ &\ddot{\xi}_\alpha(t) = \mathfrak{g}\,\nabla\boldsymbol{\phi}_0(t,\xi_\alpha(t)) \quad \text{for every } \alpha. \end{aligned} \tag{I3.18}$$

Therefore the planet moves with $t \mapsto \xi_\alpha(t)$ in a central gravitational field. In the following statement the potential of the Sun $\boldsymbol{\phi}_0$ and the position of the planet ξ_α have no index.

3.3 Kepler's laws of planetary motion. The mass of the Sun is concentrated on the point $\{0\}$. The planet is modeled as a mass point $\{\xi(t)\}$ at time t with mass m and satisfies

$$m\ddot{\xi} = \mathbf{f}\,, \quad \mathbf{f}(t) = m\mathfrak{g}\nabla\boldsymbol{\phi}(t,\xi(t))\,, \tag{I3.19}$$

where $\boldsymbol{\phi}$ is the gravitational potential of the Sun, given by

$$\operatorname{div}(-\nabla[\boldsymbol{\phi}]) = m_0\boldsymbol{\mu}_0\,. \tag{I3.20}$$

It is assumed that $\xi(t) \neq 0$. Then (with some exceptions of one dimensional movement in the positive or negative direction to the Sun) the equations of Keplerian motion apply, that is, the movement is in a plane spanned by an orthonormal system $\{e_1, e_2\}$ with the representation

$$\xi(t) = r(\varphi(t))\big(\cos\varphi(t)\,e_1 + \sin\varphi(t)\,e_2\big)$$

and

$$\begin{aligned} r(\varphi) &= \frac{p}{1 + e\cdot\cos\varphi}\,, \\ \dot{\varphi} &= \frac{d}{r(\varphi)^2}\,, \quad d^2 = p\,Gm_0 > 0\,. \end{aligned}$$

The independent quantities are $p > 0$ and e. If $|e| < 1$ the planet makes a periodic movement. (See also exercise 7.17.)

Proof. We have to solve the system (I3.18). The first differential equation is (I3.20), and with the boundary condition $\boldsymbol{\phi}(t,x) \to 0$ as $|x| \to \infty$ it has the solution

$$\boldsymbol{\phi}(t,x) = \frac{m_0}{4\pi|x|} \qquad \text{hence} \qquad \nabla\boldsymbol{\phi}(t,x) = -\frac{m_0}{4\pi}\frac{x}{|x|^3}\,.$$

The second equation is (I3.19)

$$\ddot{\xi}(t) = \mathfrak{g}\nabla\boldsymbol{\phi}(t,\xi(t))\,,$$

i.e. with $\mathfrak{g} = 4\pi G$

$$\ddot{\xi}(t) = -Gm_0\frac{\xi(t)}{|\xi(t)|^3}\,. \tag{I3.21}$$

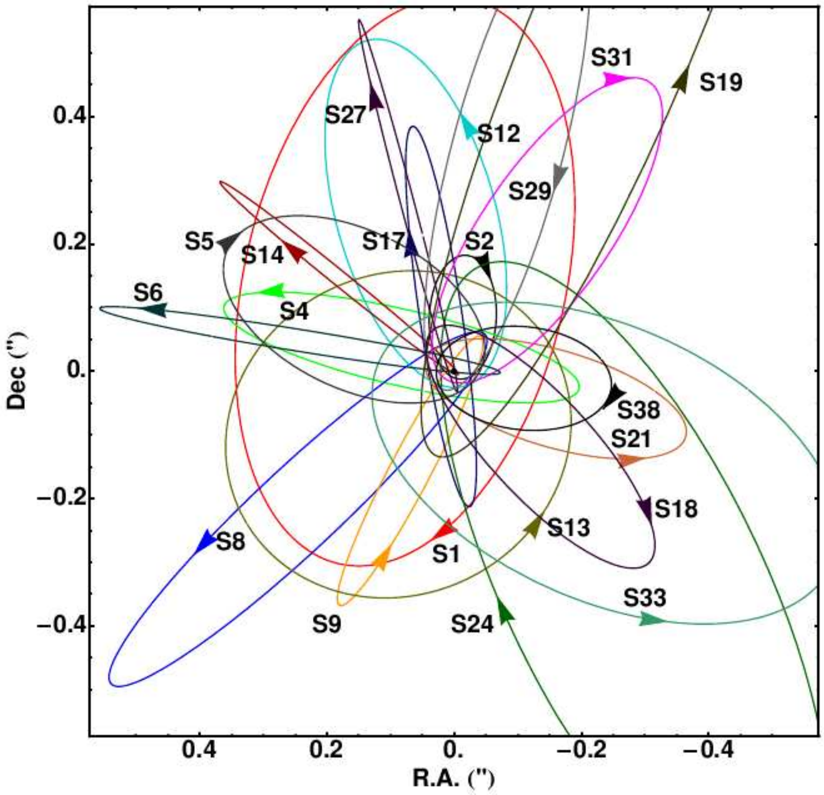


Fig. 12: "Elliptical orbits of stars at the galactic center. The massive black hole is at coordinate (0,0). Star S2 has an orbital period of about 15 years" from Department of Physics and Astronomy (California State L.A.). See also Fig. 14.

By assumption $\xi(t)$ is non-zero. From the differential equation it follows that $\dot{\xi}(t)$ has at most finitely many zeros. So we can assume that $\xi(0) \neq 0$ and $\dot{\xi}(0) \neq 0$.

1. Step. We show that we only need to treat the two-dimensional case. We denote with H that subspace which contains 0, $\xi(0)$, und $\dot{\xi}(0)$. We decompose

$$\xi(t) = \underbrace{x(t)}_{\in H} + \underbrace{y(t)}_{\in H^\perp} .$$

Then y satisfies the differential equation

$$\ddot{y}(t) = -Gm_0 \frac{y(t)}{|\xi(t)|^3}, \quad y(0) = 0, \quad \dot{y}(0) = 0$$

($|\xi(t)|^3$ is in the denominator). Because of the homogeneous initial condition it follows from the differential equation that $y = 0$. Consequently $\xi(t) \in H$.

Then H is a hyperplane provided $\dot{\xi}(0)$ and $\xi(0)$ are linearly independent. If not, then H is one-dimensional and $\xi(t)$ goes to 0 or infinity.

2. Step. We assume that (I3.21) applies and that the motion is two-dimensional, hence without loss of generality $t \mapsto \xi(t) \in \mathbb{R}^2$. Then we can introduce locally in time polar coordinates

$$\xi(t) = r(\varphi(t))\mathrm{e}^{\mathrm{i}\varphi(t)}$$

that means, r is a function of φ. Then with

$$c := \sqrt{Gm_0}$$

one computes

$$\begin{aligned}
-\frac{c^2}{r^2}\mathrm{e}^{\mathrm{i}\varphi} &= -c^2\frac{\xi(t)}{|\xi(t)|^3} = \ddot{\xi} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\mathrm{e}^{\mathrm{i}\varphi}) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\varphi}(r\prime_\varphi + \mathrm{i}r)\mathrm{e}^{\mathrm{i}\varphi}) \\
&= (\ddot{\varphi}(r\prime_\varphi + \mathrm{i}r) + \dot{\varphi}^2(r\prime_{\varphi\varphi} - r + 2\mathrm{i}r\prime_\varphi))\mathrm{e}^{\mathrm{i}\varphi}
\end{aligned}$$

and therefore

$$\ddot{\varphi}(r\prime_\varphi + \mathrm{i}r) + \dot{\varphi}^2(r\prime_{\varphi\varphi} - r + 2\mathrm{i}r\prime_\varphi) = -\frac{c^2}{r^2}\,.$$

Real part and imaginary part result in the two equations

$$\begin{aligned}
&\ddot{\varphi}r + 2\dot{\varphi}^2 r\prime_\varphi = 0\,, \\
&\ddot{\varphi}r\prime_\varphi + \dot{\varphi}^2(r\prime_{\varphi\varphi} - r) = -\frac{c^2}{r^2}\,.
\end{aligned} \tag{I3.22}$$

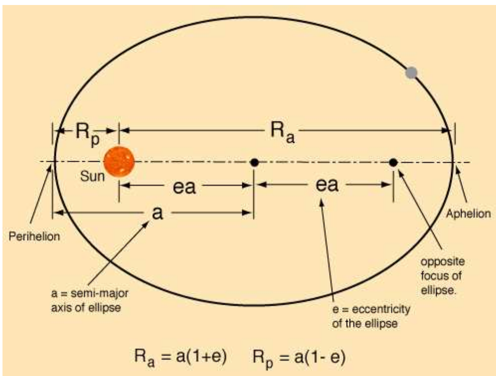


Fig. 13: "All planets move in elliptical orbits, with the Sun at one focus" from [hyperphysics.phy-astr.gsu.edu/hbase/kepler.html]

3. Step. Solution of the first equation in (I3.22).
If $\dot{\varphi} \neq 0$, the first equation can be written as

$$\frac{\ddot{\varphi}}{\dot{\varphi}} + 2\frac{r\prime_\varphi \dot{\varphi}}{r} = 0\,,$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log|\dot{\varphi}| + 2\log r(\varphi)) = 0\,,$$

therefore with a constant $d \neq 0$

$$\dot{\varphi} = \frac{d}{r(\varphi)^2}\,. \tag{I3.23}$$

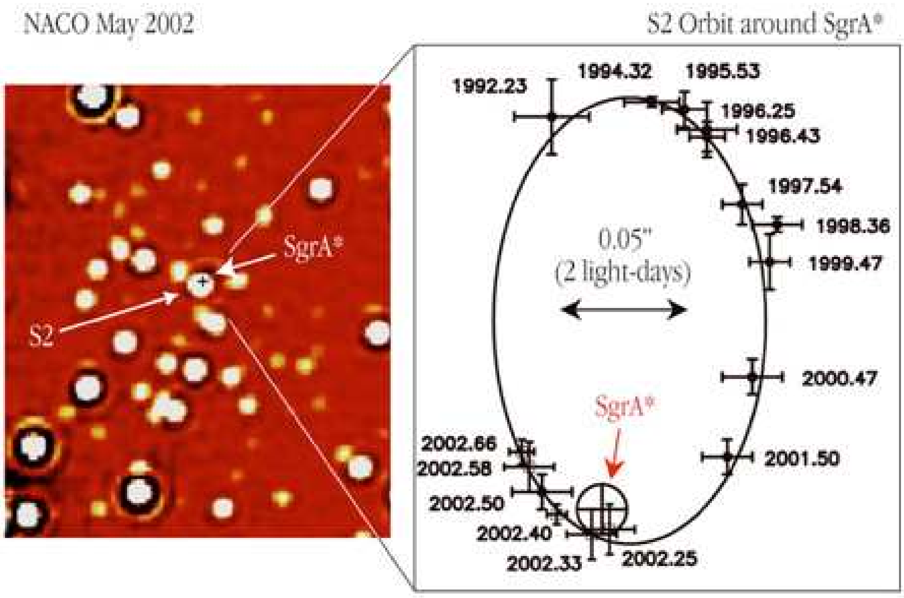


Fig. 14: "*Left:* An image of Sgr A* and S2 from the 8.2m VLT YEPUN telescope at the ESO Paranal Observatory. *Right:* The orbit of S2 around Sgr A*, highlighting the last close encounter, in 2002" from ESO. "The next encounter of S2 with Sgr A* will occur in 2018" from www.chandra.si.edu Chandra X-Ray Observatory. See also Fig. 12.

4. Step. Solution of the second equation in (I3.22).
We get from (I3.23)

$$\ddot{\varphi} = -\frac{2d}{r(\varphi)^3}r_{\prime\varphi}(\varphi)\dot{\varphi} = -\frac{2r_{\prime\varphi}}{r}\dot{\varphi}^2$$

If we plug this into the second equation, we obtain

$$-\frac{c^2}{r^2} = \ddot{\varphi}r_{\prime\varphi} + \dot{\varphi}^2(r_{\prime\varphi\varphi} - r) = \dot{\varphi}^2\Big(-\frac{2r_{\prime\varphi}^2}{r} + r_{\prime\varphi\varphi} - r\Big)\,,$$

that means with (I3.23)

$$-\frac{c^2}{d^2}r^2 = r_{\prime\varphi\varphi} - \frac{2r_{\prime\varphi}^2}{r} - r\,. \tag{I3.24}$$

This is a ordinary differential equation of second order in $\varphi \mapsto r(\varphi)$. If we now set, with $p \neq 0$ and a given $e \in \mathbb{R}$,

$$r(\varphi) = \frac{p}{1 + e \cdot \cos\varphi}\,, \tag{I3.25}$$

then it is

$$\begin{aligned} r'_{\varphi} &= \frac{p\,e\sin\varphi}{(1+e\cos\varphi)^2} = \frac{e\sin\varphi}{p}r^2 \\ r'_{\varphi\varphi} &= \frac{e\cos\varphi}{p}r^2 + \frac{2e\sin\varphi}{p}\,r\,r'_{\varphi} \\ &= \Big(\frac{p}{r}-1\Big)\frac{r^2}{p} + \frac{2e^2\sin^2\varphi}{p^2}\,r^3 \\ &= r - \frac{r^2}{p} + \frac{2e^2\sin^2\varphi}{p^2}\,r^3 = r - \frac{r^2}{p} + \frac{2r'^2_{\varphi}}{r} \end{aligned}$$

and thus becomes (I3.24) to

$$-\frac{c^2}{d^2}r^2 = r'_{\varphi\varphi} - \frac{2r'^2_{\varphi}}{r} - r = -\frac{r^2}{p}\,.$$

This is equivalent to the condition

$$p = \frac{d^2}{c^2}\,. \tag{I3.26}$$

The equations (I3.23), (I3.25) and (I3.26) are the Kepler motion (siehe [21, Kepler's laws]). □

Sogar im Zentrum der Milchstrasse ist die Bewegung der Sterne um das Zentrum ("Schwarzes Loch", SgrA*) nahe einer Kepler Bewegung, wie Fig. 12 und Fig. 14 zeigt. Eine Abweichung davon ist wie bei den Planeten des Sonnensystems (insbesondere die Bewegung des Merkur) eine Perihelbewegung, verursacht durch die Gravitation der übrigen Planeten (siehe [21, N-body problem]). Die Bestimmung der Position der Sterne von Aufnahmen derselben ist nichttrivial, die Schritte, die dabei gebraucht werden, sind in Gillessen et. al. [43] dargestellt, see also [44], wo auch eine Abschätzung der Masse von SgrA* gegeben wird.

Collection of mass points

We now consider a collection of mass points (or particles) with interacting forces. We obtain in the distributional momentum conservation a special matrix Π, that means the interacting forces have in part a form which allows them to be reformulated, so that they are expressed as a term under the divergence operator (see [16, 2.2 and 2.4] and [19, section 7]).

3.4 Multiple mass points. We consider N mass points with mass m_α at the position $t \mapsto \xi^\alpha(t)$ for $\alpha = 1, \dots, N$. They should satisfy the ordinary differential equation

$$m_\alpha \ddot{\xi}^\alpha(t) = -\sum_{\beta:\beta\neq\alpha} F_{\alpha\beta}\big(\xi^\alpha(t) - \xi^\beta(t)\big) + f_\alpha(t) \tag{I3.27}$$

in t, where mappings $F_{\alpha\beta} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ for $\alpha \neq \beta$ are given with

$$F_{\alpha\beta}(-z) = -F_{\beta\alpha}(z)\,. \tag{I3.28}$$

We consider a t-region in which the mass points are disjoint, i.e. they do not meet EX:Kollektion von Massenpunkten). With velocities $v_\alpha(t, \xi^\alpha(t)) = \dot{\xi}^\alpha(t)$ the following mass-momentum equations hold

$$\begin{aligned} &\partial_t\Big(\sum_\alpha m_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) + \mathrm{div}_x\Big(\sum_\alpha m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) = 0\,, \\ &\partial_t\Big(\sum_\alpha m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) + \mathrm{div}_x\Big(\sum_\alpha m_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha} \\ &\qquad -\frac{1}{2}\sum_{\alpha,\beta:\alpha\neq\beta} F_{\alpha\beta}(\xi^\alpha - \xi^\beta)\,\big(\xi^\alpha - \xi^\beta\big)^{\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha,\xi^\beta}\Big) = \sum_\alpha f_\alpha \boldsymbol{\mu}_{\xi^\alpha}\,. \end{aligned} \tag{I3.29}$$

Here the distributions $\boldsymbol{\mu}_{\xi^\alpha}$ and $\boldsymbol{\mu}_{\xi^\alpha,\xi^\beta}$ are given for test functions ζ by

$$\begin{aligned} \langle\, \zeta\,, \boldsymbol{\mu}_{\xi^\alpha} \,\rangle &:= \int_{\mathbb{R}} \zeta(t, \xi^\alpha(t))\,\mathrm{d}t \qquad \text{(as in 2.8)}, \\ \langle\, \zeta\,, \boldsymbol{\mu}_{\xi^\alpha,\xi^\beta} \,\rangle &:= \int_{\mathbb{R}} \int_0^1 \zeta(t, (1-s)\xi^\alpha(t) + s\xi^\beta(t))\,\mathrm{d}s\,\mathrm{d}t\,. \end{aligned}$$

We see that in the ordinary differential equation $m_\alpha \ddot{\xi}^\alpha = \mathbf{f}^\alpha$ the force term

$$\mathbf{f}^\alpha := -\sum_{\beta:\beta\neq\alpha} F_{\alpha\beta}\big(\xi^\alpha - \xi^\beta\big) + f_\alpha$$

consists of two terms, where the first one can be written as "internal" term

$$-\frac{1}{2}\sum_{\alpha,\beta:\alpha\neq\beta} F_{\alpha\beta}(\xi^\alpha - \xi^\beta)\,\big(\xi^\alpha - \xi^\beta\big)^{\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha,\xi^\beta}$$

in the flux of the momentum equation. Therefore we call

$$\sum_{\beta:\beta\neq\alpha} F_{\alpha\beta}\big(\xi^\alpha - \xi^\beta\big)$$

an "internal force density".

Proof. For a single mass point, the mass m_α is constant and it holds

$$m_\alpha \ddot{\xi}^\alpha = \mathbf{f}^\alpha := -\sum_{\beta:\beta\neq\alpha} F_{\alpha\beta}\big(\xi^\alpha - \xi^\beta\big) + f_\alpha\,.$$

Therefore follows as in 3.1 that this is equivalent to

$$\begin{aligned}
&\partial_t(m_\alpha \boldsymbol{\mu}_{\xi^\alpha}) + \mathrm{div}_x(m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}) = 0\,,\\
&\partial_t(m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}) + \mathrm{div}_x(m_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha}) = \mathbf{f}^\alpha \boldsymbol{\mu}_{\xi^\alpha}\,.
\end{aligned}$$

By forming the sum, of course, it follows

$$\begin{aligned}
&\partial_t\Big(\sum_\alpha m_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) + \mathrm{div}_x\Big(\sum_\alpha m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) = 0\,,\\
&\partial_t\Big(\sum_\alpha m_\alpha v_\alpha \boldsymbol{\mu}_{\xi^\alpha}\Big) + \mathrm{div}_x\Big(\sum_\alpha m_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} \boldsymbol{\mu}_{\xi^\alpha}\Big) = \sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\xi^\alpha}\,.
\end{aligned}$$

It remains to rewrite the force term. (This manipulation is not so apparent in the existing literature.) Now

$$\sum_\alpha \mathbf{f}^\alpha \boldsymbol{\mu}_{\xi\alpha} = -\sum_\alpha F_\alpha \boldsymbol{\mu}_{\xi\alpha} + \sum_\alpha f_\alpha \boldsymbol{\mu}_{\xi\alpha}$$

where

$$F_\alpha := \sum_{\beta:\beta\neq\alpha} F_{\alpha\beta}(\xi^\alpha - \xi^\beta)\,.$$

We show that the F_α-expressions can be written as term under the divergence, hence we call it an "internal force" term. Thus we have to show that for $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$

$$\begin{aligned}
&\sum_\alpha \langle\, \zeta\,,\, F_\alpha \boldsymbol{\mu}_{\xi^\alpha}\,\rangle = \langle\, \mathrm{D}\zeta\,,\, M\,\rangle = -\langle\, \zeta\,,\, \mathrm{div}_x M\,\rangle\,,\\
&M := \frac{1}{2}\sum_{\alpha,\beta:\alpha\neq\beta} F_{\alpha\beta}(\xi^\alpha - \xi^\beta)\,(\xi^\alpha - \xi^\beta)^{\mathrm{T}}\,\boldsymbol{\mu}_{\xi^\alpha,\xi^\beta}\,.
\end{aligned} \tag{I3.30}$$

To prove this, we let $\widetilde{F}_{\alpha\alpha} := 0$ and $\widetilde{F}_{\alpha\beta} := F_{\alpha\beta}(\xi^\alpha - \xi^\beta)$ for $\alpha \neq \beta$, so that by assumption (I3.28)

$$F_{\alpha\beta}(\xi^\alpha - \xi^\beta) = -F_{\beta\alpha}(\xi^\beta - \xi^\alpha) \text{ for } \alpha\neq\beta,$$

or $\widetilde{F}_{\alpha\beta} = -\widetilde{F}_{\beta\alpha}$ for all α and β. Thus we obtain

$$
\begin{aligned}
&\sum_\alpha \langle\, \zeta\,,\, F_\alpha \boldsymbol{\mu}_{\xi^\alpha} \,\rangle = \sum_{\alpha,\beta} \Big\langle\, \zeta\,,\, \widetilde{F}_{\alpha\beta} \boldsymbol{\mu}_{\xi^\alpha} \,\Big\rangle = \sum_{\alpha,\beta} \int_{\mathbb{R}} \zeta(t,\xi^\alpha(t)) \bullet \widetilde{F}_{\alpha\beta}(t)\,\mathrm{d}t \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}} \frac{1}{2}\Big(\zeta(t,\xi^\alpha(t)) \bullet \widetilde{F}_{\alpha\beta}(t) + \zeta(t,\xi^\beta(t)) \bullet \widetilde{F}_{\beta\alpha}(t)\Big)\,\mathrm{d}t \\
&= \sum_{\alpha,\beta} \int_{\mathbb{R}} \frac{1}{2}\Big(\zeta(t,\xi^\alpha(t)) - \zeta(t,\xi^\beta(t))\Big) \bullet \widetilde{F}_{\alpha\beta}(t)\,\mathrm{d}t \\
&= \frac{1}{2} \sum_{\alpha,\beta} \int_{\mathbb{R}} \int_0^1 \Big(\mathrm{D}\zeta(t,(1-s)\xi^\beta(t) + s\xi^\alpha(t))(\xi^\alpha(t) - \xi^\beta(t))\Big) \bullet \widetilde{F}_{\alpha\beta}(t)\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad\qquad (\text{since } (Dz_1) \bullet z_2 = D \bullet (z_2 {z_1}^{\mathrm{T}}),\ D \text{ matrix},\ z_1, z_2 \text{ vectors}) \\
&= \frac{1}{2} \sum_{\alpha,\beta} \int_{\mathbb{R}} \int_0^1 \mathrm{D}\zeta(t,(1-s)\xi^\beta(t) + s\xi^\alpha(t)) \bullet \big(\widetilde{F}_{\alpha\beta}(t)\,(\xi^\alpha(t) - \xi^\beta(t))^{\mathrm{T}}\,\big)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2} \sum_{\alpha,\beta} \Big\langle\, \mathrm{D}\zeta\,,\, \widetilde{F}_{\alpha\beta}\,(\xi^\alpha - \xi^\beta)^{\mathrm{T}}\, \boldsymbol{\mu}_{\xi^\alpha,\xi^\beta} \,\Big\rangle = \langle\, \mathrm{D}\zeta\,,\, M \,\rangle \\
&= \sum_{j,k} \langle\, \partial_j \zeta_k\,,\, M_{kj} \,\rangle = -\sum_{j,k} \langle\, \zeta_k\,,\, \partial_j M_{kj} \,\rangle = -\langle\, \zeta\,,\, \mathrm{div}_x M \,\rangle\ .
\end{aligned}
$$

Consequently the assertion follows. □

Hence we have seen, that the mass and momentum conservation plays an essential role even for models with particles. We have also seen how Newton's mechanics, if it is interpreted using distributions, is part of continuum mechanics.

Fluid equations

Now we come back to the general systen of conservation laws (I3.1). If ϱ is the total mass, it is usually $\mathbf{J} = 0$ and $\mathbf{r} = 0$, the term $\widetilde{\mathbf{f}}$, which we called "general force density", now becomes the "force density" $\mathbf{f}$ (see (I3.2), for an explanation see the mass-momentum balance in section II.3). We then obtain the

(Special))Mass-momentum conservation:

$$
\begin{aligned}
&\partial_t \varrho + \mathrm{div}(\varrho v) = 0\ , \\
&\partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}
\end{aligned}
\tag{I3.31}
$$

ϱ mass density, v velocity,

$\Pi = (\Pi_{ij})_{i,j=1,\dots,n}$ pressure tensor,

$\mathbf{f} = (\mathbf{f}_i)_{i=1,\dots,n}$ force density.

It is very important for this mass-momentum balance how the pressure tensor Π is defined. Please, compare the distributional representation in (I3.29), where the pressure tensor also is determined by the forces which act between the molecules. We will now model the movement of fluids (for a single fluid we dont need a distributional mass-momentum law, unless in certain limit situations), where Π is given by $\Pi = p\mathrm{Id} - S$ with a pressure p and a stress tensor S. The stress tensor is assumed to be linear in $(\mathrm{D}v)^{\mathrm{S}}$. (These properties are derived in Chapter II, and the inequalities in (I3.32) follow from the entropy principle, see Chapter III.) The force term $\mathbf{f}$ is now usually an external force. We obtain the following

(Compressible) Navier-Stokes equations:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f} \,,$$

$$\begin{aligned} &\Pi = p\mathrm{Id} - S \text{ pressure tensor}, \quad p \text{ pressure},\\ &S = a(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}) + b(\operatorname{div} v)\,\mathrm{Id}\\ &\quad = 2a\,(\mathrm{D}v)^{\mathrm{S}} + b(\operatorname{div} v)\,\mathrm{Id} \quad \text{tension tensor},\\ &a > 0 \text{ and } b + \frac{2a}{n} \geq 0 \quad \text{viscosity coefficients},\\ &\mathbf{f} \text{ force density}, \quad (\text{sometimes } a = \mu,\ b = \lambda\ ^{18}) \end{aligned} \tag{I3.32}$$

It should be noted that this special representation of the pressure tensor Π and the tension S is a consequence of II.4.12, it is

$$\begin{aligned} S &= 2a\,(\mathrm{D}v)^{\mathrm{S}} + b(\operatorname{div} v)\,\mathrm{Id}\\ &= 2a\,\Big(\underbrace{(\mathrm{D}v)^{\mathrm{S}} - \frac{1}{n}\operatorname{div}(v)\mathrm{Id}}_{\text{trace free}}\Big) + \Big(b + \frac{2a}{n}\Big)\operatorname{div}(v)\mathrm{Id}\,, \end{aligned}$$

hence the positivity of the coefficients becomes clear. This positivity we will later, in connection with the entropy principle, study in detail (see e.g. III.2.5). The pressure p is a function of ϱ (among other things) if we regard a compressible fluid (see Section IV.2). Another version of the momentum balance comes from the following computations, which are true

[18] Lamé introduces his coefficients in the theory of elasticity. Mathematicians use his name because of analogy of the physical terms, see [Wikipedia: Lamé parameters]: "For example, the parameter μ is referred to in fluid dynamics as the dynamic viscosity of a fluid; whereas in the context of elasticity, μ is called the shear modulus."

in general,

$$\begin{aligned}&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}})\\ &= (\underbrace{\partial_t\varrho + \operatorname{div}(\varrho v)}_{=0})v + \varrho(\partial_t v + v\bullet\nabla v)\,,\end{aligned}$$

and

$$\operatorname{div}\Pi = \operatorname{div}(p\mathrm{Id} - S) = \nabla p - \operatorname{div}S\,.$$

Thus the (compressible) Navier-Stokes equations are equivalent to

(Compressible) Navier-Stokes equations:

$$\begin{aligned}&\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,,\\ &\varrho(\partial_t v + v\bullet\nabla v) + \nabla p - \operatorname{div}S = \mathbf{f}\end{aligned} \tag{I3.33}$$

p pressure (e.g. a function of ϱ),

$S = a(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}) + b(\operatorname{div}v)\,\mathrm{Id}\,,$

$\mathbf{f}$ force density, for the other quantities see (I3.32).

We test the equations by looking at the example of a centrifuge.

3.5 Centrifuge. We model a centrifuge by an infinitely long pipe in $\mathbb{R}^n$, $n = 3$, that is $\{x \in \mathbb{R}^3\,;\ |(x_1, x_2)| < R\}$, and we denote by r the distance from the axis $r = \sqrt{x_1^2 + x_2^2}$. We consider ***stationary solutions***, that means solutions that do not depend on the time variable.[19] We make the ansatz

$$\begin{aligned}\varrho = \varrho(r)\,,\quad p = p(r)\,,\quad \mathbf{f} = 0\,,\\ v(x) = \omega\cdot(-x_2, x_1, 0)\,.\end{aligned} \tag{I3.34}$$

Then the compressible Navier-Stokes equation (see (I3.33)) in the stationary case is equivalent to the differential equation

$$\partial_r p = \omega^2 r\varrho\,. \tag{I3.35}$$

That is, the pressure increases with the radius r. Hier schaut der Beobachter von außen der rotierenden Zentrifuge zu. Dreht sich der Beobachter mit der rotierenden Flüssigkeit und schreibt er die Situation in seinen Koordinaten auf, muss er die gleiche Physik beschreiben, d.h. für ihn muss ebenfalls der Druck mit dem Radius ansteigen. Dass dem wirklich so ist, wird in Beispiel II.3.10 nachgerechnet.

Proof. The equations (I3.33) are in the stationary case

$$\begin{aligned}\operatorname{div}(\varrho v) = 0\,,\\ \varrho v\bullet\nabla v + \nabla p - \operatorname{div}S = 0\,.\end{aligned} \tag{I3.36}$$

[19]This is the general definition of “stationary”.

It is

$$v = \omega \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \quad \text{also} \quad \mathrm{D}v = \omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $(\mathrm{D}v)^{\mathrm{S}} = 0$ and $\mathrm{div}v = 0$, hence $S = 0$. Moreover

$$\nabla(p(r)) = \partial_r p \nabla r = \frac{\partial_r p}{r}(x_1, x_2, 0).$$

It follows, we have the same computation for $\nabla(\varrho(r))$,

$$\begin{aligned} \mathrm{div}(\varrho v) &= \varrho\,\mathrm{div}v + (\nabla\varrho)\bullet v \\ &= \omega\frac{\partial_r \varrho}{r}(x_1, x_2, 0)\bullet(-x_2, x_1, 0) = 0 \end{aligned}$$

and the momentum equation becomes

$$\varrho v\bullet\nabla v + \nabla p = 0.$$

We compute [20]

$$\begin{aligned} \varrho v\bullet\nabla v &= \sum_{i=1}^{3} \varrho v_i \partial_{x_i} v \\ &= \varrho\omega^2(-x_2\mathbf{e}_2 - x_1\mathbf{e}_1) = -\varrho\omega^2(x_1, x_2, 0), \end{aligned}$$

hence it follows that

$$0 = \varrho v\bullet\nabla v + \nabla p = (-\varrho\omega^2 + \frac{\partial_r p}{r})(x_1, x_2, 0),$$

that is

$$\frac{1}{r}\partial_r p = \omega^2\varrho.$$

□

3.6 Different materials. We discuss different constitutive relations for p and its consequences for the identity (I3.35).

(1) Let $\varrho = \text{const} > 0$ and let p be an arbitrary free variable. Then it follows from (I3.35) that

$$p = \frac{\omega^2\varrho}{2}r^2 + \text{const}.$$

(2) Let $p = c\varrho^\gamma$ with $\gamma > 1$. Then (I3.35) is

$$\varrho = \left(\frac{(\gamma-1)\omega^2}{2c\gamma}r^2 + \text{const}\right)^{\frac{1}{\gamma-1}}$$

[20] It is $\mathbf{e}_i = (\delta_{ij})_{j=1,\ldots,n}$ the i-th basis vector of $\mathbb{R}^n$.

(3) Let $p = c\varrho$. Then (I3.35) is

$$\varrho = \exp\left(\frac{\omega^2}{2c}r^2 + \text{const}\right).$$

(4) Let $p = \varrho f'_\varrho(\varrho) - f(\varrho)$ with a given function f. Then (I3.35) is

$$f'_\varrho(\varrho) = \frac{\omega^2}{2}r^2 + \text{const}.$$

(5) Let $p = c_1\varrho + c_2\varrho^\gamma$ mit $\gamma > 1$. Then p is as in (4) if

$$f(\varrho) = c_1\varrho(\log\varrho - 1) + \frac{c_2}{\gamma - 1}\varrho^\gamma + \text{const}\cdot\varrho - c_0\,.$$

Hence p is as in (4) So (I3.35) is equivalent to the formula in (4).
Remark: The function f is the "internal free energy", see Section III.5.Note, that f is a convex function and f'_ϱ monotone increasing. Constitutive functions you also find in [39, 1.3.2 and 1.4].

Proof (1). This follows by integrating (I3.35). □

Proof (2). It is

$$\omega^2 r = \frac{c}{\varrho}\partial_r(\varrho^\gamma) = c\gamma\varrho^{\gamma-2}\partial_r\varrho = \partial_r(h(\varrho))\,,$$

if

$$h'_\varrho(\varrho) = c\gamma\varrho^{\gamma-2} \quad \text{hence} \quad h(\varrho) = c\frac{\gamma}{\gamma-1}\varrho^{\gamma-1} + \text{const}\,.$$

Therefore

$$\partial_r\left(h(\varrho) - \frac{\omega^2}{2}r^2\right) = 0\,,$$

that means

$$\frac{c\gamma}{\gamma-1}\varrho^{\gamma-1} + \text{const} = h(\varrho) = \frac{\omega^2}{2}r^2 + \text{const}.$$

From this it follows the result. □

Proof (3). It is

$$\omega^2 r = \frac{c}{\varrho}\partial_r\varrho = c\,\partial_r(\log\varrho)$$

and thus

$$\partial_r\left(c\log\varrho - \frac{\omega^2}{2}r^2\right) = 0\,,$$

that is,

$$\log\varrho = \frac{\omega^2}{2c}r^2 + \text{const}.$$

□

Proof (4). It is $p'_\varrho = \varrho f'_{\varrho\varrho}$, and thus

$$\partial_r p = p'_\varrho \partial_r \varrho = \varrho f'_{\varrho\varrho}(\varrho)\partial_r \varrho \,.$$

Since $\partial_r p = \omega^2 r \varrho$, we have

$$f'_{\varrho\varrho}(\varrho)\partial_r \varrho = \omega^2 r \,.$$

That means

$$\partial_r \left(f'_\varrho(\varrho) - \frac{\omega^2}{2} r^2 \right) = 0 \,,$$

and from this follows the assertion. □

Proof (5). It is

$$f'_\varrho(\varrho) = c_1 \log \varrho + \frac{c_2 \gamma}{\gamma - 1} \varrho^{\gamma - 1} + \text{const}$$

hence

$$\varrho f'_\varrho(\varrho) - f(\varrho) = c_0 + c_1 \varrho + c_2 \varrho^\gamma = p.$$

So p is as in (4). □

Incompressible fluids

In many applications it is assumed that the fluid is incompressible, that is, $\varrho = \varrho_0 = \text{const} > 0$. In this case, the mass conservation reduces to

$$0 = \partial_t \varrho + \operatorname{div}(\varrho v) = \varrho_0 \operatorname{div} v \,,$$

hence it is $\operatorname{div} v = 0$ and, with $(\mathrm{D}v)^{\mathrm{S}} = \frac{1}{2}\big(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}\big)$,

$$S = 2a\,(\mathrm{D}v)^{\mathrm{S}} + b \underbrace{\operatorname{div} v}_{=0} \mathrm{Id} = 2a\,(\mathrm{D}v)^{\mathrm{S}} \,.$$

Thus we obtain from (I3.33) the following for the incompressible Navier-Stokes equations

Incompressible Navier-Stokes equation:

$$\operatorname{div} v = 0 \,,$$

$$\varrho_0(\partial_t v + v \bullet \nabla v) + \nabla p - \operatorname{div} S = \mathbf{f} \tag{I3.37}$$

$\varrho_0 > 0$ constant, v velocity, p pressure,

$S = a(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}) = 2a\,(\mathrm{D}v)^{\mathrm{S}}$ tension tensor,

$a > 0$ viscosity coefficient, $\mathbf{f}$ force density.

If in addition $a = \text{const}$, then

$$2\,\mathrm{div}((\mathrm{D}v)^{\mathrm{S}}) = \Big(\sum_{j=1}^{n} \partial_j(\partial_j v_k + \partial_k v_j)\Big)_{k=1,\dots,n}$$

$$= \Delta v + \Big(\sum_{j=1}^{n} \partial_k\partial_j v_j\Big)_{k=1,\dots,n} = \Delta v + \nabla(\,\mathrm{div}\,v)\,,$$

therefore

$$\mathrm{div}S = \mathrm{div}(2a\,(\mathrm{D}v)^{\mathrm{S}}) = 2a\,\mathrm{div}((\mathrm{D}v)^{\mathrm{S}}) = a\Delta v\,.$$

Hence, if $a = \text{const}$, then (I3.37) is equivalent to

Special Navier-Stokes equation:

$$\mathrm{div}\,v = 0\,,$$

$$\varrho_0(\partial_t v + v\bullet\nabla v) + \nabla p - a\Delta v = \mathbf{f} \tag{I3.38}$$

quantities as in (I3.37),

$a = \text{const} > 0$ viscosity coefficient.

This is the version which is often treated.

Wir geben nun die Poiseuille Strömung als Beispiel für eine inkompressible zähe Strömung an (siehe auch [8, 5.1 Laminare Rohrströmung], wo beliebige Rohrquerschnitte behandelt werden).

3.7 Poiseuille-Strömung durch ein Rohr. Das Rohr ist gegeben durch

$$D := \{x \in \mathbb{R}^3\,;\ \sqrt{x_1^2 + x_2^2} < R\}\,.$$

Wir betrachten die stationäre inkompressible Navier-Stokes Gleichung ohne äußere Kraft, d.h. $\mathbf{f} = 0$. Die Gleichungen lauten dann

$$\begin{aligned} &\mathrm{div}\,v = 0\,,\\ &\varrho v\bullet\nabla v + \mathrm{div}\Pi = 0\,,\\ &\Pi = p\mathrm{Id} - S \end{aligned} \tag{I3.39}$$

in D, mit Druck und Spannungstensor wie in (I3.38). Wir betrachten Strömungen mit konstanter Viskosität, also $a = \text{const}$, und der Randbedingung

$$v = 0 \text{ auf } \partial D. \tag{I3.40}$$

Eine stationäre Lösung der Massen- und Impulserhaltung mit dieser Randbedingung ist gegeben durch

$$v(x) = \frac{c}{4}\cdot\big(R^2 - (x_1^2 + x_2^2)\big)\,\mathbf{e}_3\,,$$

$$p(x) = -cax_3 + \text{const}\,,$$

wobei c eine Konstante ist und a der konstante Viskositätskoeffizient.

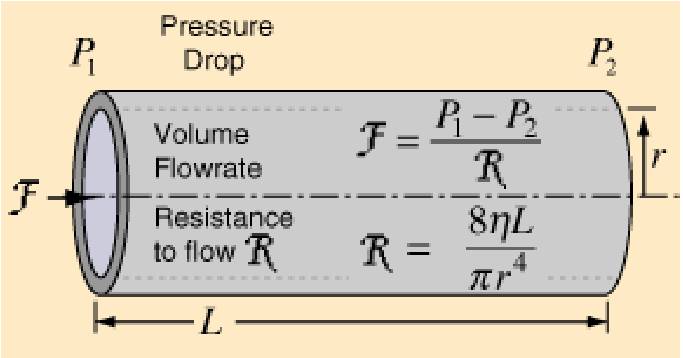


Fig. 15: From [hyperphysics.phy-astr.gsu.edu/hbase/ppois.html]

Ist also $a > 0$ und strömt die Flüssigkeit in Richtung $\mathbf{e}_3$, d.h. $c > 0$, so fällt der Druck p in Strömungsrichtung. Also muss bei einem langen Rohr die Flüssigkeit mit hohem Druck eingeführt werden, um eine stationäre Strömung zu erreichen.

Proof. Sei (v, p) eine Lösung von (I3.39) in D mit der Randbedingung (I3.40), und für die wir die Darstellung

$$v(x) = v_3(x_1, x_2)\,\mathbf{e}_3$$

annehmen. Dann gilt

$$\operatorname{div} v = \partial_{x_3} v_3 = 0\,,$$

also ist für die Massenerhaltung wegen $\varrho = \text{const}$

$$\operatorname{div}(\varrho v) = \varrho \operatorname{div} v = 0\,.$$

Es folgt weiter $S = 2a\,(\mathrm{D}v)^{\mathrm{S}}$, d.h. der Divergenzterm von S verschwindet, und daher gilt für die Impulserhaltung, da $a = \text{const}$,

$$\begin{aligned} 0 &= \varrho v \bullet \nabla v + \nabla p - \operatorname{div} S \\ &= \underbrace{\varrho v_3 \partial_{x_3} v}_{=0} + \nabla p - \operatorname{div} S = \nabla p - 2a \operatorname{div}(\mathrm{D}v)^{\mathrm{S}}\,. \end{aligned}$$

Nun ist

$$\mathrm{D}v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 v_3 & \partial_2 v_3 & 0 \end{bmatrix}, \quad (\mathrm{D}v)^{\mathrm{S}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & \partial_1 v_3 \\ 0 & 0 & \partial_2 v_3 \\ \partial_1 v_3 & \partial_2 v_3 & 0 \end{bmatrix},$$

also gilt

$$\operatorname{div}(\mathrm{D}v)^{\mathrm{S}} = \frac{1}{2}(\Delta_{(x_1,x_2)} v_3)\mathbf{e}_3\,,$$

somit

$$\nabla p = 2a \,\mathrm{div}\,(\mathrm{D}v)^{\mathrm{S}} = a(\Delta_{(x_1,x_2)} v_3)\mathbf{e}_3\,.$$

Daraus schließen wir zunächst, dass $\partial_1 p = 0$ und $\partial_2 p = 0$, also ist $p = \widehat{p}(x_3)$. Dann wird die Impulserhaltung

$$\underbrace{\partial_3 p}_{\text{Funktion von } x_3} = \underbrace{a\Delta_{(x_1,x_2)} v_3}_{\text{Funktion von } (x_1,x_2)}\,,$$

also gilt mit einer Konstanten c

$$\partial_3 p = -ca \quad \text{und} \quad \Delta_{(x_1,x_2)} v_3 = -c\,.$$

Daraus ergibt sich $p = -cax_3 + \mathrm{const}$. Und wegen der Randbedingung an v folgt aus der Differentialgleichung die quadratische Darstellung von v_3, d.h. $v_3 = \frac{c}{4} \cdot (R^2 - (x_1^2 + x_2^2))$. □

Proof der Formel in Fig. 15. The flow rate $\mathcal{F}$ is the total flow through a cross section to the pipe, that is,

$$\mathcal{F} := \int_{B_R(0)} v(x')\bullet\mathbf{e}_3 \,\mathrm{dL}^2(x') = \frac{c}{4}\int_{B_R(0)} (R^2 - |x'|^2)\,\mathrm{dL}^2(x')$$
$$= \frac{c}{4} \cdot 2\pi \int_0^R r(R^2 - r^2)\,\mathrm{d}r = \frac{c\pi}{8} R^4\,.$$

The pressure difference for a pipe of length L is

$$P_1 - P_2 = \Big[-cax_3 + \mathrm{const} \Big]_{x_3=L}^{x_3=0} = caL\,,$$

hence

$$\mathcal{F} = \frac{c\pi}{8} R^4 = \frac{\pi(P_1 - P_2)}{8aL} R^4 = \frac{\pi R^4 (P_1 - P_2)}{8aL} = \frac{P_1 - P_2}{\mathcal{R}}\,,$$

so that for the resistance

$$\mathcal{R} := \frac{P_1 - P_2}{\mathcal{F}} = \frac{8aL}{\pi R^4}\,,$$

where a is the viscosity coefficient. In Fig. 15 we have $\eta = a$. □

In der Regel hat die stationäre Navier-Stokes Gleichung (bei gegebenen Randbedingungen) mehrere Lösungen. Die hier gezeigte stationäre Lösung der Navier-Stokes Gleichung ist stabil für kleine Geschwindigkeiten bzw. große Viskositäten. Für große Geschwindigkeiten bzw. kleine Viskositäten bildet sich physikalisch eine Randschicht aus, die in der Regel nicht stationär ist. Also existiert dann noch eine weitere instationäre "stabile Lösung". Will man das Problem numerisch behandeln, so ist also auf eine genügend feine Diskretisierung nahe des Randes zu achten, und ggf. muss eine mathematisch formulierte Randschicht benutzt werden (siehe zum Beispiel die Prandtl'sche Randschicht in Abschnitt IV.15), damit man eine genauere Darstellung der Strömung erhält.

4 Interfaces

We encounter daily (see Fig. 16) different media which touch each other, for example, a ship in the water, a drop of water, a leaf in the air, and stones on the ground. This situation is described mathematically by a surface that separates the medium 1 and the medium 2. The main observation is that also in this case the conservation of mass and the conservation of momentum applies. So we have to deal here again with the situation that we can use the concept of distributions. Only with the help of distributions the conservation laws are effectively writable and easy to understand.

We give in this section only examples that do not have differential equations on the interface such as surface tension or surfactant (this is treated for example in the advanced lecture [22]). These simpler cases occur for example in connection with hyperbolic equations, see Section IV.4 where the Euler equations are treated. We assume that the media are fluids or gases. Thus

Fig. 16: Interface of water and air

we have

$$\mathcal{U} := \Omega^1 \cup \Gamma \cup \Omega^2 \subset \mathbb{R} \times \mathbb{R}^n ,$$

where $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ is an open spacetime subset. The sets Ω^m are open sets containing the fluid or gas, and the common interface is Γ, which we assume to be a C^2-surface. Hence

$$t \mapsto \Omega^m_t := \{x \in \mathbb{R}^n ;\ (t,x) \in \Omega^m\} \subset \mathbb{R}^n$$

is the set, which is occupied by the fluid or gas at the given time moment t,

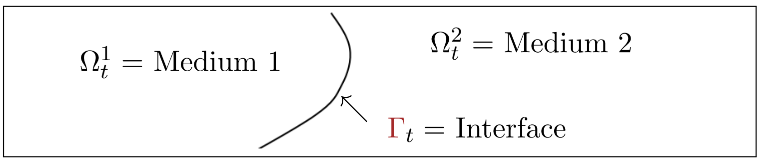


Fig. 17: Two adjoining media

and the ***interface*** at that time t is

$$t \mapsto \Gamma_t := \{x \in \mathbb{R}^n ;\ (t,x) \in \Gamma\} \subset \mathbb{R}^n ,$$

which by the way also should be a C^2-surface. In general $t \mapsto \Gamma_t$ can depend on time, and then there exists a normal vector $v_\Gamma(t,x) \in \mathbb{R}^n$ which indicates how fast the interface is moving in time:

4.1 Normal velocity. Zu jedem Punkt $x \in \Gamma_t$ gibt es genau einen Vektor

$$v_\Gamma(t,x) \in T_x(\Gamma_t)^\perp\,,$$

der im Normalenraum von Γ_t liegt und der die folgende Eigenschaft hat: Ist $\widetilde{t} \mapsto \xi(\widetilde{t}) \in \Gamma_{\widetilde{t}}$ die Bewegung eines Massenpunktes auf Γ, so gilt für $x = \xi(t)$

$$v_\Gamma(t,x) = P_{(t,x)} v(t,x)\,, \quad v(t,\xi(t)) := \dot{\xi}(t).$$

Dabei ist $P_{(t,x)}\!:\mathbb{R}^n \to T_x(\Gamma_t)^\perp$ die orthogonale Projektion. *Remark:* Im stationären Fall ist auch Γ_t von der Zeit t unabhängig, also ist dann $v_\Gamma(t,x) = 0$.

Hierbei ist also $v(t,\xi(t)) \in \Gamma_t$ die Geschwindigkeit des Massenpunktes und die Aussage besagt, dass $v(t,x) - v_\Gamma(t,x) \in T_x(\Gamma_t)$. Der Massenpunkt $t \mapsto \xi(t) \in \Gamma_t$ kann sich also beliebig auf Γ bewegen.

Proof. Let $x \in \Gamma_t$ and define the normal space by $N_x := T_x(\Gamma_t)^\perp$. For small $\delta > 0$ the set $\Gamma_{t+\delta}$ intersects $\{x + \nu\,;\ \nu \in N_x\}$ in exactly one point $\{x_\delta\}$. Since the set Γ is C^1 there is exactly one vector $v_\Gamma(t,x)$ such that

$$x_\delta = x + \delta v_\Gamma(t,x) + \mathcal{o}(\delta)\,.$$

Now if $\widetilde{t} \mapsto \xi(\widetilde{t}) \in \Gamma_{\widetilde{t}}$ is the movement of a mass point and $x = \xi(t)$ it follows again from the C^1-property, that $P_{(t,x)}(\xi(t+\delta) - x_\delta) = \mathcal{o}(\delta)$. Hence

$$\begin{aligned} P_{(t,x)}(\dot{\xi}(t)) &= \frac{1}{\delta} P_{(t,x)}(\xi(t+\delta) - \xi(t)) + \mathcal{o}(1) \\ &= \frac{1}{\delta} P_{(t,x)}(x_\delta - x) + \mathcal{o}(1) = P_{(t,x)}(v_\Gamma(t,x)) + \mathcal{o}(1) \end{aligned}$$

wich gives $P_{(t,x)}(\dot{\xi}(t)) = v_\Gamma(t,x)$. □

The balance between the two media Ω^m, $m = 1,2$, is given by conservation laws, which are the distributional version of the mass and momentum equation. With given quantities (ϱ^m, v^m, Π^m) they read

$$\begin{aligned} &\partial_t[\textstyle\sum_m \varrho^m \mathcal{X}_{\Omega^m}] + \operatorname{div}[\textstyle\sum_m \varrho^m v^m \mathcal{X}_{\Omega^m}] = 0 \\ &\partial_t[\textstyle\sum_m \varrho^m v^m \mathcal{X}_{\Omega^m}] \\ &+ \operatorname{div}[\textstyle\sum_m (\varrho^m v^m v^{m\mathrm{T}} + \Pi^m)\mathcal{X}_{\Omega^m}] = [\textstyle\sum_m \mathbf{f}^m \mathcal{X}_{\Omega^m}]\,. \end{aligned} \tag{I4.1}$$

These are the equations we consider without additional terms on the interface Γ.

In the following examples we will use this distributional differential equations (I4.1) and we make the assumption of a stationary solution[21] therefore

$$\Omega^m = \mathbb{R} \times D^m \text{ hence } \Gamma = \mathbb{R} \times \boldsymbol{S}\,, \tag{I4.2}$$

with $D^m, \boldsymbol{S} \subset \mathbb{R}^n$. Furthermore, we assume that there no mass exchange on the surface $\boldsymbol{S}$, hence v^m on $\boldsymbol{S}$ points in a tangential direction of $\boldsymbol{S}$. Under these assumptions (I4.1) is equivalent to

$$\begin{aligned} &\mathrm{div}[\varrho^m v^m \mathcal{X}_{D^m}] = 0 \text{ for each } m, \\ &\mathrm{div}[\textstyle\sum_m (\varrho^m v^m v^{m\mathrm{T}} + \Pi^m)\mathcal{X}_{D^m}] = [\textstyle\sum_m \mathbf{f}^m \mathcal{X}_{D^m}]\,. \end{aligned} \tag{I4.3}$$

We present three examples, which are all very classical.

Principle of Archimedes

As first example we consider a body in water. The set $D^1 = D^b$ represents the body and $D^2 = D^w$ the water. One case is that the body is totally under water.

4.2 Archimedes' principle. "Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object." From [Wikipedia: Archimedes' principle].

How this has to be understood one sees in the following proofs, in particular it is of interest how the mass-momentum system enters the argumentation.

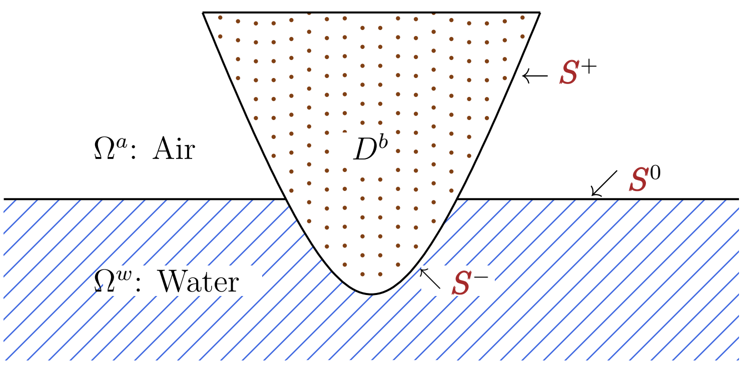


Fig. 18: A partial submerged body D^b with boundary $\boldsymbol{S} := \partial D^b$

[21] According to the definition this says that also the velocities are independent of time.

Proof if the body is comletely submerged. Wir betrachten den ***statischen Zustand***, d.h. den stationären Zustand mit Geschwindigkeit 0 [22] und es sei $n = 3$. Der Körper, repräsentiert durch $D^1 = D^b$, sei ganz eingetaucht in Wasser $D^2 = D^w$, und der Rand dazwischen sei $\boldsymbol{S}$. Die Massenerhaltungen in (I4.3) sind dann trivial und die Impulserhaltung in (I4.3) lautet

$$\operatorname{div}\left[\sum_{m=1,2} \Pi^m \mathcal{X}_{D^m}\right] = \left[\sum_{m=1,2} \mathbf{f}^m \mathcal{X}_{D^m}\right].$$

Wir schreiben dies mit Testfunktionen $\zeta \in \mathscr{D}(\mathbb{R}^3;\mathbb{R}^3)$

$$\begin{aligned} 0 &= \left\langle \zeta\,,\, -\operatorname{div}\left[\sum_{m=1,2} \Pi^m \mathcal{X}_{D^m}\right] + \left[\sum_{m=1,2} \mathbf{f}^m \mathcal{X}_{D^m}\right] \right\rangle_{\mathscr{D}(\mathbb{R}^3)} \\ &= \sum_m \int_{D^m} \big(\mathrm{D}\zeta : \Pi^m + \zeta \bullet \mathbf{f}^m\big)\, \mathrm{dL}^3 \\ &= \sum_m \int_{D^m} \zeta \bullet \big(-\operatorname{div}\Pi^m + \mathbf{f}^m\big)\, \mathrm{dL}^3 + \int_{\boldsymbol{S}} \zeta \bullet \left(\sum_m \Pi^m \nu_{D^m}\right) \mathrm{dH}^2 . \end{aligned} \tag{I4.4}$$

Dies ist äquivalent zu, wobei ν eine Normale an $\boldsymbol{S}$ sei,

$$\operatorname{div}\Pi^m = \mathbf{f}^m \text{ in } D^m, \quad (\Pi^2 - \Pi^1)\nu = 0 \text{ auf } \boldsymbol{S}\,,$$

also, wenn wir die andere Bezeichnung verwenden,

$$\operatorname{div}\Pi^w = \mathbf{f}^w \text{ in } D^w, \quad \operatorname{div}\Pi^b = \mathbf{f}^b \text{ in } D^b, \quad (\Pi^w - \Pi^b)\nu = 0 \text{ auf } \boldsymbol{S}.$$

Im statischen Zustand ist für das Wasser $\Pi^w = p^w \mathrm{Id}$ mit dem Druck p^w und für die Kräfte nehmen wir nur die Schwerkraft und diese approximieren wir linear, d.h. $\mathbf{f}^w = \varrho^w \mathbf{g}$ sowie $\mathbf{f}^b = \varrho^b \mathbf{g}$ mit $\mathbf{g} = -g_{\mathrm{Erde}} \mathbf{e}_3$, wobei g_{Erde} die Gravitationskonstante auf der Erdoberfläche sei. Hier sind ϱ^w und ϱ^b die Dichten, wobei wir annehmen, dass

$$\varrho^w = \varrho^o = \text{const}.$$

Nun ist in D^w

$$\nabla p^w = \operatorname{div}\Pi^w = \mathbf{f}^w = \varrho^w \mathbf{g} = -\varrho^w g_{\mathrm{Erde}} \mathbf{e}_3$$

und daher (bis auf eine Additionskonstante, die dann auch beim Körper addiert werden müsste)

$$p^w = -\varrho^w g_{\mathrm{Erde}} x_3 \,.$$

Da $\varrho^w = \varrho^o$ in D^w und $\varrho^o = \text{const}$ eine Zahl ist, können wir mit ihr eine globale Funktion

$$p^o(x) := -\varrho^o g_{\mathrm{Erde}} x_3$$

[22] das heißt, wir betrachten ein Equilibrium

definieren. Es ist also $p^o = p^w$ auf Ω^w. Wir erhalten

$$\begin{aligned}
&\int_{D^w} (\mathrm{D}\zeta \mathbin{:} \Pi^w + \zeta \bullet \mathbf{f}^w)\,\mathrm{dL}^3 = \int_{D^w} (\mathrm{D}\zeta \mathbin{:} (p^w \mathrm{Id}) + \zeta \bullet \nabla p^w)\,\mathrm{dL}^3 \\
&= \int_{D^w} (\,\mathrm{div}\zeta \cdot p^w + \zeta \bullet \nabla p^w)\,\mathrm{dL}^3 = \int_{D^w} \mathrm{div}(p^w \zeta)\,\mathrm{dL}^3 \\
&= \int_{\boldsymbol{S}} p^w \zeta \bullet \nu_{D^w}\,\mathrm{dH}^2 = \int_{\boldsymbol{S}} p^o \zeta \bullet \nu_{D^w}\,\mathrm{dH}^2 = -\int_{\boldsymbol{S}} p^o \zeta \bullet \nu_{D^b}\,\mathrm{dH}^2 \\
&= -\int_{D^b} \mathrm{div}(p^o \zeta)\,\mathrm{dL}^3 = -\int_{D^b} \mathrm{D}\zeta \mathbin{:} (p^o \mathrm{Id})\,\mathrm{dL}^3 + \int_{D^b} \zeta \bullet (-\nabla p^o)\,\mathrm{dL}^3 \,,
\end{aligned}$$

wobei $-\nabla p^o = \varrho^o g_{\mathrm{Erde}} \mathbf{e}_3$. Somit ist, wenn wir dies in (I4.4) einsetzen,

$$\begin{aligned}
0 &= \int_{D^b} (\mathrm{D}\zeta \mathbin{:} \Pi^b + \zeta \bullet \mathbf{f}^b)\,\mathrm{dL}^3 + \int_{D^w} (\mathrm{D}\zeta \mathbin{:} \Pi^w + \zeta \bullet \mathbf{f}^w)\,\mathrm{dL}^3 \\
&= \int_{D^b} (\mathrm{D}\zeta \mathbin{:} (\Pi^b - p^o \mathrm{Id}) + \zeta \bullet (\mathbf{f}^b - \nabla p^o))\,\mathrm{dL}^3 \,.
\end{aligned}$$

Wähle nun die Testfunktion gleich 1 auf D^b und erhalte

$$0 = \underbrace{\int_{D^b} \mathbf{f}^b \,\mathrm{dL}^3}_{\substack{\text{Total force} \\ \text{on the body}}} - \underbrace{\int_{D^b} \nabla p^o \,\mathrm{dL}^3}_{\substack{\text{Weight of the} \\ \text{displaced fluid}}} \,.$$

□

Proof if the body floats on the water. Der Körper, repräsentiert durch $D^1 = D^b$, befinde sich nur teilweise im Wasser (siehe Fig. 18). Dann ist

$$\boldsymbol{S}^0 := \{x\,;\; x_3 = 0\} \setminus D^b \in \mathbb{R}^3$$

das Interface zwischen Wasser D^w und Luft D^a und

$$D^2 = D^w \cup \boldsymbol{S}^0 \cup D^a \,.$$

Also besteht $\boldsymbol{S} := \partial D^b$ aus zwei Teilen, dem Rand zur Luft $\boldsymbol{S}^+ := \boldsymbol{S} \cap \overline{D}^a$ und dem Rand zum Wasser $\boldsymbol{S}^- := \boldsymbol{S} \cap \overline{D}^w$. Die beiden Mengen werden getrennt durch $\boldsymbol{S} \cap \boldsymbol{S}^0$. Es ist dann (wir nehmen dasselbe wir im vorigen Beweis an, aber jetzt für die drei Gebiete D^b, D^w, D^a)

$$\begin{aligned}
&\int_{D^w} (\mathrm{D}\zeta \mathbin{:} \Pi^w + \zeta \bullet \mathbf{f}^w)\,\mathrm{dL}^3 + \int_{D^a} (\mathrm{D}\zeta \mathbin{:} \Pi^a + \zeta \bullet \mathbf{f}^a)\,\mathrm{dL}^3 \\
&= \int_{D^w} \mathrm{div}(p^w \zeta)\,\mathrm{dL}^3 + \int_{D^a} \mathrm{div}(p^a \zeta)\,\mathrm{dL}^3 \\
&= \int_{\boldsymbol{S}^-} p^w \zeta \bullet \nu_{D^w}\,\mathrm{dH}^2 + \int_{\boldsymbol{S}^+} p^a \zeta \bullet \nu_{D^a}\,\mathrm{dH}^2
\end{aligned}$$

(es sind $p^w = 0$, $p^a = 0$, $p^o = 0$ auf $\{x_3 = 0\}$), und das erste Integral ist gleich

$$= -\int_{S^-} p^o \zeta \bullet \nu_{D^b} \,\mathrm{dH}^2 = -\int_{D^b \cap \{x_3<0\}} \operatorname{div}(p^o \zeta) \,\mathrm{dL}^3$$
$$= \int_{D^b \cap \{x_3<0\}} \big(\mathrm{D}\zeta \mathbin{:} (-p^o \mathrm{Id}) + \zeta \bullet (-\nabla p^o)\big) \,\mathrm{dL}^3 \,.$$

Also erhalten wir insgesamt

$$0 = \int_{D^b} (\mathrm{D}\zeta \mathbin{:} \Pi^b + \zeta \bullet \mathbf{f}^b) \,\mathrm{dL}^3 + \int_{D^w} (\mathrm{D}\zeta \mathbin{:} \Pi^w + \zeta \bullet \mathbf{f}^w) \,\mathrm{dL}^3$$
$$+ \int_{D^a} (\mathrm{D}\zeta \mathbin{:} \Pi^a + \zeta \bullet \mathbf{f}^a) \,\mathrm{dL}^3$$
$$= \int_{D^b} (\mathrm{D}\zeta \mathbin{:} (\Pi^b - \mathcal{X}_{\{x_3<0\}} p^o \mathrm{Id}) + \zeta \bullet (\mathbf{f}^b - \mathcal{X}_{\{x_3<0\}} \nabla p^o)) \,\mathrm{dL}^3$$
$$- \int_{S^+} p^a \zeta \bullet \nu_{D^b} \,\mathrm{dH}^2 \,.$$

Man lässt jetzt $\varrho^a \to 0$ (und dami auch $p^a \to 0$) gehen oder man formt auch das zweite Randintegral um. Man wählt dann wieder eine Testfunktion, die gleich 1 auf D^b ist, und erhält dann

$$0 = \underbrace{\int_{D^b} \mathbf{f}^b \,\mathrm{dL}^3}_{\substack{\text{Total force}\\ \text{on the body}}} - \underbrace{\int_{D^b} \mathcal{X}_{\{x_3<0\}} \nabla p^o \,\mathrm{dL}^3}_{\substack{\text{Weight of the}\\ \text{displaced fluid}}} \,.$$

□

<u>Fluid with free surface</u>

As second example we want to consider a rotating fluid with free surface, that is $D_f := D^1$ and $D_g := D^2$, where D_f is occupied by the water and D_g by the air (gas).

4.3 Stationary liquid with a surface. We consider a stationary solution of an (in)compressible fluid with free boundary to a gas as just above described, where there is no mass exchange between the two phases (no evaporation or condensation).

Assertion: The differential equations are

$$\left.\begin{aligned}&\operatorname{div}(\varrho v) = 0\,,\\ &\operatorname{div}(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f}\end{aligned}\right\}\ \text{in } D_f\,,$$
$$p_g = \text{const locally in } D_g\,,$$
$$\left.\begin{aligned}&v\bullet\nu = 0\,,\\ &(p - p_g)\nu = S\nu\end{aligned}\right\}\ \text{on } \partial D_f = \boldsymbol{S}\,,$$

where ν is a normal unit vector of $\boldsymbol{S}$.

Proof. The conservation of mass is for the liquid (no mass exchange with the gas)

$$\operatorname{div}[\varrho v \mathcal{X}_{D_f}] = 0\,,$$

thus for test functions $\zeta \in \mathscr{D}(\mathbb{R}^n;\mathbb{R})$

$$\begin{aligned}0 &= \big\langle\, \zeta\,,\, -\operatorname{div}[\varrho v \mathcal{X}_{D_f}]\,\big\rangle = \int_{D_f} \nabla\zeta(x)\bullet(\varrho v)(x)\,\mathrm{d}x\\ &= \int_{\partial D_f} \zeta(x)\nu_{D_f}(x)\bullet(\varrho v)(x)\,\mathrm{dH}^{n-1}(x) - \int_{D_f}\zeta(x)\operatorname{div}(\varrho v)(x)\,\mathrm{d}x\,.\end{aligned}$$

From this it follows $\operatorname{div}(\varrho v) = 0$ in D_f and $\nu_{D_f}\bullet v = 0$ on ∂D_f. The conservation of momentum is true for vector-valued test functions $\zeta \in \mathscr{D}(\mathbb{R}^n;\mathbb{R}^n)$

$$\begin{aligned}0 &= \big\langle\, \zeta\,,\, -\operatorname{div}\big[(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S)\mathcal{X}_{D_f} + p_g\mathrm{Id}\mathcal{X}_{D_g}\big] + \big[\mathbf{f}\mathcal{X}_{D_f}\big]\,\big\rangle\\ &= \big\langle\, \mathrm{D}\zeta\,,\, \big[(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S)\mathcal{X}_{D_f}\big] + \big[p_g\mathrm{Id}\mathcal{X}_{D_g}\big]\,\big\rangle + \big\langle\, \zeta\,,\, \big[\mathbf{f}\mathcal{X}_{D_f}\big]\,\big\rangle\\ &= \int_{D_f}\Big(\mathrm{D}\zeta\boldsymbol{:}(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S) + \zeta\bullet\mathbf{f}\Big)\,\mathrm{dL}^n + \int_{D_g}\mathrm{D}\zeta\boldsymbol{:}(p_g\mathrm{Id})\,\mathrm{dL}^n\,.\end{aligned}$$

Since $\nu_{D_g} = -\nu_{D_f}$, this is

$$\begin{aligned}&= \int_{D_f}\zeta\bullet\big(-\operatorname{div}(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S) + \mathbf{f}\big)\,\mathrm{dL}^n - \int_{D_g}\zeta\bullet\nabla p_g\,\mathrm{dL}^n\\ &\qquad + \int_{\partial D_f}\zeta\bullet(\varrho v\,v^{\mathrm{T}} + (p - p_g)\mathrm{Id} - S)\nu_{D_f}\,\mathrm{dH}^{n-1}\,,\end{aligned}$$

from which the remaining differential equations in D_f and D_g, and the interface equation follow, i.e.

$$\begin{aligned}&\operatorname{div}(\varrho v\,v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f} \text{ on } D_f,\\ &\nabla p_g = 0 \text{ on } D_g,\\ &(\varrho v\,v^{\mathrm{T}} + (p - p_g)\mathrm{Id} - S)\nu_{D_f} = 0 \text{ on } \partial D_f.\end{aligned}$$

Since we have already shown that $v \bullet \nu_{D_f} = 0$, the equation on ∂D_f is equivalent to

$$(p - p_g)\nu_{D_f} - S\nu_{D_f} = 0.$$

□

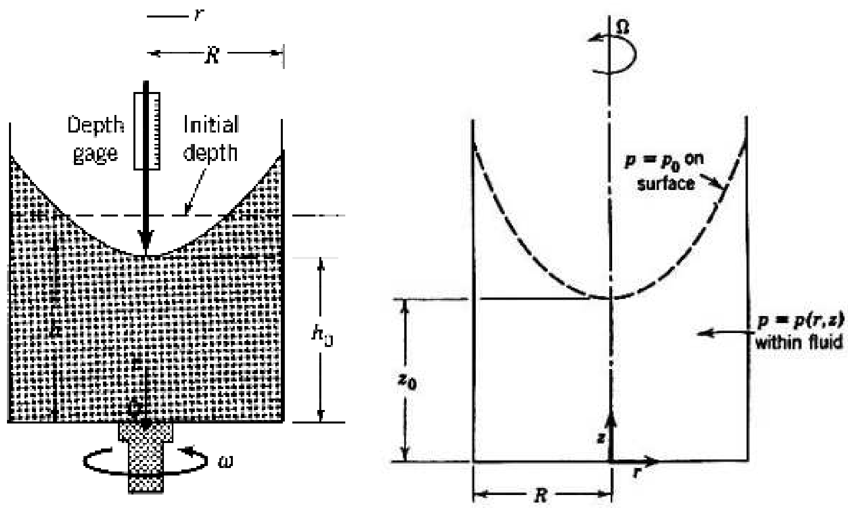


Fig. 19: The surface is a paraboloid

4.4 The parabolic shape of the surface. We consider the problem in 4.3 with an incompressible fluid having stress tensor S as in (I3.33), and

$$v(x) = \omega \cdot (-x_2, x_1, 0) \qquad \text{(rotating liquid)},$$
$$\mathbf{f}(x) = \varrho g_{\text{Earth}}(0, 0, -1) \qquad \text{(Earth's gravity linearized)}$$
$$\text{with } \varrho = \varrho_0 = \text{const}\,, \quad g_{\text{Earth}} = 9.81\frac{m}{s^2}\ \big(= 9.80665\frac{m}{s^2}\big)\,.$$

This is exactly a solution of 4.3, if

$$v \bullet \nu = 0\,, \quad p = p_g = \text{const on } \boldsymbol{S},$$

$\boldsymbol{S}$ is given by a paraboloid in vertical direction:

$$x_3 = \frac{\omega^2}{2g}(x_1^2 + x_2^2) + \text{const}$$

Hint: It ist $\Gamma = \mathbb{R} \times \boldsymbol{S}$ time independent and ν is a normal on $\boldsymbol{S}$.

Thus: The shape of the boundary is given by the rotation of the liquid in connection with the gravity, and especially by the distributional mass and momentum balance. (For comparison see the publication [134].)

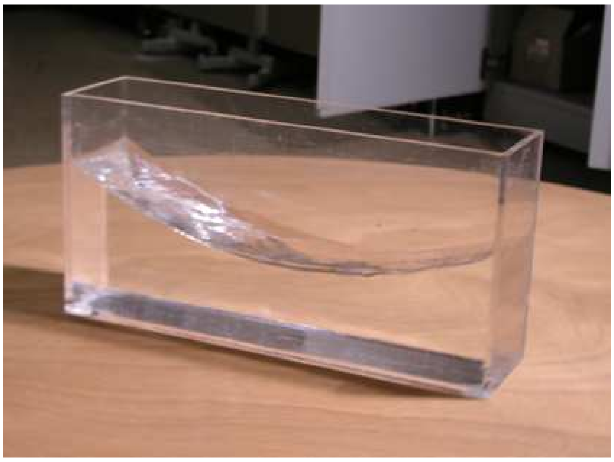


Fig. 20: Rotation of water

Proof. We first consider the general case of a compressible flow, and we use the equations in 4.3, which have been proved before. Specifically for the given v it is $(\mathrm{D}v)^{\mathrm{S}} = 0$ (see the proof of 3.5) and $\operatorname{div} v = \operatorname{trace}(\mathrm{D}v)^{\mathrm{S}} = 0$. This proves $S = 0$. The boundary of $\boldsymbol{S}$ is orthogonal to the direction of the flow ($v \bullet \nu_{\boldsymbol{S}} = 0$ on $\boldsymbol{S}$), therefore it must be rotationally symmetric (because of the mass conservation of the liquid). Due to $\operatorname{div}(\varrho v\, v^{\mathrm{T}}) = \varrho v \bullet \nabla v + v \bullet \operatorname{div}(\varrho v)$, the equations to be solved are

$$\begin{aligned} &\varrho v \bullet \nabla v + \nabla p = \mathbf{f} \text{ in } D_f, \\ &p - p_g = 0 \text{ on } \boldsymbol{S}, \end{aligned}$$

where p_g is a constant, the pressure of the gas. So we compute p from the differential equation and then we set $p = p_g$ on $\boldsymbol{S}$. Now the force of gravity is $\mathbf{f} = -\varrho g \mathbf{e}_3$ with $g = g_{\mathrm{Erde}}$ (as an approximation), thus

$$\varrho v \bullet \nabla v + \nabla p = -\varrho g \mathbf{e}_3 \,.$$

Due to $v \bullet \nabla v = -\omega^2(x_1, x_2, 0)$ (see the proof of 3.5) we obtain

$$\nabla p = \varrho \begin{bmatrix} \omega^2 x_1 \\ \omega^2 x_2 \\ -g \end{bmatrix} . \tag{I4.5}$$

In the incompressible case $\varrho = \text{const}$ it follows

$$p(x) = \varrho \left(\frac{\omega^2}{2}(x_1^2 + x_2^2) - g x_3 \right) + \text{const},$$

hence we obtain for $x \in \boldsymbol{S}$

$$g x_3 = \frac{\omega^2}{2}(x_1^2 + x_2^2) - \frac{p_g - \text{const}}{\varrho}.$$

The constant is used to determine the height of $\boldsymbol{S}$ in the center. □

For a compressible fluid, the last equations will be different, so the analysis described above has to be done again (see Exercise 7.21).

Schwerkraft eines Planeten

Als weiteres Beispiel für eine stationäre Lösung betrachten wir das Schwerkraftfeld eines inkompressiblen Planeten auf sich selbst, wobei wir hier annehmen, dass der Planet nicht rotiert. Wir machen neben der Gravitationsgleichung wieder von der stationären distributionellen Schreibweise der Navier-Stokes Gleichungen Gebrauch, und zwar ist $D^1 = \mathrm{B}_R(0)$, und $D^2 = \mathbb{R}^3 \setminus \overline{\mathrm{B}_R(0)}$ ist ein Vakuum.

4.5 Schwerkraftfeld eines inkompressiblen Planeten. Wir betrachten die inkompressiblen Navier-Stokes Gleichungen im $\mathbb{R}^3$ und eine stationäre Lösung mit $v = 0$. Ein nicht rotierender inkompressibler Planet werde modelliert durch eine Kugel mit homogener Massenverteilung

$$\varrho = \varrho_0 \mathcal{X}_{\mathrm{B}_R(0)} ,$$

wobei $\varrho_0 > 0$ eine Konstante sei. Außerhalb des Planeten sei Vakuum vorhanden. Dann erfüllen das Schwerepotential $\boldsymbol{\phi}$ in 2.16 und der Druck p die Gleichungen

$$\begin{aligned} \operatorname{div}(-[\nabla\boldsymbol{\phi}]) &= [\varrho] , \\ \nabla[p\mathcal{X}_{\mathrm{B}_R(0)}] &= \mathbf{f} := \mathfrak{g}[\varrho\nabla\boldsymbol{\phi}] . \end{aligned}$$

Hierbei ist $\mathbf{f}$ die Kraftdichte auf den Planeten selbst.
Behauptung: Es ist dabei

$$p = \mathfrak{g}\varrho_0(\boldsymbol{\phi} - \boldsymbol{\phi}_0)$$

und $\boldsymbol{\phi}_0$ der konstante Wert von $\boldsymbol{\phi}$ auf $\partial\mathrm{B}_R(0)$, siehe auch Fig. 7.
Bemerkung: Newton hat den Fall $v \neq 0$ betrachtet, siehe IV.16.5.

Es handelt sich hier also um die Gravitationsgleichung (I2.10) zusammen mit der Massen- und Impulserhaltung (I4.3) im $\mathbb{R}^3$ mit $v = 0$ im Planeten $D^1 = \mathrm{B}_R(0)$ und dem Vakuuum D^2 mit verschwindendem Druck, d.h.

$$\begin{aligned} \operatorname{div}[\Pi^1 \mathcal{X}_{D^1}] &= [\mathbf{f}^1 \mathcal{X}_{D^1}] , \\ \Pi^1 = p\mathrm{Id} , \quad \mathbf{f}^1 &= \mathfrak{g}\varrho\nabla\boldsymbol{\phi} , \end{aligned}$$

Proof. Da die Dichte des Planeten konstant ist, schreibt sich die Impulserhaltung als

$$-\nabla[p\mathcal{X}_{\mathrm{B}_R(0)}] + [\mathcal{X}_{\mathrm{B}_R(0)}\nabla(\mathfrak{g}\varrho_0\boldsymbol{\phi})] = 0 ,$$

oder mit Testfunktionen $\zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$

$$\begin{aligned}
0 &= \left\langle \zeta\,,\, -\nabla[p\mathcal{X}_{\mathrm{B}_R(0)}] + [\mathcal{X}_{\mathrm{B}_R(0)}\nabla(\mathfrak{g}\varrho_0\boldsymbol{\phi})]\right\rangle \\
&= \left\langle \operatorname{div}\zeta\,,\,[p\mathcal{X}_{\mathrm{B}_R(0)}]\right\rangle + \left\langle \zeta\,,\,[\mathcal{X}_{\mathrm{B}_R(0)}\nabla(\mathfrak{g}\varrho_0\boldsymbol{\phi})]\right\rangle \\
&= \int_{\mathrm{B}_R(0)} \big(\operatorname{div}\zeta\cdot p + \zeta\bullet\nabla(\mathfrak{g}\varrho_0\boldsymbol{\phi})\big)\,\mathrm{dL}^n \\
&= \int_{\partial\mathrm{B}_R(0)} \zeta\bullet\nu_{\mathrm{B}_R(0)}\,p\,\mathrm{dH}^{n-1} + \int_{\mathrm{B}_R(0)} \zeta\bullet\big(-\nabla p + \nabla(\mathfrak{g}\varrho_0\boldsymbol{\phi})\big)\,\mathrm{dL}^n\,.
\end{aligned}$$

Dies ist äquivalent dazu, dass $p = 0$ auf $\partial\mathrm{B}_R(0)$ und

$$\nabla(p - \mathfrak{g}\varrho_0\boldsymbol{\phi}) = 0 \text{ in } \mathrm{B}_R(0)\,,$$

also $p - \mathfrak{g}\varrho_0\boldsymbol{\phi} = \text{const}$ in $\mathrm{B}_R(0)$. Da $\boldsymbol{\phi} = \boldsymbol{\phi}_0 = \text{const}$ auf $\partial\mathrm{B}_R(0)$ (dies folgt, weil hier $\boldsymbol{\phi}$ radialsymmetrisch ist), folgt die Behauptung. □

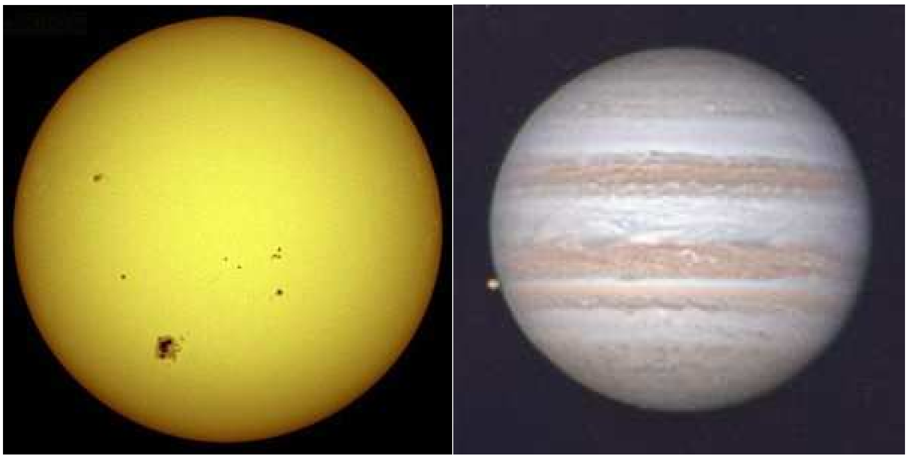


Fig. 21: *Left:* Sun at 7 Jun 1992 from [Wikipedia: Sonne]. *Right:* Jupiter and Io at 2 Jan 2013 from SuW 3|2013 (Photo by Thorsten Edelmann).

Referenzen: Dieses Ergebnis wird in [21, Gravitation] weiter behandelt, siehe die dort gemachten Literaturangaben. Falls der Planet rotiert, wird er durch die Rotation zu einem Spheroid. Dies wurde von Newton gezeigt und dessen Beweis wird in IV.16.5 präsentiert. Zum kompressiblen Fall siehe auch den Abschnitt IV.16.

Die Schwerkraft der Erde auf irgendeinen Körper wurde in (I3.11) definiert, es ist unter Vernachlässigung des Schwerepotential von Körpern ungleich der Erde die Massendichte mal der Beschleunigung a, wobei

$$a = \mathfrak{g}\nabla\boldsymbol{\phi}_{\mathrm{Earth}}\,.$$

Also ist nach 2.16 für Punkte x außerhalb des Erdinnneren (wenn der Erdmittelpunkt zu 0 normiert wird)

$$\nabla \phi_{\mathrm{Earth}}(x) = -\frac{M_{\mathrm{Earth}}}{4\pi} \frac{x}{|x|^3} \,. \tag{I4.6}$$

Damit erhält man für x auf dem Rande der Erde

$$\nabla \phi_{\mathrm{Earth}}(x) = -\frac{M_{\mathrm{Earth}}}{4\pi R_{\mathrm{Earth}}^2} e(x) \quad \text{mit } e(x) = \frac{x}{|x|} \,.$$

Hierbei ist die Erde mit einer Kugelapproximation mit Radius R_{Earth} beschrieben, was natürlich eine grobe Vereinfachung darstellt, zumal wir auch von einer Gleichverteilung der Masse ausgehen (siehe [Wikipedia: Erdmasse], [Wikipedia: Erdradius], [Wikipedia: Gewichtskraft], und IV.16.5, dort wird die Schwerkraft innerhalb eines rotierenden inkompressiblen Planeten betrachtet, was zu einer Abplattung führt). Also ist in Näherung

$$\begin{aligned} &M_{\mathrm{Earth}} = 5.9736 \cdot 10^{24} kg \,, \\ &R_{\mathrm{Earth}} = 6371.0\, km \,(\text{approximativ}) \,, \\ &G = 6.67384 \cdot 10^{-11} \frac{m^3}{kg\, s^2} \,, \end{aligned}$$

wobei G eine genauer Wert ist. Nun folgt für x auf der Erdoberfläche

$$\mathfrak{g} \nabla \phi_{\mathrm{Earth}}(x) = -\frac{\mathfrak{g} M_{\mathrm{Earth}}}{4\pi R_{\mathrm{Earth}}^2} e(x) = -\frac{G \cdot M_{\mathrm{Earth}}}{R_{\mathrm{Earth}}^2} e(x) = -g_0 e(x)$$

mit $g_0 = 9.825 \frac{m}{s^2}$, was dem Wert

$$g_{\mathrm{Earth}} = 9.81 \frac{m}{s^2} \,(= 9.80665 \frac{m}{s^2})$$

bis auf 0.153% genau nahe kommt. Hierbei muss berücksichtigt werden, dass alleine die Abplattung der Erde 0.335% betrifft (siehe die Abplattung in IV.16.5, da die Erde rotiert). Es sei bemerkt, dass die hier gemachten Vereinfachungen nicht bezüglich der physikalischen Gesetze Massen- und Impulsbilanz getroffen wurden, sondern in Hinsicht auf eine geometrische Vereinfachung zur einfacheren Berechnung der Lösung.

5 Change of coordinates

We describe a coordinate transform which can be used for the transformation of

- physical coordinates into e.g. polar coordinates (see 5.4)
- reference coordinates into physical coordinates (see section 6)
- observer coordinates in section II.1 and chapter VI

where the last two are motivated by physical reasons, and the first one only by mathematical reasons. We consider a spacetime domain $\mathcal{U} \subset \mathbb{R}^{1+n}$ with coordinates $y \in \mathcal{U}$ and we consider L^∞_{loc}-fluxes q_i^k, $i = 0, \dots, n$, and L^∞_{loc}-functions $\mathbf{r}^k$ of a so-called ***divergence system*** in the domain $\mathcal{U}$

$$\boxed{\sum_{i=0}^{n} \partial_{y_i}[q_i^k] = [\mathbf{r}^k] \quad \text{for } k = 1, \dots, N} \tag{I5.1}$$

in $\mathscr{D}'(\mathcal{U})$. We suppose a C^1-transformation Y of the coordinates $y^* \in \mathcal{U}^*$ into the given coordinates $y \in \mathcal{U}$ is given by

$$y = Y(y^*) \text{ with positive Jacobian.} \tag{I5.2}$$

Further, we suppose that a matrix

$$\boxed{Z = (Z_{kl})_{k,l=1,\dots,N} \text{ is invertible and in } C^1(\mathcal{U}^*; \mathbb{R}^{N\times N})} \tag{I5.3}$$

is given, and that quantities q_j^{*l} and $\mathbf{r}^{*l}$ are defined by q_j^l and $\mathbf{r}^l$ in the following way

$$\boxed{\begin{aligned} & q_i^k \circ Y = \frac{1}{J} \sum_{j,l} Y_{i}{}'_{j} Z_{kl} q_j^{*l} , \quad J := \det \mathrm{D}_{y^*} Y > 0 , \\ & \mathbf{r}^k \circ Y = \frac{1}{J} \left(\sum_{j,l} Z_{kl}{}'_{j} q_j^{*l} + \sum_{l} Z_{kl} \mathbf{r}^{*l} \right) , \\ & \text{for all } i = 0, \dots, n \text{ and } k = 1, \dots, N, \\ & \text{where } j \text{ runs from } 0 \text{ to } n, \text{ and } l \text{ from } 1 \text{ to } N. \end{aligned}} \tag{I5.4}$$

Note, that the transformation rule (I5.4) gives a bijektive correspondence between

$$\left(q_i^k\right)_{ik} \text{ and } \left(q_j^{*l}\right)_{jl}.$$

Note also, that the last transformation rule in (I5.4) involves derivatives of the matrix Z. The following is true.

5.1 Main invariance theorem. The system (I5.1) is invariant under the transformation of quantities described in (I5.4). Hence, if (I5.1) in $\mathscr{D}'(\mathcal{U})$ is satisfied and the quantities q_j^{*l} and $\mathbf{r}^{*l}$ for $j = 0, \ldots, n$ and $l = 1, \ldots, N$ fulfill (I5.4), then

$$\sum_{j=0}^{n} \partial_{y_j^*}[q_j^{*k}] = [\mathbf{r}^{*k}] \text{ for } k = 1, \ldots, N \tag{I5.5}$$

in $\mathscr{D}'(\mathcal{U}^*)$.

This follows from the following statement, which also shows, that the transformation of the differential equation is due to a transformation rule for test functions. This transformation rule involves a matrix Z, which is so far an arbitrary varying invertible matrix.

5.2 Property. If $\zeta \in C_0^1(\mathcal{U};\mathbb{R}^N)$ and $\zeta^* \in C_0^1(\mathcal{U}^*;\mathbb{R}^N)$ are test functions which correspond by

$$\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y \qquad (\text{that is } \zeta\circ Y = Z^{-\mathrm{T}}\,\zeta^*) \tag{I5.6}$$

then the transformation rule (I5.4) implies

$$\begin{aligned} &\sum_{l=1}^{N} \Big\langle \zeta_l^*\,,\, -\sum_{j=0}^{n} \partial_{y_j^*}[q_j^{*l}] + [\mathbf{r}^{*l}] \Big\rangle_{\mathscr{D}(\mathcal{U}^*)} \\ &= \sum_{k=1}^{N} \Big\langle \zeta_k\,,\, -\sum_{i=0}^{n} \partial_{y_i}[q_i^{k}] + [\mathbf{r}^{k}] \Big\rangle_{\mathscr{D}(\mathcal{U})} \end{aligned} \tag{I5.7}$$

The expressions concerning distributions are defined for C_0^1-functions, since q_j^{*l}, $\mathbf{r}^{*l}$ and q_i^k, $\mathbf{r}^k$ are locally bounded functions.

Proof. The test functions satisfy (I5.6), that is for $l = 1, \ldots, N$

$$\zeta_l^* = \textstyle\sum_{k=1}^N Z_{kl}\zeta_k\circ Y\,.$$

Taking the derivative with respect to y_j^*, $j = 0, \ldots, n$, we obtain

$$\begin{aligned} \partial_{y_j^*}\zeta_l^* &= \textstyle\sum_{k=1}^N \partial_{y_j^*}(Z_{kl}\zeta_k\circ Y) \\ &= \textstyle\sum_{k=1}^N\sum_{i=0}^n Z_{kl} Y_i{}'_{y_j^*}(\partial_{y_i}\zeta_k)\circ Y + \sum_{k=1}^N Z_{kl}{}'_{y_j^*}\zeta_k\circ Y\,. \end{aligned}$$

With this we obtain

$$\begin{aligned} &\sum_l \Big\langle \zeta_l^*\,,\, -\sum_j \partial_{y_j^*}[q_j^{*l}] + [\mathbf{r}^{*l}] \Big\rangle \\ &= \sum_{l,j} \Big\langle \partial_{y_j^*}\zeta_l^*\,,\, [q_j^{*l}] \Big\rangle + \sum_l \Big\langle \zeta_l^*\,,\, [\mathbf{r}^{*l}] \Big\rangle \\ &= \sum_{k,l,j} \Big\langle \sum_i Z_{kl} Y_i{}'_{y_j^*}(\partial_{y_i}\zeta_k)\circ Y + Z_{kl}{}'_{y_j^*}\zeta_k\circ Y\,,\, [q_j^{*l}] \Big\rangle \\ &\quad + \sum_{k,l} \Big\langle Z_{kl}\zeta_k\circ Y\,,\, [\mathbf{r}^{*l}] \Big\rangle \end{aligned}$$

$$= \sum_{k,i} \Big\langle (\partial_{y_i}\zeta_k)\circ Y \,,\, \underbrace{\sum_{lj} Z_{kl} Y_i{}'_{y^*_j}[q^{*l}_j]}_{=[J\,q^k_i\circ Y]} \Big\rangle$$

$$+ \sum_k \Big\langle \zeta_k\circ Y \,,\, \underbrace{\sum_{lj} Z_{kl}{}'_{y^*_j}[q^{*l}_j] + \sum_l Z_{kl}[\mathbf{r}^{*l}]}_{=[J\,\mathbf{r}^k\circ Y]} \Big\rangle$$

using (I5.4), and this is

$$\begin{aligned}
&= \sum_{k,i} \Big\langle (\partial_{y_i}\zeta_k)\circ Y \,,\, [J\,q^k_i\circ Y] \Big\rangle + \sum_k \Big\langle \zeta_k\circ Y \,,\, [J\,\mathbf{r}^k\circ Y] \Big\rangle \\
&= \sum_{k,i} \int_{\mathcal{U}^*} (\partial_{y_i}\zeta_k)\circ Y\; q^k_i\circ Y\; J\,\mathrm{dL}^{n+1} + \sum_k \int_{\mathcal{U}^*} \zeta_k\circ Y\; \mathbf{r}^k\circ Y\; J\,\mathrm{dL}^{n+1} \\
&= \sum_{k,i} \int_{\mathcal{U}} (\partial_{y_i}\zeta_k)\, q^k_i\,\mathrm{dL}^{n+1} + \sum_k \int_{\mathcal{U}} \zeta_k\,\mathbf{r}^k\,\mathrm{dL}^{n+1} \\
&= \sum_{k,i} \Big\langle \partial_{y_i}\zeta_k \,,\, [q^k_i] \Big\rangle + \sum_k \Big\langle \zeta_k \,,\, [\mathbf{r}^k] \Big\rangle \\
&= \sum_k \Big\langle \zeta_k \,,\, -\sum_i \partial_{y_i}[q^k_i] + [\mathbf{r}^k] \Big\rangle \,.
\end{aligned}$$

We have seen, that the factor J enters because of the transformation of an L^{n+1}-integral. □

References: This theorem one finds in [19, §5 Objectivity of Differential Equations].

We summarize and obtain in the general case

Invariance of the divergence system with respect to Z:

$$\sum_{i=0}^{n} \partial_{y_i}[q^k_i] = [\mathbf{r}^k] \text{ for } k = 1,\dots,N$$

related to the transformation $\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y$

if the following ***transformation rules*** are satisfied:

$$q^k_i\circ Y = \frac{1}{J}\sum_{j=0}^{n}\sum_{l=1}^{N} Y_i{}'_j Z_{kl} q^{*l}_j \,, \quad J := \det \mathrm{D}_{y^*}Y > 0$$

$$\mathbf{r}^k\circ Y = \frac{1}{J}\Big(\sum_{j=0}^{n}\sum_{l=1}^{N} Z_{kl}{}'_j q^{*l}_j + \sum_{l=1}^{N} Z_{kl}\mathbf{r}^{*l}\Big)$$

(I5.8)

5.3 Hinweis. Zur Transformationsformel sei Folgendes gesagt. Da nun im regulären Fall die Differentialgleichung in (I5.1)

$$\sum_{i=0}^{n} \partial_{y_i} q_i^k = \mathbf{r}^k \text{ für } k = 1, \ldots, N$$

lautet, ergibt sich die natürliche Frage, ob die Transformationsformel (I5.4) für $\mathbf{r}^k$ nicht durch diese Differentialgleichung und die Transformationsformel (I5.4) für die q_i^k direkt hergeleitet werden kann. Die Antwort ist: Natürlich ist dies so, siehe den Beweis 1. Es wird dabei auch klar, warum der Beweis mit Testfunktionen, siehe auch den Beweis 2, dem anderen Beweis vorgezogen wurde.

Proof 1. Die Transformationsformel für die q_i^k in (I5.4) war

$$q_i^k \circ Y = \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Y_{i\,'\bar{j}} Z_{kl} q_{\bar{j}}^{*l} .$$

Aus jeder Transformationsformel kann eine Formel für die Ableitung hergeleitet werden, man braucht die Formel nur zu differenzieren. Also ist für $j = 0, \ldots, n$

$$\sum_{\bar{i}=0}^{n} q_{i\,'\bar{i}}^k \circ Y Y_{\bar{i}\,'j} = (q_i^k \circ Y)_{'j} = \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl\,'j} q_{\bar{j}}^{*l} Y_{i\,'\bar{j}}$$
$$+ \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} q_{\bar{j}}^{*l} \Big(\frac{1}{J} Y_{i\,'\bar{j}} \Big)_{'j} + \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} Y_{i\,'\bar{j}} q_{\bar{j}\,'j}^{*l} .$$

Wir schreiben dies in Matrixform

$$\mathrm{D}q^k \circ Y \mathrm{D}Y = \frac{1}{J} \sum_{j,\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl\,'j} q_{\bar{j}}^{*l} (\mathrm{D}Y \mathbf{e}_{\bar{j}}) \otimes \mathbf{e}_j$$
$$+ \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} q_{\bar{j}}^{*l} \mathrm{D}\Big(\frac{Y_{'\bar{j}}}{J} \Big) + \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} \mathrm{D}Y \mathrm{D}q^{*l} ,$$

so dass also

$$\mathrm{D}q^k \circ Y = \frac{1}{J} \sum_{j,\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl\,'j} q_{\bar{j}}^{*l} \mathrm{D}Y \mathbf{e}_{\bar{j}} \otimes \mathbf{e}_j \, \mathrm{D}Y^{-1}$$
$$+ \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} q_{\bar{j}}^{*l} \mathrm{D}\Big(\frac{Y_{'\bar{j}}}{J} \Big) \mathrm{D}Y^{-1} + \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} \mathrm{D}Y \mathrm{D}q^{*l} \, \mathrm{D}Y^{-1} .$$

Dies ist also die Transformationsformel für die Ableitung $\mathrm{D}q$. Uns interessiert die Spur dieser Identität

$$\mathrm{trace}_{\,y} \mathrm{D}q^k = \sum_{i=0}^{n} q_{i\,'i}^k .$$

Hier meint "$\mathrm{trace}_{\,y}$" die Spur in allen y_i-Koordinaten, also für $i = 0, \ldots, n$ (in der klassischen Physik ist $y = (t, x)$). Dies ergibt

$$\mathrm{trace}_{\,y} \mathrm{D}q^k \circ Y = \frac{1}{J} \sum_{j,\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl\,'j} q_{\bar{j}}^{*l} \mathrm{trace}_{\,y} \big(\mathrm{D}Y \mathbf{e}_{\bar{j}} \otimes \mathbf{e}_j \, \mathrm{D}Y^{-1} \big)$$
$$+ \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} q_{\bar{j}}^{*l} \mathrm{trace}_{\,y} \Big(\mathrm{D}\Big(\frac{Y_{'\bar{j}}}{J} \Big) \mathrm{D}Y^{-1} \Big) + \frac{1}{J} \sum_{\bar{j}=0}^{n} \sum_{l=1}^{N} Z_{kl} \mathrm{trace}_{\,y} \big(\mathrm{D}Y \mathrm{D}q^{*l} \, \mathrm{D}Y^{-1} \big) .$$

Nun gilt:

(1) Für jede Matrix M ist $\mathrm{trace}_{\,y} \big(\mathrm{D}Y M \, \mathrm{D}Y^{-1} \big) = \mathrm{trace}_{\,y} M$.

(2) Für $j = 0, \ldots, n$ ist $\mathrm{trace}_{\,y} \big(\mathrm{D}\big(\frac{Y_{'j}}{J} \big) \mathrm{D}Y^{-1} \big) = 0$.

Somit reduziert sich die Formel zu

$$\mathrm{trace}_{\,y}\mathrm{D}q^k\circ Y = \frac{1}{J}\sum_{\bar{j}=0}^{n}\sum_{l=1}^{N} Z_{kl}{}_{,\bar{j}}q^{*l}_{\bar{j}} + \frac{1}{J}\sum_{\bar{j}=0}^{n}\sum_{l=1}^{N} Z_{kl}\mathrm{trace}_{\,y}\mathrm{D}q^{*l}\,,$$

was die Formel in (I5.4) für $\mathbf{r}^k = \mathrm{trace}_{\,y}\mathrm{D}q^k$ ist. □

Proof der Identität (1). Es ist

$$\mathrm{trace}_{\,y}\big(\mathrm{D}Y M\,\mathrm{D}Y^{-1}\big) = \sum_{i=0}^{n}\mathbf{e}_i\bullet(\mathrm{D}Y M\,\mathrm{D}Y^{-1}\,\mathbf{e}_i)$$
$$= \sum_{k,l,i=0}^{n} M_{kl}\mathbf{e}_i\bullet(\mathrm{D}Y\mathbf{e}_k\otimes\mathbf{e}_l\,\mathrm{D}Y^{-1})\mathbf{e}_i = \sum_{k,l,i=0}^{n} M_{kl}\mathbf{e}_i\bullet(\mathrm{D}Y\mathbf{e}_k)\cdot\mathbf{e}_l\bullet(\mathrm{D}Y^{-1}\,\mathbf{e}_i)$$
$$= \sum_{k,l,i=0}^{n} M_{kl}\mathbf{e}_i\bullet(\mathrm{D}Y\mathbf{e}_k)\cdot(\mathrm{D}Y^{-\mathrm{T}}\,\mathbf{e}_l)\bullet\mathbf{e}_i = \sum_{k,l=0}^{n} M_{kl}(\mathrm{D}Y\mathbf{e}_k)\bullet(\mathrm{D}Y^{-\mathrm{T}}\,\mathbf{e}_l)$$
$$= \sum_{k,l=0}^{n} M_{kl}(\mathrm{D}Y^{-1}\,\mathrm{D}Y\mathbf{e}_k)\bullet\mathbf{e}_l = \sum_{k,l=0}^{n} M_{kl}\mathbf{e}_k\bullet\mathbf{e}_l = \sum_{k=0}^{n} M_{kk} = \sum_{k=0}^{n}\mathbf{e}_k\bullet(M\mathbf{e}_k)\,,$$

which is $\mathrm{trace}_{\,y}M$. □

Proof der Identität (2). Wir setzen $A = \mathrm{D}Y$ also $J = \det A$. Die Adjunkte von A erfüllt

$$A\,\mathrm{adj}\,(A) = (\det A)\,\mathrm{Id}$$

und die Jacobi'sche Formel ist

$$\partial_i\det A = \mathrm{trace}\,(\mathrm{adj}\,(A)\partial_i A)\,.$$

Damit folgt

$$\mathrm{trace}\,\Big(\mathrm{D}\Big(\frac{Y_{,i}}{J}\Big)\mathrm{adj}\,(\mathrm{D}Y)\Big) = \sum_{kl}\Big(\frac{Y_{k\,,i}}{J}\Big)_{,l}(\mathrm{adj}\,(\mathrm{D}Y))_{lk}$$
$$= \frac{1}{J^2}\sum_{kl}\big(JA_{kl\,,i}\mathrm{adj}\,(\mathrm{D}Y)_{lk} - J_{,l}A_{k\,,i}\mathrm{adj}\,(\mathrm{D}Y)_{lk}\big) = \frac{1}{J^2}\big(JJ_{,i} - \sum_{l}J_{,l}J\delta_{li}\big) = 0\,,$$

was zu beweisen war. □

Proof 2. Dies ist eine andere Version des Beweises mit Testfunktionen $\zeta\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^N)$. Es ist

$$\sum_{i=0}^{n} J(\zeta_k q^k_{i\,,i})\circ Y = \sum_{i=0}^{n} J(\partial_i(\zeta_k q^k_i))\circ Y - \sum_{i=0}^{n} J(\partial_i\zeta_k\,q^k_i)\circ Y\,.$$

Unter Benutzung der Formel (I5.4) für q^k_i ist der zweite Summand gleich

$$\sum_{i=0}^{n} J(\partial_i\zeta_k\,q^k_i)\circ Y = \sum_{\bar{k}=1}^{N}\sum_{i,\bar{j}=0}^{n}\partial_i\zeta_k\circ Y\,Y_{i\,,\bar{j}}Z_{k\bar{k}}q^{*\bar{k}}_{\bar{j}} = \sum_{\bar{k}=1}^{N}\sum_{\bar{j}=0}^{n}\partial_{\bar{j}}(\zeta_k\circ Y)\,Z_{k\bar{k}}q^{*\bar{k}}_{\bar{j}}$$
$$= \sum_{\bar{k}=1}^{N}\sum_{\bar{j}=0}^{n}\partial_{\bar{j}}(\zeta_k\circ Y\,Z_{k\bar{k}}q^{*\bar{k}}_{\bar{j}}) - \sum_{\bar{k}=1}^{N}\sum_{\bar{j}=0}^{n}\zeta_k\circ Y\partial_{\bar{j}}(Z_{k\bar{k}}q^{*\bar{k}}_{\bar{j}})\,.$$

Indem wir die Testfunktionen

$$\zeta^*_{\bar{k}} := \sum_{k=1}^{N} Z_{k\bar{k}}\zeta_k\circ Y$$

definieren, erhalten wir insgesamt

$$\sum_{k=1}^{N}\sum_{i=0}^{n} J(\zeta_k q^k_{i\,,i})\circ Y = \sum_{k,\bar{k}=1}^{N}\sum_{\bar{j}=0}^{n}\zeta_k\circ Y\partial_{\bar{j}}(Z_{k\bar{k}}q^{*\bar{k}}_{\bar{j}})$$
$$+\sum_{k=1}^{N}\sum_{i=0}^{n} J(\partial_i(\zeta_k q^k_i))\circ Y + \sum_{\bar{k}=1}^{N}\sum_{\bar{j}=0}^{n}\partial_{\bar{j}}(\zeta^*_{\bar{k}}q^{*\bar{k}}_{\bar{j}})\,.$$

Integrating we see that the last two terms vanish, and we end up with the equations

$$\sum_{i=0}^{n} J q^k_{i}{}'_i \circ Y = \sum_{\bar{k}=1}^{N} \sum_{\bar{j}=0}^{n} \partial_{\bar{j}} (Z_{k\bar{k}} q^{*\bar{k}}_{\bar{j}}) = \sum_{\bar{k}=1}^{N} \sum_{\bar{j}=0}^{n} (Z_{k\bar{k}}{}'_{\bar{j}} q^{*\bar{k}}_{\bar{j}} + Z_{k\bar{k}} q^{*\bar{k}}_{\bar{j}}{}'_{\bar{j}}) \,,$$

was die zu zeigende Identität in (I5.4) für $\mathbf{r}^k = \sum_{i=0}^{n} q^k_{i}{}'_i$ ist. □

In this chapter we are involved with classical physics where the coordinates are $y = (t, x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$. Here the situation is special in the sense, that only x is transformed, that is,

$$\begin{bmatrix} t \\ x \end{bmatrix} = y = Y(y^*) = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} t^* + \mathrm{a} \\ X(t^*, x^*) \end{bmatrix} \tag{I5.9}$$

with a constant $\mathrm{a} \in \mathbb{R}$, where the coordinates $y^* = (t^*, x^*) \in \mathcal{U}^*$ are transfored into the coordinates $y = (t, x) \in \mathcal{U}$ with a positive Jacobian matrix

$$J = \det \mathrm{D}_{y^*} Y = \det \mathrm{D}_{x^*} X > 0\,.$$

We call the functions q^k_0 now $u^k := q^k_0$ and the entire system is given by L^∞_{loc}-solutions u^k, q^k_i, $i = 1, \ldots, n$, and L^∞_{loc}-functions $\mathbf{r}^k$ of the following conservation laws

$$\partial_t [u^k] + \sum_{i=1}^{n} \partial_{x_i} [q^k_i] = [\mathbf{r}^k] \text{ for } k = 1, \ldots, N \text{ in } \mathscr{D}'(\mathcal{U}). \tag{I5.10}$$

Wir werden dies in diesem Kapitel anwenden für

- $N = 1$, $Z = 1$ auf Zylinderkoordinaten in 5.4,
- $N = n + 1$, $Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & \mathrm{D}X \end{bmatrix}$ auf die rotierende Erde in 5.5,
- $N = n + 1$, $Z = \mathrm{Id}$ auf deformierbare Körper in Abschnitt 6,

und in Abschnitt II.3 für

- $N = 1$ auf die Massenbilanz mit $Z = 1$,
- $N = n + 1$ auf die Masse-Impuls-Bilanz mit

$$Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & \mathrm{D}X \end{bmatrix},$$

- $N = n + 2$ auf die Masse-Impuls-Energie-Bilanz mit

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & \mathrm{D}X & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathrm{T}} Q & 1 \end{bmatrix}.$$

The matrix Z in (I5.3) is an arbitrary invertible matrix. Either this matrix is given, or one chooses Z so that the equation (I5.7) for $(u^*, q^*, \mathbf{r}^*)$ is the wanted equation in (t^*, x^*). In any case the transformation (I5.4) becomes, where now i and j run only from 1 to n,

Invariance of the divergence system with respect to Z:

$$\partial_t[u^k] + \sum_{i=1}^{n} \partial_{x_i}[q_i^k] = [\mathbf{r}^k] \text{ for } k = 1, \dots, N$$

related to the transformation $\zeta^* = Z^{\mathrm{T}} \zeta \circ Y$

with Y as in (I5.9)

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if the following ***transformation rules*** are satisfied:

$$u^k \circ Y = \frac{1}{J} \sum_l Z_{kl} u^{*l}, \quad J := \det \mathrm{D}_{x^*} X > 0$$

$$q_i^k \circ Y = \frac{1}{J} \Big(\sum_l \dot{X}_i Z_{kl} u^{*l} + \sum_{j,l} X_{i'j} Z_{kl} q_j^{*l} \Big)$$

$$\mathbf{r}^k \circ Y = \frac{1}{J} \Big(\sum_l \dot{Z}_{kl} u^{*l} + \sum_{j,l} Z_{kl'j} q_j^{*l} + \sum_l Z_{kl} \mathbf{r}^{*l} \Big)$$

for all $i = 1, \dots, n$ and $k = 1, \dots, N$,

where j now runs from 1 to n, and l from 1 to N (I5.11)

This follows from (I5.8), since in this classical sense for $i, j = 1, \dots, n$

$$Y_{0'0} = 1, \quad Y_{0'j} = 0, \quad Y_{i'0} = \dot{X}_i, \quad Y_{i'j} = X_{i'j}.$$

We mention that if we write the terms with u^k, $k = 1, \dots, N$, in vectorial form $u = (u^1, \dots, u^N)$, the transformation rule for u is

$$u \circ Y = \tfrac{1}{J} Z u^*,$$

that is, in classical physics the matrix Z can be considered as the transformation rule for u. In this section we give two examples for the general transformation rule. As first example we take the cylindrical coordinates from 1.10.

5.4 Cylindrical coordinates. We consider a transformation of the coordinates (r, θ, z) in the physical coordinates $x = (x_1, x_2, x_3)$ given by

$$\begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \tau(t, r, \theta, z) := \begin{bmatrix} t \\ r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}.$$

The time variable is unchanged. We consider a solution of the conservation law $\partial_t u + \mathrm{div}_x q = \mathbf{r}$ which we write as

$$\partial_t u + \partial_{x_1} q_1 + \partial_{x_2} q_2 + \partial_{x_3} q_3 = \mathbf{r}\,.$$

If we define the transformed quantities as in 1.10 then

$$\partial_t(r \cdot \underline{u}) + \partial_r(r \cdot \underline{q_r}) + \partial_\theta \underline{q_\theta} + \partial_z(r \cdot \underline{q_z}) = r \cdot \underline{\mathbf{r}}\,,$$

that is, we get the differential equation in (I1.18).

Proof. In (I5.9) we let $Y = \tau$. Then it follows in the new coordinates that

$$\partial_t u^* + \mathrm{div}_{(r,\theta,z)} q^* = \mathbf{r}^*\,,$$

if $(u^*, q^*, \mathbf{r}^*)$ are defined as in (I5.11) with $N = 1$ and $Z = 1$, that is,

$$\begin{aligned} u{\circ}Y &= \frac{1}{J} u^*\,, \quad J = \det \mathrm{D}X = r,\\ q{\circ}Y &= \frac{1}{J} \mathrm{D}X q^*\\ \mathbf{r}{\circ}Y &= \frac{1}{J}\mathbf{r}^*\,. \end{aligned}$$

Here we have used that X is independent of t. We use for the transformation of the flux

$$\mathrm{D}X = [\, X'_r \quad X'_\theta \quad X'_z\,] = \begin{bmatrix} \cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{e}_r = \widetilde{\mathbf{e}}_r(r,\theta,z), \quad \mathbf{e}_\theta = \widetilde{\mathbf{e}}_\theta(r,\theta,z), \quad \mathbf{e}_z = \widetilde{\mathbf{e}}_z(r,\theta,z),$$

$$\widetilde{\mathbf{e}}_r(r,\theta,z) = \begin{bmatrix} \cos\theta\\ \sin\theta\\ 0 \end{bmatrix}, \quad \widetilde{\mathbf{e}}_\theta(r,\theta,z) = \begin{bmatrix} -\sin\theta\\ \cos\theta\\ 0 \end{bmatrix}, \quad \widetilde{\mathbf{e}}_z(r,\theta,z) = \begin{bmatrix} 0\\ 0\\ 1 \end{bmatrix}.$$

Further we give two versions.
First Version: From the above equation we obtain

$$q^* = J\,(DX)^{-1}\, q{\circ}Y\,.$$

In 1.10 we used the representation (I1.16) $q = q_r\mathbf{e}_r + q_\theta\mathbf{e}_\theta + q_z\mathbf{e}_z$ which implies $q{\circ}Y = q_r{\circ}Y\widetilde{\mathbf{e}}_r + q_\theta{\circ}Y\widetilde{\mathbf{e}}_\theta + q_z{\circ}Y\widetilde{\mathbf{e}}_z$ and hence

$$\begin{bmatrix} q_1^*\\ q_2^*\\ q_3^* \end{bmatrix} = q^* = J\,(DX)^{-1}\, q{\circ}Y = J\,(DX)^{-1}\,[\widetilde{\mathbf{e}}_r\;\widetilde{\mathbf{e}}_\theta\;\widetilde{\mathbf{e}}_z] \begin{bmatrix} q_r{\circ}Y\\ q_\theta{\circ}Y\\ q_z{\circ}Y \end{bmatrix}$$

$$= r \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} [\widetilde{\mathbf{e}}_r \; \widetilde{\mathbf{e}}_\theta \; \widetilde{\mathbf{e}}_z] \begin{bmatrix} q_r\circ Y \\ q_\theta\circ Y \\ q_z\circ Y \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} q_r\circ Y \\ q_\theta\circ Y \\ q_z\circ Y \end{bmatrix} = \begin{bmatrix} r\,q_r\circ Y \\ q_\theta\circ Y \\ r\,q_z\circ Y \end{bmatrix}$$

or

$$q_1^* = r\cdot q_r\circ Y\,, \quad q_2^* = q_\theta\circ Y\,, \quad q_3^* = r\cdot q_z\circ Y\,.$$

Second Version: With the representation of the flux $q = q_r\widehat{\mathbf{e}}_r + q_\theta\widehat{\mathbf{e}}_\theta + q_z\widehat{\mathbf{e}}_z$ defined in (I1.16), where $(\widehat{\mathbf{e}}_r, \widehat{\mathbf{e}}_\theta, \widehat{\mathbf{e}}_z) = (\widetilde{\mathbf{e}}_r, \widetilde{\mathbf{e}}_\theta, \widetilde{\mathbf{e}}_z)\circ Y^{-1}$, follows

$$q_r\circ Y\widetilde{\mathbf{e}}_r + q_\theta\circ Y\widetilde{\mathbf{e}}_\theta + q_z\circ Y\widetilde{\mathbf{e}}_z = q\circ Y$$

$$= \frac{1}{r}\mathrm{D}Xq^* = \frac{1}{r}\mathrm{D}X \begin{bmatrix} q_1^* \\ q_2^* \\ q_3^* \end{bmatrix} = \frac{1}{r}\big(q_1^*\widetilde{\mathbf{e}}_r + rq_2^*\widetilde{\mathbf{e}}_\theta + q_3^*\widetilde{\mathbf{e}}_z\big)\,.$$

This gives

$$q_1^* = r\cdot q_r\circ Y\,, \quad q_2^* = q_\theta\circ Y\,, \quad q_3^* = r\cdot q_z\circ Y\,.$$

In both versions we obtain

$$\begin{aligned} &\mathrm{div}_{(r,\theta,z)}q^* = \partial_r q_1^* + \partial_\theta q_2^* + \partial_z q_3^* \\ &= \partial_r(r\cdot q_r\circ Y) + \partial_\theta(q_\theta\circ Y) + \partial_z(r\cdot q_z\circ Y)\,. \end{aligned}$$

Since $u^* = r\cdot u\circ Y$ and $\mathbf{r}^* = r\cdot\mathbf{r}\circ Y$, the assertion follows under usage of the underlines quantities in 1.10. □

Analogously one can treat polar coordinates in $\mathbb{R}^n$, see IV.8.3 for the case $n = 3$. Our second example has a physical background, the transformation is an "observer transformation" (definition in section II.1).

5.5 Air flow on the Earth. We model the Earth as a ball and consider the outside of the Earth

$$D := \{x^* \in \mathbb{R}^3\,;\ |x^*| > R\}.$$

In $\mathbb{R}\times D$ we consider the conservation of mass and momentum

$$\begin{aligned} &\partial_{t^*}\varrho^* + \mathrm{div}_{x^*}(\varrho^* v^*) = 0\,, \\ &\partial_{t^*}(\varrho^* v^*) + \mathrm{div}_{x^*}(\varrho^* v^* v^{*\mathrm{T}} + \Pi^*) = \mathbf{f}^*\,. \end{aligned}$$

The pressure tensor can have the representation $\Pi^* = p^*\mathrm{Id} - S^*$ with p^* and S^* as in (I3.32). The force term $\mathbf{f}^* = \mathbf{f}_0^*$ is the gravitational force of the Earth. Other forces, like the gravitational force of the Sun, are neglected here. The coordinates are based on the fact that we look at the Earth from outside, therefore we have nontrivial boundary conditions

$$v^*(t^*,x^*) = v_0^*(x^*) \text{ for } x^* \in \partial D\,,$$
$$v_0^*(x^*) = \omega\cdot\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1^*\\ x_2^*\\ x_3^*\end{bmatrix} = \omega\cdot\begin{bmatrix}-x_2^*\\ x_1^*\\ 0\end{bmatrix},$$

where ω is the angular velocity of the Earth. *Remark:* We do not consider boundary conditions on v^* on the high atmosphere, this is another problem.

We introduce now x as the coordinates of the Earth, that is, these coordinates rotate with the Earth,

$$t = t^*,\quad x = Q(t^*)x^*,$$
$$Q(t^*) = \begin{bmatrix}\cos(\omega t^*) & \sin(\omega t^*) & 0\\ -\sin(\omega t^*) & \cos(\omega t^*) & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Assertion: Define ϱ, v, Π by

$$\varrho(t,x) = \varrho^*(t^*,x^*)\,,$$
$$v(t,x) = \dot{Q}(t^*)x^* + Q(t^*)v^*(t^*,x^*)\,,$$
$$\Pi(t,x) = Q(t^*)\Pi^*(t^*,x^*)\,Q(t^*)^{\mathrm{T}}\,,$$

then the boundary condition becomes

$$v(t,x) = 0 \text{ for } x \in \partial D,$$

and the same set of conservation laws

$$\partial_t\varrho + \mathrm{div}_x(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \mathrm{div}_x(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,$$

in $\mathbb{R}\times D$ is satisfied, where now

$$\mathbf{f}(t,x) = \varrho(t,x)\cdot(\underbrace{\omega^2 Ix}_{\textit{Centrifugal force}} + \underbrace{2\omega Av(t,x)}_{\textit{Coriolis force}}) + \mathbf{f}_0(t,x)\,,$$

$$\text{where } I = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ A = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ \mathbf{f}_0 := (Q\mathbf{f}_0^*)\circ Y^{-1}\,.$$

Hence $\mathbf{f}$ contains additional terms. These terms are the centrifugal force and the Coriolis force.

The “fictitious forces”, by which we mean the just described centrifugal forces and Coriolis forces, have an important impact in daily life. They are present for every on Earth living observer, although they are neglected most times. We mention the “Foucault pendulum” [131], see also [21, Foucault Pendulum], the experiment was carried out 1851 in Paris, which proved the existence of Coriolis force on Earth. Mathematically the connection between coordinates (t, x) and (t^*, x^*) is given by

$$(t, x) = Y(t, x^*) = (t, Q(t)x^*)\,,$$

or

$$\begin{aligned} t &= t^*\,, \\ x &= X(t^*, x^*) = Q(t^*)x^*\,. \end{aligned} \tag{I5.12}$$

This is a transformation as in (I5.9). The term $\mathbf{f}_0 := Q\mathbf{f}_0^*\circ Y^{-1}$ is, for example, the gravitational force. Since it depends only on the distance $|x|$ from the center 0 of the Earth and $|x| = |x^*|$, its modulus is unchanged.

Proof of the differential equation. The differential equations in the variables $(\varrho^*, v^*, \Pi^*, \mathbf{f}^*)$ are

$$\begin{aligned} &\partial_{t^*}\varrho^* + \mathrm{div}_{x^*}(\varrho^* v^*) = 0\,, \\ &\partial_{t^*}(\varrho^* v^*) + \mathrm{div}_{x^*}(\varrho^* v^* v^{*\mathrm{T}} + \Pi^*) = \mathbf{f}^*\,, \end{aligned}$$

which also can be written as

$$\partial_{t^*}\begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \mathrm{div}_{x^*}\begin{bmatrix} \varrho^* v^* \\ \varrho^* v^* v^{*\mathrm{T}} + \Pi^* \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f}^* \end{bmatrix}\,,$$

or

$$\partial_{t^*}\underbrace{\begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix}}_{=:\, u^*} + \sum_{j\geq 1}\partial_{x_j^*}\underbrace{\begin{bmatrix} \varrho^* v_j^* \\ \varrho^* v_j^* v^* + (\Pi_{lj}^*)_l \end{bmatrix}}_{=:\, q_j^*} = \underbrace{\begin{bmatrix} 0 \\ \mathbf{f}^* \end{bmatrix}}_{=:\, \mathbf{r}^*}\,,$$

that is as equation as in (I5.1). We know from theorem 5.1 that equation

$$\partial_t u + \sum_{i\geq 1}\partial_{x_i} q_i = \mathbf{r} \tag{I5.13}$$

follows, if the quantities $(u, q_1, \dots, q_n, \mathbf{r})$ are determined by (I5.11), where here we set

$$Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} \quad \text{for mass-momentum equation.}$$

We obtain, since $J = \det \mathrm{D}X = 1$,

$$
\begin{aligned}
u\circ Y = \; & Zu^* = \begin{bmatrix} 1 & 0 \\ \dot X & Q \end{bmatrix} \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} = \begin{bmatrix} \varrho^* \\ \Big(\varrho^*(\dot X_k + \sum_j Q_{kj} v_j^*) \Big)_{k\geq 1} \end{bmatrix} , \\
q_i\circ Y = \; & \dot X_i Z u^* + \sum_{j\geq 1} X_i{}'_j Z q_j^* \\
= \; & \begin{bmatrix} 1 & 0 \\ \dot X & Q \end{bmatrix} \left(\dot X_i \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \sum_{j\geq 1} Q_{ij} \begin{bmatrix} \varrho^* v_j^* \\ \varrho^* v_j^* v^* + \big(\Pi^*_{lj}\big)_l \end{bmatrix} \right) \\
= \; & \begin{bmatrix} \varrho^*(\dot X_i + \sum_j Q_{ij} v_j^*) \\ \varrho^*(\dot X_i + \sum_j Q_{ij} v_j^*)(\dot X + Q v^*) + \sum_{j\geq 1} Q_{ij} Q \,\big(\Pi^*_{lj}\big)_l \end{bmatrix} .
\end{aligned}
$$

Therefore defining ϱ and v by

$$
\partial_t \underbrace{\begin{bmatrix} \varrho \\ \varrho v \end{bmatrix}}_{:= u} + \textstyle\sum_{i\geq 1} \partial_{x_i} \underbrace{\begin{bmatrix} \varrho v_i \\ \varrho v_i v + (\Pi_{li})_l \end{bmatrix}}_{:= q_i} = \underbrace{\begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix}}_{:= \mathbf{r}} , \tag{I5.14}
$$

which is possible since $q_i^0 = u_i$ for $i \geq 1$ and the first component of $\mathbf{r}$ is 0 as we shall see from (I5.11)

$$
\begin{aligned}
\mathbf{r}\circ Y = \; & \dot Z u^* + \sum_{j\geq 1} Z'_j q_j^* + Z\mathbf{r}^* \\
= \; & \begin{bmatrix} 0 & 0 \\ \ddot X & \dot Q \end{bmatrix} \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \sum_j \begin{bmatrix} 0 & 0 \\ \dot X'_j & 0 \end{bmatrix} \begin{bmatrix} \varrho^* v_j^* \\ \varrho^* v_j^* v^* + \big(\Pi^*_{lj}\big)_l \end{bmatrix} \\
& + \begin{bmatrix} 1 & 0 \\ \dot X & Q \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{f}^* \end{bmatrix} \\
= \; & \begin{bmatrix} 0 \\ \varrho^*(\ddot X + 2\dot Q v^*) + Q\mathbf{f}^* \end{bmatrix} ,
\end{aligned}
$$

since $\dot X'_j = \big(\dot Q_{kj}\big)_k$ is a consequence of (I5.12). Hence the force $\mathbf{f}$ is defined by (see also (II3.18))

$$
\mathbf{f}\circ Y = \varrho^*(\ddot X + 2\dot Q v^*) + Q\mathbf{f}^* . \tag{I5.15}
$$

We see that $\mathbf{f}$ is nonzero, even if $\mathbf{f}^*$ is zero. Then the equation (I5.14) is

$$
\partial_t \begin{bmatrix} \varrho \\ \varrho v \end{bmatrix} + \mathrm{div}_x \begin{bmatrix} \varrho v \\ \varrho v \, v^{\mathrm{T}} + \Pi \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{f} \end{bmatrix} ,
$$

where $\mathbf{f}$ is given by (I5.15). In order to compute $\mathbf{f}$ we define $\mathbf{f}_0$ by

$$
\mathbf{f}_0 := (Q\mathbf{f}^*)\circ Y^{-1} , \quad \mathbf{f}^* = \mathbf{f}_0^*
$$

and then

$$\mathbf{f} = (\varrho^*(\ddot{X} + 2\dot{Q}v^*))\circ Y^{-1} + \mathbf{f}_0 \,.$$

We want to write the first term on the right-hand side in terms of ϱ and v. If we use the formulas in (I5.14), we get

$$\varrho^* = \varrho\circ Y \,, \quad v^* = Q^{\mathrm{T}}\, v\circ Y - Q^{\mathrm{T}}\, \dot{X} \,,$$

and therefore

$$\varrho^*(\ddot{X} + 2\dot{Q}v^*) \;\; = \varrho\circ Y\big((\ddot{X} - 2\dot{Q}\,Q^{\mathrm{T}}\,\dot{X}) + 2\dot{Q}\,Q^{\mathrm{T}}\, v\circ Y\big) \,,$$

thus

$$\mathbf{f} = \underbrace{\varrho\,(\ddot{X} - 2\dot{Q}\,Q^{\mathrm{T}}\,\dot{X})\circ Y^{-1}}_{=\,\omega^2 I x} + 2\varrho\,\underbrace{(\dot{Q}\,Q^{\mathrm{T}})\circ Y^{-1}}_{=\,\omega A}\, v + \mathbf{f}_0 \,,$$

where we have to compute the matrices I and A. Now for our special transformation $x = X(t,x^*) = Q(t)x^*$:

$$\begin{aligned}
&\dot{X} = \dot{Q}x^* = \dot{Q}\,Q^{\mathrm{T}}\,x \,, \quad \ddot{X} = \ddot{Q}x^* = \ddot{Q}\,Q^{\mathrm{T}}\,x \\
&\ddot{X} - 2\dot{Q}\,Q^{\mathrm{T}}\,\dot{X} = (\ddot{Q}\,Q^{\mathrm{T}} - 2(\dot{Q}\,Q^{\mathrm{T}})^2)x \,, \\
&Q(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \\
&\dot{Q}(t) = \omega \begin{bmatrix} -\sin(\omega t) & \cos(\omega t) & 0 \\ -\cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix} , \\
&Q(t)^{\mathrm{T}} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} , \\
&\dot{Q}\,Q^{\mathrm{T}} = \omega \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: \omega A \,, \\
&\ddot{Q} = -\omega^2 \begin{bmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 0 \end{bmatrix} , \\
&\ddot{Q}\,Q^{\mathrm{T}} = -\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: -\omega^2 I \,, \quad A^2 = -I \,, \\
&\ddot{Q}\,Q^{\mathrm{T}} - 2(\dot{Q}\,Q^{\mathrm{T}})^2 = -\omega^2 I - 2(\omega A)^2 = \omega^2 I \,.
\end{aligned}$$

The result follows. □

Proof of the boundary condition. Für $x \in \partial D$, also auch $x^* \in \partial D$, falls x und x^* zusammenhängen wie in (I5.12), gilt

$$\begin{aligned} v(t,x) &= \dot{Q}(t^*)x^* + Q(t^*)v^*(t^*,x^*) \\ &= \dot{Q}(t^*)x^* + Q(t^*)v_0^*(x^*) \\ &= \omega \begin{bmatrix} -\sin(\omega t^*) & \cos(\omega t^*) & 0 \\ -\cos(\omega t^*) & -\sin(\omega t^*) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} \\ &\quad + \omega \begin{bmatrix} \cos(\omega t^*) & \sin(\omega t^*) & 0 \\ -\sin(\omega t^*) & \cos(\omega t^*) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_2^* \\ x_1^* \\ 0 \end{bmatrix} = 0\,. \end{aligned}$$

That x are in fact the coordinates on the Earth can be seen in the following way: In the (x_1^*, x_2^*)-system define

$$e_1^*(t^*) := \mathrm{e}^{\mathrm{i}\omega t^*} = \begin{bmatrix} \cos(\omega t^*) \\ \sin(\omega t^*) \end{bmatrix}, \quad e_2^*(t^*) := \mathrm{i}e_1^*(t^*) = \begin{bmatrix} -\sin(\omega t^*) \\ \cos(\omega t^*) \end{bmatrix}.$$

Then the orthonormal system $\{e_1^*, e_2^*\}$ is in fact turning with the Earth and $(x_1^*, x_2^*) = x_1 e_1^* + x_2 e_2^*$ with

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} x^* \bullet e_1^* \\ x^* \bullet e_2^* \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \bullet e_1^* & \mathbf{e}_2 \bullet e_1^* \\ \mathbf{e}_1 \bullet e_2^* & \mathbf{e}_2 \bullet e_2^* \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{e}_1 \bullet e_1^* & \mathbf{e}_2 \bullet e_1^* \\ \mathbf{e}_1 \bullet e_2^* & \mathbf{e}_2 \bullet e_2^* \end{bmatrix} = \begin{bmatrix} \cos(\omega t^*) & \sin(\omega t^*) \\ -\sin(\omega t^*) & \cos(\omega t^*) \end{bmatrix}.$$

□

Die Physik ist für unterschiedliche Geschwindigkeiten dieselbe, wenn diese verschiedenen Geschwindigkeiten durch eine Beobachtertransformation auseinander hervorgehen, oder anders ausgedrückt, die Physik hängt nicht vom Beobachter ab. Wenn wir v^* durch v ersetzen, muss $\mathbf{f}^*$ durch $\mathbf{f}$ ersetzt werden, und die Transformation zwischen $\mathbf{f}^*$ und $\mathbf{f}$ enthält sowohl ϱ als auch v bzw. ϱ^* und v^*. Bei einem Beobachterwechsel müssen alle physikalischen Größen angepasst werden. Also ist insbesondere die Zentrifugalkraft und die Corioliskraft da oder nicht, je nachdem, welchen Beobachter man darstellt. Was bei einem mit der Erde rotierenden Koordinatensystem die Corioliskraft tut, ist bei einem von außen gegebenen Koordinatensystem durch die Randbedingung für v auf einer gekrümmten Fläche, hier die Oberfläche der Kugel, gegeben.

We shall consider "observer transformations" systematically in section II.1. Not all transformations, which we have treated here in this section, are observer transformations, as we have seen in example 5.4, where cylindrical coordinates have been considered. This is also true for the "reference coordinates", which are considered in the next section 6.

6 Reference coordinates

The mass and momentum conservation for deformable bodies apply (initially) in physical space, and in these coordinates the laws of conservation are

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}_x(\varrho v) = \mathbf{r}\ , \\ &\partial_t(\varrho v) + \mathrm{div}_x(\varrho v\, v^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}} := \mathbf{f} + \mathbf{r} v\,. \end{aligned} \tag{I6.1}$$

This is the mass and momentum balance in physical space, with ϱ as mass density, v as velocity, Π as pressure tensor, and $\mathbf{r}$ as mass production rate, $\mathbf{f}$ as classical force density. The identity between $\mathbf{f}$ and $\widetilde{\mathbf{f}} = \mathbf{f} + \mathbf{r}v$ has been discussed after (I3.2). Therefore we start with the same basic equations as for fluids in section 3. But in contrast to fluids the velocity v is not an independent variable. It is rather assumed that the velocity v is given by the motion of a body, and the constitutive function Π depends on the deformation of this body. The body is described by reference coordinates:

Reference coordinates:

$$\varphi :]t_1, t_2[\times \mathcal{B} \to \mathbb{R}^n\,, \quad \underline{x} \mapsto x = \varphi(t, \underline{x})$$

an isomorphism with $\det \mathrm{D}_{\underline{x}}\varphi > 0\,,$

$$(t, \underline{x}) \mapsto (t, x) = \tau(t, \underline{x}) := (t, \varphi(t, \underline{x})) \tag{I6.2}$$

$\mathcal{B} \subset \mathbb{R}^n$ the unperturbed body,

$v(t, x) := \partial_t \varphi(t, \underline{x})$ if $x = \varphi(t, \underline{x})\,,$

v the velocity.

Here $\mathcal{B}$ is the "unperturbed body" and the image in the physical space is $\Omega_t = \{\varphi(t, \underline{x})\,;\ \underline{x} \in \mathcal{B}\}$, where $\Omega_t := \{x \in \mathbb{R}^n\,;\ (t, x) \in \Omega\}$. And $v = \partial_t \varphi$ can be considered as the velocity of a "single particle", although the physical atoms and molecules undergo a thermal movement. It is assumed that this thermal movement in the problems treated here leaves the order of molecules unchanged. Mathematically, the equations (I6.1) are transformed to a system in $\mathcal{B}$ with the help of section 5.

We start with the following theorem which shows that a system of differential equations in physical coordinates is equivalent to a corresponding sytem in reference coordinates. This is a consequence of the general "Main invariance theorem" 5.1 and shows that the theory of elasticity is a special form of the theory of mass and momentum in physical coordinates.

6.1 Theorem. Let $K \in \mathbb{N}$. Under the transformation τ a system of equations

$$\partial_t u^k + \mathrm{div}_x q^k = r^k \text{ for } k = 1, \ldots, K \quad \text{in } \Omega$$

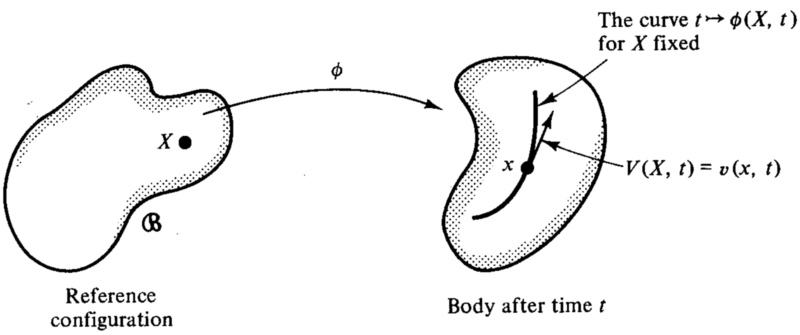


Fig. 22: The map $\underline{x} \mapsto \varphi(t, \underline{x})$ (from Marsden & Hughes [55]) which is called the "push-forward" operation whereas the inverse is called "pull-back".

is transformed into the "same" system

$$\partial_t \underline{u}^k + \operatorname{div}_{\underline{x}} \underline{q}^k = \underline{r}^k \text{ for } k = 1, \dots, K \quad \text{in } \mathcal{B},$$

if the transformed quantities are given by

$$\boxed{\begin{aligned} \underline{u}^k &= J\, u^k \circ \tau\,, \quad \underline{r}^k = J\, r^k \circ \tau\,, \\ \underline{q}^k &= J\, F^{-1} \left(q^k \circ \tau - u^k \circ \tau\, V\right) \end{aligned}}$$

for $k = 1, \dots, K$, where $V := \partial_t \varphi$ is the velocity, $F := \mathrm{D}_{\underline{x}} \varphi$ the deformation gradient, and $J := \det F > 0$.

Proof. We use the system for (u, q, r) in (I5.11) with transformation $Y = \tau$, hence $X = \varphi$, and matrix $Z = \mathrm{Id}$. Then it transforms to $\partial_t \underline{u}^k + \operatorname{div}_{\underline{x}} \underline{q}^k = \underline{r}^k$ for $k = 1, \dots, K$, where the functions in the transformed system are called $(\underline{u}, \underline{q}, \underline{r})$ and by (I5.11) they are defined by

$$\begin{aligned} u^k \circ \tau &= \frac{1}{J} \underline{u}^k\,, \quad J := \det \mathrm{D}_{x^*} \varphi > 0\,, \\ q_i^k \circ \tau &= \frac{1}{J} \big(\partial_t \varphi_i \cdot \underline{u}^k + \sum_j \partial_{\underline{x}_j} \varphi_i \cdot \underline{q}_j^k \big)\,, \\ r^k \circ \tau &= \frac{1}{J} \underline{r}^k \end{aligned}$$

for $k = 1, \dots, K$. With $V := (\partial_t \varphi_i)_i$ and $F := \left(\partial_{\underline{x}_j} \varphi_i\right)_{ij}$ one obtains that $J\, u^k \circ \tau = \underline{u}^k$, $J\, r^k \circ \tau = \underline{r}^k$, and $J\, q^k \circ \tau = \underline{u}^k V + F \underline{q}^k$, hence

$$J \left(q^k \circ \tau - u^k \circ \tau V\right) = F \underline{q}^k\,,$$

the assertion. □

References: We refer to Ciarlet [31] and Marsden & Hughes [55] for readers, who want to have a comparison with existing literature. Also we refer to the classical paper of Truesdell & Noll [67].

We obtain the following theorem.

6.2 Theorem. The differential equations (I6.1) read in reference coordinates (I6.2)

$$\begin{aligned} \partial_t \underline{\varrho} &= \underline{\mathbf{r}}\,, \\ \partial_t(\underline{\varrho} V) - \operatorname{div}_{\underline{x}} P &= \widetilde{\underline{\mathbf{f}}} := \underline{\mathbf{f}} + \underline{\mathbf{r}} V \end{aligned} \tag{I6.3}$$

where the transformed quantities are given in (I6.4).

Mass and momentum:

$$\partial_t \underline{\varrho} = \underline{\mathbf{r}}\,,$$
$$\partial_t(\underline{\varrho} V) - \operatorname{div}_{\underline{x}} P = \widetilde{\underline{\mathbf{f}}} := \underline{\mathbf{f}} + \underline{\mathbf{r}} V$$

$$\begin{aligned} &V := \partial_t \varphi = (\partial_t \varphi_i)_i = v \circ \tau \quad \text{velocity}, \\ &F := \mathrm{D}_{\underline{x}} \varphi = \Big(\partial_{\underline{x}_j} \varphi_i \Big)_{ij} \quad \text{deformation gradient}, \\ &J := \det F > 0 \quad \text{determinant}, \\ &P := J \cdot (-\Pi \circ \tau)\, F^{-\mathrm{T}} \\ &\qquad\qquad \text{(first) Piola-Kirchhoff stress tensor}, \\ &\underline{\varrho} := J \cdot (\varrho \circ \tau) \quad \text{reference mass density}, \\ &\underline{\mathbf{r}} := J \cdot (\mathbf{r} \circ \tau) \quad \text{reference rate}, \\ &\underline{\mathbf{f}} := J \cdot (\mathbf{f} \circ \tau) \quad \text{reference force density}. \end{aligned} \tag{I6.4}$$

We use the general coordinate transformation in 5.1 to prove this.

Proof (First Version). We write the system (I6.1) in the form

$$\partial_t \begin{bmatrix} \varrho \\ \varrho v \end{bmatrix} + \operatorname{div}_x \begin{bmatrix} \varrho v \\ \varrho v\, v^{\mathrm{T}} + \Pi \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix},$$

or to have the form of section 5

$$\partial_t \underbrace{\begin{bmatrix} \varrho \\ \varrho v \end{bmatrix}}_{=:\, u} + \sum_{i=1}^n \partial_{x_i} \underbrace{\begin{bmatrix} \varrho v_i \\ \varrho v_i v + (\Pi_{ki})_k \end{bmatrix}}_{=:\, q_i} = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix},$$

which is (I5.11) with $N = n+1$,

$$\partial_t u + \sum_{i=1}^n \partial_{x_i} q_i = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix},$$

where $Y = \tau$, $X = \varphi$, and we choose for the matrix $Z = \mathrm{Id}$. Then the transformed equations read

$$\partial_t \underline{u} + \sum_{j=1}^{n} \partial_{\underline{x}_j} \underline{q}_j = \begin{bmatrix} \underline{\mathbf{r}} \\ \underline{\widetilde{\mathbf{f}}} \end{bmatrix} , \tag{I6.5}$$

if the quantities in this transformed equation satisfy the transformation rules in (I5.11), which are

$$\begin{aligned} u\circ\tau &= \frac{1}{J}\underline{u} , \quad J := \det \mathrm{D}_{\underline{x}} X > 0 , \\ q_i\circ Y &= \frac{1}{J}\Big(\dot{X}_i \underline{u} + \sum_j X_{i\,'j} \underline{q}_j\Big) , \\ \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix}\circ\tau &= \frac{1}{J}\begin{bmatrix} \underline{\mathbf{r}} \\ \underline{\widetilde{\mathbf{f}}} \end{bmatrix} . \end{aligned} \tag{I6.6}$$

Since $Y = \tau$, $X = \varphi$, we compute

$$\begin{aligned} J = \det \mathrm{D}_{\underline{x}}\varphi = \det F > 0 , \\ \dot{X} = \partial_t \varphi = V , \quad X_{i\,'j} = \varphi_{i\,'\underline{x}_j} = F_{ij} . \end{aligned} \tag{I6.7}$$

Then we get for the first line of (I6.6), since $V = v\circ\tau$,

$$\underline{u} = J\,u\circ\tau = J\begin{bmatrix} \varrho\circ\tau \\ \varrho\circ\tau\, v\circ\tau \end{bmatrix} = J\,\varrho\circ\tau\begin{bmatrix} 1 \\ v\circ\tau \end{bmatrix} = \underline{\varrho}\begin{bmatrix} 1 \\ V \end{bmatrix}$$

and for the second line of (I6.6)

$$q_i\circ\tau = \frac{1}{J}\Big(V_i \underline{u} + \sum_j F_{ij} \underline{q}_j\Big) ,$$

hence

$$\begin{aligned} &\textstyle\sum_j F_{ij} \underline{q}_j = J\, q_i\circ\tau - V_i \underline{u} \\ &= J\begin{bmatrix} \varrho v_i \\ \varrho v_i v + (\Pi_{ki})_k \end{bmatrix}\circ\tau - V_i\begin{bmatrix} \underline{\varrho} \\ \underline{\varrho} V \end{bmatrix} = J\begin{bmatrix} 0 \\ (\Pi_{ki}\circ\tau)_k \end{bmatrix} , \end{aligned}$$

and therefore

$$\underline{q}_j = J\begin{bmatrix} 0 \\ \sum_i (F^{-1})_{ji}\,(\Pi_{ki}\circ\tau)_k \end{bmatrix} = \begin{bmatrix} 0 \\ \big(J\,(\Pi\circ\tau\, F^{-\mathrm{T}})_{kj}\big)_k \end{bmatrix} = \begin{bmatrix} 0 \\ -\,(P_{kj})_k \end{bmatrix} .$$

Thus (I6.5) becomes

$$\partial_t\begin{bmatrix} \underline{\varrho} \\ \underline{\varrho} V \end{bmatrix} + \sum_j \partial_{\underline{x}_j}\begin{bmatrix} 0 \\ -\,(P_{kj})_k \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{r}} \\ \underline{\widetilde{\mathbf{f}}} \end{bmatrix} ,$$

which is the assertion. The identity $\widetilde{\mathbf{f}} = \mathbf{f} + \mathbf{r}v$ on the right side of the momentum equation becomes

$$\underline{\widetilde{\mathbf{f}}} = J\,\widetilde{\mathbf{f}}\circ\tau = J(\mathbf{f}\circ\tau + \mathbf{r}\circ\tau\, v\circ\tau) = \underline{\mathbf{f}} + (J\mathbf{r}\circ\tau) v\circ\tau = \underline{\mathbf{f}} + \underline{\mathbf{r}}\, V$$

using the definitions. □

The second version is an explicit computation, where the general statements in section 5 are not used.

Proof (Second Version). The first equation in (I6.1) in its weak form reads

$$\int_\Omega (\partial_t\eta\cdot\varrho + \nabla\eta\bullet(\varrho v) + \eta\cdot\mathbf{r})\,\mathrm{dL}^{n+1} = 0$$

for $\eta\in C_0^\infty(\Omega;\mathbb{R})$. We transform this into an integral over $]t_1,t_2[\times\mathcal{B} = \tau^{-1}(\Omega)$. Defining

$$\widetilde{\eta}(t,\underline{x}) = \eta(t,\varphi(t,\underline{x})),\quad \tau(t,\underline{x}) = (t,\varphi(t,\underline{x})),$$

we obtain

$$\partial_t\widetilde{\eta} = (\partial_t\eta)\circ\tau + (\nabla\eta)\circ\tau\bullet\partial_t\varphi = (\partial_t\eta + v\bullet\nabla\eta)\circ\tau,$$

and the weak equation becomes, since $J = \det \mathrm{D}_{\underline{x}}\varphi = \big|\det \mathrm{D}_{(t,\underline{x})}\tau\,\big|$,

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} (\partial_t\widetilde{\eta}\cdot J\varrho\circ\tau + \widetilde{\eta}\cdot J\mathbf{r}\circ\tau)\,\mathrm{dL}^n\,\mathrm{dL}^1 = 0.$$

In its strong version this is

$$\boxed{\partial_t(J\varrho\circ\tau) = J\,\mathbf{r}\circ\tau.}$$

The second equation in (I6.1) is for $\zeta\in C_0^\infty(\Omega;\mathbb{R}^n)$

$$\int_\Omega\Big(\partial_t\zeta\bullet(\varrho v) + \sum_{i,j=1}^n \partial_j\zeta_i\cdot(\varrho v_i v_j + \Pi_{ij}) + \zeta\bullet\widetilde{\mathbf{f}}\Big)\,\mathrm{dL}^{n+1} = 0.$$

Transforming this via $\widetilde{\zeta}(t,\underline{x}) = \zeta(t,\varphi(t,\underline{x}))$, so that

$$\partial_t\widetilde{\zeta} = (\partial_t\zeta + v\bullet\nabla\zeta)\circ\tau,\quad \mathrm{D}_{\underline{x}}\widetilde{\zeta} = (\mathrm{D}_x\zeta)\circ\tau\mathrm{D}_{\underline{x}}\varphi,$$

we see that the above integral equals

$$\begin{aligned}
&= \int_\Omega\Big((\partial_t\zeta + \sum_{j=1}^n v_j\partial_j\zeta)\bullet(\varrho v) + (\mathrm{D}_x\zeta)\bullet\Pi + \zeta\bullet\widetilde{\mathbf{f}}\Big)\,\mathrm{dL}^{n+1}\\
&= \int_{t_1}^{t_2}\int_{\mathcal{B}}\Big(\partial_t\widetilde{\zeta}\bullet(J(\varrho v)\circ\tau) + J\cdot(\mathrm{D}_{\underline{x}}\widetilde{\zeta}\,(\mathrm{D}_{\underline{x}}\varphi)^{-1})\bullet(\Pi\circ\tau) + \widetilde{\zeta}\bullet(J\widetilde{\mathbf{f}}\circ\tau)\Big)\\
&\qquad\qquad \mathrm{dL}^n\,\mathrm{dL}^1\\
&= \int_{t_1}^{t_2}\int_{\mathcal{B}}\Big(\partial_t\widetilde{\zeta}\bullet(J(\varrho\circ\tau)V) + (\mathrm{D}_{\underline{x}}\widetilde{\zeta})\bullet(J(\Pi\circ\tau)\,(\mathrm{D}_{\underline{x}}\varphi)^{-\mathrm{T}}) + \widetilde{\zeta}\bullet(J\widetilde{\mathbf{f}}\circ\tau)\Big)\\
&\qquad\qquad \mathrm{dL}^n\,\mathrm{dL}^1\,.
\end{aligned}$$

In its strong version this is

$$\partial_t(J(\varrho\circ\tau)V) + \mathrm{div}_{\underline{x}}(J(\Pi\circ\tau)\,(\mathrm{D}_{\underline{x}}\varphi)^{-\mathrm{T}}) = J\widetilde{\mathbf{f}}\circ\tau\,,$$

which is the momentum equation in reference coordinates. □

As we see, the pressure tensor in the reference coordinates is

$$\Pi\circ\tau = -\frac{1}{J}\cdot P F^{\mathrm{T}} \,. \tag{I6.8}$$

Thus if the Piola-Kirchhoff tensor P depends only on the first derivatives F of the deformation so also does $\Pi\circ\tau$. So far we have derived the general system of deformable bodies.

***Large Deformation (General theory)*:**

$$\left.\begin{aligned} &\partial_t \underline{\varrho} = \underline{\mathbf{r}} \\ &\partial_t(\underline{\varrho} V) - \mathrm{div}_{\underline{x}} P = \underline{\mathbf{f}} + \underline{\mathbf{r}} V \end{aligned}\right\} \text{ in }]t_1,t_2[\times\mathcal{B} \tag{I6.9}$$

$\underline{\varrho}$ reference density,

$V = \partial_t\varphi$ velocity, where φ as in (I6.2),

P first Piola-Kirchhoff stress tensor.

$\underline{\mathbf{f}}$ force density (it is $\widetilde{\underline{\mathbf{f}}} = \underline{\mathbf{f}} + \underline{\mathbf{r}} V$).

The following equations are elementary.

6.3 Lemma. It is

$$\partial_t F = \mathrm{D}_{\underline{x}} V \,,$$
$$\partial_t(\underline{\varrho} V) = \underline{\mathbf{r}} V + \underline{\varrho}\,\partial_t V \,.$$

Proof. The second one is the product rule and the mass equation and the first one holds because $\partial_t F = \partial_t \mathrm{D}_{\underline{x}}\varphi = \mathrm{D}_{\underline{x}}\partial_t\varphi = \mathrm{D}_{\underline{x}} V$. □

If $\mathbf{r} = 0$ then the first equation of (I6.4) says that $\underline{\varrho}$ is independent of time. Then we have the standard version of a continously deformable body consisting only of the conservation of momentum, that is, the mass density $\underline{x} \mapsto \underline{\varrho}(\underline{x})$ is given and we have to solve only for the function φ (the velocity V and the tensor P are expressed by derivatives of φ):

***Large Deformation (Classical theory)*:**

$$\underline{\varrho}\partial_t V - \mathrm{div}_{\underline{x}} P = \underline{\mathbf{f}} \text{ in }]t_1,t_2[\times\mathcal{B}, \tag{I6.10}$$

$\underline{\varrho} = \underline{\varrho}(\underline{x})$ reference density (since $\underline{\mathbf{r}} = 0$),

$V = \partial_t\varphi$ velocity, where φ as in (I6.2),

P first Piola-Kirchhoff stress tensor.

For further studies on deformable bodies see the section IV.5 about elasticity. Here we study now rigid bodies.

Rigid bodies

The simplest example of a body is that $F^{\mathrm{T}} F$ is constant, or more general that it is not dependent on time. For this special case we show:

6.4 Lemma. Let $\mathcal{B}$ be connected. Then the following are equivalent:

(1) For the deformation gradient it holds that $F(t,\underline{x})^{\mathrm{T}} F(t,\underline{x}) = C(\underline{x})$, where $C(\underline{x})$ is independent of time.

(2) The velocity gradient fulfills $\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}} = 0$, i.e. it is antisymmetric.

(3) The velocity vector is $v(t,x) = A(t)x + b(t)$ with an antisymmetric matrix function A and a vector function b.

Proof (1)⇒(2). The property that C is independent of t is equivalent to

$$0 = \partial_t(F^{\mathrm{T}} F) = (\partial_t F)^{\mathrm{T}} F + F^{\mathrm{T}} \partial_t F = F^{\mathrm{T}} (M^{\mathrm{T}} + M)F$$

with $M := (\partial_t F) F^{-1}$. Thus the fact that C is independent of t is equivalent to $M^{\mathrm{T}} + M = 0$, i.e. the antisymmetry of M. Now

$$\partial_t F = \Big(\partial_t \partial_{\underline{x}_j} \varphi_i\Big)_{ij} = \Big(\partial_{\underline{x}_j} (\partial_t \varphi_i)\Big)_{ij} = \Big(\partial_{\underline{x}_j} V_i\Big)_{ij} = \mathrm{D}V \tag{I6.11}$$

and due to $V(t,\underline{x}) = v(t,\varphi(t,\underline{x})) = (v \circ \tau)(t,\underline{x})$ it is

$$\partial_{\underline{x}_j} V_i = \sum_k (\partial_{x_k} v_i) \circ \tau \cdot \partial_{\underline{x}_j} \varphi_k = \sum_k (\partial_{x_k} v_i) \circ \tau \cdot F_{kj} = ((\mathrm{D}v \circ \tau) F)_{ij} \,,$$

hence

$$\partial_t F = \mathrm{D}V = ((\mathrm{D}v) \circ \tau) F \,.$$

Therefore $M = (\partial_t F) F^{-1} = (\mathrm{D}v) \circ \tau$, that is

$$\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}} = (M + M^{\mathrm{T}}) \circ \tau^{-1} = 0 \,,$$

the assertion. □

Proof (3)⇒(1). The identity

$$V(t,\underline{x}) := \partial_t \varphi(t,\underline{x}) = v(t,\varphi(t,\underline{x})) = A(t)\varphi(t,\underline{x}) + b(t)$$

yields by (I6.11)

$$\partial_t F = \mathrm{D}_{\underline{x}} V = A(t) \mathrm{D}_{\underline{x}} \varphi = A(t) F \,.$$

This implies

$$\partial_t(F^{\mathrm{T}} F) = (\partial_t F)^{\mathrm{T}} F + F^{\mathrm{T}} \partial_t F = (AF)^{\mathrm{T}} F + F^{\mathrm{T}} AF = F^{\mathrm{T}} (A^{\mathrm{T}} + A) F = 0 \,,$$

hence $F^{\mathrm{T}} F$ is independent of time. □

Proof (2)⇒(3). With repeated use of the antisymmetry of $\mathrm{D}v$ we obtain

$$\begin{aligned}\partial_{ij}v_k &= \partial_i\,(\partial_j v_k)\\ &= -\partial_i\,(\partial_k v_j) + \partial_i\,\underbrace{(\partial_j v_k + \partial_k v_j)}_{=0}\\ &= -\partial_i\,(\partial_k v_j) = -\partial_k\,(\partial_i v_j)\\ &= \partial_k\,(\partial_j v_i) - \partial_k\,\underbrace{(\partial_i v_j + \partial_j v_i)}_{=0}\\ &= \partial_k\,(\partial_j v_i) = \partial_j\,(\partial_k v_i)\\ &= -\partial_j\,(\partial_i v_k) + \partial_j\,\underbrace{(\partial_k v_i + \partial_i v_k)}_{=0}\\ &= -\partial_j\,(\partial_i v_k) = -\partial_i\,(\partial_j v_k) = -\partial_{ij}v_k\,,\end{aligned}$$

Thus we have shown that $\partial_{ij}v_k = 0$ for all (i,j,k). Since the domain that we consider is simply connected, it follows from $\mathrm{D}^2 v = 0$ that v is affine linear in x, thus it holds

$$v(t,x) = M(t)x + b(t) \text{ with a matrix } M \text{ and a vector } b.$$

Then $M = \mathrm{D}v$ is antisymmetric. In the same way the statement (2) follows directly from the fact that v is affine linear in x with an anti-symmetric matrix. □

A body for which C depends only on $\underline{x}$, can be transformed to a body, in which this matrix is the identity Id.

6.5 Transformation of $\boldsymbol{\mathcal{B}}$. Let C be a matrix dependent only on $\underline{x}$. Furthermore, let $h:\mathcal{B}^* \to \mathcal{B}$ be a mapping, see Fig. 23, with

$$H(\underline{x}^*) := \mathrm{D}_{\underline{x}^*}h(\underline{x}^*)\,, \quad H(\underline{x}^*)^{\mathrm{T}}\,C(\underline{x})H(\underline{x}^*) = \mathrm{Id}\,, \quad \underline{x} = h(\underline{x}^*)\,.$$

Then, if φ is a deformation, define φ^* by

$$\varphi^*(t,\underline{x}^*) := \varphi(t,h(\underline{x}^*))\,.$$

It follows that the deformation gradient of φ^* is

$$F^*(t,\underline{x}^*) = F(t,h(\underline{x}^*))H(\underline{x}^*)\,. \tag{I6.12}$$

Further if φ is a deformation as in 6.4, i.e. $F(t,\underline{x})^{\mathrm{T}}\,F(t,\underline{x}) = C(\underline{x})$ for all t, then

$$C^* := F^{*\mathrm{T}}\,F^* = \mathrm{Id}\,.$$

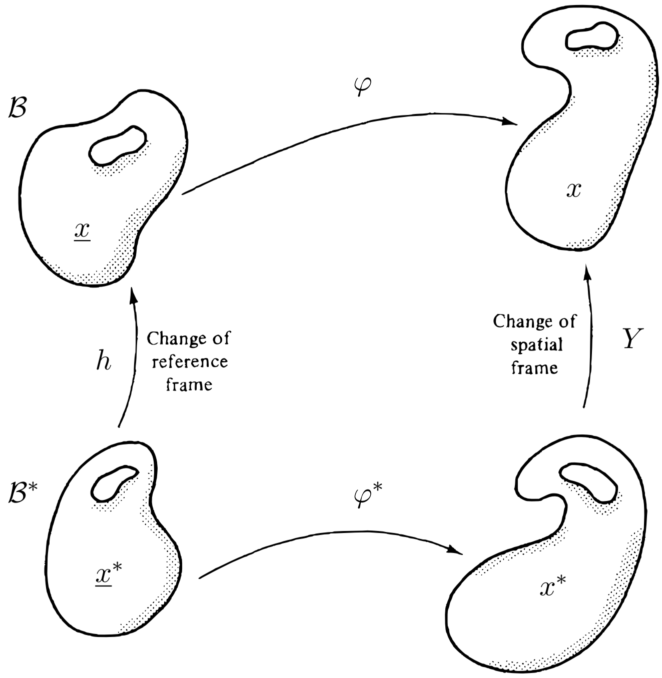


Fig. 23: Different reference coordinates, picture from [55]

Proof. Using (I6.12) one has

$$C^* = F^{*\mathrm{T}} F^* = (FH)^{\mathrm{T}} FH = H^{\mathrm{T}} F^{\mathrm{T}} FH = H^{\mathrm{T}} CH = \mathrm{Id}\,.$$

□

Hence we define (see Marsden & Hughes [55, CH.2: 3.6])

6.6 Rigid bodies. A body for which the deformation gradient fulfills

$$\begin{gathered} F^{\mathrm{T}} F = \mathrm{Id}\,, \\ \underline{x} \mapsto F(t_0, \underline{x}) \text{ is constant for some } t_0, \end{gathered} \tag{I6.13}$$

is called ***rigid body***. A rigid body moves in physical space by translation and rotation only, that is, there is a position $\xi(t)$ and an orthonormal system $\{e_i(t)\,;\ i = 1, \ldots, n\}$ with

$$\varphi(t, \underline{x}) = \xi(t) + \sum_{i=1}^{n} \underline{x}_i e_i(t)\,. \tag{I6.14}$$

Addendum: The orthonormal system $t \mapsto \{e_i(t)\,;\ i = 1, \ldots, n\}$ is also called a ***moving frame***, since it depends on time.

Proof. We show that both characterizations are equivalent. The orthonormal system defines a matrix $Q(t)$ by

$$Q(t)z = \sum_{i=1}^{n} z_i e_i(t) \text{ for } z \in \mathbb{R}^n\,,$$

and this implies that

$$(Q(t)^{\mathrm{T}} Q(t))_{ij} = \mathbf{e}_i\bullet(Q(t)^{\mathrm{T}} Q(t)\mathbf{e}_j) = (Q(t)\mathbf{e}_i)\bullet(Q(t)\mathbf{e}_j) = e_i\bullet e_j = \delta_{ij}\,,$$

i.e. $Q(t)^{\mathrm{T}} Q(t) = \mathrm{Id}$. Thus $Q(t)$ is an orthonormal transformation, and equation (I6.14) becomes

$$\varphi(t,\underline{x}) = Q(t)\underline{x} + \xi(t)\,. \tag{I6.15}$$

We show that this is for given t_0 equivalent to

$$\begin{aligned}&\partial_t\varphi(t,\underline{x}) = A(t)\varphi(t,\underline{x}) + b(t) \text{ where } A(t) \text{ is antisymmetric,}\\ &\varphi(t_0,\underline{x}) = Q_0\underline{x} + \xi_0 \text{ for some } \xi_0 \text{ and some orthonormal } Q_0.\end{aligned} \tag{I6.16}$$

In fact, it follows from (I6.15) that

$$\partial_t\varphi(t,\underline{x}) = \dot{Q}(t)\underline{x} + \dot{\xi}(t) = \dot{Q}(t)\,Q(t)^{\mathrm{T}}\,\varphi(t,\underline{x}) + \dot{\xi}(t) - \dot{Q}(t)\,Q(t)^{\mathrm{T}}\,\xi(t)\,,$$

hence it holds (I6.16) with $b = \dot{\xi} - \dot{Q}\,Q^{\mathrm{T}}\,\xi$ and $A = \dot{Q}\,Q^{\mathrm{T}}$, which is antisymmetric, since Q is orthonormal. Conversely, if statement (I6.16) holds, then it follows that for any matrix $M(t)$ and any vector $\xi(t)$

$$\begin{aligned}&\partial_t(\varphi - M\underline{x} - \xi) = \partial_t\varphi - \dot{M}\underline{x} - \dot{\xi} = A\varphi + b - \dot{M}\underline{x} - \dot{\xi}\\ &= A(\varphi - M\underline{x} - \xi) - (\dot{M} - AM)\underline{x} - (\dot{\xi} - A\xi - b)\,.\end{aligned}$$

Now we determine M and ξ from the ordinary differential equations

$$\begin{aligned}&\dot{M} = AM \text{ with } M(t_0) = Q_0\,,\\ &\dot{\xi} = A\xi + b \text{ with } \xi(t_0) = \xi_0\,.\end{aligned}$$

Then

$$\begin{aligned}&\partial_t(\varphi - M\underline{x} - \xi) = A(\varphi - M\underline{x} - \xi)\,,\\ &\varphi(t_0,\underline{x}) - M(t_0)\underline{x} - \xi(t_0) = 0\,,\end{aligned}$$

and therefore $\varphi(t,\underline{x}) = M(t)\underline{x} + \xi(t)$ for all t, i.e. (I6.15) holds, if $M(t)$ is an orthonormal transformation. To show this, we realize that $M(t_0) = Q_0$ is orthonormal, therefore $M(t_0)^{\mathrm{T}}\,M(t_0) - \mathrm{Id} = 0$ and

$$\begin{aligned}&(M^{\mathrm{T}}\,M - \mathrm{Id})^{\dot{}} = \dot{M}^{\mathrm{T}}\,M + M^{\mathrm{T}}\,\dot{M}\\ &= (AM)^{\mathrm{T}}\,M + M^{\mathrm{T}}\,AM = M^{\mathrm{T}}\,A^{\mathrm{T}}\,M + M^{\mathrm{T}}\,AM\\ &= M^{\mathrm{T}}\,(\underbrace{A + A^{\mathrm{T}}}_{=0})M\,.\end{aligned}$$

As a result we have $M(t)^{\mathrm{T}}\,M(t) - \mathrm{Id} = 0$, and thus, $M(t)$ is orthonormal.

Therefore it is shown that the formulation for φ in the theorem is equivalent to (I6.16). We now show that (I6.16) is equivalent to the first representation of a rigid body in the theorem. The first line in (I6.16) is equivalent to

$$v(t,\varphi(t,\underline{x})) = A(t)\varphi(t,\underline{x}) + b(t)$$

or to

$$v(t,x) = A(t)x + b(t)\,.$$

Now it follows from 6.4, that this is equivalent to $F(t,\underline{x})^{\mathrm{T}} F(t,\underline{x}) = C(\underline{x})$, and the second line in (I6.16) is equivalent to $F(t_0,\underline{x}) = \mathrm{D}\varphi(t_0,\underline{x}) = Q_0$, which is independent of $\underline{x}$. This matrix must be an orthonormal transformation, since $F(t_0,\underline{x})^{\mathrm{T}} F(t_0,\underline{x}) = \mathrm{Id}$ by the statement in 6.4. ☐

For further studies see the section II.5 on objectivity and section IV.5 about Elasticity.

7 Exercises

Conservation laws

7.1 Definition der Ableitung. Die Ableitung einer Funktion f auf $\mathbb{R}^N$ mit Werten in $\mathbb{R}^M$ wird in der Regel als lineare Abbildung $\mathrm{D}f(x){:}\mathbb{R}^N \to \mathbb{R}^M$ mit

$$f(y) = f(x) + \mathrm{D}f(x)(y-x) + \mathcal{O}(y-x)$$

definiert. Es ist dann für $z \in \mathbb{R}^N$

$$\mathrm{D}f(x)z = \begin{bmatrix} \partial_1 f_1 & \cdots & \partial_N f_1 \\ \vdots & & \vdots \\ \partial_1 f_M & \cdots & \partial_N f_M \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_M \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}.$$

Das ist in Übereinstimmung mit der Definition $\mathrm{D}f(x)$ als Matrix in Abschnitt 1. Vergleiche dies mit der Definition des Gradienten $\nabla f(x)$.

7.2 Identitäten für Ableitungen. Für ein Vektorfeld q im $\mathbb{R}^n$ gilt

$$\partial_{x_i} q = (\mathrm{D}q)\mathbf{e}_i \text{ für } i = 1,\dots,n\,,$$
$$\partial_e q = (\mathrm{D}q)e \text{ für } e \in \mathbb{R}^n\,,$$
$$\partial_{x_i} q_k = \mathbf{e}_k \bullet (\mathrm{D}q)\mathbf{e}_i \text{ für } k,i = 1,\dots,n.$$

7.3 Übung. Sei $\{e_1(x),\dots,e_n(x)\}$ eine Orthonormalbasis des $\mathbb{R}^n$, die von x abhängt. Zeige, dass für stetig differenzierbares

(1) $p{:}\mathbb{R}^n \to \mathbb{R}$ gilt

$$\nabla p = \sum_{i=1}^n (\partial_{e_i} p)\, e_i\,,$$

(2) $v{:}\mathbb{R}^n \to \mathbb{R}^n$ gilt

$$\mathrm{D}v = \sum_{i=1}^n (\partial_{e_i} v)\, {e_i}^{\mathrm{T}} = \sum_{i,j=1}^n (e_i \bullet \partial_{e_j} v)\, e_i\, {e_j}^{\mathrm{T}}\,.$$

Hinweis: Benutze die entsprechende Aussage für die Divergenz.

7.4 Übung. Sei $\{e_1(x), \dots, e_n(x)\}$ eine Orthonormalbasis des $\mathbb{R}^n$, die von x abhängt. Zeige, dass für ein Vektorfeld $q \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ für jede Funktion $\zeta \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ gilt

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \partial_{e_i}\zeta \cdot q \bullet e_i \, \mathrm{dL}^n = -\sum_{i=1}^n \int_{\mathbb{R}^n} \zeta e_i \bullet \partial_{e_i} q \, \mathrm{dL}^n \,.$$

Bemerkung: Es gilt im Allgemeinen

$$\int_{\mathbb{R}^n} \partial_{e_i}(\zeta \cdot q \bullet e_i) \, \mathrm{dL}^n \neq 0 \,.$$

7.5 Übung. Let $\sigma, \omega \in C^1(\mathbb{R})$ and with $n = 2$ $(\mathbb{R}^2 \cong \mathbb{C})$

$$\varrho(t,x) = \sigma(|x|) \ , \quad v(t,x) = \mathrm{i}\omega(|x|)x \ .$$

Show, that $\partial_t \varrho + \operatorname{div}(\varrho v) = 0$ in $\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})$.

7.6 Übung. Beweise die Gleichungen in 1.4

7.7 Beispiel. Es sei $v(t,x) = s(t,x)x$, d.h. die Geschwindigkeit zeigt

- vom Ursprung weg, falls $s \geq 0$.
- in Richtung Ursprung, falls $s \leq 0$.

Ist dann

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0 \quad \text{und} \quad \varrho \geq 0 \ , \tag{I7.1}$$

so folgt für die (bis auf eine Konstante) mittlere Gesamtmasse in $\mathrm{B}_\varepsilon(0)$

$$m_\varepsilon(t) := \frac{1}{\varepsilon^n} \int_{\mathrm{B}_\varepsilon(0)} \varrho(t,x) \, \mathrm{d}x$$

dass (es sei $\dot{m}_\varepsilon$ die Ableitung von m_ε) nach dem Gauß'schen Satz

$$\begin{aligned}
\dot{m}_\varepsilon(t) &= \frac{1}{\varepsilon^n} \int_{\mathrm{B}_\varepsilon(0)} \partial_t \varrho(t,x) \, \mathrm{d}x = -\frac{1}{\varepsilon^n} \int_{\mathrm{B}_\varepsilon(0)} \operatorname{div}(\varrho(t,x) v(t,x)) \, \mathrm{d}x \\
&= -\frac{1}{\varepsilon^n} \int_{\partial \mathrm{B}_\varepsilon(0)} \varrho(t,x) v(t,x) \bullet \nu_{\mathrm{B}_\varepsilon(0)}(t,x) \, \mathrm{d}x \\
&= -\frac{1}{\varepsilon^{n-1}} \int_{\partial \mathrm{B}_\varepsilon(0)} \underbrace{\varrho(t,x) s(t,x)}_{\text{hat Vorzeichen}} \, \mathrm{d}x \ ,
\end{aligned}$$

hence $\dot{m}_\varepsilon(t) \geq 0$ in case $s \leq 0$, whereas $\dot{m}_\varepsilon(t) \leq 0$ in case $s \geq 0$.

Distributions

7.8 Distributionsableitung. Ist $u \in C^1(U)$, so ist

$$\partial_i [u] = [\partial_i u] \ .$$

7.9 Distribution und Funktion. (Siehe Fig. 5.) Sei $(\varphi_\varepsilon)_{\varepsilon>0}$ eine ***Dirac-Folge***, d.h.

$$\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^N)\,, \quad \varphi_\varepsilon \geq 0\,, \quad \int_{\mathbb{R}^N} \varphi_\varepsilon \,\mathrm{dL}^N = 1$$

und $[\varphi_\varepsilon] \to \boldsymbol{\delta}_0$ für $\varepsilon \to 0$, wobei $\boldsymbol{\delta}_0$ die Dirac-Distribution ist. Zeige: Ist g eine lokal integrierbare Funktion, so gilt für fast alle y

$$g(y) = \lim_{\varepsilon\to 0} \langle\, \varphi_\varepsilon(\bullet - y)\,,\, [g]\,\rangle_{\mathscr{D}(\mathbb{R}^N)}\ .$$

Hinweis: Es ist

$$\langle\, \varphi_\varepsilon(\bullet - y)\,,\, [g]\,\rangle_{\mathscr{D}} - g(y) = \int_{\mathbb{R}^N} \varphi_\varepsilon(y'-y)(g(y') - g(y))\,\mathrm{d}y'\,.$$

7.10 Übung. Sei ξ wie in 2.9 und

$$\varrho_\varepsilon(t,x) := \eta_\varepsilon(x - \xi(t))\,, \quad v(t,x) := \dot{\xi}(t)\,.$$

Zeige:

(1) $\partial_t \varrho_\varepsilon + \mathrm{div}\,(\varrho_\varepsilon v) = 0$

(2) As $\varepsilon \to 0$ this converges in distributional sense to the movement of a mass point with ξ and

$$m = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \eta_\varepsilon(y)\,\mathrm{d}y\ .$$

7.11 Übung. Sei $\boldsymbol{\mu}_\xi$ wie in 2.8. Weiter seien differenzierbare $t \mapsto m(t) \in \mathbb{R}$, $t \mapsto \mathbf{r}(t) \in \mathbb{R}$ und $t \mapsto v(t) \in \mathbb{R}^n$ gegeben, welche die distributionelle Massenerhaltung

$$\partial_t(m\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) = \mathbf{r}\boldsymbol{\mu}_\xi$$

erfüllen. Zeige, dass dann

$$\dot{m} = \mathbf{r}\ , \quad v = \dot{\xi}\,.$$

7.12 Exercise. Sei $\alpha > 0$. Dann gilt punktweise in $\mathbb{R}^n \setminus \{0\}$

$$\mathrm{div}\frac{x}{|x|^\alpha} = \frac{n-\alpha}{|x|^\alpha}\,.$$

7.13 Heat operator. Let

$$F(t,x) := \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\dfrac{|x|^2}{4t}\right) & \text{for } t > 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

(1) It converges

$$[F(t,\bullet)] \to \boldsymbol{\delta}_0 \text{ für } t \searrow 0 \text{ pointwise in } \mathscr{D}'(\mathbb{R}^n)\,.$$

(2) It is F the ***fundamental solution*** of the heat equation

$$\partial_t[F] - \Delta[F] = \boldsymbol{\delta}_{(0,0)} \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)\,.$$

Remark: Es ist $-\Delta[F] = \boldsymbol{\delta}_{(0,0)} - \partial_t[F]$ nicht von der Gestalt in 2.10. *Notation:* Die physikalische Bedeutung des Begriffes “Wärmeleitungsgleichung” findet sich in IV.2.6.

Conservation of momentum

7.14 Divergence.

(1) For an orthonormal bases $\{e_1(t,x),\ldots,e_n(t,x)\}$ of the Euclidean space $\mathbb{R}^n$ it holds for matrices M

$$\mathrm{div}\, M = \sum_{i=1}^{n} (\partial_{e_i} M) e_i \,,$$

see the analogy in 1.3.

(2) For vectors a and b it holds

$$\mathrm{div}\,(ab) = \mathrm{div}(b)a + (\mathrm{D}a)b \,.$$

7.15 Kollision von Massepunkten. Seien ξ^α, $\alpha = 1,2$, zwei Massepunkte, die wie in 3.2 kollidieren. Es sei

$$v_+^1 = av_-^1 + bv_-^2 \,,$$
$$v_+^2 = cv_-^1 + dv_-^2 \,.$$

Gebe den Bereich der Koeffizienten an, der generisch für eine Kollision erlaubt ist, und berechne die Massen m_+^α, m_-^α aus diesen Koeffizienten bei vorgegebener Gesamtmasse m.

7.16 Nachlesen. Vergleiche die Herleitung der Kepler-Bewegung in 3.3 mit der Herleitung in [Wikipedia: Kepler Gesetze].

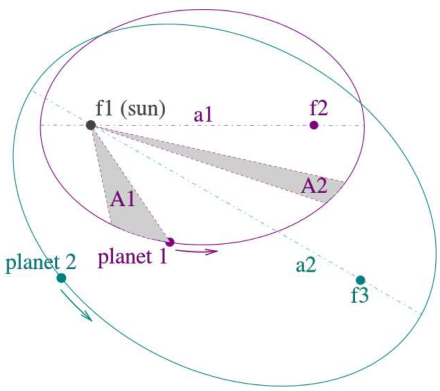


Fig. 24: “Illustration of Kepler’s three laws with two planetary orbits” from Wikipedia.

7.17 Kepler’s laws. Show that the differential equations in 3.3 imply the three Kepler laws:
1. The Law of Orbits. All planets move in elliptical orbits, with the Sun at one focus.
2. The Law of Areas. A line that connects a planet to the Sun sweeps out equal areas in equal times.
3. The Law of Periods. The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.
(From hyperphysics.phy-astr.gsu.edu/hbase/kepler.html)

7.18 Freie Energie. Für den Druck nimm an

$$p(\varrho) = \varrho f'_{\varrho}(\varrho) - f(\varrho)\,.$$

Zeige, dass dann

$$\frac{p}{\varrho^2} = \Big(\frac{f(\varrho)}{\varrho}\Big)'_{\varrho}\,.$$

Berechne die innere freie Energie f, wenn

(1) $p = c\varrho^{\gamma}$ with $\gamma > 1$.

(2) $p = c\varrho$.

Definition: Es ist f die innere freie Energie und $\frac{f}{\varrho}$ die spezifische freie innere Energie.

Solution. Es ist

$$\Big(\frac{f(\varrho)}{\varrho}\Big)'_{\varrho} = \frac{f(\varrho)'_{\varrho}}{\varrho} - \frac{f(\varrho)}{\varrho^2} = \frac{\varrho f(\varrho)'_{\varrho} - f(\varrho)}{\varrho^2} = \frac{p}{\varrho^2}\,.$$

Zum Beweis von (1) nehmen wir $c = 1$ an. Es ist

$$\Big(\frac{f}{\varrho}\Big)'_{\varrho} = \varrho^{\gamma-2} \quad\Longrightarrow\quad \frac{f}{\varrho} = \frac{1}{\gamma-1}\varrho^{\gamma-1} + \mathrm{const} \quad\Longrightarrow\quad f = \frac{1}{\gamma-1}\varrho^{\gamma} + \mathrm{const}\cdot\varrho\,.$$

Auch in (2) nehmen wir $c = 1$ an. Es ist

$$\Big(\frac{f}{\varrho}\Big)'_{\varrho} = \frac{1}{\varrho} \quad\Longrightarrow\quad \frac{f}{\varrho} = \log\varrho + \mathrm{const} \quad\Longrightarrow\quad f = \varrho\log\varrho + \mathrm{const}\cdot\varrho\,.$$

□

7.19 Erdatmosphäre. Die Erdatmosphäre sei modelliert als Masse-Impulserhaltung in $\mathbb{R}^3 \setminus \mathrm{B}_R(0)$, wobei als Kraftterm die Gravitation der Erde diene, deren Masse M_{Erde} gleichmäßig in $\mathrm{B}_R(0)$ verteilt sei.

(1) Es gelte $\Pi = p\mathrm{Id} - S$ und

$$p = \widehat{p}(\varrho) = \varrho f'_{\varrho}(\varrho) - f(\varrho) \text{ mit der inneren freien Energie } f,$$
$$S = \widehat{S}(\varrho, (\mathrm{D}v)^{\mathrm{S}}), \quad \widehat{S}(\varrho, 0) = 0.$$

Stelle die Gleichungen für Masse und Impuls auf.

(2) Zeige, dass für eine stationäre Lösung mit $v = 0$, die nur von $r = |x|$ abhängt, gilt, dass

$$\mu(R) = \mu(\infty) + \frac{GM_{Erde}}{R}\,,$$

wobei $\mu(r) = f'_{\varrho}(\varrho(x))$ ist. *Definition:* Die Variable μ heißt chemisches Potential (siehe IV.11.9).

Solution (2). Ist ϱ zeitunabhängig und $v = 0$, so ist die Massenerhaltung trivial erfüllt und die Impulserhaltung lautet

$$\nabla p = \mathfrak{g}\varrho\nabla\phi_{\mathrm{Erde}}\,.$$

Es ist nach (I4.6)

$$\nabla p = \mathfrak{g}\varrho\nabla\phi_{\mathrm{Erde}} = -\mathfrak{g}\varrho\frac{M_{\mathrm{Erde}}}{\sigma_n}\frac{x}{|x|^n}\,.$$

Wegen $p(\varrho) = \varrho f'_{\varrho}(\varrho) - f(\varrho)$, ist $p'_{\varrho} = \varrho f'_{\varrho\varrho}$, d.h.

$$\nabla p = \varrho f'_{\varrho\varrho}\nabla\varrho = \varrho\nabla f'_{\varrho} = \varrho\nabla\mu \text{ wenn } \mu := f'_{\varrho},$$

und damit

$$\nabla\mu = -\frac{\mathfrak{g}M_{\text{Erde}}}{\sigma_n}\frac{x}{|x|^n}\,.$$

Da die Funktionen nur von r abhängen, ist $\nabla\mu = \mu'_r\frac{x}{|x|}$, also

$$\mu'_r = -\frac{\mathfrak{g}M_{\text{Erde}}}{\sigma_n}\frac{1}{r^{n-1}}\,. \tag{I7.2}$$

Die Behauptung folgt durch Integration. Es ist $n = 3$, beachte dass dann

$$\frac{\mathfrak{g}}{\sigma_n(n-2)} = G\,.$$

Aus der Darstellung (I7.2) folgt, dass $\mu(\infty)$ existiert. □

7.20 Fluchtgeschwindigkeit. Wie groß muss die Geschwindigkeit mindestens sein, um ein Schwerefeld (eines Punktes in $\{0\}$ mit der Masse m) zu verlassen?

Interfaces

7.21 Parabelform der Oberfläche. Es sei $p = \widehat{p}(\varrho) := c\varrho^\gamma$ mit Konstanten $\gamma > 1$ und $c > 0$. Weiter seien v und $\mathbf{f}$ wie in 4.4 gegeben. Zeige, dass dann die Konklusion von 4.4 richtig ist, d.h. die Oberfläche der kompressiblen Flüssigkeit ist ein Paraboloid.

Solution. By (I4.5)

$$\nabla p = \varrho\begin{bmatrix}\omega^2 x_1\\ \omega^2 x_2\\ -g\end{bmatrix}.$$

This has been shown for general compressible fluids. Since $\widehat{p}(\varrho) = \varrho f'_\varrho(\varrho) - f(\varrho)$ and then $p'_\varrho(\varrho) = \varrho f'_{\varrho\varrho}(\varrho)$ we compute

$$\nabla\Big(\frac{\omega^2}{2}(x_1^2+x_2^2) - gx_3\Big) = \begin{bmatrix}\omega^2 x_1\\ \omega^2 x_2\\ -g\end{bmatrix} = \frac{1}{\varrho}\nabla p = \frac{p'_\varrho(\varrho)}{\varrho}\nabla\varrho = f'_{\varrho\varrho}(\varrho)\nabla\varrho = \nabla(f'_\varrho(\varrho))\,.$$

Hence

$$f'_\varrho(\varrho) = \frac{\omega^2}{2}(x_1^2+x_2^2) - gx_3 + \text{const}\,.$$

On the surface Γ we have $\widehat{p}(\varrho) = p_g = \text{const}$. Therefore we conclude $\varrho = \varrho_0 = \text{const}$, if $p'_\varrho > 0$, and

$$gx_3 = \frac{\omega^2}{2}(x_1^2+x_2^2) - f'_\varrho(\varrho_0) + \text{const}\,.$$

This holds also in the special case $p = \widehat{p}(\varrho) = c\varrho^\gamma$, in which case

$$f(\varrho) = \frac{c}{\gamma-1}\varrho^\gamma.$$

□

II Objectivity

Restrictions for the description of of physical processes come from the ***principle of objectivity*** (also called ***frame indifference***). It consists of the following:

- The value of physical quantities depend on the observer.
- The type of a physical quantity is given by a transformation rule.
- The description of a physical process has to be independent of the observer.

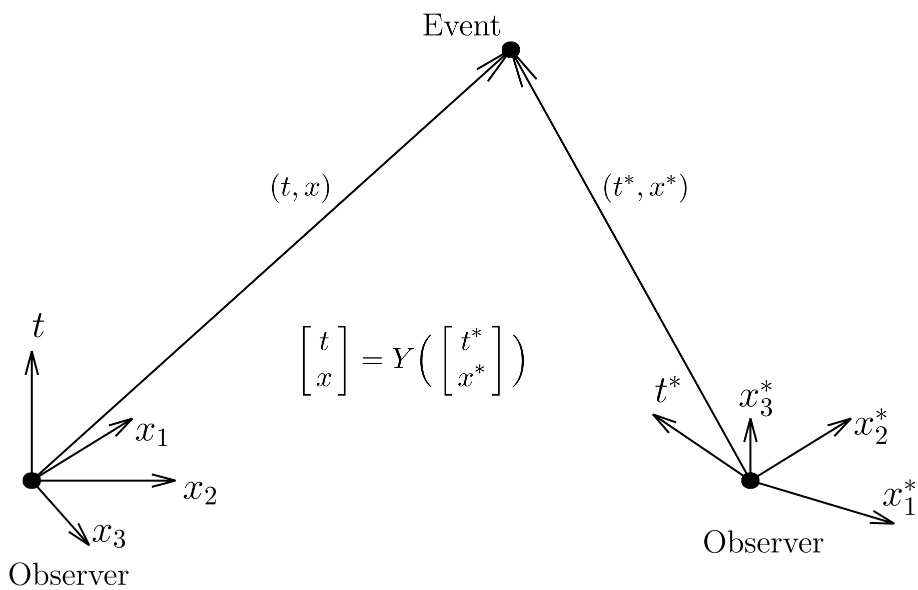


Fig. 1: Physical spacetime

The last property is "objectivity" and states, that the description of physical processes has to be the same for all observers. This applies to formulations with differential equations, see section 3, as well as to formulations with constitutive relations or constraints, see section 4. The first property is the well known "relativity", an classical example being the Doppler effect.

The second property is "rationality" saying, that it is possible to describe analytically, how quantities change under observer transformations.

Therefore the description of any situation in classical continuum physics has to be ***frame indifferent***, and this description includes everything like differential equations, constitutive relations, the domain of definition, positivity of functions, et cetera. In order to formulate the principle of objectivity, one has to specify how coordinates transform. A general observer transformation is a bijective mapping

$$y = Y(y^*)\,,$$

which transforms the spacetime coordinates $y^* \in \mathbb{R}^{n+1}$ of one observer into the spacetime coordinates $y \in \mathbb{R}^{n+1}$ of the other observer.

In this lecture we restrict mainly, except in chapter VI, to classical continuum physics, that is, we consider Newton transformations, see section 1. We add a special section 2, where we introduce Lorentz transformations, which are different from the classical ones. In both cases we deal with spacetime coordinates $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ for one observer and for the other observer they are called $y^* = (t^*, x^*) \in \mathbb{R} \times \mathbb{R}^n$, and the mapping Y between them is of the form

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\Big(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\Big) = \begin{bmatrix} T(t^*, x^*) \\ X(t^*, x^*) \end{bmatrix}. \tag{II0.1}$$

In this situation, if the other observer is located in his coordinate system at the origin $\{(t^*, x^*)\,;\ x^* = 0,\, t^* \in \mathbb{R}\}$, he is seen from the observer with (t, x) coordinates at

$$\xi(t) := x = X(t^*, 0) \text{ for } t = T(t^*, 0)\,,$$

that is, $\{(t, \xi(t))\,;\ t \in \mathbb{R}\}$ is the position of the other observer. This implies, that he moves in the (t, x) coordinates with velocity

$$\dot{\xi} = \frac{\dot{X}}{\dot{T}} \text{ with appropriate arguments.}$$

In detail: We compute

$$\frac{\mathrm{d}}{\mathrm{d}t^*} X(t^*, 0) = \frac{\mathrm{d}}{\mathrm{d}t^*} \xi(T(t^*, 0)) = \frac{\mathrm{d}}{\mathrm{d}t} \xi(t) \frac{\mathrm{d}}{\mathrm{d}t^*} T(t^*, 0)\,,$$

and since $\partial_{t^*} T > 0$ the assertion is valid.
In contrast to Lorentz transformations, where $T(t^*, x^*)$ depends linearly on x^*, in classical transformations $T(t^*, x^*)$ is independent of x^*.

The objectivity is the main subject in this chapter. We mention that all statements proved so far are fully consistent with the principle of classical objectivity. This you can easily verify.

In general, the formulation of a class of physical processes has to be frame independent, that is, it has to be the same for all observers. This applies to all statements which are "published" and in particular the following statements are independent of the observer:

- Conservation laws (see section 3). Conservation laws have been introdced in section I.1 and they are the basis for the dynamics in physical studies.

- Constitutive functions (see section 4). Constitutive function arise as description of particular materials and they are part of the underlying conservation laws.).

Die Objektivität von Erhaltungsgleichungen hat Konsequenzen für die beteiligten physikalischen Größen, d.h. physikalische Größen sind dadurch definiert, wie sie in welchen Erhaltungssätzen auftreten. Dies bestimmt die Struktur der konstitutiven Funktionen, eine Herangehensweise, die hier also anders ist als in der bisherigen Standardliteratur.

1 Classical observers transformations

Classical observers transformations have the significant property that $T(t^*, x^*)$ depends only on t^*, and this dependence is linear with factor 1.

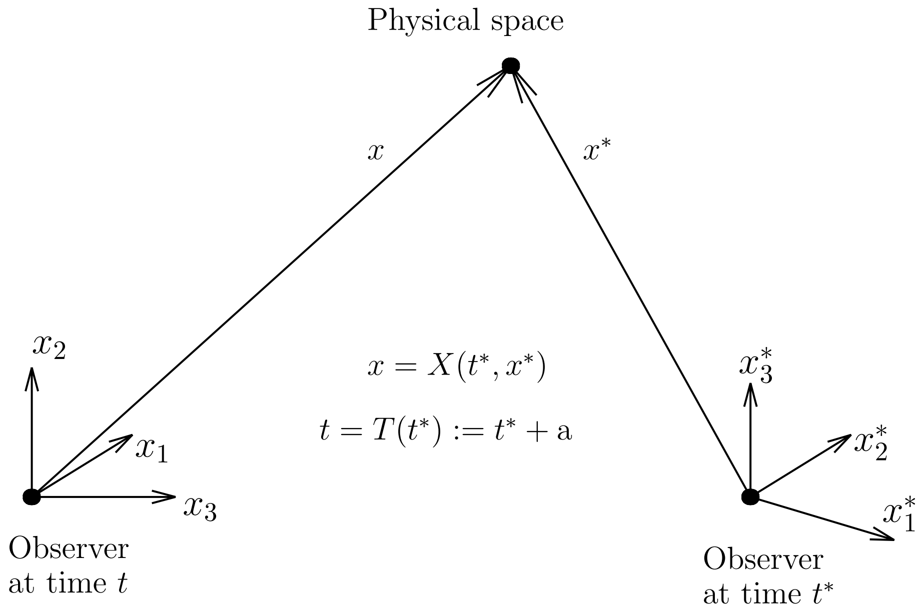


Fig. 2: Classical observers

The simplest case is a Galilei transformation, a linear spacetime transformation defined by a ***Galilei matrix***

$$\mathbf{G}(V,Q) := \begin{bmatrix} 1 & 0 \\ V & Q \end{bmatrix}, \tag{II1.1}$$

where $V \in \mathbb{R}^n$ is a "velocity" and Q an orthonormal $n \times n$-transformation with positive determinant, that is a transformation satisfying $Q^{\mathrm{T}} Q = \mathrm{Id}$ and $\det Q = 1$. This matrix reflects, how the coordinate systems are rotated towards each other. The set of these matrices will be denoted by $\mathcal{G}_{\mathrm{Galilei}}$. Each matrix defines a linear transformation.

1.1 Galilei transformation. A ***Galilei transformation***, $Y \in \mathcal{T}_{\mathrm{Galilei}}$, is given by a linear coordinate transformation Y of coordinates (t^*, x^*) into coordinates (t, x) given by

$$\begin{aligned} \begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) &:= \begin{bmatrix} t_0 \\ x_0 \end{bmatrix} + \mathbf{G}(V,Q)\begin{bmatrix} t^* - t_0^* \\ x^* - x_0^* \end{bmatrix} \\ &= \begin{bmatrix} t^* + (t_0 - t_0^*) \\ Qx^* + (t^* - t_0^*)V + x_0 - Qx_0^* \end{bmatrix} \end{aligned} \tag{II1.2}$$

with two given points (t_0, x_0) and (t_0^*, x_0^*), and a matrix $\mathbf{G}(V,Q) \in \mathcal{G}_{\mathrm{Galilei}}$.

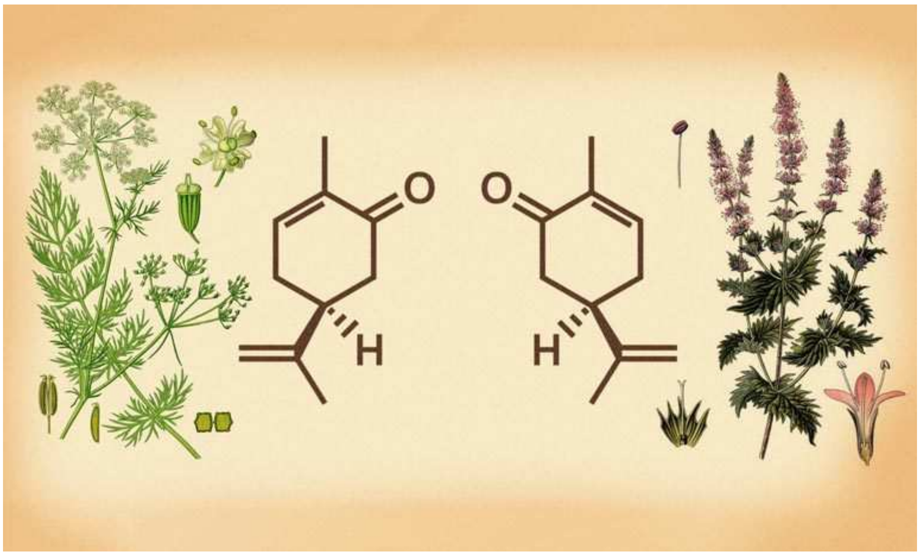


Fig. 3: “Chirotope in der Praxis: Moleküle mit der gleichen chemischen Formel sind chiral, wenn sich das Bild der einen nicht durch Drehung mit dem anderen zur Deckung bringen lässt. ⋯ Chirale Moleküle können ganz unterschiedliche Wirkungen haben. So duftet die linksdrehende Form der Substanz Carvon nach Kümmel, die rechtsdrehende hingegen nach Minze” from [KlarText 2018]

The mapping Y is thus determined by the matrix $\mathbf{G}(V,Q)$ and the two vectors (t_0, x_0) and (t_0^*, x_0^*). Why is $\det Q = 1$ assumed for Q? The answer is in Fig. 3, there the left molecule is not a rotation of the right molecule, if the molecules are seen in a 3D-version..

1.2 Group property. The set of Galilei'an matrices $\mathcal{G}_{\text{Galilei}}$ is a group, where $\mathbf{G}(0, \text{Id})$ is the unity, and where the inverse is given by

$$\mathbf{G}(V,Q)^{-1} = \mathbf{G}(-Q^{\mathrm{T}}\,V, Q^{\mathrm{T}})\,.$$

The same holds also for the set $\mathcal{T}_{\text{Galilei}}$ of Galilei transformations (II1.2), where the unit is given by $V = 0$, $Q = \text{Id}$, and $(t_0, x_0) = (t_0^*, x_0^*)$. (See also 7.1.)

Proof. Are $\mathbf{G}(V_1, Q_1)$ and $\mathbf{G}(V_2, Q_2)$ be two matrices, the matrix multiplication is given by

$$\begin{aligned}
&\mathbf{G}(V_1,Q_1)\mathbf{G}(V_2,Q_2) = \\
&= \begin{bmatrix} 1 & 0 \\ V_1 & Q_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ V_2 & Q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ V_1 + Q_1 V_2 & Q_1 Q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ V & Q \end{bmatrix} \\
&= \mathbf{G}(V,Q),
\end{aligned}$$

if

$$V = V_1 + Q_1 V_2 \quad \text{and} \quad Q = Q_1 Q_2 ,$$

a matrix Q, which again is orthonormal and has determinant 1.

If we set specifically $\mathbf{G}(V,Q) = \mathbf{G}(0,\mathrm{Id}) = \mathrm{Id}$, hence $V_1 + Q_1 V_2 = V = 0$ and $Q_1 Q_2 = Q = \mathrm{Id}$, so we have

$$V_2 = -{Q_1}^{\mathrm{T}} V_1 \quad \text{and} \quad Q_2 = {Q_1}^{\mathrm{T}} .$$

Thus $\mathbf{G}(-{Q_1}^{\mathrm{T}} V_1, {Q_1}^{\mathrm{T}})$ is the right inverse of $\mathbf{G}(V_1, Q_1)$. Similarly, it follows that $\mathbf{G}(-{Q_2}^{\mathrm{T}} V_2, {Q_2}^{\mathrm{T}})$ is the left inverse of $\mathbf{G}(V_2, Q_2)$. □

We now consider the general nonlinear classical observer transformations, which allow accelerations and rotations.

1.3 Newton transformation. We denote this set by $\mathcal{T}_{\mathrm{Newton}}$. A map Y is called a ***Newtonian transformation*** of the coordinates $y^* = (t^*, x^*)$ into coordinates $y = (t, x)$, if

Newton's transformation:

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\Big(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\Big) = \begin{bmatrix} T(t^*) \\ X(t^*, x^*) \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Q(t^*)x^* + \mathrm{b}(t^*) \end{bmatrix}$$

(t, x) coordinates of one observer, (II1.3)

(t^*, x^*) coordinates of onother observer,

$\mathrm{a} \in \mathbb{R}\,, \quad t^* \mapsto \mathrm{b}(t^*) \in \mathbb{R}^n\,, \quad t^* \mapsto Q(t^*) \in \mathbb{R}^{n\times n}\,,$

$Q(t^*)$ orthonormal with $\det Q(t^*) = 1$.

with a number $\mathrm{a} \in \mathbb{R}$ and smooth functions $\mathrm{b} : \mathbb{R} \to \mathbb{R}^n$ and $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ satisfying $Q^{\mathrm{T}} Q = \mathrm{Id}$ and $\det Q = 1$. *Remark:* By smooth we mean in general C^2.

The derivative of a Newton transformation Y is

$$\mathrm{D}_{(t^*,x^*)} Y = (Y_{k\,'l})_{k,l=0,\dots,n} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} ,$$

where $\dot{X}$ denotes the time derivative

$$\dot{X}(t^*, x^*) := \frac{\partial}{\partial t^*} X(t^*, x^*) = \dot{Q}(t^*)x^* + \dot{\mathrm{b}}(t^*)$$

and similar $\dot{Q}$ and $\dot{\mathrm{b}}$. Clearly, the following lemma holds.

1.4 Lemma. If Y is a Newton transformation, then for (t_0^*, x_0^*) the linear approximation of Y,

$$\begin{bmatrix} t^* \\ x^* \end{bmatrix} \mapsto \begin{bmatrix} t \\ x \end{bmatrix} = Y(t_0^*, x_0^*) + \mathrm{D}_{(t^*,x^*)} Y(t_0^*, x_0^*) \begin{bmatrix} t^* - t_0^* \\ x^* - x_0^* \end{bmatrix}$$

is a Galilei'an transformation. Every Galilei transformation is a Newton transformation.

Proof. The derivative of a Newton transformation at (t_0^*, x_0^*) is

$$\mathrm{D}_{(t^*,x^*)}Y(t_0^*, x_0^*) = \begin{bmatrix} 1 & 0 \\ \dot{Q}(t_0^*)x_0^* + \dot{\mathrm{b}}(t_0^*) & Q(t_0^*) \end{bmatrix},$$

which for fixed (t_0^*, x_0^*) is a Galilei'an matrix, since $Q(t_0^*)$ is an orthonormal transformation with determinant 1. □

About the "relative velocity" $V := \dot{X}$ we make the following remark, see [Wikipedia: Doppler effect] and as a comparison 2.2. The equation (II1.4) justifies the notion made for V.

1.5 Doppler effect. Beobachtet ein Beobachter ein Objekt mit $t \mapsto \xi(t)$ und ein anderer Beobachter dasselbe Objekt mit $t^* \mapsto \xi^*(t^*)$ so gilt nach der Gleichung (II1.3)

$$\begin{bmatrix} t \\ \xi(t) \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ \xi^*(t^*) \end{bmatrix}\right) = \begin{bmatrix} t^* + \mathrm{a} \\ X(t^*, \xi^*(t^*)) \end{bmatrix},$$

also für die Geschwindigkeiten des beobachteten Objektes

$$\dot{\xi}(t) = \underbrace{\dot{X}(t^*, \xi^*(t^*))}_{=:\, V(t^*, \xi^*(t^*))} + \underbrace{\mathrm{D}_{x^*}X(t^*, \xi^*(t^*))}_{=\, Q(t^*)}\, \dot{\xi}^*(t^*)$$

somit

$$\dot{\xi}(t^* + \mathrm{a}) = V(t^*, \xi^*(t^*)) + Q(t^*)\dot{\xi}^*(t^*)\,. \tag{II1.4}$$

Speziell: Ist der *-Beobachter an der Position $\xi^*(t^*) = 0$ so ist er vom Standardbeobachter aus gesehen an der Position $\xi(t)$ mit

$$\dot{\xi}(t) = V\,.$$

Ist der Standardbeobachter an der Position $\xi(t) = 0$ so ist er vom *-Beobachter aus gesehen an der Position $\xi^*(t^*)$ mit

$$\dot{\xi}^*(t^*) = -Q^{\mathrm{T}}\, V\,.$$

Also the class of nonlinear Newton transformations satifies the

1.6 Group property. The set $\mathcal{T}_{\text{Newton}}$ of Newton transformations in 1.3 satifies the group property. The inverse of the function Y in 1.3 is given by

$$\begin{bmatrix} t^* \\ x^* \end{bmatrix} = Y^{-1}\left(\begin{bmatrix} t \\ x \end{bmatrix}\right) = \begin{bmatrix} t - \mathrm{a} \\ Q(t-\mathrm{a})^{\mathrm{T}}\,(x - \mathrm{b}(t-\mathrm{a})) \end{bmatrix},$$

that is, $Y^* = Y^{-1}$ with

$$\mathrm{a}^* = -\mathrm{a}\,, \quad Q^*(t) = Q(t-\mathrm{a})^{\mathrm{T}}\,, \quad \mathrm{b}^*(t) = -Q(t-\mathrm{a})^{\mathrm{T}}\,\mathrm{b}(t-\mathrm{a}).$$

Proof. If two mappings $(t_1, x_1) = N_{12}(t_2, x_2)$ and $(t_2, x_2) = N_{23}(t_3, x_3)$ of Newton type,

$$\begin{bmatrix} t_1 \\ x_1 \end{bmatrix} = N_{12}\Big(\begin{bmatrix} t_2 \\ x_2 \end{bmatrix}\Big) = \begin{bmatrix} t_2 + \mathrm{a}_{12} \\ Q_{12}(t_2)x_2 + \mathrm{b}_{12}(t_2) \end{bmatrix},$$
$$\begin{bmatrix} t_2 \\ x_2 \end{bmatrix} = N_{23}\Big(\begin{bmatrix} t_3 \\ x_3 \end{bmatrix}\Big) = \begin{bmatrix} t_3 + \mathrm{a}_{23} \\ Q_{23}(t_3)x_3 + \mathrm{b}_{23}(t_3) \end{bmatrix},$$

are given, then

$$\begin{aligned} \begin{bmatrix} t_1 \\ x_1 \end{bmatrix} &= N_{12}\circ N_{23}\Big(\begin{bmatrix} t_3 \\ x_3 \end{bmatrix}\Big) \\ &= \begin{bmatrix} t_3 + \mathrm{a}_{23} + \mathrm{a}_{12} \\ Q_{12}(t_2)(Q_{23}(t_3)x_3 + \mathrm{b}_{23}(t_3)) + \mathrm{b}_{12}(t_3 + \mathrm{a}_{23}) \end{bmatrix} \\ &= \begin{bmatrix} t_3 + \mathrm{a}_{13} \\ Q_{13}(t_3)x_3 + \mathrm{b}_{13}(t_3) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} &\mathrm{a}_{13} := \mathrm{a}_{23} + \mathrm{a}_{12} \\ &Q_{13}(t_3) := Q_{12}(t_3 + \mathrm{a}_{23})Q_{23}(t_3) \\ &\mathrm{b}_{13}(t_3) := Q_{12}(t_3 + \mathrm{a}_{23})\mathrm{b}_{23}(t_3) + \mathrm{b}_{12}(t_3 + \mathrm{a}_{23}). \end{aligned}$$

Now Q_{13} again is an orthonormal transformation with determinant 1. Therefore $N_{13} := N_{12}\circ N_{23}$ is a Newton transformation. And $N_{13} = \mathrm{Id}$, if

$$\begin{bmatrix} t_3 + \mathrm{a}_{23} + \mathrm{a}_{12} \\ Q_{12}(t_2)(Q_{23}(t_3)x_3 + \mathrm{b}_{23}(t_3)) + \mathrm{b}_{12}(t_3 + \mathrm{a}_{23}) \end{bmatrix} = \begin{bmatrix} t_3 \\ x_3 \end{bmatrix},$$

that is

$$\begin{aligned} &\mathrm{a}_{23} + \mathrm{a}_{12} = 0\,, \\ &Q_{12}(t_2)\mathrm{b}_{23}(t_3) + \mathrm{b}_{12}(t_3 + \mathrm{a}_{23}) = 0\,, \\ &Q_{12}(t_2)Q_{23}(t_3) = \mathrm{Id}\,, \end{aligned}$$

which is equivalent to the assertion. □

We have seen that linear approximations of Newtonian transformations are Galilei transformations. And this is indeed, as we shall see, a characterization of Newtonian transformations.

1.7 Theorem. If a C^1-transformation $Y : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ at every point (t_0^*, x_0^*) has a derivative $\mathrm{D}_{(t^*,x^*)}Y(t_0^*, x_0^*)$, which is a Galilei'an Matrix, then Y is a Newton transformation.

Proof. We consider transformations $Y : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, whose derivative is given by

$$\mathrm{D}_{(t^*,x^*)}Y(t^*, x^*) = \mathbf{G}(V(t^*, x^*), Q(t^*, x^*)). \tag{II1.5}$$

If we write

$$Y(t^*, x^*) = \begin{bmatrix} T(t^*, x^*) \\ X(t^*, x^*) \end{bmatrix},$$

we see that (II1.5) means

$$\begin{bmatrix} T_{\prime t^*} & (\mathrm{D}_{x^*}T)^{\mathrm{T}} \\ X_{\prime t^*} & \mathrm{D}_{x^*}X \end{bmatrix} = \mathrm{D}_{(t^*,x^*)}Y = \mathbf{G}(V, Q) = \begin{bmatrix} 1 & 0 \\ V & Q \end{bmatrix},$$

that is

$$\begin{aligned} T_{\prime t^*} &= 1, \quad \mathrm{D}_{x^*}T = 0, \\ X_{\prime t^*} &= V, \quad \mathrm{D}_{x^*}X = Q. \end{aligned} \tag{II1.6}$$

Here Q is an orthonormal transformation with $Q^{\mathrm{T}} Q = \mathrm{Id}$, that is

$$\sum_{i\geq 1} Q_{ik}Q_{il} = \delta_{kl} \quad \text{for } k, l = 1, \ldots, n.$$

It follows, that

$$A_{mkl} := \sum_{i\geq 1} X_{i\,\prime km}X_{i\,\prime l} \quad \text{for } k, l, m = 1, \ldots, n$$

satisfies

$$\begin{aligned} A_{mkl} + A_{mlk} &= \sum_{i\geq 1} X_{i\,\prime km}X_{i\,\prime l} + \sum_{i\geq 1} X_{i\,\prime k}X_{i\,\prime lm} \\ &= \Big(\sum_{i\geq 1} X_{i\,\prime k}X_{i\,\prime l}\Big)_{\prime m} = \Big(\sum_{i\geq 1} Q_{ik}Q_{il}\Big)_{\prime m} = (\delta_{kl})_{\prime m} = 0\,, \end{aligned}$$

that is, A_{mkl} is antisymmetric in k and l. Since A_{mkl} is symmetric in m and k, it follows from a known lemma (see Lemma 1.8 below), that the matrix $C = (A_{kji})_{i,j,k=1,\ldots,n}$ is zero. Hence, since also $QQ^{\mathrm{T}} = \mathrm{Id}$ as shown in the proof of I.1.3, that is

$$\sum_{l\geq 1} Q_{il}Q_{jl} = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n,$$

we derive for m and k, since we showed that $A_{mkl} = 0$ for all l,

$$0 = \sum_{l} A_{mkl}Q_{jl} = \sum_{l}\sum_{i} X_{i\,\prime km}Q_{il}Q_{jl} = \sum_{i} X_{i\,\prime km}\delta_{ij} = X_{j\,\prime km},$$

that is $\mathrm{D}^2_{x^*}X = 0$. Therefore $(t^*, x^*) \mapsto X(t^*, x^*)$ is (affin) linear in x^*, that is

$$X(t^*, x^*) = \bar{b}(t^*) + \bar{Q}(t^*)x^*$$

with a vector $\bar{b}(t^*)$ and a matrix $\bar{Q}(t^*)$.

Then (II1.6) says $Q(t^*, x^*) = \mathrm{D}_{x^*}X(t^*, x^*) = \bar{Q}(t^*)$ is an orthonormal matrix, and (II1.6) also says that $T(t^*, x^*) = t^* + \bar{a}$ with a constant $\bar{a}$. This results in

$$Y(t^*, x^*) = \begin{bmatrix} t^* + \bar{a} \\ \bar{b}(t^*) + \bar{Q}(t^*)x^* \end{bmatrix},$$

from which the result follows. □

1.8 Fundamental lemma. Suppose $C = (C_{ijk})_{i,j,k=1,\dots,N}$ is a 3-matrix in $\mathbb{R}^N$ with $N \geq 1$, which is antisymmetric in the first two indices, and symmetric in the last two indices. Then $C = 0$.

Proof. It is

$$C_{lkj} = -C_{klj} = -C_{kjl} = C_{jkl}\,,$$
$$C_{lkj} = C_{ljk} = -C_{jlk} = -C_{jkl}\,.$$

Hence $C_{jkl} = 0$. □

2 Lorentz transformations

References: See [Wikipedia: History of Lorentz transformations] and also the paper by Lorentz [113] for historical reasons. The proof of the Thomas rotation in 2.5 can be found in [68]. In general see chapter VI for relativistic physics.

A Lorentz transformation is a linear transformation given by a Lorentz matrix. A ***Lorentz matrix*** is defined for $c > 0$ and for $V \in \mathbb{R}^n$ with $|V| < c$ by

$$\begin{aligned} \mathbf{L}_c(V,Q) &:= \begin{bmatrix} \gamma & \dfrac{\gamma}{c^2} V^{\mathrm{T}} Q \\ \gamma V & \left(\mathrm{Id} + \dfrac{\gamma^2}{c^2(\gamma+1)} V V^{\mathrm{T}}\right) Q \end{bmatrix}, \\ \gamma &:= \frac{1}{\sqrt{1 - \frac{|V|^2}{c^2}}} \geq 1. \end{aligned} \tag{II2.1}$$

Here $Q \in \mathbb{R}^{n\times n}$ is an orthonormal transformation with positive determinant, that is $Q^{\mathrm{T}} Q = \mathrm{Id}$ and $\det Q = 1$. The matrix has the decomposition

$$\mathbf{L}_c(V,Q) = \mathbf{L}_c(V,\mathrm{Id})\mathbf{L}_c(0,Q) = \mathbf{L}_c(V,\mathrm{Id}) \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$$

into a velocity part and a rotation. In components the velocity part has the form

$$\mathbf{L}_c(V,\mathrm{Id}) = \begin{bmatrix} \gamma & \dfrac{\gamma}{c^2} V_1 & \dfrac{\gamma}{c^2} V_2 & \dfrac{\gamma}{c^2} V_3 \\ \gamma V_1 & 1 + \dfrac{\gamma^2}{c^2(\gamma+1)} V_1^2 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_1 V_2 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_1 V_3 \\ \gamma V_2 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_1 V_2 & 1 + \dfrac{\gamma^2}{c^2(\gamma+1)} V_2^2 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_2 V_3 \\ \gamma V_3 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_1 V_3 & \dfrac{\gamma^2}{c^2(\gamma+1)} V_2 V_3 & 1 + \dfrac{\gamma^2}{c^2(\gamma+1)} V_3^2 \end{bmatrix}.$$

We also write

$$\mathbf{L}_c(V,Q) = \begin{bmatrix} \gamma & \frac{\gamma}{c^2} V^{\mathrm{T}} Q \\ \gamma V & \mathbf{B}_c(V)Q \end{bmatrix}, \tag{II2.2}$$

where

$$\begin{aligned} \mathbf{B}_c(V) := & \ \mathrm{Id} + \frac{\gamma^2}{c^2(\gamma+1)} V V^{\mathrm{T}} \quad \text{for } |V| < c\,, \\ = & \ \mathrm{Id} + (\gamma - 1)\frac{V V^{\mathrm{T}}}{|V|^2} \quad \text{for } 0 < |V| < c\,, \end{aligned} \tag{II2.3}$$

which is true because

$$\frac{\gamma - 1}{|V|^2} = \frac{\gamma^2 - 1}{(\gamma+1)|V|^2} = \frac{\gamma^2}{c^2(\gamma+1)}\,.$$

The matrix $\mathbf{B}_c(V)$ is the so-called "boost" in the direction of the "relative velocity" V, that is

$$\begin{aligned} &\mathbf{L}_c(V,\mathrm{Id}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma V \end{bmatrix}, \\ &\mathbf{L}_c(V,\mathrm{Id}) \begin{bmatrix} 0 \\ V \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{c^2}|V|^2 \\ (1 + (\gamma-1))V \end{bmatrix} = \begin{bmatrix} \gamma - \frac{1}{\gamma} \\ \gamma V \end{bmatrix}, \\ &\mathbf{L}_c(V,\mathrm{Id}) \begin{bmatrix} 0 \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ e \end{bmatrix} \quad \text{for } e \bullet V = 0\,. \end{aligned} \tag{II2.4}$$

We have that $\det \mathbf{L}_c(V,Q) = 1$ (see 7.3). The class of Lorentz matrices give rise to linear observer transformations (this is the analogon to 1.2).

2.1 Lorentz transformation. A ***Lorentz transformation*** in $\mathcal{T}_{\mathrm{Lorentz}}$ is given by a linear coordinate transformation Y of coordinates (t^*, x^*) into coordinates (t, x) given by

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) := \begin{bmatrix} t_0 \\ x_0 \end{bmatrix} + \mathbf{L}_c(V,Q) \begin{bmatrix} t^* - t_0^* \\ x^* - x_0^* \end{bmatrix} \tag{II2.5}$$

with two given points (t_0, x_0) and (t_0^*, x_0^*). The mapping Y is therefore determined by the matrix $\mathbf{L}_c(V,Q)$ and two vectors (t_0, x_0) and (t_0^*, x_0^*).

2.2 Bewegung eines Objektes. Beobachtet ein Beobachter ein Objekt mit $t \mapsto \xi(t)$ und ein anderer Beobachter dasselbe Objekt mit $t^* \mapsto \xi^*(t^*)$ so gilt nach der Transformation (II2.5)

$$\begin{bmatrix} t \\ \xi(t) \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ \xi^*(t^*) \end{bmatrix}\right) = \begin{bmatrix} T(t^*, \xi^*(t^*)) \\ X(t^*, \xi^*(t^*)) \end{bmatrix} = \begin{bmatrix} \gamma & \frac{\gamma}{c^2} V^{\mathrm{T}} Q \\ \gamma V & \mathbf{B}_c(V)Q \end{bmatrix} \begin{bmatrix} t^* \\ \xi^*(t^*) \end{bmatrix},$$

wobei die zwei Vektoren $(t_0, x_0) = 0$ und $(t_0^*, x_0^*) = 0$ sein sollen.

(1) Dies ist äquivalent zu

$$t = \gamma t^* + \frac{\gamma}{c^2} V \bullet Q\xi^*(t^*)\,,$$
$$\xi(t) = \gamma t^* V + \mathbf{B}_c(V) Q\xi^*(t^*)\,.$$

Vorsicht: Es sind im Allgemeinen nicht ξ und ξ^* Funktionen.

(2) Wenn t eine monoton wachsende Funktion von t^* ist, so gilt für die "Geschwindigkeiten" des Objekts

$$\dot{\xi}(t) = \frac{V + \frac{1}{\gamma}\mathbf{B}_c(V)Q\dot{\xi}^*(t^*)}{1 + \frac{1}{c^2} V \bullet Q\dot{\xi}^*(t^*)}\,,$$

wobei der Nenner positiv sein soll, was für $|\dot{\xi}^*| \leq c$ erfüllt ist.

(3) Wenn $v = \dot{\xi}(t)$ und $v^* = \dot{\xi}^*(t^*)$ mit $|v^*| \leq c$ sind, so folgt mit $\bar{v} := Qv^*$ aus (2)

$$v = \frac{V + \frac{1}{\gamma}\mathbf{B}_c(V)\bar{v}}{1 + \frac{1}{c^2} V \bullet \bar{v}}\,.$$

Beachte: Dies ist die relativistische "Addition von Geschwindigkeiten", siehe [Wikipedia: Velocity-addition formula].

(4) Der *-Beobachter an der Position $\xi^*(t^*) = 0$ ist vom Standardbeobachter aus gesehen an der Position $\xi(t)$ mit

$$\dot{\xi}(t) = V\,,$$

der Standardbeobachter an der Position $\xi(t) = 0$ ist vom *-Beobachter aus gesehen an der Position $\xi^*(t^*)$ mit

$$\dot{\xi}^*(t^*) = -Q^{\mathrm{T}} V\,,$$

Es ist also wichtig für die "Geschwindigkeiten", welche Zeit gemeint ist (siehe dazu auch [23, Abschnitt II.5]).

Proof (2). Compute the t^*-derivative of $\xi(T(t^*, \xi^*(t^*))) = X(t^*, \xi^*(t^*))$ and obtain $\dot{\xi}(t)\frac{\mathrm{d}}{\mathrm{d}t^*}T = \frac{\mathrm{d}}{\mathrm{d}t^*}X$. □

There are also different forms of a Lorentz matrix, which are valid for all vectors $\widetilde{V} \in \mathbb{R}^n$ and $\bar{V} \in \mathbb{R}^n$.

2.3 Remark. For $V \in \mathbb{R}^n$ with $|V| < c$ we used

$$\gamma := \frac{1}{\sqrt{1 - \frac{|V|^2}{c^2}}} > 1. \tag{II2.6}$$

Then $\widetilde{V} \in \mathbb{R}^n$ runs over all of $\mathbb{R}^n$, defined by

$$\widetilde{V} := \frac{1}{\sqrt{1 - |\frac{V}{c}|^2}} V = \gamma V.$$

Also $\bar{V} \in \mathbb{R}^n$ runs over all of $\mathbb{R}^n$, defined by

$$\bar{V} := \frac{1}{c}\widetilde{V} = \frac{\gamma}{c} V.$$

(1) With the definition of $\widetilde{V}$

$$V = \frac{1}{\sqrt{1 + \left|\frac{\widetilde{V}}{c}\right|^2}} \widetilde{V} = \frac{1}{\gamma}\widetilde{V}, \quad \gamma = \sqrt{1 + \left|\frac{\widetilde{V}}{c}\right|^2}$$

and it follows, that for all orthonormal $Q \in \mathbb{R}^{n\times n}$

$$\widetilde{\mathbf{L}}_c(\widetilde{V}, Q) := \mathbf{L}_c\Big(\frac{\widetilde{V}}{\sqrt{1 + \left|\frac{\widetilde{V}}{c}\right|^2}}, Q\Big) = \begin{bmatrix} \gamma & \frac{1}{c^2}\widetilde{V}^{\mathrm{T}} Q \\ \widetilde{V} & \Big(\mathrm{Id} + \frac{1}{c^2(\gamma+1)}\widetilde{V}\,\widetilde{V}^{\mathrm{T}}\Big) Q \end{bmatrix}.$$

(2) With the definition of $\bar{V}$

$$V = \frac{c}{\sqrt{1 + |\bar{V}|^2}} \bar{V} = \frac{c}{\gamma}\bar{V}, \quad \gamma = \sqrt{1 + |\bar{V}|^2}$$

and it follows, that for all orthonormal $Q \in \mathbb{R}^{n\times n}$

$$\begin{aligned}
\mathbf{L}_c\Big(\frac{c\bar{V}}{\sqrt{1 + |\bar{V}|^2}}, Q\Big) &= \begin{bmatrix} \gamma & \frac{1}{c}\bar{V}^{\mathrm{T}} Q \\ c\bar{V} & \Big(\mathrm{Id} + \frac{1}{\gamma+1}\bar{V}\,\bar{V}^{\mathrm{T}}\Big) Q \end{bmatrix} \\
&= \mathbf{I}_{\frac{1}{c}} \mathbf{L}(\bar{V}) \mathbf{I}_c \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}, \\
\mathbf{L}(\bar{V}) &:= \begin{bmatrix} \bar{\gamma} & \bar{V}^{\mathrm{T}} \\ \bar{V} & \mathrm{Id} + \frac{1}{\bar{\gamma}+1}\bar{V}\,\bar{V}^{\mathrm{T}} \end{bmatrix}, \\
\mathbf{I}_a &:= \begin{bmatrix} a & 0 \\ 0 & \mathrm{Id} \end{bmatrix}, \quad \bar{\gamma} = \sqrt{1 + |\bar{V}|^2} = \gamma.
\end{aligned}$$

Proof (1). With $\widetilde{V} = \gamma V$ we compute

$$\begin{aligned}
\mathbf{L}_c(V, Q) &= \begin{bmatrix} \gamma & \frac{\gamma}{c^2} V^{\mathrm{T}} Q \\ \gamma V & \Big(\mathrm{Id} + (\gamma - 1)\frac{V\,V^{\mathrm{T}}}{|V|^2}\Big) Q \end{bmatrix} \\
&= \begin{bmatrix} \gamma & \frac{1}{c^2}\widetilde{V}^{\mathrm{T}} Q \\ \widetilde{V} & \Big(\mathrm{Id} + (\gamma - 1)\frac{\widetilde{V}\,\widetilde{V}^{\mathrm{T}}}{|\widetilde{V}|^2}\Big) Q \end{bmatrix}
\end{aligned}$$

and

$$\frac{\gamma - 1}{|\widetilde{V}|^2} = \frac{1}{c^2(\gamma+1)} \text{ since } \gamma^2 = 1 + \frac{|\widetilde{V}|^2}{c^2}.$$

□

Proof (2). With $\bar{V} = \frac{\gamma}{c} V$ we compute

$$\mathbf{L}_c(V,Q) = \begin{bmatrix} \gamma & \dfrac{\gamma}{c^2} V^{\mathrm{T}} Q \\ \gamma V & \left(\mathrm{Id} + (\gamma-1)\dfrac{V\,V^{\mathrm{T}}}{|V|^2}\right) Q \end{bmatrix} = \begin{bmatrix} \gamma & \dfrac{1}{c} \bar{V}^{\mathrm{T}} Q \\ c\bar{V} & \left(\mathrm{Id} + (\gamma-1)\dfrac{\bar{V}\,\bar{V}^{\mathrm{T}}}{|\bar{V}|^2}\right) Q \end{bmatrix}$$

and

$$\frac{\gamma-1}{|\bar{V}|^2} = \frac{1}{\gamma+1} \text{ since } \gamma^2 = 1 + |\bar{V}|^2 .$$

□

The linear coordinate transformations given by Lorentz transformations converge towards Galilei transformations, if c tends to infinity, see lemma 2.4. There are also cases, where a Lorentz matrix coincides with a Galilei matrix, this is the case when only a rotation is applied:

$$\mathbf{L}_c(0,Q) = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \mathbf{G}(0,Q) .$$

2.4 Lemma. For every V, $\widetilde{V}$, and Q it holds

$$\mathbf{L}_c(V,Q) \to \mathbf{G}(V,Q) \quad \text{as } c \to \infty,$$
$$\widetilde{\mathbf{L}}_c(\widetilde{V},Q) \to \mathbf{G}(\widetilde{V},Q) \quad \text{as } c \to \infty.$$

Proof. This follows immediately. □

Für die linearen Lorentz-Transformationen gilt die folgende

2.5 Group property. The set of Lorentz matrices $\mathcal{G}_{\text{Lorentz}}$ is a group, where $\mathbf{L}_c(0,\mathrm{Id})$ is the identity and the inverse of $\mathbf{L}_c(V,\mathrm{Id})$ is given by

$$\mathbf{L}_c(V,Q)^{-1} = \mathbf{L}_c(-Q^{\mathrm{T}} V, Q^{\mathrm{T}}) .$$

The same holds also for the set $\mathcal{T}_{\text{Lorentz}}$ of Lorentz transformations (II2.5). where the unit is given by $V = 0$, $Q = \mathrm{Id}$, and $(t_0, x_0) = (t_0^*, x_0^*)$.
Hinweis: For a non-explicit proof see section [23, I.3].

Proof. We show the result by explicit computation. It is elementary to see that

$$\mathbf{L}_c(V,Q) = \mathbf{L}_c(V,\mathrm{Id})\mathbf{L}_c(0,Q),$$
$$\mathbf{L}_c(V,Q) = \mathbf{L}_c(0,Q)\mathbf{L}_c(Q^{\mathrm{T}} V,\mathrm{Id}),$$
$$\mathbf{L}_c(0,Q_1)\mathbf{L}_c(0,Q_2) = \mathbf{L}_c(0,Q_1 Q_2).$$

After the essential Thomas-identity in (II2.7) is shown, this completes the axioms of a semigroup.

In particular this implies, denoting

$$B = \mathbf{B}_c(V) = \mathrm{Id} + \frac{\gamma^2}{c^2(\gamma+1)} V\,V^{\mathrm{T}} \;,$$

that (note, that γ has the same value for V and $-V$)

$$\begin{aligned}
&\mathbf{L}_c(V,Q)\mathbf{L}_c(-Q^{\mathrm{T}}\,V,Q^{\mathrm{T}})\\
&= \mathbf{L}_c(V,\mathrm{Id})\; \underbrace{\mathbf{L}_c(0,Q)\mathbf{L}_c(0,Q^{\mathrm{T}})}_{=\,\mathrm{Id}}\; \mathbf{L}_c(-V,\mathrm{Id})\\
&= \mathbf{L}_c(V,\mathrm{Id})\mathbf{L}_c(-V,\mathrm{Id})\\
&= \begin{bmatrix} \gamma & \frac{\gamma}{c^2}V^{\mathrm{T}} \\ \gamma V & B \end{bmatrix} \begin{bmatrix} \gamma & -\frac{\gamma}{c^2}V^{\mathrm{T}} \\ -\gamma V & B \end{bmatrix}\\
&= \begin{bmatrix} \gamma^2\Big(1-\frac{|V|^2}{c^2}\Big) & \frac{\gamma}{c^2}(-\gamma V + BV)^{\mathrm{T}} \\ \gamma(\gamma V - BV) & BB - \frac{\gamma^2}{c^2}V\,V^{\mathrm{T}} \end{bmatrix} = \mathrm{Id},
\end{aligned}$$

which is true because

$$\begin{aligned}
&\gamma^2\Big(1-\frac{|V|^2}{c^2}\Big) = 1,\\
&BV = \gamma V,\\
&BB = \Big(\mathrm{Id} + \frac{\gamma^2}{c^2(\gamma+1)}V\,V^{\mathrm{T}}\Big)^2\\
&= \mathrm{Id} + \frac{\gamma^2}{c^2(\gamma+1)}\Big(2 + \frac{|V|^2\gamma^2}{c^2(\gamma+1)}\Big)V\,V^{\mathrm{T}} = \mathrm{Id} + \frac{\gamma^2}{c^2}V\,V^{\mathrm{T}}
\end{aligned}$$

since

$$2 + \frac{|V|^2\gamma^2}{c^2(\gamma+1)} = 2 + \Big(1 - \frac{1}{\gamma^2}\Big)\frac{\gamma^2}{\gamma+1} = \gamma + 1\,.$$

Thus we have shown, that $\mathbf{L}_c(-Q^{\mathrm{T}}\,V,Q^{\mathrm{T}})$ is the right inverse. It is also the left inverse, because

$$\begin{aligned}
&\mathbf{L}_c(-Q^{\mathrm{T}}\,V,Q^{\mathrm{T}})\mathbf{L}_c(V,Q)\\
&= \mathbf{L}_c(0,Q^{\mathrm{T}})\; \underbrace{\mathbf{L}_c(-V,\mathrm{Id})\mathbf{L}_c(V,\mathrm{Id})}_{=\,\mathrm{Id}\ \text{(see above)}}\; \mathbf{L}_c(0,Q)\\
&= \mathbf{L}_c(0,Q^{\mathrm{T}})\mathbf{L}_c(0,Q) = \mathbf{L}_c(0,Q^{\mathrm{T}}\,Q) = \mathrm{Id}.
\end{aligned}$$

Therefore $\mathbf{L}_c(V,Q)^{-1} = \mathbf{L}_c(-Q^{\mathrm{T}}\,V,Q^{\mathrm{T}})$.

Now the nontrivial step is to show that

$$\mathbf{L}_c(U,\mathrm{Id})\mathbf{L}_c(V,\mathrm{Id}) = \mathbf{L}_c(W,\mathrm{Id})\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{T}(U,V) \end{bmatrix}\;, \tag{II2.7}$$

where W depends on U and V and $\mathbf{T}(U,V)$ is the ***Thomas rotation*** (1926), which can be found in [68],

$$W = \left(\gamma_V + \frac{V\bullet U}{c^2(\gamma_U+1)}\right)U + V\,,$$
$$\mathbf{T}(U,V) = Q \text{ with } Q \text{ orthonormal.}$$

We prove this identity by using the representation of Lorentz transformation with arbitrary vectors in $\mathbb{R}^3$. The assertion (II2.7) then reads

$$\begin{gathered}\widetilde{\mathbf{L}}_c(\widetilde{U},\mathrm{Id})\widetilde{\mathbf{L}}_c(\widetilde{V},\mathrm{Id}) = \widetilde{\mathbf{L}}_c(\widetilde{W},\mathrm{Id})\begin{bmatrix}1 & 0\\ 0 & \widetilde{\mathbf{T}}(\widetilde{U},\widetilde{V})\end{bmatrix},\\ \widetilde{W} := \gamma_{\widetilde{V}}\widetilde{U} + B(\widetilde{U})\widetilde{V}\,,\\ \widetilde{\mathbf{T}}(\widetilde{U},\widetilde{V}) := B(\widetilde{W})^{-1}\left(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\right),\\ \gamma_{\widetilde{U}} := \sqrt{1+\frac{|\widetilde{U}|^2}{c^2}} \text{ and similar } \gamma_{\widetilde{V}},\ \gamma_{\widetilde{W}},\\ B(\widetilde{U}) := \mathrm{Id} + \frac{1}{c^2(\gamma_{\widetilde{U}}+1)}\widetilde{U}\,\widetilde{U}^{\mathrm{T}} \text{ and similar } B(\widetilde{V}),\ B(\widetilde{W}).\end{gathered} \tag{II2.8}$$

Inserting the definition of the Lorentz matrix in 2.3(1) we have to prove the identity

$$\begin{bmatrix}\gamma_{\widetilde{U}} & \frac{1}{c^2}\widetilde{U}^{\mathrm{T}}\\ \widetilde{U} & B(\widetilde{U})\end{bmatrix}\begin{bmatrix}\gamma_{\widetilde{V}} & \frac{1}{c^2}\widetilde{V}^{\mathrm{T}}\\ \widetilde{V} & B(\widetilde{V})\end{bmatrix} = \begin{bmatrix}\gamma_{\widetilde{W}} & \frac{1}{c^2}\widetilde{W}^{\mathrm{T}}\\ \widetilde{W} & B(\widetilde{W})\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & M\end{bmatrix},$$

where M denotes a certain matrix which has to be shown to be the Thomas rotation. Performing the matrix multiplication on both sides this identity becomes

$$\begin{bmatrix}\gamma_{\widetilde{U}}\gamma_{\widetilde{V}} + \dfrac{\widetilde{U}\bullet\widetilde{V}}{c^2} & \dfrac{1}{c^2}\left(\gamma_{\widetilde{U}}\widetilde{V} + B(\widetilde{V})\widetilde{U}\right)^{\mathrm{T}}\\ \gamma_{\widetilde{V}}\widetilde{U} + B(\widetilde{U})\widetilde{V} & B(\widetilde{U})B(\widetilde{V}) + \dfrac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}\gamma_{\widetilde{W}} & \frac{1}{c^2}\widetilde{W}^{\mathrm{T}}M\\ \widetilde{W} & B(\widetilde{W})M\end{bmatrix}.$$

The last row we take as definition for $\widetilde{W}$ and M

$$\begin{aligned}&\widetilde{W} := \gamma_{\widetilde{V}}\widetilde{U} + B(\widetilde{U})\widetilde{V}\\ &M := B(\widetilde{W})^{-1}\left(B(\widetilde{U})B(\widetilde{V}) + \tfrac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\right).\end{aligned} \tag{II2.9}$$

Here the inverse of $B(\widetilde{W})$ exists, since this matrix is positive definite. Therefore it remains to show that

$$\begin{gathered}\gamma_{\widetilde{U}}\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} = \gamma_{\widetilde{W}}\,,\\ M(\gamma_{\widetilde{U}}\widetilde{V} + B(\widetilde{V})\widetilde{U}) = \widetilde{W}\,,\\ M \text{ is an orthogonal transformation with } \det M = 1.\end{gathered} \tag{II2.10}$$

In fact, the second statement gives with the fact that M is an orthogonal transformation, that is $M^{\mathrm{T}} = M^{-1}$,

$$M^{\mathrm{T}}\,\widetilde{W} = \gamma_{\widetilde{U}}\widetilde{V} + B(\widetilde{V})\widetilde{U}\,.$$

Altogether this would complete the proof. □

Here is a proof of the three statements in (II2.10).

Proof 1. Statement. Zu zeigen ist

$$\gamma_{\widetilde{U}}\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} = \sqrt{1 + \frac{|\widetilde{W}|^2}{c^2}}\,.$$

Da die linke Seite positiv ist, denn es gilt

$$\left|\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}\right| = \left|\frac{\widetilde{U}}{c}\right|\cdot\left|\frac{\widetilde{V}}{c}\right| \le \sqrt{\Big(1 + \frac{|\widetilde{U}|^2}{c^2}\Big)\Big(1 + \frac{|\widetilde{V}|^2}{c^2}\Big)} = \gamma_{\widetilde{U}}\gamma_{\widetilde{V}}\,,$$

reicht es

$$\Big(\gamma_{\widetilde{U}}\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2}\Big)^2 = 1 + \frac{|\widetilde{W}|^2}{c^2}$$

zu zeigen. Nach der Definition von $\widetilde{W}$ und $B(\widetilde{U})$ ist

$$\begin{aligned}
1 + \frac{|\widetilde{W}|^2}{c^2} &= 1 + \frac{1}{c^2}\left|\gamma_{\widetilde{V}}\widetilde{U} + B(\widetilde{U})\widetilde{V}\right|^2 \\
&= 1 + \frac{1}{c^2}\left|\Big(\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\Big)\widetilde{U} + \widetilde{V}\right|^2 \\
&= 1 + \frac{|\widetilde{V}|^2}{c^2} + \frac{2}{c^2}\Big(\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\Big)\widetilde{U}\bullet\widetilde{V} + \frac{|\widetilde{U}|^2}{c^2}\Big(\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\Big)^2 \\
&= \gamma_{\widetilde{V}}^2 + \frac{2}{c^2}\Big(\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\Big)\widetilde{U}\bullet\widetilde{V} + (\gamma_{\widetilde{U}}^2 - 1)\Big(\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\Big)^2.
\end{aligned}$$

Dies ist das gewünschte Ergebnis, denn mit Umformungen ist dies

$$\begin{aligned}
&= \gamma_{\widetilde{V}}^2 + \frac{2\gamma_{\widetilde{V}}}{c^2}\widetilde{U}\bullet\widetilde{V} + \frac{2}{c^4(\gamma_{\widetilde{U}}+1)}(\widetilde{U}\bullet\widetilde{V})^2 \\
&\qquad + (\gamma_{\widetilde{U}}^2 - 1)\gamma_{\widetilde{V}}^2 + \frac{2(\gamma_{\widetilde{U}}^2 - 1)\gamma_{\widetilde{V}}}{c^2(\gamma_{\widetilde{U}}+1)}\widetilde{U}\bullet\widetilde{V} + \frac{\gamma_{\widetilde{U}}^2 - 1}{c^4(\gamma_{\widetilde{U}}+1)^2}(\widetilde{U}\bullet\widetilde{V})^2 \\
&= \gamma_{\widetilde{U}}^2\gamma_{\widetilde{V}}^2 + \frac{2\gamma_{\widetilde{V}}}{c^2}(1 + \gamma_{\widetilde{U}} - 1)\widetilde{U}\bullet\widetilde{V} + \frac{1}{c^4}\underbrace{\Big(\frac{2}{\gamma_{\widetilde{U}}+1} + \frac{\gamma_{\widetilde{U}} - 1}{\gamma_{\widetilde{U}}+1}\Big)}_{=1}(\widetilde{U}\bullet\widetilde{V})^2 \\
&= \Big(\gamma_{\widetilde{U}}\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2}\Big)^2.
\end{aligned}$$

□

Proof 2. Statement. Es muss

$$\begin{aligned}
&M(\gamma_{\widetilde{U}}\widetilde{V} + B(\widetilde{V})\widetilde{U}) = \widetilde{W} \\
&M := B(\widetilde{W})^{-1}\Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\,,
\end{aligned}$$

gezeigt werden, also

$$\Big(B(\widetilde{U})B(\widetilde{V})+\frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\Big(\gamma_{\widetilde{U}}\widetilde{V}+B(\widetilde{V})\widetilde{U}\Big)=B(\widetilde{W})\widetilde{W}\,. \tag{II2.11}$$

Nun gilt für die rechte Seite

$$\begin{aligned}B(\widetilde{W})\widetilde{W}&=\Big(\mathrm{Id}+\frac{1}{c^2(\gamma_{\widetilde{W}}+1)}\widetilde{W}\,\widetilde{W}^{\mathrm{T}}\Big)\\&=\underbrace{\Big(\ 1+\frac{|\widetilde{W}|^2}{c^2(\gamma_{\widetilde{W}}+1)}\ \Big)}_{=\gamma_{\widetilde{W}}}\widetilde{W}=\gamma_{\widetilde{W}}\widetilde{W}\,,\end{aligned}$$

und genauso $B(\widetilde{U})\widetilde{U}=\gamma_{\widetilde{U}}\widetilde{U}$ und $B(\widetilde{V})\widetilde{V}=\gamma_{\widetilde{V}}\widetilde{V}$. Und die linke Seite von (II2.11) ist gleich

$$\begin{aligned}&=\gamma_{\widetilde{U}}B(\widetilde{U})\underbrace{B(\widetilde{V})\widetilde{V}}_{=\gamma_{\widetilde{V}}\widetilde{V}}+B(\widetilde{U})B(\widetilde{V})^2\widetilde{U}\\&+\gamma_{\widetilde{U}}\underbrace{\frac{|\widetilde{V}|^2}{c^2}}_{=\gamma_{\widetilde{V}}^2-1}\widetilde{U}+\frac{1}{c^2}(\widetilde{V}\bullet(B(\widetilde{V})\widetilde{U}))\widetilde{U}\,.\end{aligned}$$

Wir berechnen

$$\widetilde{V}\bullet(B(\widetilde{V})\widetilde{U})=(B(\widetilde{V})^{\mathrm{T}}\,\widetilde{V})\bullet\widetilde{U}=(B(\widetilde{V})\widetilde{V})\bullet\widetilde{U}=\gamma_{\widetilde{V}}\widetilde{V}\bullet\widetilde{U}=\gamma_{\widetilde{V}}\widetilde{U}\bullet\widetilde{V}\,,$$

und da

$$\begin{aligned}&B(\widetilde{V})^2=(\mathrm{Id}+a\widetilde{V}\,\widetilde{V}^{\mathrm{T}})^2\qquad(\text{wobei } a:=\tfrac{1}{c^2(\gamma_{\widetilde{V}}+1)})\\&=\mathrm{Id}+(2a+|\widetilde{V}|^2a^2)\widetilde{V}\,\widetilde{V}^{\mathrm{T}}\qquad(\text{wegen } |\widetilde{V}|^2=c^2(\gamma_{\widetilde{V}}^2-1))\\&=\mathrm{Id}+\frac{1}{c^2}\widetilde{V}\,\widetilde{V}^{\mathrm{T}}\ ,\end{aligned}\tag{II2.12}$$

ist

$$\begin{aligned}B(\widetilde{U})B(\widetilde{V})^2\widetilde{U}&=B(\widetilde{U})\Big(\mathrm{Id}+\frac{1}{c^2}\widetilde{V}\,\widetilde{V}^{\mathrm{T}}\Big)\widetilde{U}\\&=B(\widetilde{U})\widetilde{U}+\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}B(\widetilde{U})\widetilde{V}=\gamma_{\widetilde{U}}\widetilde{U}+\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}B(\widetilde{U})\widetilde{V}\,,\end{aligned}$$

und daher wird die linke Seite von (II2.11) gleich

$$\begin{aligned}&=\gamma_{\widetilde{U}}\gamma_{\widetilde{V}}B(\widetilde{U})\widetilde{V}+B(\widetilde{U})B(\widetilde{V})^2\widetilde{U}\\&+\gamma_{\widetilde{U}}(\gamma_{\widetilde{V}}^2-1)\widetilde{U}+\frac{1}{c^2}(\widetilde{V}\bullet(B(\widetilde{V})\widetilde{U}))\widetilde{U}\\&=\gamma_{\widetilde{U}}\gamma_{\widetilde{V}}B(\widetilde{U})\widetilde{V}+\gamma_{\widetilde{U}}\widetilde{U}+\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}B(\widetilde{U})\widetilde{V}\\&+\Big(\gamma_{\widetilde{U}}(\gamma_{\widetilde{V}}^2-1)+\frac{1}{c^2}\gamma_{\widetilde{V}}\widetilde{U}\bullet\widetilde{V}\Big)\widetilde{U}\\&=\Big(\gamma_{\widetilde{U}}\gamma_{\widetilde{V}}+\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}\Big)B(\widetilde{U})\widetilde{V}+\Big(\gamma_{\widetilde{U}}\gamma_{\widetilde{V}}^2+\frac{1}{c^2}\gamma_{\widetilde{V}}\widetilde{U}\bullet\widetilde{V}\Big)\widetilde{U}\\&=\Big(\gamma_{\widetilde{U}}\gamma_{\widetilde{V}}+\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}\Big)\Big(B(\widetilde{U})\widetilde{V}+\gamma_{\widetilde{V}}\widetilde{U}\Big)=\gamma_{\widetilde{W}}\Big(B(\widetilde{U})\widetilde{V}+\gamma_{\widetilde{V}}\widetilde{U}\Big)=\gamma_{\widetilde{W}}\widetilde{W}\,.\end{aligned}$$

□

Proof 3. Statement. Wir haben zu zeigen, dass M eine orthogonale Transformation mit Determinante 1 ist, d.h.

$$(Mx_1)\bullet(Mx_2) = x_1\bullet x_2$$

ist, oder dazu äquivalent $M^{\mathrm{T}}\, M = \mathrm{Id}$. Dazu äquivalent ist (siehe Beweis von I.1.3)

$$M\, M^{\mathrm{T}} = \mathrm{Id}\,.$$

Mit (II2.9) muss also gelten

$$\mathrm{Id} = B(\widetilde{W})^{-1}\Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)^{\mathrm{T}} B(\widetilde{W})^{-\mathrm{T}}$$

oder

$$B(\widetilde{W})\,B(\widetilde{W})^{\mathrm{T}} = \Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)^{\mathrm{T}}\,. \tag{II2.13}$$

Die linke Seite ist (wie (II2.12) für $\widetilde{V}$)

$$B(\widetilde{W})\,B(\widetilde{W})^{\mathrm{T}} = B(\widetilde{W})^2 = \mathrm{Id} + \frac{1}{c^2}\widetilde{W}\,\widetilde{W}^{\mathrm{T}}\,.$$

Da

$$\widetilde{W} = \gamma_{\widetilde{V}}\widetilde{U} + B(\widetilde{U})\widetilde{V} = \underbrace{\Big(\,\gamma_{\widetilde{V}} + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2(\gamma_{\widetilde{U}}+1)}\,\Big)}_{=:\,b}\widetilde{U} + \widetilde{V}\,,$$

ist

$$\begin{aligned} B(\widetilde{W})\,B(\widetilde{W})^{\mathrm{T}} &= \mathrm{Id} + \frac{1}{c^2}\widetilde{W}\,\widetilde{W}^{\mathrm{T}} \\ &= \mathrm{Id} + \frac{1}{c^2}(b\widetilde{U}+\widetilde{V})(b\widetilde{U}^{\mathrm{T}}+\widetilde{V}^{\mathrm{T}}) \\ &= \mathrm{Id} + \frac{1}{c^2}\Big(b^2\widetilde{U}\,\widetilde{U}^{\mathrm{T}} + b\widetilde{U}\,\widetilde{V}^{\mathrm{T}} + b\widetilde{V}\,\widetilde{U}^{\mathrm{T}} + \widetilde{V}\,\widetilde{V}^{\mathrm{T}}\Big)\,. \end{aligned} \tag{II2.14}$$

Mit

$$a_{\widetilde{U}} := \frac{1}{\gamma_{\widetilde{U}}+1}\,, \quad a_{\widetilde{V}} := \frac{1}{\gamma_{\widetilde{V}}+1}$$

ist

$$\begin{aligned} & B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}} \\ &= \Big(\mathrm{Id} + \frac{a_{\widetilde{U}}}{c^2}\widetilde{U}\,\widetilde{U}^{\mathrm{T}}\Big)\Big(\mathrm{Id} + \frac{a_{\widetilde{V}}}{c^2}\widetilde{V}\,\widetilde{V}^{\mathrm{T}}\Big) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}} \\ &= \mathrm{Id} + \frac{1}{c^2}\Big(a_{\widetilde{U}}\widetilde{U}\,\widetilde{U}^{\mathrm{T}} + a_{\widetilde{V}}\widetilde{V}\,\widetilde{V}^{\mathrm{T}} + \underbrace{\Big(\,\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}a_{\widetilde{U}}a_{\widetilde{V}} + 1\,\Big)}_{=:\,d}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\,. \end{aligned}$$

Also ist die rechte Seite von (II2.13)

$$\begin{aligned} &= \Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\Big(B(\widetilde{U})B(\widetilde{V}) + \frac{1}{c^2}\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)^{\mathrm{T}} \\ &= \Big(\mathrm{Id} + \frac{1}{c^2}\Big(a_{\widetilde{U}}\widetilde{U}\,\widetilde{U}^{\mathrm{T}} + a_{\widetilde{V}}\widetilde{V}\,\widetilde{V}^{\mathrm{T}} + d\widetilde{U}\,\widetilde{V}^{\mathrm{T}}\Big)\Big) \\ &\qquad \Big(\mathrm{Id} + \frac{1}{c^2}\Big(a_{\widetilde{U}}\widetilde{U}\,\widetilde{U}^{\mathrm{T}} + a_{\widetilde{V}}\widetilde{V}\,\widetilde{V}^{\mathrm{T}} + d\widetilde{V}\,\widetilde{U}^{\mathrm{T}}\Big)\Big) \\ &= \mathrm{Id} \; + \frac{1}{c^2}\Big(2a_{\widetilde{U}} + \frac{|\widetilde{U}|^2}{c^2}a_{\widetilde{U}}^2 + 2\frac{\widetilde{U}\bullet\widetilde{V}}{c^2}da_{\widetilde{U}} + d^2\frac{|\widetilde{V}|^2}{c^2}\Big)\widetilde{U}\,\widetilde{U}^{\mathrm{T}} \\ &\qquad + \frac{1}{c^2}\Big(2a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2}a_{\widetilde{V}}^2\Big)\widetilde{V}\,\widetilde{V}^{\mathrm{T}} \\ &\qquad + \frac{1}{c^2}\Big(d + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2}a_{\widetilde{U}}a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2}da_{\widetilde{V}}\Big)\Big(\widetilde{U}\,\widetilde{V}^{\mathrm{T}} + \widetilde{V}\,\widetilde{U}^{\mathrm{T}}\Big)\,. \end{aligned}$$

This is the same as in (II2.14) if we prove

$$\begin{aligned}
1 &= 2a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2} a_{\widetilde{V}}^2 \,, \\
b &= d + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2} d a_{\widetilde{V}} \,, \\
b^2 &= 2a_{\widetilde{U}} + \frac{|\widetilde{U}|^2}{c^2} a_{\widetilde{U}}^2 + 2\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} d a_{\widetilde{U}} + d^2 \frac{|\widetilde{V}|^2}{c^2} \,.
\end{aligned}$$

Indem wir die Definitionen einsetzen, ist

$$\begin{aligned}
&2a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2} a_{\widetilde{V}}^2 = 2a_{\widetilde{V}} + (\gamma_{\widetilde{V}}^2 - 1) a_{\widetilde{V}}^2 = 1 \,, \\
&d + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} + \frac{|\widetilde{V}|^2}{c^2} d a_{\widetilde{V}} \\
&= 1 + 2\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} + (\gamma_{\widetilde{V}}^2 - 1) a_{\widetilde{V}} \Big(1 + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}}\Big) \\
&= 1 + (\gamma_{\widetilde{V}} - 1) + \frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} \Big(2 + (\gamma_{\widetilde{V}} - 1)\Big) = b
\end{aligned}$$

und

$$\begin{aligned}
&2a_{\widetilde{U}} + \frac{|\widetilde{U}|^2}{c^2} a_{\widetilde{U}}^2 + 2\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} d a_{\widetilde{U}} + d^2 \frac{|\widetilde{V}|^2}{c^2} \\
&= 2a_{\widetilde{U}} + (\gamma_{\widetilde{U}}^2 - 1) a_{\widetilde{U}}^2 + d\Big(2\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} + \Big(\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} + 1\Big)(\gamma_{\widetilde{V}}^2 - 1)\Big) \\
&= 1 + d\Big(\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} (2 + a_{\widetilde{V}} (\gamma_{\widetilde{V}}^2 - 1)) + (\gamma_{\widetilde{V}}^2 - 1)\Big) \\
&= 1 + \Big(\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} a_{\widetilde{V}} + 1\Big)\Big(\frac{\widetilde{U}\bullet\widetilde{V}}{c^2} a_{\widetilde{U}} (\gamma_{\widetilde{V}} + 1)) + (\gamma_{\widetilde{V}}^2 - 1)\Big) = b^2 \,.
\end{aligned}$$

□

The definitions suggest that the Lorentz transformations represent the relativistic analogue of the Galilean transformations. But this is not true, because the Lorentz transformations do not generate non-linear transformations (see [23, Section I.4]) in contrast to Galilean transformations as shown in 1.7.

3 Objectivity of balance laws

In general, the formulation of a class of physical processes has to be frame independent with respect to a given group of observer transformations. That is, it has to be the same for all observers, whose coordinates transform by an element in $\mathcal{T}$. We require that in particular the following is independent of the observer:

- Conservation laws (Conservation laws have been introduced in section I.1 and they are the basis for the dynamics in physical studies.)
- Constitutive functions (Constitutive functions arise as description of particular materials and they are part of the underlying conservation laws. They will be considered in the next section 4.)

We consider in this section some special conservation laws, where for simplicity we restrict ourselves to classical solutions. We will now give a short repetition of parts of section I.5, which is based on the "Main invariance theorem" I.5.1. As transformation group Y we choose the Newton group $\mathcal{T}_{\text{Newton}}$.[1] For a transformation Y in this group the vector X and the matrix Q are defined in (II1.3) and it holds for the Jacobian J of Y

$$J = \det \mathrm{D}_{(t^*,x^*)} Y = \det \mathrm{D}_{x^*} X = \det Q = 1\,,$$

hence the transformation rules in (I5.11) take the simpler form (II3.5). With $N \in \mathbb{N}$ we consider in $\mathcal{U}$ the following differential equations (see (I5.1))

$$\partial_t u^k + \mathrm{div}_x q^k = \mathbf{r}^k \text{ for } k = 1, \dots, N\,, \tag{II3.1}$$

where u^k, q_i^k and $\mathbf{r}^k$ are given functions for $k = 1, \dots, N$. This can be written in vector notation $u = \big(u^k\big)_k$, $q = \big(q^k\big)_k$, $\mathbf{r} = \big(\mathbf{r}^k\big)_k$,

$$\partial_t u + \sum_{i=1}^{n} \partial_{x_i} q_i = \mathbf{r}\,. \tag{II3.2}$$

Hence for test functions $\zeta \in C_0^\infty(\mathcal{U}; \mathbb{R}^N)$

$$\sum_{k=1}^{N} \int_{\mathcal{U}} (\partial_t \zeta_k u^k + \nabla_x \zeta_k \bullet q^k + \zeta_k \mathbf{r}^k)\, \mathrm{dL}^{n+1} = 0\,. \tag{II3.3}$$

The objectivity of certain conservation laws is defined by a transformation rule for test functions: Let the coordinate system of the observer be $y = (t, x)$ and the coordinates of another observer $y^* = (t^*, x^*)$. These coordinates are

[1] We will choose here the group of Newtonian transformations, therefore we will not agree with some literature in which frame indifference is required only for Galileian transformations. Of course, our definition requires more conditions on the physical quantities, as for the Coriolis force or the stress tensor. which we had introduced in 1.3.

related by a transformation $y = Y(y^*)$. Now, if $y \mapsto \zeta(y) \in \mathbb{R}^N$ is the test function of the system for the first observer and $y^* \mapsto \zeta^*(y^*) \in \mathbb{R}^N$ the test function of the other observer, then the following transformation rule defines the type of the system of conservation laws:

Transformation rule for test functions of a conservation law:

$$\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y \qquad (\text{or } \zeta\circ Y = Z^{-\mathrm{T}}\,\zeta^*), \tag{II3.4}$$

$Z = (Z_{kl})_{k,l=1,\dots,N}$ invertible.

Hence the ***objectivity of the system of conservation laws*** (II3.1) is defined with respect to a function $Z:\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{N\times N}$. Using (I5.11) this is satisfied if the following holds:

Transformation formula:

$$\begin{aligned} u^k\circ Y &= \sum_l Z_{kl}u^{*l}\,, \\ q_i^k\circ Y &= \sum_l \dot{X}_i Z_{kl}u^{*l} + \sum_{l,j} Q_{ij}Z_{kl}q_j^{*l}\,, \\ \mathbf{r}^k\circ Y &= \sum_l \dot{Z}_{kl}u^{*l} + \sum_{l,j} Z_{kl}{}'_{j}q_j^{*l} + \sum_l Z_{kl}\mathbf{r}^{*l}\,, \end{aligned} \tag{II3.5}$$

for all $k = 1,\dots,N$ and $i = 1,\dots,n$,

where l runs from 1 to N, and j from 1 to n.

We have to choose Z appropriately in order to get information about certain systems of conservation laws. In this section we choose for the

Mass balance: $Z = 1$

Mass-momentum balance: $Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & \mathrm{D}X \end{bmatrix}$

Mass-momentum-energy balance: $Z = \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & \mathrm{D}X & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathrm{T}}Q & 1 \end{bmatrix}$

Transformation formula:

$$\begin{aligned} u\circ Y &= Zu^*\,, \\ q_i\circ Y &= \dot{X}_i Zu^* + \sum_j Q_{ij}Zq_j^*\,, \\ \mathbf{r}\circ Y &= \dot{Z}u^* + \sum_j Z{}'_{j}q_j^* + Z\mathbf{r}^* \end{aligned}$$

Now we introduce different classes of conservation laws which differ only in the matrix Z.

Scalar laws

To apply these general considerations we start with a single equation

$$\partial_t u + \operatorname{div} q = \mathbf{r}\,, \tag{II3.6}$$

which in the weak form for $\zeta \in C_0^\infty(\mathcal{U};\mathbb{R})$ reads

$$0 = \int_{\mathcal{U}} (\partial_t \zeta \cdot u + \nabla\zeta \bullet q + \zeta \cdot \mathbf{r})\, \mathrm{dL}^{n+1}\,.$$

3.1 Scalar equation (Definition). The equation (II3.6) is called a ***scalar equation*** whenever the weak form transforms according to the law

$$\zeta^* = \zeta \circ Y \text{ for real-valued test functions,}$$

where the coordinate transformation of observers is $(t, x) = Y(t^*, x^*)$. Hence $Z = 1$. This is the case (see (I5.11) or (II3.5)) if

$$\begin{aligned} u \circ Y &= u^*\,, \\ q \circ Y &= u^* \dot{X} + Q q^*\,, \\ \mathbf{r} \circ Y &= \mathbf{r}^*\,. \end{aligned} \tag{II3.7}$$

Proof. By (II3.5) we obtain for $N = 1$ and $Z = 1$ that $\zeta^* = \zeta \circ Y$, and therefore $u \circ Y = u^*$ and $q_i \circ Y = \dot{X}_i u^* + \sum_j Q_{ij} q_j^*$ as well as $\mathbf{r} \circ Y = \mathbf{r}^*$. $\square$

This is the occasion for the following definitions, where we mention that these definitions do not cover the transformation rule for the flux in (II3.7).

3.2 Objective tensors. The classical definitions are, if any two observers are in connection by means of the transformation $(t, x) = Y(t^*, x^*)$:

(1) A real variable $(t, x) \mapsto u(t, x) \in \mathbb{R}$ is called an ***objective scalar*** if

$$u \circ Y = u^*\,,$$

where $(t^*, x^*) \mapsto u^*(t^*, x^*) \in \mathbb{R}$ is the real function for the other observer.

(2) A vectorial variable $(t, x) \mapsto q(t, x) \in \mathbb{R}^n$ is called ***objectiv vector*** if

$$q \circ Y = Q q^*\,,$$

where $(t^*, x^*) \mapsto q^*(t^*, x^*) \in \mathbb{R}^n$ is the vector for the other observer.

(3) A tensor $(t, x) \mapsto M(t, x) \in \mathbb{R}^{n\times n}$ is called ***objective tensor*** if

$$M \circ Y = Q M^* Q^{\mathrm{T}}\,,$$

where $(t^*, x^*) \mapsto M^*(t^*, x^*) \in \mathbb{R}^{n\times n}$ is the tensor for the other observer.

(4) In general, if $(t,x) \mapsto T_{i_1,\dots,i_m}(t,x) \in \mathbb{R}$ for $i_1,\dots,i_m \in \{1,\dots,n\}$ are the components of a m-tensor, then T is called an ***objective m-tensor*** if

$$T_{i_1,\dots,i_m}\circ Y = \sum_{j_1,\dots,j_m=1}^{n} \Big(\prod_{k=1}^{m} Q_{i_k j_k}\Big) T^*_{j_1,\dots,j_m} \text{ for } i_1,\dots,i_m = 1,\dots,n\,,$$

where $(t^*,x^*) \mapsto T^*(t^*,x^*)$ is the m-tensor for the other observer.

The mass-momentum system is a scalar equation and of the form

$$\partial_t \varrho + \operatorname{div}\widetilde{J} = \mathbf{r}\,, \tag{II3.8}$$

As we know the mass density ϱ (it stands for u) and the production rate $\mathbf{r}$ are objective scalars. It remains to interpret the flux term $\widetilde{J}$ (it stands for q) which satisfies the transformation rule in (II3.7):

$$\widetilde{J}\circ Y = \varrho^* \dot{X} + Q\widetilde{J}^*\,. \tag{II3.9}$$

In this connection we introduce the velocity v.[2]

3.3 Velocity (Definition). A vectorial variable v is called ***velocity*** if

$$v\circ Y = \dot{X} + Qv^*$$

is satisfied for different observers transforming with $(t,x) = Y(t^*,x^*)$.
Remark: We also could define

$$\begin{bmatrix} 1 \\ v \end{bmatrix} = \mathrm{D}Y \begin{bmatrix} 1 \\ v^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} \begin{bmatrix} 1 \\ v^* \end{bmatrix} = \begin{bmatrix} 1 \\ \dot{X} + Qv^* \end{bmatrix}$$

where the first identity holds also for the Lorentz case. This definition of $(1,v)$ as 4-velocity shows that the transformation rule of $(1,v)$ is linear with matrix $\mathrm{D}Y$.

Now if v is a velocity, it follows that ϱv satisfies the transformation rule

$$\begin{aligned} &(\varrho v)\circ Y = (\varrho\circ Y)(v\circ Y) \\ &= \varrho^*(\dot{X} + Qv^*) = \varrho^*\dot{X} + Q(\varrho^* v^*)\,, \end{aligned}$$

i.e. the same transformation rule as $\widetilde{J}$ in (II3.9). With

$$\mathbf{J} := \widetilde{J} - \varrho v \quad \text{or} \quad \widetilde{J} = \varrho v + \mathbf{J} \tag{II3.10}$$

we now perform the difference of the transformation formula for $\widetilde{J}$ and the one for ϱv and obtain

$$\begin{aligned} &\mathbf{J}\circ Y = \widetilde{J}\circ Y - (\varrho v)\circ Y \\ &= (\varrho^*\dot{X} + Q\widetilde{J}^*) - \varrho^*(\dot{X} + Qv^*) = Q(\widetilde{J}^* - \varrho^* v^*) = Q\mathbf{J}^*\,, \end{aligned}$$

that is, $\mathbf{J}$ is an objective vector. Thus we can say

[2] We will give here the definition of a velocity by means of a transformation formula. It is clear that we do not agree with the comment in some literature that the velocity is not be objective. In fact this is due to an incomplete use of the concept of objectivity.

3.4 Mass equation. The general ***mass equation***

$$\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}$$

is a scalar equation. This is satisfied if the following holds:

ϱ and $\mathbf{r}$ are objective scalars (see Definition 3.2(1)),
v is a velocity (see Definition 3.3),
$\mathbf{J}$ is an objective vector (see Definition 3.2(2)),

where $\mathbf{r}$ is the ***mass production*** and $\mathbf{J}$ the ***mass diffusion***.

In case a second velocity $\bar{v}$ is given the difference $v - \bar{v}$ satisfies, see 3.3,

$$(v - \bar{v})\circ Y = Q(v^* - \bar{v}^*),$$

that is, $v - \bar{v}$ is an objective vector. Hence for the flux of the mass equation

$$\widetilde{\mathbf{J}} = \varrho v + \mathbf{J} = \varrho \bar{v} + \bar{\mathbf{J}}\,, \quad \bar{\mathbf{J}} := \varrho(v - \bar{v}) + \mathbf{J}\,, \tag{II3.11}$$

where $\bar{\mathbf{J}}$ is again an objective vector. Another scalar equation is due to gravity.

3.5 Gravitation law. The ***gravitational law***

$$\operatorname{div}(-\nabla \boldsymbol{\phi}) = \varrho$$

is a scalar equation. This is consistent with the fact that we know already that the mass density ϱ is an objective scalar. Furthermore, the ***gravitational potential***

$$\boldsymbol{\phi} \text{ is an objective scalar,}$$

a definition which is made here.

Proof. The differential equation is

$$\partial_t \underbrace{0}_{u :=} + \operatorname{div}(\underbrace{-\nabla \boldsymbol{\phi}}_{q :=}) = \underbrace{\varrho}_{\mathbf{r} :=}$$

Since $u = 0$ one has to show that $q = -\nabla \boldsymbol{\phi}$ is an objective vector. Since $\boldsymbol{\phi}$ is an objective scalar, that is,

$$\boldsymbol{\phi}\circ Y = \boldsymbol{\phi}^*$$

one computes the derivative of this equation. It is

$$\partial_{x_j^*} \boldsymbol{\phi}^* = \partial_{x_j^*}(\boldsymbol{\phi}\circ Y) = \sum_{i=1}^{n} (\partial_{x_i} \boldsymbol{\phi})\circ Y \cdot \partial_{x_j^*} Y_i\,,$$

and since $\partial_{x_j^*} Y_i = Y_i{'}_j = Q_{ij}$ one obtains

$$\partial_{x_j^*}\boldsymbol{\phi}^* = \sum_{i=1}^{n} Q_{ij}(\partial_{x_i}\boldsymbol{\phi})\circ Y \quad \text{or} \quad \nabla_{x^*}\boldsymbol{\phi}^* = Q^{\mathrm{T}}\big((\nabla_x\boldsymbol{\phi})\circ Y\big)\,.$$

Since $Q^{\mathrm{T}} = Q^{-1}$ this is equivalent to $(\nabla_x\boldsymbol{\phi})\circ Y = Q\nabla_{x^*}\boldsymbol{\phi}^*$. (This conclusion is true for every objective scalar.) □

Mass-momentum laws

As another example we consider mass and momentum.

3.6 Mass-momentum equation (Definition). This system of equations has the general form

$$\partial_t\varrho + \mathrm{div}\widetilde{J} = \mathbf{r}\,,$$
$$\partial_t(\varrho v) + \mathrm{div}\widetilde{\Pi} = \widetilde{\mathbf{f}}\,,$$

and is called objective, if it transforms between two observers, which are given by a coordinate transformation $(t,x) = Y(t^*,x^*)$, according to the general law

$$\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y$$

for test functions $\zeta : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{1+n}$ with the matrix

$$Z := \mathrm{D}_{(t^*,x^*)}Y = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}. \tag{II3.12}$$

Here besides the quantities of the mass equation ϱv is the ***Momentum*** and $\widetilde{\Pi}$ the ***momentum flux*** and $\widetilde{\mathbf{f}}$ die ***general force density***.
Attention: The system is an arbitrary system of $1{+}n$ balance laws. The only constraint of this system is due to the transformation law for test functions. The matrix Z in (II3.12) defines this system as a mass-momentum system. Everything else is a consequence of this definition.

The system can also be written as

$$\partial_t \underbrace{\begin{bmatrix} \varrho \\ \varrho v \end{bmatrix}}_{=:\,u} + \sum_{i=1}^{n}\partial_{x_i} \underbrace{\begin{bmatrix} \widetilde{J}_i \\ \big(\widetilde{\Pi}_{ki}\big)_k \end{bmatrix}}_{=:\,q_i} = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix}$$

and is objective, if with the coordinate transformation $(t,x) = Y(t^*,x^*)$ the following holds (see (II3.5) or (I5.11))

$$\begin{bmatrix} \varrho \\ \varrho v \end{bmatrix} \circ Y = Z \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{J}_i \\ \left(\widetilde{\Pi}_{ki}\right)_k \end{bmatrix} \circ Y = \dot{X}_i Z \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \sum_{j=1}^{n} Q_{ij} Z \begin{bmatrix} \widetilde{J}_j^* \\ \left(\widetilde{\Pi}_{lj}^*\right)_l \end{bmatrix} \text{ für } i = 1, \dots, n,$$

$$\begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \end{bmatrix} \circ Y = Z_{,t^*} \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} + \sum_{j=1}^{n} Z_{,x_j^*} \begin{bmatrix} \widetilde{J}_j^* \\ \left(\widetilde{\Pi}_{lj}^*\right)_l \end{bmatrix} + Z \begin{bmatrix} \mathbf{r}^* \\ \widetilde{\mathbf{f}}^* \end{bmatrix}.$$

Since

$$Z = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}, \quad Z_{,t^*} = \begin{bmatrix} 0 & 0 \\ \ddot{X} & \dot{Q} \end{bmatrix}, \quad Z_{,x_j^*} = \begin{bmatrix} 0 & 0 \\ \left(\dot{Q}_{kj}\right)_k & 0 \end{bmatrix},$$

we obtain for the first of the three identities

$$\varrho \circ Y = \varrho^* ,$$
$$(\varrho v) \circ Y = \varrho^* \dot{X} + Q(\varrho^* v^*) .$$

The first of these equations, namely that ϱ is an objective scalar, we know from the mass equation, which is part of our system. For the second of the three identities we obtain for $k, i = 1, \dots, n$

$$\widetilde{J}_i \circ Y = \varrho^* \dot{X}_i + \sum_{j=1}^{n} Q_{ij} \widetilde{J}_j^* ,$$

$$\widetilde{\Pi}_{ki} \circ Y = \dot{X}_i \Big(\dot{X}_k \varrho^* + \sum_{l=1}^{n} Q_{kl} \varrho^* v_l^* \Big) + \sum_{j=1}^{n} Q_{ij} \Big(\dot{X}_k \widetilde{J}_j^* + \sum_{l=1}^{n} Q_{kl} \widetilde{\Pi}_{lj}^* \Big) .$$

The first equation is identical with an equation of the mass conservation (see the proof of 3.4). Therefore it is not surprising, that $\mathbf{J} := \widetilde{J} - \varrho v$ is an objective vector (as in the proof of 3.4). The second equation is

$$\widetilde{\Pi}_{ki} \circ Y = \varrho^* \dot{X}_k \dot{X}_i + \varrho^* \dot{X}_i \sum_{l=1}^{n} Q_{kl} v_l^* + \dot{X}_k \sum_{j=1}^{n} Q_{ij} \widetilde{J}_j^* + \sum_{j=1}^{n} \sum_{l=1}^{n} Q_{kl} Q_{ij} \widetilde{\Pi}_{lj}^* ,$$

or in matrix notation

$$\begin{aligned} \widetilde{\Pi} \circ Y &= \varrho^* \dot{X} \dot{X}^{\mathrm{T}} + \varrho^* (Q v^*) \dot{X}^{\mathrm{T}} + \dot{X} \big(Q \widetilde{J}^*\big)^{\mathrm{T}} + Q \widetilde{\Pi}^* Q^{\mathrm{T}} \\ &= (\dot{X} + Q v^*) \big(\varrho^* \dot{X}\big)^{\mathrm{T}} + \dot{X} \big(Q \widetilde{J}^*\big)^{\mathrm{T}} + Q \widetilde{\Pi}^* Q^{\mathrm{T}} . \end{aligned} \tag{II3.13}$$

We compare this with the known transformation behavior of $v \widetilde{J}^{\mathrm{T}}$,

$$\begin{aligned} (v \widetilde{J}^{\mathrm{T}}) \circ Y &= (\dot{X} + Q v^*) \big(\varrho^* \dot{X} + Q \widetilde{J}^*\big)^{\mathrm{T}} \\ &= (\dot{X} + Q v^*) \big(\varrho^* \dot{X}\big)^{\mathrm{T}} + \dot{X} \big(Q \widetilde{J}^*\big)^{\mathrm{T}} + \underbrace{(Q v^*) \big(Q \widetilde{J}^*\big)^{\mathrm{T}}}_{= Q (v^* \widetilde{J}^{*\mathrm{T}}) Q^{\mathrm{T}}} . \end{aligned}$$

Comparing this with (II3.13) we derive that

$$(\widetilde{\Pi} - v\,\widetilde{J}^{\mathrm{T}})\circ Y = Q(\widetilde{\Pi}^* - v^*\,\widetilde{J}^{*\mathrm{T}})\,Q^{\mathrm{T}}\ ,$$

that is,

$$\Pi := \widetilde{\Pi} - v\,\widetilde{J}^{\mathrm{T}} = \widetilde{\Pi} - v\,(\varrho v + \mathbf{J})^{\mathrm{T}}$$

is an objective tensor. Finally, we have to consider the source terms. They become, using the above identities for the derivative of Z,

$$\begin{aligned}
\begin{bmatrix} \mathbf{r}\circ Y \\ \widetilde{\mathbf{f}}\circ Y \end{bmatrix} = {} & \begin{bmatrix} 0 & 0 \\ \ddot{X} & \dot{Q} \end{bmatrix} \begin{bmatrix} \varrho^* \\ \varrho^* v^* \end{bmatrix} \\
& + \sum_{j=1}^{n} \begin{bmatrix} 0 & 0 \\ \Big(\dot{Q}_{kj}\Big)_k & 0 \end{bmatrix} \begin{bmatrix} \widetilde{J}_j^* \\ \Big(\widetilde{\Pi}_{lj}^*\Big)_l \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} \begin{bmatrix} \mathbf{r}^* \\ \widetilde{\mathbf{f}}^* \end{bmatrix} \\
= {} & \begin{bmatrix} 0 \\ \varrho^*(\ddot{X} + \dot{Q} v^*) \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{Q}\widetilde{J}^* \end{bmatrix} + \begin{bmatrix} \mathbf{r}^* \\ \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^* \end{bmatrix},
\end{aligned}$$

that is, $\mathbf{r}\circ Y = \mathbf{r}^*$, and

$$\begin{aligned}
\widetilde{\mathbf{f}}\circ Y = {} & \varrho^*(\ddot{X} + \dot{Q}v^*) + \dot{Q}\widetilde{J}^* + \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^* \\
= {} & \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}\mathbf{J}^* + \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^*\,,
\end{aligned}$$

hence

$$\widetilde{\mathbf{f}}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}\mathbf{J}^* + \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^*\,. \tag{II3.14}$$

Therefore we can state the following theorem.

3.7 Theorem. The mass-momentum system, that means the system in 3.6, with

$$\widetilde{J} = \varrho v + \mathbf{J}\,, \quad \widetilde{\Pi} = \varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi\,,$$

therefore with arbitrary $\mathbf{J}$ and Π, reads

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,.
\end{aligned} \tag{II3.15}$$

This system is objective (see the definition 3.6), if

ϱ and $\mathbf{r}$ are objective scalars,

v is a velocity,

$\mathbf{J}$ is an objective vector,

Π is an objective tensor,

where Π is the general ***pressure tensor*** and $\widetilde{\mathbf{f}}$ is the ***general force density***, that is, the transformation rule (II3.14) is satisfied (which contains both $\mathbf{r}$ and $\mathbf{J}$). For the definition of a ***classical force density*** as part of $\widetilde{\mathbf{f}}$ see 3.8.

The transformation rule (II3.14) does not allow a vanishing force for all observers, even for the special case that $\mathbf{r}$ and $\mathbf{J}$ vanish, since then the inhomogeneous term is $\varrho^*(\ddot{X} + 2\dot{Q}v^*)$ containing centrifugal and Coriolis forces (see example I.5.5). These forces are also called “fictitious forces” (*de:* “Scheinkräfte”, *fr:* “forces translucides”, *it:* “forze traslucidi”, *pl:* “sily przeswitujace”). The equation (II3.14) can also be written by reordering the terms of the right-hand side as

$$\widetilde{\mathbf{f}}\circ Y = \varrho^*\ddot{X} + \dot{Q}(2\varrho^* v^* + \mathbf{J}^*) + \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^* \,.$$

Here the term $2\varrho^* v^* + \mathbf{J}^*$ is the sum of the term under the time derivative of the momentum equation and the term under the space derivative of the mass equation. Now, if w is a velocity, that is $w\circ Y = \dot{X} + Qw^*$, and consequently $(\mathrm{D}w\circ Y)Q = \dot{Q} + Q\mathrm{D}w^*$ for the derivative, then $\mathbf{r}w + \mathrm{D}w\mathbf{J}$ satifies the following transformation rule using the fact that $\mathbf{J}$ is an objective vector,

$$\begin{aligned}(\mathbf{r}w + \mathrm{D}w\mathbf{J})\circ Y &= \mathbf{r}^*(\dot{X} + Qw^*) + \mathrm{D}w\circ Y Q\mathbf{J}^* \\ &= \mathbf{r}^*(\dot{X} + Qw^*) + (\dot{Q} + Q\mathrm{D}w^*)\mathbf{J}^* \\ &= \mathbf{r}^*\dot{X} + \dot{Q}\mathbf{J}^* + Q(\mathbf{r}^* w^* + \mathrm{D}w^*\mathbf{J}^*) \,.\end{aligned}$$

Subtracting this from the transformation rule (II3.14) for $\widetilde{\mathbf{f}}$ we obtain

$$(\widetilde{\mathbf{f}} - (\mathbf{r}w + \mathrm{D}w\mathbf{J}))\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q(\widetilde{\mathbf{f}}^* - (\mathbf{r}^* w^* + \mathrm{D}w^*\mathbf{J}^*)) \,,$$

that is,

$$\widetilde{\mathbf{f}} - (\mathbf{r}w + \mathrm{D}w\mathbf{J})$$

satisfies the tranformation rule in (II3.18), if $\mathbf{f}$ is defined as in (II3.17) and if $w = v$ is taken as standard velocity.

3.8 Classical Force (Definition). The mass-momentum system (II3.15) (equivalent to 3.6) is equivalent to

$$\boxed{\begin{aligned}\overset{\circ}{\varrho} + \varrho\,\mathrm{div}\,v &= \mathbf{r} - \mathrm{div}\mathbf{J} \,, \\ \varrho\overset{\circ}{v} + \mathrm{div}\Pi &= \mathbf{f} \,,\end{aligned}} \qquad \text{(II3.16)}$$

where $\mathbf{f}$ is the ***(classical) force density*** given by

$$\boxed{\mathbf{f} := \widetilde{\mathbf{f}} - (\mathbf{r}v + \mathrm{D}v\mathbf{J}) \,.} \qquad \text{(II3.17)}$$

The classical force density $\mathbf{f}$ satisfies the following transformation formula

$$\boxed{\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^* \,.} \qquad \text{(II3.18)}$$

Notation: It is $\overset{\circ}{h} := \partial_t h + v\bullet\nabla h$ for every function h.[3]
Remark: In fact, the mass-momentum system (II3.15) for smooth functions is equivalent to 3.6, but for distributions please go back to the original system (II3.15).

Proof. The equivalence of mass conservations follows from

$$\partial_t\varrho + \operatorname{div}(\varrho v) = (\partial_t + v\bullet\nabla)\varrho + \varrho \operatorname{div} v = \overset{\circ}{\varrho} + \varrho \operatorname{div} v\,.$$

The momentum conservation we write as

$$\begin{aligned}
\widetilde{\mathbf{f}} &= \partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) \\
&= (\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}))v + \varrho(\partial_t v + v\bullet\nabla v) + \mathrm{D}v\mathbf{J} + \operatorname{div}\Pi \\
&= \mathbf{r}v + \mathrm{D}v\mathbf{J} + \varrho(\partial_t v + v\bullet\nabla v) + \operatorname{div}\Pi \\
&= \mathbf{r}v + \mathrm{D}v\mathbf{J} + \varrho\overset{\circ}{v} + \operatorname{div}\Pi\,,
\end{aligned}$$

where we have used

$$\operatorname{div}(v\,\mathbf{J}^{\mathrm{T}}) = (\operatorname{div}\mathbf{J})v + \mathrm{D}v\mathbf{J} = \big(\operatorname{div}\mathbf{J} + \mathbf{J}\bullet\nabla\big)v\,.$$

This gives the equivalence of the momentum equations by defining the force as in the statement (II3.17). The transformation rule for the force $\mathbf{f}$ has been shown above. □

Usually one considers the system (II3.15) in the case $\mathbf{J} = 0$. Then this system becomes

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v) = \mathbf{r}\,, \\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{r}v + \mathbf{f}\,,
\end{aligned}$$

where $\mathbf{f}$ is the “classical force” in 3.8. If one considers the total mass then usually $\mathbf{r} = 0$, and in this case the total mass is “conserved”.

3.9 Inertial systems. A coordinate system is called inertial system if there is no external force in the total momentum balance. It is never clear whether such systems exist, therefore one always is well advised to have a closer look to the special situation. Inertial systems, as defined, exist only as an approximation, the error can of course be very small. For example, there is always the gravity field of an (unknown) celestial body present.

[3] It is $\overset{\circ}{h} := \partial_t h + v\bullet\nabla h = \partial_{(1,v)}h = (1,v)\bullet(\partial_t,\nabla)h$ for every function h. In the literature one usually writes $\dot{h}$ instead of $\overset{\circ}{h}$, also one finds $\mathrm{D}_t h$ (see e.g. [4, Definition 5.1]), $\frac{\mathrm{d}}{\mathrm{d}t}h$ or $\mathrm{D}_v h$. If you don't mind the other use of the dot as time derivative, use $\dot{h}$. *Attention:* In most abbreviations there is no reference to v.

In this book there are some examples with several observers where one observer says to have chosen an inertial system. We recommend section IV.1 and in the current section 3.10. In the second example we go back to I.3.5 where the observer is chosen outside of the centrifuge. Here we introduce a second observer which is turning with the centrifuge. He writes the situation in his coordinates, that is, the centrifuge has velocity zero. Nevertheless, as we shall see, the physics is the same for him.

3.10 Example. In I.3.5 we dealt with a centrifuge and we considered stationary solutions of the mas-momentum system

$$\begin{aligned} \operatorname{div}_x(\varrho v) &= 0\,, \\ \varrho v \bullet \nabla_x v + \nabla_x p - \operatorname{div}_x S &= \mathbf{f}\,. \end{aligned} \tag{II3.19}$$

The pressure tensor S we took from the Navier-Stokes equation (I3.32).
External observer: The observer which we had considered in I.3.5 we now call the *-observer with *-equations, that is, everything in I.3.5 is marked with a '*'. We had assumed that the centrifuge for him is

$$v^* = \omega \begin{bmatrix} -x_2^* \\ x_1^* \\ 0 \end{bmatrix} = -\omega A x^*\,, \quad \mathbf{f}^* = 0\,,$$

where the matrix A is as in I.5.5 Therefore, his coordinate system is an inertial system and the pressure p^* and ϱ^* depend only on the distance to the axis of the centrifuge and the values of v^* imply $S^* = 0$. Consequently, the equations (II3.19) written with '*' become

$$\varrho^* v^* \bullet \nabla_{x^*} v^* + \nabla_{x^*} p^* = 0\,. \tag{II3.20}$$

Now $v^* \bullet \nabla_{x^*} v^* = -\omega^2(x_1^*, x_2^*, 0) = -\omega^2 I x^*$ where ω is the angular velocity of the rotating cylinder, and I is from I.5.5. Hence (II3.20) is equivalent to $\nabla_{x^*} p^* = \varrho^* \omega^2 I x^*$.
Interior observer: Der Beobachter, der sich in der Achse mit der Zentrifuge mitdreht, hat auch das System (II3.19) zu betrachten, und zwar gilt für ihn

$$v = 0\,, \quad \mathbf{f} \circ Y = \varrho^*(\ddot{X} + 2\dot{Q} v^*)$$

wegen $\mathbf{f}^* = 0$, eine Gleichung die in (II3.18) bei der klassischen Kraft auftrat. Hier ist Y eine klassische Beobachtertransformation mit $t = t^*$ und $x = X(t^*, x^*) = Q(t^*) x^*$. Also ist er kein Inertialsystem. Nun ist für ihn auch $S = 0$ wegen $v = 0$, und die Differentialgleichungen (II3.19) werden zu

$$\nabla_x p = \mathbf{f}\,. \tag{II3.21}$$

Wegen $\ddot{Q} = -\omega^2 I Q$ und $\dot{Q} A = -\omega I Q$ folgt

$$\mathbf{f} \circ Y = \varrho^*(\ddot{X} + 2\dot{Q} v^*) = \varrho^*(\ddot{Q} - 2\omega \dot{Q} A) x^* = \varrho^* \omega^2 I Q x^*\,,$$

wegen $\varrho^* = \varrho\circ Y$ somit $\mathbf{f} = \varrho\omega^2 Ix$. Also ist $\nabla_x p = \varrho\omega^2 Ix$.
Observer comparison: Die Beobachter haben also das gleiche physikalische Resultat $\nabla_{x^*} p^* = \varrho^*\omega^2 Ix^*$ bzw. $\nabla_x p = \varrho\omega^2 Ix$. Es sei noch bemerkt, dass der Druck ein objektiver Skalar ist, denn $\Pi\circ Y = Q\Pi^* Q^{\mathrm{T}}$ und $\Pi = p\mathrm{Id}$ sowie $\Pi^* = p^*\mathrm{Id}$ implizieren $p\circ Y = p^*$.

In (II3.16) we have written the term $\mathrm{div}\mathbf{J}$ on the right-hand side, which then became $\mathbf{r} - \mathrm{div}\mathbf{J}$. In the same way we can put in the momentum conservation the term $\mathrm{div}\Pi$ on the right-hand side, which then becomes $\widetilde{\mathbf{f}} - \mathrm{div}\Pi$, or we can do it only for a part of this term. The latter we have already seen in Section I.3, where we have dealt with mass points (see I.3.1 for a single mass point and I.3.4 for a collection of mass points). Therefore we make the following remark on the right-hand sides.

3.11 The term J. The differential system is (II3.15), which is equivalent to

$$\begin{aligned} &\partial_t\varrho + \mathrm{div}(\varrho v) = \overline{\mathbf{r}}\,, \\ &\partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \overline{\mathbf{f}}\,, \end{aligned} \tag{II3.22}$$

if $\overline{\mathbf{r}} := \mathbf{r} - \mathrm{div}\mathbf{J}$ and $\overline{\mathbf{f}} := \widetilde{\mathbf{f}} - \mathrm{div}(v\,\mathbf{J}^{\mathrm{T}})$. Show by a direct calculation, that $\overline{\mathbf{f}}$ fulfils the transformation rule

$$\overline{\mathbf{f}}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \overline{\mathbf{r}}^*\dot{X} + Q\overline{\mathbf{f}}^*\,. \tag{II3.23}$$

Proof. The last formula given here in (II3.23) is the transformation rule as in definition 3.6, if we consider (II3.22) as mass-momentum system. The purpose is to show this formula independently from the mass-momentum system (II3.14) and from the definitions in the statement. The transformation rules for system (II3.14) include $\mathbf{r}\circ Y = \mathbf{r}^*$ and

$$\widetilde{\mathbf{f}}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + \dot{Q}\mathbf{J}^* + \mathbf{r}^*\dot{X} + Q\widetilde{\mathbf{f}}^*$$

Subtracting (II3.23) gives

$$(\widetilde{\mathbf{f}} - \overline{\mathbf{f}})\circ Y = \dot{Q}\mathbf{J}^* + \mathbf{r}^*\dot{X} - \overline{\mathbf{r}}^*\dot{X} + Q(\widetilde{\mathbf{f}}^* - \overline{\mathbf{f}}^*)\,.$$

Using the definition $\widetilde{\mathbf{f}} - \overline{\mathbf{f}} = \mathrm{div}(v\,\mathbf{J}^{\mathrm{T}})$ and $\mathbf{r}^* - \overline{\mathbf{r}}^* = \mathrm{div}\mathbf{J}^*$ in the statement, this equation reads

$$(\,\mathrm{div}(v\,\mathbf{J}^{\mathrm{T}}))\circ Y = \dot{Q}\mathbf{J}^* + (\,\mathrm{div}\mathbf{J}^*)\dot{X} + Q\,\mathrm{div}(v^*\,\mathbf{J}^{*\mathrm{T}})\,.$$

Since $\mathrm{div}(v\,\mathbf{J}^{\mathrm{T}}) = (\,\mathrm{div}\mathbf{J})v + \mathrm{D}v\mathbf{J}$, this is

$$((\,\mathrm{div}\mathbf{J})v + \mathrm{D}v\mathbf{J})\circ Y = \dot{Q}\mathbf{J}^* + (\,\mathrm{div}\mathbf{J}^*)\dot{X} + Q((\,\mathrm{div}\mathbf{J}^*)v^* + \mathrm{D}v^*\mathbf{J}^*)\,.$$

Now, $\mathrm{div}\mathbf{J}$ is an objective scalar, i.e. $(\,\mathrm{div}\mathbf{J})\circ Y = \mathrm{div}\mathbf{J}^*$, and $v\circ Y = \dot{X} + Qv^*$. Therefore the equation reduces to

$$(\mathrm{D}v\mathbf{J})\circ Y = \dot{Q}\mathbf{J}^* + Q\mathrm{D}v^*\mathbf{J}^*\,.$$

Since $\mathbf{J}$ is an objective vector, i.e. $\mathbf{J}\circ Y = Q\mathbf{J}^*$, the equation becomes

$$((\mathrm{D}v)\circ Y)Q\mathbf{J}^* = \dot{Q}\mathbf{J}^* + Q\mathrm{D}v^*\mathbf{J}^*$$

or

$$((\mathrm{D}v)\circ Y)Q = \dot{Q} + Q\mathrm{D}v^*\,.$$

This is the transformation rule for the derivative of v. ☐

Mass-momentum-energy laws

As another example, we consider the mass, momentum and energy (the energy is introduced in section III.2). For motivation, we compute the transformation behavior of $\varrho v\, v^{\mathrm{T}}$, which can be computed from the transformation rules for ϱ and v. It is

$$\begin{aligned}(\varrho v\, v^{\mathrm{T}})\circ Y &= (\varrho\circ Y)(v\circ Y)\,(v\circ Y)^{\mathrm{T}} = \varrho^*(\dot X + Qv^*)\,(\dot X + Qv^*)^{\mathrm{T}}\\ &= \varrho^*\dot X\,\dot X^{\mathrm{T}} + \varrho^*\Big(\dot X\,(Qv^*)^{\mathrm{T}} + (Qv^*)\,\dot X^{\mathrm{T}} + \underbrace{(Qv^*)\,(Qv^*)^{\mathrm{T}}}_{=Q(v^*\,v^{*\mathrm{T}})\,Q^{\mathrm{T}}}\Big)\\ &= \varrho^*\dot X\,\dot X^{\mathrm{T}} + \Big(\dot X\,(Q(\varrho^* v^*))^{\mathrm{T}} + Q(\varrho^* v^*)\,\dot X^{\mathrm{T}}\Big) + Q(\varrho^* v^*\,v^{*\mathrm{T}})\,Q^{\mathrm{T}}\end{aligned}$$

and it follows for the ***kinetic energy***

$$\frac{\varrho}{2}|v|^2 = \frac{1}{2}\mathrm{trace}\,(\varrho v\, v^{\mathrm{T}})\,,$$

that

$$(\frac{\varrho}{2}|v|^2)\circ Y = \frac{1}{2}|\dot X|^2\varrho^* + (Q^{\mathrm{T}}\,\dot X)\bullet(\varrho^* v^*) + \frac{\varrho^*}{2}|v^*|^2\,. \qquad \text{(II3.24)}$$

This gives the additional line

$$\Big[\,\frac{1}{2}|\dot X|^2 \quad \dot X^{\mathrm{T}}Q \quad 1\,\Big]$$

in the matrix Z for the energy, as we will see in the following definition.

3.12 Mass-momentum-energy equation (Definition). We consider a system of differential equations

$$\begin{aligned}&\partial_t\varrho + \mathrm{div}\widetilde{J} = \mathbf{r}\,,\\ &\partial_t(\varrho v) + \mathrm{div}\widetilde{\Pi} = \widetilde{\mathbf{f}}\,,\\ &\partial_t e + \mathrm{div}\widetilde{q} = \widetilde{g}\,.\end{aligned}$$

This system is called objective if it transformes between two observers, whose coordinates are related by $(t,x) = Y(t^*,x^*)$, with the general rule

$$\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y$$

for test functions $\zeta:\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{2+n}$, where the quadratic $(1+n+1)$-matrix Z is given by

$$Z := \begin{bmatrix} 1 & 0 & 0\\ \dot X & Q & 0\\ \frac{1}{2}|\dot X|^2 & \dot X^{\mathrm{T}}Q & 1\end{bmatrix}. \qquad \text{(II3.25)}$$

Besides the qantities of the mass-momentum equation the quantity e denotes the ***total energy***, $\widetilde{q}$ the total ***energy flux***, and $\widetilde{g}$ the total ***energy production***.

Remark: Hierbei handelt es sich wieder um ein beliebiges System von jetzt 1+n+1 Gleichungen (5 Gleichungen für $n = 3$), was nur durch das Gesetz für Testfunktionen eingeschränkt ist. Die Matrix Z in (II3.25) definiert dieses System als Masse-Impulse-Energie-System. Alles andere ist eine Folgerung aus dieser Definition.

We can write the system as $n + 2$ single scalar equations

$$\begin{aligned} \partial_t \varrho + \sum_{i=1}^{n} \partial_{x_i} \widetilde{J}_i &= \mathbf{r}\,, \\ \partial_t(\varrho v_k) + \sum_{i=1}^{n} \partial_{x_i} \widetilde{\Pi}_{ki} &= \widetilde{\mathbf{f}}_k \qquad \text{for } k = 1, \dots, n, \\ \partial_t e + \sum_{i=1}^{n} \partial_{x_i} \widetilde{q}_i &= \widetilde{g}\,, \end{aligned}$$

or with $\widetilde{\Pi}_i := \left(\widetilde{\Pi}_{ki}\right)_{k=1,\dots,n}$ by an equation for vectors

$$\partial_t \underbrace{\begin{bmatrix} \varrho \\ \varrho v \\ e \end{bmatrix}}_{=:\, u} + \sum_{i=1}^{n} \partial_{x_i} \underbrace{\begin{bmatrix} \widetilde{J}_i \\ \widetilde{\Pi}_i \\ \widetilde{q}_i \end{bmatrix}}_{=:\, q_i} = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \\ \widetilde{g} \end{bmatrix}.$$

Hence it is of the form in (II3.2) and this system is objective if the transformation rules (II3.5) (equivalent to (I5.11)) hold where we use the quadratic $(1 + n + 1)$-matrix with derivatives

$$Z := \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & Q & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathrm{T}} Q & 1 \end{bmatrix},$$

$$Z_{'0} = \begin{bmatrix} 0 & 0 & 0 \\ \ddot{X} & \dot{Q} & 0 \\ \ddot{X} \bullet \dot{X} & \ddot{X}^{\mathrm{T}} Q + \dot{X}^{\mathrm{T}} \dot{Q} & 0 \end{bmatrix}, \quad Z_{'j} = \begin{bmatrix} 0 & 0 & 0 \\ \dot{X}_{'j} & 0 & 0 \\ \dot{X} \bullet \dot{X}_{'j} & \dot{X}_{'j}{}^{\mathrm{T}} Q & 0 \end{bmatrix}$$

for $j = 1, \dots, n$, where $\dot{X}_{k'j} = \dot{Q}_{kj}$. The equations (II3.5) are

$$\begin{aligned}
\begin{bmatrix} \varrho \\ \varrho v \\ e \end{bmatrix}\circ Y &= Z\begin{bmatrix} \varrho^* \\ \varrho^* v^* \\ e^* \end{bmatrix}, \\
\begin{bmatrix} \widetilde{J}_i \\ \widetilde{\Pi}_i \\ \widetilde{q}_i \end{bmatrix}\circ Y &= \dot{X}_i Z\begin{bmatrix} \varrho^* \\ \varrho^* v^* \\ e^* \end{bmatrix} + \textstyle\sum_{j=1}^n Q_{ij} Z\begin{bmatrix} \widetilde{J}_j^* \\ \widetilde{\Pi}_j^* \\ \widetilde{q}_j^* \end{bmatrix} \qquad \text{for } i=1,\dots,n, \\
\begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \\ \widetilde{g} \end{bmatrix}\circ Y &= Z_{\prime 0}\begin{bmatrix} \varrho^* \\ \varrho^* v^* \\ e^* \end{bmatrix} + \textstyle\sum_{j=1}^n Z_{\prime j}\begin{bmatrix} \widetilde{J}_j^* \\ \widetilde{\Pi}_j^* \\ \widetilde{q}_j^* \end{bmatrix} + Z\begin{bmatrix} \mathbf{r}^* \\ \widetilde{\mathbf{f}}^* \\ \widetilde{g}^* \end{bmatrix}.
\end{aligned}$$

Due to the structure of the matrix Z (the last column of the first two rows is null) the properties of the quantities in the mass and momentum part follow as in 3.7. Therefore, we have to evaluate the energy part of the above transformation rule. The first identity gives

$$e\circ Y = \tfrac{1}{2}|\dot{X}|^2\varrho^* + \varrho^*\dot{X}\bullet(Qv^*) + e^*\,. \tag{II3.26}$$

Comparing this with the rule in (II3.24) for the kinetic energy $\frac{\varrho}{2}|v|^2$ and defining the ***internal energy*** ε by

$$\boxed{e = \varepsilon + \frac{\varrho}{2}|v|^2}$$

we obtain

$$\varepsilon\circ Y = e\circ Y - (\frac{\varrho}{2}|v|^2)\circ Y = e^* - \frac{\varrho^*}{2}|v^*|^2 = \varepsilon^*\,,$$

that is, ε is an objective scalar. The energy part of the second identity is

$$\begin{aligned}
\widetilde{q}_i\circ Y &= \dot{X}_i\Big(\frac{1}{2}|\dot{X}|^2\varrho^* + \varrho^*\dot{X}^{\mathrm{T}}Qv^* + e^*\Big) \\
&\quad + \sum_{j=1}^n Q_{ij}\Big(\frac{1}{2}|\dot{X}|^2\widetilde{J}_j^* + \dot{X}^{\mathrm{T}}Q\widetilde{\Pi}_j^* + \widetilde{q}_j^*\Big),
\end{aligned}$$

which in vector notation reads

$$\begin{aligned}
\widetilde{q}\circ Y &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^*\dot{X}^{\mathrm{T}}Qv^* + e^*)\dot{X} \\
&\quad + \tfrac{1}{2}|\dot{X}|^2 Q\widetilde{J}^* + (Q\widetilde{\Pi}^*Q^{\mathrm{T}})^{\mathrm{T}}\dot{X} + Q\widetilde{q}^*.
\end{aligned} \tag{II3.27}$$

The energy part of the third identity is

$$\begin{aligned}
\widetilde{g}\circ Y &= \varrho^*\ddot{X}\bullet\dot{X} + \varrho^*(\ddot{X}^{\mathrm{T}}Q + \dot{X}^{\mathrm{T}}\dot{Q})v^* \\
&\quad + \sum_{j=1}^n\Big(\dot{X}\bullet\dot{X}_{\prime j}\widetilde{J}_j^* + \dot{X}_{\prime j}{}^{\mathrm{T}}Q\widetilde{\Pi}_j^*\Big) \\
&\quad + \frac{1}{2}|\dot{X}|^2\mathbf{r}^* + \dot{X}^{\mathrm{T}}Q\widetilde{\mathbf{f}}^* + \widetilde{g}^*,
\end{aligned}$$

that is

$$\widetilde{g}\circ Y = \varrho^* \ddot{X}\bullet(\dot{X} + Qv^*) + (\dot{Q}^{\mathrm{T}}\dot{X})\bullet(\varrho^* v^* + \widetilde{J}^*) \\ +(Q^{\mathrm{T}}\dot{Q}):\widetilde{\Pi}^* + \tfrac{1}{2}|\dot{X}|^2\mathbf{r}^* + \dot{X}\bullet Q\widetilde{\mathbf{f}}^* + \widetilde{g}^*. \tag{II3.28}$$

Defining $\mathbf{J}$, Π, and q by

$$\begin{aligned} \widetilde{J} &= \varrho v + \mathbf{J}\,, \\ \widetilde{\Pi} &= \varrho v v^{\mathrm{T}} + v\mathbf{J}^{\mathrm{T}} + \Pi\,, \\ \widetilde{q} &= ev + \frac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}} v + q\,, \end{aligned} \tag{II3.29}$$

we derive, see the following proof, from (II3.26), (II3.27), and (II3.28), the following theorem.

3.13 Theorem. It follows that the mass-momentum-energy system 3.12, written with arbitrary terms $\mathbf{J}$, Π, and q as in (II3.29),

$$\begin{aligned} &\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \\ &\partial_t(\varrho v) + \operatorname{div}(v\,(\varrho v + \mathbf{J})^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,, \\ &\partial_t e + \operatorname{div}(ev + \tfrac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}} v + q) = \widetilde{g}\,, \end{aligned} \tag{II3.30}$$

is objective (siehe Definition 3.12), if

$$\begin{aligned} &e = \varepsilon + \frac{\varrho}{2}|v|^2\,, \quad \varepsilon \text{ is the } \textbf{\textit{internal energy}}, \\ &\varrho,\ \varepsilon,\ \mathbf{r} \text{ are objective scalars, } v \text{ is a velocity,} \\ &\mathbf{J},\ q \text{ are objective vectors, } \Pi \text{ is an objective tensor,} \end{aligned} \tag{II3.31}$$

and with $\Pi = \Pi^{sym} + \Pi^{rest}$, where Π^{sym} must be symmetric,

$$\begin{aligned} &\widetilde{\mathbf{f}} = (\mathbf{r} + \mathbf{J}\bullet\nabla)v + \mathbf{f}\,, \\ &\quad \mathbf{f} \text{ is a } \textbf{\textit{(classical) force}} \text{ (see 3.8),} \\ &\widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + v\bullet\mathrm{D}v\mathbf{J} + \mathrm{D}v:\Pi^{rest} + g\,, \\ &\quad g \text{ is an objective scalar.} \end{aligned} \tag{II3.32}$$

Here both, Π^{sym} and Π^{rest}, must be are objective tensors.
Symmetric case: If Π is symmetric, choose $\Pi^{sym} := \Pi$ and $\Pi^{rest} := 0$.
Standard case: It is $\Pi = \Pi^{\mathrm{S}} + \Pi^{\mathrm{A}}$, hence $\Pi^{sym} := \Pi^{\mathrm{S}}$ and $\Pi^{rest} := \Pi^{\mathrm{A}}$ can be set.

One can also write

$$\widetilde{g} = v\bullet\big(\frac{\mathbf{r}}{2}v + \mathbf{f}\big) + \mathrm{D}v:(v\mathbf{J}^{\mathrm{T}} + \Pi^{rest}) + g\,. \tag{II3.33}$$

(The equations (II3.32) do not mean that $\mathbf{f}$ is independent of $\mathbf{r}$ and $\mathbf{J}$ or that g is independent of $\mathbf{r}$, $\mathbf{J}$, and $\mathbf{f}$, although this is true in many examples.)

Conservation of energy: Es ist g wie in (II3.32) (siehe (II3.40) im Beweis unten) ein objektiven Skalar, somit können wir im Einvernehnmen mit dem Entropieprinzip (siehe z.B. III.1.3) sagen, dass die Wahl $g = 0$ im Entropieprinzip der ***Energieerhaltung*** entspricht.

Proof. For the properties of ϱ, v, $\mathbf{J}$, $\mathbf{r}$, Π, and $\mathbf{f}$ see 3.7 and 3.8. That ε is an objective scalar, has already been shown.

Next, insert $\widetilde{J}^*$ and $\widetilde{\Pi}^*$, as defined in (II3.29), in rule (II3.27). Since

$$\frac{1}{2}|\dot{X}|^2 Q\widetilde{J}^* = \frac{\varrho^*}{2}|\dot{X}|^2 Qv^* + \frac{1}{2}|\dot{X}|^2 Q\mathbf{J}^*$$

and

$$\begin{aligned}\left(Q\widetilde{\Pi}^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X} &= Q(\varrho^* v^* + \mathbf{J}^*)v^{*\mathrm{T}} Q^{\mathrm{T}}\dot{X} + \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X} \\ &= \dot{X}\bullet Qv^*(\varrho^* Qv^* + Q\mathbf{J}^*) + \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X}\,,\end{aligned}$$

this gives

$$\begin{aligned}\widetilde{q}\circ Y &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^*\dot{X}\bullet Qv^* + e^*)\dot{X} \\ &\quad + (\tfrac{1}{2}|\dot{X}|^2 + \dot{X}\bullet Qv^*)(\varrho^* Qv^* + Q\mathbf{J}^*) \\ &\quad + \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X} + Q\widetilde{q}^*. \\ &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^*\dot{X}\bullet Qv^*)(\dot{X} + Qv^*) + e^*\dot{X} \\ &\quad + (\tfrac{1}{2}|\dot{X}|^2 + \dot{X}\bullet Qv^*)Q\mathbf{J}^* + \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X} + Q\widetilde{q}^*.\end{aligned} \tag{II3.34}$$

From known rules one computes

$$\begin{aligned}(\Pi^{\mathrm{T}} v)\circ Y &= \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}(\dot{X} + Qv^*) \\ &= \left(Q\Pi^* Q^{\mathrm{T}}\right)^{\mathrm{T}}\dot{X} + Q(\Pi^{*\mathrm{T}} v^*),\end{aligned} \tag{II3.35}$$

$$\begin{aligned}(\tfrac{1}{2}|v|^2\mathbf{J})\circ Y &= (\tfrac{1}{2}|\dot{X}|^2 + \dot{X}\bullet Qv^* + \tfrac{1}{2}|v^*|^2)Q\mathbf{J}^* \\ &= (\tfrac{1}{2}|\dot{X}|^2 + \dot{X}\bullet Qv^*)Q\mathbf{J}^* + Q(\tfrac{1}{2}|v^*|^2\mathbf{J}^*),\end{aligned} \tag{II3.36}$$

$$\begin{aligned}&(ev)\circ Y = (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^*\dot{X}\bullet(Qv^*) + e^*)(\dot{X} + Qv^*) \\ &= (\tfrac{\varrho^*}{2}|\dot{X}|^2 + \varrho^*\dot{X}\bullet(Qv^*))(\dot{X} + Qv^*) + e^*\dot{X} + Q(e^* v^*).\end{aligned} \tag{II3.37}$$

Subtraction of (II3.35), (II3.36), (II3.37) from (II3.34) gives $q\circ Y = Qq^*$.

Finally the rule (II3.28) becomes, by again inserting $\widetilde{J}^*$ and $\widetilde{\Pi}^*$ and using the formula (II3.17), that is $\widetilde{\mathbf{f}}^* = \mathbf{f}^* + \mathbf{r}^* v^* + \mathrm{D}v^*\mathbf{J}^*$,

$$\begin{aligned}\widetilde{g}\circ Y &= \varrho^*\ddot{X}\bullet(\dot{X}+Qv^*) + (\dot{Q}^{\mathrm{T}}\dot{X})\bullet(\varrho^* v^* + \widetilde{J}^*) + (Q^{\mathrm{T}}\dot{Q}):\widetilde{\Pi}^* \\ &\qquad + \frac{1}{2}|\dot{X}|^2\mathbf{r}^* + \dot{X}\bullet Q\widetilde{\mathbf{f}}^* + \widetilde{g}^* \\ &= \varrho^*\ddot{X}\bullet(\dot{X}+Qv^*) + (\dot{Q}^{\mathrm{T}}\dot{X})\bullet(2\varrho^* v^* + \mathbf{J}^*) + (Q^{\mathrm{T}}\dot{Q}):(v^*(\varrho^* v^* + \mathbf{J}^*)^{\mathrm{T}}) \\ &\qquad + \frac{\mathbf{r}^*}{2}|\dot{X}|^2 + \dot{X}\bullet Q(\mathbf{f}^* + \mathbf{r}^* v^* + \mathrm{D}v^*\mathbf{J}^*) + (Q^{\mathrm{T}}\dot{Q}):\Pi^* + \widetilde{g}^*\,.\end{aligned}$$

Subtracting

$$\begin{aligned}(v\bullet\mathrm{D}v\mathbf{J})\circ Y &= (\dot{X}+Qv^*)\bullet(\mathrm{D}v\circ Y)Q\mathbf{J}^* \\ &= (\dot{X}+Qv^*)\bullet(\dot{Q}+Q\mathrm{D}v^*)\mathbf{J}^* \\ &= (\dot{X}+Qv^*)\bullet\dot{Q}\mathbf{J}^* + \dot{X}\bullet Q\mathrm{D}v^*\mathbf{J}^* + v^*\bullet\mathrm{D}v^*\mathbf{J}^*\end{aligned}$$

one gets using

$$(Qv^*)\bullet(\dot{Q}v^*) = v^*\bullet(Q^{\mathrm{T}}\dot{Q}v^*) = (Q^{\mathrm{T}}\dot{Q}):(v^* v^{*\mathrm{T}}) = 0 \tag{II3.38}$$

since $Q^{\mathrm{T}}\dot{Q}$ is antisymmetric, the equation

$$\begin{aligned}&(\widetilde{g} - v\bullet\mathrm{D}v\mathbf{J})\circ Y \\ &= \varrho^*\ddot{X}\bullet(\dot{X}+Qv^*) + (\dot{Q}^{\mathrm{T}}\dot{X})\bullet(2\varrho^* v^*) \\ &+ \frac{\mathbf{r}^*}{2}|\dot{X}|^2 + \dot{X}\bullet Q(\mathbf{f}^* + \mathbf{r}^* v^*) + (Q^{\mathrm{T}}\dot{Q}):\Pi^* + (\widetilde{g}^* - v^*\bullet\mathrm{D}v^*\mathbf{J}^*)\,.\end{aligned}$$

Then subtracting

$$\begin{aligned}(v\bullet\mathbf{f})\circ Y &= (\dot{X}+Qv^*)\bullet(\varrho^*(\ddot{X}+2\dot{Q}v^*) + Q\mathbf{f}^*) \\ &= \varrho^*(\dot{X}+Qv^*)\bullet(\ddot{X}+2\dot{Q}v^*) + \dot{X}\bullet Q\mathbf{f}^* + v^*\bullet\mathbf{f}^* \\ &= \varrho^*\ddot{X}\bullet(\dot{X}+Qv^*) + 2\varrho^*\dot{X}\bullet\dot{Q}v^* + \dot{X}\bullet Q\mathbf{f}^* + v^*\bullet\mathbf{f}^*\,,\end{aligned}$$

where one used again (II3.38), one is lead to

$$\begin{aligned}(\widetilde{g} - v\bullet\mathbf{f} - v\bullet\mathrm{D}v\mathbf{J})\circ Y &= \frac{\mathbf{r}^*}{2}|\dot{X}|^2 + \mathbf{r}^*\dot{X}\bullet Qv^* + (Q^{\mathrm{T}}\dot{Q})\bullet\Pi^* \\ &\qquad + (\widetilde{g}^* - v^*\bullet\mathbf{f}^* - v^*\bullet\mathrm{D}v^*\mathbf{J}^*)\,,\end{aligned}$$

and therefore using $\mathbf{r}\circ Y = \mathbf{r}^*$ and (II3.24)

$$\begin{aligned}&(\widetilde{g} - \frac{\mathbf{r}}{2}|v|^2 - v\bullet\mathbf{f} - v\bullet\mathrm{D}v\mathbf{J})\circ Y \\ &= (Q^{\mathrm{T}}\dot{Q}):\Pi^* + (\widetilde{g}^* - \frac{\mathbf{r}^*}{2}|v^*|^2 - v^*\bullet\mathbf{f}^* - v^*\bullet\mathrm{D}v^*\mathbf{J}^*)\,.\end{aligned}$$

Hence if we define

$$\bar{g} := \widetilde{g} - \frac{\mathbf{r}}{2}|v|^2 - v\bullet\mathbf{f} - v\bullet\mathrm{D}v\mathbf{J}$$

we have shown

$$\bar{g}\circ Y = (Q^{\mathrm{T}}\dot{Q}){:}\Pi^* + \bar{g}^* \,. \tag{II3.39}$$

Remark: Falls Π symmetrisch ist, also auch Π^*, so ist $(Q^{\mathrm{T}}\dot{Q}){:}\Pi^* = 0$ und deshalb $\bar{g}$ ein objektiver Skalar.
Es folgt im allgemeinen Fall, wenn wir den Tensor $\mathrm{D}v$ nehmen, wegen (II4.13) die Formel

$$(\mathrm{D}v{:}\Pi)\circ Y = (\dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^*Q^{\mathrm{T}}){:}(Q\Pi^*Q^{\mathrm{T}}) = (Q^{\mathrm{T}}\dot{Q}){:}\Pi^* + \mathrm{D}v^*{:}\Pi^* \,,$$

das heißt die gleiche Transformationsformel wie in (II3.39). Um der Aussage in dem Hinweis nahezukommen, zerlegen wir Π in zwei objektive Tensoren

$$\Pi = \Pi^{sym} + \Pi^{rest} \quad \text{wobei } \Pi^{sym} \text{ symmetrisch}$$

ist. Da Π^{sym*} symmetrisch ist, ist $(Q^{\mathrm{T}}\dot{Q}){:}\Pi^{sym*} = 0$, und daher auch

$$(\mathrm{D}v{:}\Pi^{rest})\circ Y = (\dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^*Q^{\mathrm{T}}){:}(Q\Pi^{rest*}Q^{\mathrm{T}}) = (Q^{\mathrm{T}}\dot{Q}){:}\Pi^* + \mathrm{D}v^*{:}\Pi^{rest*} \,.$$

Indem wir Differenz zu (II3.39) bilden, folgt dass

$$g := \bar{g} - \mathrm{D}v{:}\Pi^{rest} \tag{II3.40}$$

ein objektiver Skalar ist, d.h. $g\circ Y = g^*$. (Die Aufteilung Π^{rest} gleich Π und Π^{sym} gleich 0 ist zwar mathematisch möglich, aber nicht physikalisch, wenn wir darauf bestehen $g = 0$ setzen zu können.) □

We finish this section by the following remark, see section 6.

3.14 Remark. If the pressure tensor Π is symmetric, then the conservation of angular momentum is satisfied.

Proof. See the statement 6.6. □

4 Constitutive relations

A "constitutive relation" describes the dependence of a quantity on other quantities. A "constitutive function" is a rule how the value of a physical quantity is calculated from the values of other physical quantities, which in general are independent variables. Thus the physical quantity, which is described by a function, becomes a dependent variable. Constitutive relations describe various concrete materials, they are therefore necessary for the communication between observers. For example, let w be a physical quantity depending on the quantities u_i, $i = 1, \dots, N$, that is, there is a function $\widehat{w}$ connecting these quantities

$$w = \widehat{w}(u_1, \dots, u_N)\,,$$

which means

$$w(t,x) = \widehat{w}(u_1(t,x), \dots, u_N(t,x))\,.$$

We then call $\widehat{w}$ a ***constitutive function***. Regarding $\widehat{w}$ we postulate the following.

4.1 Definition. A constitutive function is called ***objective*** if it is the same function for all observers. So, for example, let us assume the coordinates of two observers are related by $(t,x) = Y(t^*,x^*)$, and w, u_1,..., u_N are quantities w.r.t. one observer, and w^*, u_1^*,..., u_N^* are the "same quantities" (as defined e.g. by 3.2) w.r.t. the other observer. Then the equation

$$w(t,x) = \widehat{w}(u_1(t,x), \dots, u_N(t,x))$$

for the first observer is equivalent to the equation

$$w^*(t^*,x^*) = \widehat{w}(u_1^*(t^*,x^*), \dots, u_N^*(t^*,x^*))$$

for the second observer. *Observe:* $\widehat{w}$ is the same for the two observers.

This definition is essential for the physical treatment of problems. Constitutive functions must therefore be objective, i.e. they have to be independent of the observer. The physical quantities have different values for different observers, but the relationships between them are the same, that means, constitutive functions that express these relations have to be the same for all observers.

Referenzen: For statements about the independence of constitutive functions see Eck & Garcke & Knabner [4, 5.8 Beobachterunabhängigkeit], Greve [5, 1.4 Transformationseigenschaften], REF–??–.

Now we give some examples of constitutive functions.

Objective scalars

As simplest example let us take objective scalars, for example, the production term of a scalar equation.

4.2 Example. If p is an objective scalar, and $u_1, \dots, u_N$ are objective scalars, then any constitutive relation between p and $(u_1, \dots, u_N)$ is objective. In other words, if the constitutive function $\widehat{p}$ is given by

$$p = \widehat{p}(u_1, u_2, \dots, u_N), \tag{II4.1}$$

then it is the same function for all observers.

Proof. Let p^*, $u_1^*, \dots, u_N^*$ be the quantities with respect to an observer whose coordinates are transformed with $(t, x) = Y(t^*, y^*)$. Then it holds

$$p \circ Y = p^* , \quad u_j \circ Y = u_j^* . \tag{II4.2}$$

From the constitutive relation (II4.1), which states

$$p(t, x) = \widehat{p}(u_1(t, x), u_2(t, x), \dots, u_N(t, x)) ,$$

follows that (set $(t, x) = Y(t^*, x^*)$)

$$p \circ Y(t^*, x^*) = \widehat{p}(u_1 \circ Y(t^*, x^*), u_2 \circ Y(t^*, x^*), \dots, u_N \circ Y(t^*, x^*)) ,$$

and hence, keepting (II4.2) in mind,

$$p^*(t^*, x^*) = \widehat{p}(u_1^*(t^*, x^*), u_2^*(t^*, x^*), \dots, u_N^*(t^*, x^*)) ,$$

that is, $\widehat{p}$ stays the same function. □

4.3 Example. Let p and $u_1, \dots, u_N$ objective scalars and let $\widehat{p}$ be with

$$p(t, x) = \widehat{p}(t, x, u_1(t, x), u_2(t, x), \dots, u_N(t, x)) . \tag{II4.3}$$

Then $\widehat{p}$ is an objective function if and only if $\widehat{p}$ is independent of t and x.

Proof. Aus den Voraussetzungen folgt für $(t, x) = Y(t^*, x^*)$

$$\begin{aligned} &\widehat{p}((t^*, x^*), u^*(t^*, x^*)) = p^* = p \circ Y \\ &= \widehat{p}(Y(t^*, x^*), u(Y(t^*, x^*))) = \widehat{p}(Y(t^*, x^*), u^*(t^*, x^*)) , \end{aligned}$$

wobei $u^* = (u_1^*, u_2^*, \dots, u_N^*)$. Wegen $Y(t^*, x^*) = (t^* + \mathrm{a}, Q(t^*)x^* + \mathrm{b}(t^*))$ folgt also

$$\widehat{p}(t^*, x^*, u^*(t^*, x^*)) = \widehat{p}(t^* + \mathrm{a}, Q(t^*)x^* + \mathrm{b}(t^*), u^*(t^*, x^*)) .$$

Indem wir nun bei gegebenem (t^*, x^*) den Wert a beliebig wählen, folgt dass $\widehat{p}$ unabhängig von t^* ist. Wenn wir dann $Q(t^*) = \mathrm{Id}$ und $\mathrm{b}(t^*)$ beliebig wählen, folgt die Unabhängigkeit von $\widehat{p}$ von x^*. □

Also inequalities are covered by the principle of objectivity.

4.4 Inequality. If u is an objective scalar, then

$$u > 0$$

is an objective inequality. The inequality $u > 0$ means actually

$$\forall\ (t,x) \in \mathbb{R} \times \mathbb{R}^n :\ u(t,x) > 0\,.$$

Remark: Transforms u (affine) linear, that is, $u{\circ}Y = Au^* + B$, then the inequality is objective only if $B = 0$ and $A > 0$.

Proof. If Y is an observer transformation, then $u^* = u{\circ}Y > 0$, hence the inequality is objective. □

Objective vectors

Now we turn to objective vectors where we take as example the objective flux vector **J** in the mass conservation.

4.5 Lemma. Assume the following is true:

$$J = \widehat{J}(\nabla\varrho)\,,$$

where ϱ is an objective scalar and J an objective vector. Then $\widehat{J}$ is objective, that means, it is the same function for all observers, if it is satisfied that

$$\widehat{J}(Qq) = Q\widehat{J}(q) \tag{II4.4}$$

for all $q \in \mathbb{R}^n$ and for all orthonormal transformations $Q \in \mathbb{R}^{n\times n}$ with determinant $\det Q = 1$. *Supplement:* Under these assumptions the constitutive function

$$\widehat{J}(q) := -a(|q|)q \text{ for all } q \in \mathbb{R}^n$$

is objective, where a is a given real function.

Proof of the supplement. For all orthonormal Q we have the identity $|Qq| = |q|$. Hence

$$\widehat{J}(Qq) = -a(|Qq|)Qq = Q(-a(|Qq|)q) = Q(-a(|q|)q) = Q\widehat{J}(q)\,.$$

Hence (II4.4) is true. □

Proof. Let Y be an observer transformation, then it holds

$$\varrho{\circ}Y = \varrho^*\,, \quad J{\circ}Y = QJ^*\,.$$

Now we calculate the gradient by (because the time component of Y does not depend on x^*)

$$\partial_{x_j^*}\varrho^* = \partial_{x_j^*}(\varrho\circ Y) = (\partial_t\varrho)\circ Y \underbrace{Y_{0}{}'_{x_j^*}}_{=0} + \sum_{i=1}^{n}(\partial_{x_i}\varrho)\circ Y \underbrace{Y_{i}{}'_{x_j^*}}_{=Q_{ij}}$$

and hence in vector notation $\nabla\varrho^* = Q^{\mathrm{T}}(\nabla\varrho\circ Y)$ or

$$(\nabla\varrho)\circ Y = Q\nabla\varrho^* \,. \tag{II4.5}$$

So $\nabla\varrho$ is an objective vector. Now let (II4.4) be true. Then the equation $J(t,x) = \widehat{J}(\nabla\varrho(t,x))$ implies, since J is an objective Vector, that

$$QJ^* = J\circ Y = \widehat{J}((\nabla\varrho)\circ Y) = \widehat{J}(Q\nabla\varrho^*) = Q\widehat{J}(\nabla\varrho^*)\,,$$

due to the assumption (II4.4) on $\widehat{J}$. Hence it follows for the *-observer

$$J^* = \widehat{J}(\nabla\varrho^*)\,, \quad \text{i.e.} \quad J^*(t^*,x^*) = \widehat{J}(\nabla\varrho^*(t^*,x^*))$$

therefore $\widehat{J}$ is the constitutive function also for the *-observer, hence $\widehat{J}$ is objective. *Remark:* Also the reverse of this conclusion is true, see 7.10.

□

This result is now used in the mass conservation as diffusion term.

4.6 Diffusion (Example). Let ϱ be an objective scalar, v a velocity, and a a scalar function. Two phrasings:

(1) Then the scalar differential equation

$$\partial_t\varrho + \operatorname{div}\big(\varrho v - a(\varrho,|\nabla\varrho|)\nabla\varrho\big) = 0$$

is objective, which means that it has the same form for all observers.

(2) You can also say

$$\partial_t\varrho + \operatorname{div}\big(\varrho v - a\nabla\varrho\big) = 0\,,$$
$$a = \widehat{a}(\varrho,|\nabla\varrho|)$$

where, of course by 4.2, $\widehat{a}$ is an objective function.

Note: The mass diffusion was established by Adolf Fick in 19th century (see [Wikipedia: Fick's laws of diffusion]). For the sign of a, that is $a \geq 0$, we have to wait for the entropy principle in section III.1. See also Hutter & Wang [9, 17.4.1 Diffusion].

Proof. It is $|\nabla\varrho|$ like ϱ an objective scalar and therefore $\widehat{a}$ is a function depending on objective scalars, therefore $\widehat{a}$ is the same for all observers by 4.2. Therefore in the mass equation $\partial_t\varrho + \operatorname{div}\big(\varrho v + \mathbf{J}\big) = 0$, see 3.4, the vector $\mathbf{J} = -a\nabla\varrho$ is objective, □

Wie geben nun ein Beispiel zum Zusammenhang zwischen objektiven Vektoren und objektiven Tensoren an. Wie wir sehen werden, werden um so mehr Größen bei konstitutiven Funktionen benötigt, je detaillierter wir in die Beschreibung von physikalischen Vorgängen einsteigen. (Siehe auch Aufgabe 7.14.)

4.7 Lemma. Let $(t,x) \mapsto \{e_1(t,x),\dots,e_n(t,x)\}$ be an orthonormal system of $\mathbb{R}^n$, i.e.

$$e_i{\bullet}e_j = \delta_{i,j} \text{ for } i,j = 1,\dots,n \tag{II4.6}$$

with e_i being objective vectors. Further, let λ_{ij} be objective scalars. If Π is an objective tensor, then

$$\Pi = \sum_{ij} \lambda_{ij} e_i {e_j}^{\mathrm{T}}$$

is an objective representation of Π. *Remark:* The fact that the e_i are objective vectors is consistent with the statement (II4.6). *Example:* The matrix $\Pi = p\mathrm{Id}$ with an objective scalar p has this representation.

It holds $e_i{\bullet}(\Pi e_j) = \lambda_{ij}$ for all i and j, that is, Π has the following representation with respect to the basis $\{e_1,\dots,e_n\}$

$$\big(e_i{\bullet}(\Pi e_j)\big)_{i,j=1,\dots,n} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{bmatrix}$$

Proof. It is $\Pi{\circ}Y = Q\Pi^* Q^{\mathrm{T}}$, $e_i{\circ}Y = Qe_i^*$, and $\lambda{\circ}Y = \lambda^*$. Due to

$$e_i^*{\bullet}e_j^* = (Q^{\mathrm{T}}\, e_i{\circ}Y){\bullet}(Q^{\mathrm{T}}\, e_j{\circ}Y) = e_i{\circ}Y{\bullet}e_j{\circ}Y = \delta_{i,j}$$

the system is also an orthonormal system for the new observer. From this it follows

$$\begin{aligned} \sum_{ij} \lambda_{ij}^* e_i^* {e_j^*}^{\mathrm{T}} &= Q^{\mathrm{T}} \sum_{ij} \lambda_{ij}^* Q e_i^* \,(Q e_j^*)^{\mathrm{T}}\, Q \quad \big(\text{since } Q^{\mathrm{T}} Q = \mathrm{Id}\big) \\ &= Q^{\mathrm{T}} \Big(\sum_{ij} \lambda_{ij} e_i {e_j}^{\mathrm{T}} \Big){\circ}Y Q = Q^{\mathrm{T}}\, \Pi{\circ}Y Q = Q^{\mathrm{T}}\, Q\Pi^* Q^{\mathrm{T}}\, Q = \Pi^* \,. \end{aligned}$$

Thus the representation is objective. □

4.8 Allgemeiner. In 4.7 kann man natürlich außer Tensoren Π auch andere Größen durch ein objektives Orthonormalsystem darstellen. So ist für einen objektiven Vektor J

$$J = \sum_i \lambda_i e_i$$

eine objektive Darstellung, falls λ_i objektive Skalare sind. *Definition:* Ein objektives Orthonormalsystem ist ein Orthonormalsystem aus objektiven Vektoren.

This is a representation of the objective tensor Π or an objective vector J by an objective orthonormal system. Such representations can be used to describe inhomogeneous materials. Take for example an objective vector J

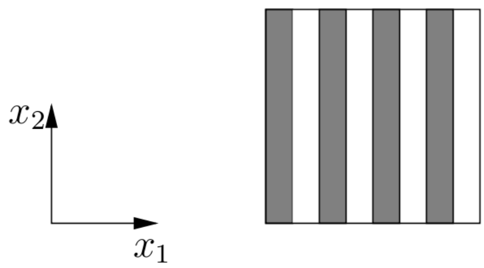


Fig. 4: “Geschichtetes Material” (aus dem Buch [4])

of the form

$$J = -D\nabla\varrho \tag{II4.7}$$

with a matrix function D (conditions will be derived) and an objective scalar ϱ. What are the conditions on D so that equation (II4.7) is the same for all observers? For a *-observer we have $J^* = -D^*\nabla\varrho^*$. Since $\nabla\varrho$ is an objective vector, we get

$$J\circ Y = -(D\nabla\varrho)\circ Y = -D\circ Y\, Q\nabla\varrho^* \,.$$

On the other hand, since J is an objective vector,

$$J\circ Y = QJ^* = -QD^*\nabla\varrho^* \,,$$

hence we obtain

$$D\circ Y\, Q\nabla\varrho^* = QD^*\nabla\varrho^* \,.$$

This is satisfied, if $D\circ Y\, Q = QD^*$ or

$$D\circ Y = QD^* Q^{\mathrm{T}} \,, \tag{II4.8}$$

that is, the transformation rule of an objective tensor. (If $D = \mathrm{const}$ and the constant is the same constant for all observers, it follows (if $n \geq 3$) that $D = a\mathrm{Id}$, a a scalar, see 4.14(4). The same holds if D depends on a finite number of objective scalars, see exercise 7.11.) Now, if $n = 2$, take D as a matrix

$$D = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} ,$$

see Fig. 4. This is for an observer with (x_1, x_2)-coordinates. If we consider another observer with (x_1^*, x_2^*)-coordinates we have to use the matrix D^* with the property in (II4.8), that is, if

$$Q = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \text{ then } D^* = \begin{bmatrix} a\cos^2\vartheta + \sin^2\vartheta & (a-1)\cos\vartheta\sin\vartheta \\ (a-1)\cos\vartheta\sin\vartheta & a\sin^2\vartheta + \cos^2\vartheta \end{bmatrix}.$$

In particular, if

$$Q = \pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ then } D^* = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}.$$

This is because the material stays at the same physical place but the coordinates differ. So we can say

4.9 Lemma. Let ϱ be an objective scalar and J an objective vector. If D satisfies the transformation rule (II4.8) then the equation

$$J = -D\nabla\varrho$$

is objective, that is, is the same for all observers.

Proof. If $J + D\nabla\varrho = 0$ we compute

$$\begin{aligned} 0 &= (J + D\nabla\varrho)\circ Y = J\circ Y + D\circ Y(\nabla\varrho)\circ Y \\ &= QJ^* + (QD^*Q^{\mathrm{T}})Q\nabla\varrho^* = Q(J^* + D^*\nabla\varrho^*)\,. \end{aligned}$$

Hence $0 = J^* + D^*\nabla\varrho^*$. □

Therefore, the vector J depends on D and $\nabla\varrho$, where D is an objective tensor and $\nabla\varrho$ is an objective vector. Hence making the matrix D public we get the following lemma.

4.10 Lemma. Let ϱ be an objective scalar, J an objective vector, and D an objective tensor. Assume

$$J = \widehat{J}(D, \nabla\varrho)\,.$$

Then the function $(M, q) \mapsto \widehat{J}(M, q)$ is objective, that is the same function for all observers, if for all orthonormal matrices Q with positive determinant and all M and q

$$\widehat{J}(QMQ^{\mathrm{T}}, Qq) = Q\widehat{J}(M, q)\,. \tag{II4.9}$$

Example: $\widehat{J}(M, q) = -Mq$.

We see that in order to have the same function $\widehat{J}$ for all observers it is neccessary to add the matrix D as argument to the function $\widehat{J}$.

Proof. We assume that $J = \widehat{J}(D, \nabla\varrho)$, that is, $\widehat{J}$ is the constitutive function for this observer. Now, another observer has the quantities J^*, D^*, ϱ^*. If Y is the corresponding observer transformation, we take Q from this transformation and set $M = D^*$ and $q = \nabla\varrho^*$ in (II4.9). We get

$$\widehat{J}(QD^*Q^{\mathrm{T}}, Q\nabla\varrho^*) = Q\widehat{J}(D^*, \nabla\varrho^*)\,. \tag{II4.10}$$

If we use the known transformation rules for D and $\nabla\varrho$, which are

$$QD^*Q^{\mathrm{T}} = D{\circ}Y \text{ and } Q\nabla\varrho^* = (\nabla\varrho){\circ}Y\,,$$

we obtain

$$\widehat{J}(D, \nabla\varrho){\circ}Y = Q\widehat{J}(D^*, \nabla\varrho^*)\,,$$

and the left side is, since J is an objective vector,

$$\widehat{J}(D, \nabla\varrho){\circ}Y = J{\circ}Y = QJ^*$$

and it follows $J^* = \widehat{J}(D^*, \nabla\varrho^*)$.

Remark: For the reverse direction of the assertion, we assume that $\widehat{J}$ is the same function for all observers. Then (II4.10) follows arguing in the reverse direction. Now, if for all processes in consideration D^* can be any matrix and $\nabla\varrho^*$ any vector, the equation (II4.9) follows. $\square$

A formulation with an orthonormal system is also possible: Let $n = 2$ and realize, that for the observer with (t, x)-coordinates

$$D = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = ae_1{e_1}^{\mathrm{T}} + e_2{e_2}^{\mathrm{T}} \tag{II4.11}$$

if $\{e_1, e_2\}$ is the standard orthonormal system with respect to the (t, x)-coordinates, this is of the form in 4.7 (for Π). If we consider the other observer with (x_1^*, x_2^*)-coordinates we have to use

$$D^* = ae_1^*{e_1^*}^{\mathrm{T}} + e_2^*{e_2^*}^{\mathrm{T}}\,,$$
$$e_1^* = \begin{bmatrix} \cos\vartheta \\ -\sin\vartheta \end{bmatrix}, \quad e_2^* = \begin{bmatrix} \sin\vartheta \\ \cos\vartheta \end{bmatrix}, \text{ if } \quad Q = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}.$$

So in general one considers (II4.11) with objective vectors e_i, $i = 1, 2$. Also lemma 4.10 can be adopted to orthogonal systems. So, for example, if $\{e_1, e_2\}$ is an orthonormal system of objective vectors, the representation

$$J = -D\nabla\varrho\,,$$
$$D = ae_1{e_1}^{\mathrm{T}} + e_2{e_2}^{\mathrm{T}}\,, \quad a = \widehat{a}(|\nabla\varrho|)\,,$$

defines an objective vector J. The considerations are relevant, if one has, for example, several bodies in a fluid.

Objective tensors

The objectivity considerations also apply to tensors. This is important for tensors in the theory of elasticity, see section 5, and in the theory of fluid dynamics, where we consider the tensor Π. The objectivity considerations for this tensor will be very important.

4.11 Lemma. Let ϱ and ε be objective scalars and v a velocity. Let Π be an objective tensor and assume

$$\Pi = \widehat{\Pi}(\varrho, \varepsilon, v, \nabla\varrho, \nabla\varepsilon, \mathrm{D}v)$$

with an objective function $\widehat{\Pi}$. Then $\widehat{\Pi}$ is independent of v and depends only on the symmetric part of $\mathrm{D}v$. Therefore $\widehat{\Pi}$ can be chosen as

$$\Pi = \widehat{\Pi}(\varrho, \varepsilon, \nabla\varrho, \nabla\varepsilon, (\mathrm{D}v)^{\mathrm{S}})\,.$$

Remark: There are of course even more conclusions.

Proof. For the velocity v there holds

$$v_i \circ Y = \dot{X}_i + \sum_j Q_{ij} v_j^*\,.$$

We calculate the derivatives with respect to x_k^*, and if we take into account that $X_{j\,{}'x_k^*} = Q_{jk}$ and $\dot{X}_{i\,{}'x_k^*} = \dot{Q}_{jk}$, we get

$$\begin{aligned}
&\sum_{j=1}^{n} (\partial_{x_j} v_i)\circ Y\, Q_{jk} = \partial_{x_k^*}(v_i \circ Y) \\
&= \dot{X}_{i\,{}'x_k^*} + \sum_j \partial_{x_k^*}(Q_{ij} v_j^*) = \dot{Q}_{ik} + \sum_j Q_{ij} \partial_{x_k^*} v_j^*\,,
\end{aligned}$$

that is, in matrix notation

$$(\mathrm{D}v)\circ Y\, Q = \dot{Q} + Q\mathrm{D}v^*\,. \tag{II4.12}$$

If we multiply this from the right with the matrix Q^{T}

$$\boxed{(\mathrm{D}v)\circ Y = \dot{Q}\,Q^{\mathrm{T}} + Q\mathrm{D}v^*\,Q^{\mathrm{T}}} \tag{II4.13}$$

with an antisymmetric matrix $\dot{Q}\,Q^{\mathrm{T}}$.[4] If we plug this into $\Pi\circ Y = Q\Pi^* Q^{\mathrm{T}}$, then, by exploiting that $\widehat{\Pi}$ is objective, we get

$$\begin{aligned}
&Q\widehat{\Pi}(\varrho^*, \varepsilon^*, v^*, \nabla\varrho^*, \nabla\varepsilon^*, \mathrm{D}v^*)\,Q^{\mathrm{T}} = Q\Pi^*\,Q^{\mathrm{T}} = \Pi\circ Y \\
&= \widehat{\Pi}(\varrho\circ Y, \varepsilon\circ Y, v\circ Y, (\nabla\varrho)\circ Y, (\nabla\varepsilon)\circ Y, (\mathrm{D}v)\circ Y) \\
&= \widehat{\Pi}(\varrho^*, \varepsilon^*, \dot{X} + Qv^*, Q\nabla\varrho^*, Q\nabla\varepsilon^*, \dot{Q}\,Q^{\mathrm{T}} + Q\mathrm{D}v^*\,Q^{\mathrm{T}})\,.
\end{aligned}$$

[4] $\dot{Q}\,Q^{\mathrm{T}} = (Q\,Q^{\mathrm{T}})^{\boldsymbol{\cdot}} - Q\,(Q^{\mathrm{T}})^{\boldsymbol{\cdot}} = -Q\,\dot{Q}^{\mathrm{T}} = -(\dot{Q}\,Q^{\mathrm{T}})^{\mathrm{T}}$

At a certain point (t_0^*, x_0^*) we can choose the transformation Y such that the value of $c = \dot{X}$ at this point can be arbitrary, and such that $A = \dot{Q}Q^{\mathrm{T}}$ is an arbitrary antisymmetric matrix, with simultaneous selection of $Q = \mathrm{Id}$ at this point. (In detail: First choose $Q(t_0^*) = \mathrm{Id}$ and let Q for $t^* \neq t_0^*$ be so that $\dot{Q}(t_0^*) = A$. Now we have the formula $\dot{X}(t^*, x^*) = \dot{Q}(t^*)x^* + \dot{\mathrm{b}}(t^*)$ for all (t^*, x^*). Then choose $\mathrm{b}(t_0^*)$ so that $\dot{X}(t_0^*, x_0^*) = c$.) We get at the point (t_0^*, x_0^*) (with abbreviations $\varepsilon^* = \varepsilon^*(t_0^*, x_0^*)$, $\nabla\varepsilon^* = \nabla\varepsilon^*(t_0^*, x_0^*)$ etc.)

$$\widehat{\Pi}(\varrho^*, \varepsilon^*, v^*, \nabla\varrho^*, \nabla\varepsilon^*, \mathrm{D}v^*) = \widehat{\Pi}(\varrho^*, \varepsilon^*, c + v^*, \nabla\varrho^*, \nabla\varepsilon^*, A + \mathrm{D}v^*)$$

for all c in $\mathbb{R}^n$ and all antisymmetric matrices A. Now look at this equation. If we choose especially $c = -v^*$ and $A = -(\mathrm{D}v^*)^{\mathrm{A}}$, then we obtain, due to $\mathrm{D}v^* = (\mathrm{D}v^*)^{\mathrm{S}} + (\mathrm{D}v^*)^{\mathrm{A}}$,

$$\begin{aligned}\widehat{\Pi}(\varrho^*, \varepsilon^*, v^*, \nabla\varrho^*, \nabla\varepsilon^*, \mathrm{D}v^*) &= \widehat{\Pi}(\varrho^*, \varepsilon^*, 0, \nabla\varrho^*, \nabla\varepsilon^*, (\mathrm{D}v^*)^{\mathrm{S}}) \\ &=: \widetilde{\Pi}(\varrho^*, \varepsilon^*, \nabla\varrho^*, \nabla\varepsilon^*, (\mathrm{D}v^*)^{\mathrm{S}})\end{aligned}$$

with a new constitutive function $\widetilde{\Pi}$. This is the assertion. □

4.12 Constitutive function for liquids. For the pressure tensor the following representation is objective:

$$\begin{aligned}\Pi &= p\mathrm{Id} - S\,, \\ S &= a\big(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}\big) + b\ \mathrm{div}(v)\mathrm{Id} \\ &= 2a\ (\mathrm{D}v)^{\mathrm{S}} + b\ \mathrm{div}(v)\mathrm{Id} \\ &= 2\eta\ (\underbrace{(\mathrm{D}v)^{\mathrm{S}} - \frac{1}{n}\mathrm{div}(v)\mathrm{Id}}_{\text{trace free}}) + \zeta\ \mathrm{div}(v)\mathrm{Id}\,, \\ & a := \eta\,, \quad b := \zeta - \frac{2}{n}\eta\ \Big(= \zeta - \frac{2}{3}\eta \text{ for } n = 3\Big)\,,\end{aligned}$$

where p, η and ζ are dependent on $(\varrho, \varepsilon, |\,(\mathrm{D}v)^{\mathrm{S}}\,|)$.Here

p is the pressure of the liquid (an objective scalar),

η, ζ are the viscosity coefficients.[5]

Proof. Due to (II4.13) we have

$$\mathrm{D}v\circ Y = \underbrace{\dot{Q}Q^{\mathrm{T}}}_{\text{skew-symmetric}} + Q\mathrm{D}v^*Q^{\mathrm{T}}$$

[5]The notation η und ζ you find in Landau & Lifschitz [10, (15,3)-(15,4)]

and thus for the symmetric part $(\mathrm{D}v)^{\mathrm{S}}$ of $\mathrm{D}v$ it is

$$(\mathrm{D}v)^{\mathrm{S}}\circ Y = Q\,(\mathrm{D}v^*)^{\mathrm{S}}\,Q^{\mathrm{T}}\ , \tag{II4.14}$$

that is, $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor.

Now it applies for the Euclidean norm of the matrices $M, N \in \mathbb{R}^{n\times n}$ representing an objective tensor that $M\circ Y = QM^*Q^{\mathrm{T}}$ and $N\circ Y = QN^*Q^{\mathrm{T}}$, and hence [6]

$$(M{:}N)\circ Y = (QM^*Q^{\mathrm{T}}){:}(QN^*Q^{\mathrm{T}}) = \sum_{ij}(QM^*Q^{\mathrm{T}})_{ij}(QN^*Q^{\mathrm{T}})_{ij}$$
$$= \sum_{ijklpq} Q_{ik}M^*_{kl}Q_{jl}Q_{ip}N^*_{pq}Q_{jq} = \sum_{klpq}\delta_{kp}M^*_{kl}N^*_{pq}\delta_{lq} = \sum_{kl}M^*_{kl}N^*_{kl} = M^*{:}N^*$$

and therefore

$$|M|\circ Y = \sqrt{M{:}M} = \sqrt{M^*{:}M^*} = |M^*|\,.$$

It follows that the Euclidean norm of $|\,(\mathrm{D}v)^{\mathrm{S}}\,|$ is an objective scalar,

$$|\,(\mathrm{D}v)^{\mathrm{S}}\,|\circ Y = |\,(\mathrm{D}v^*)^{\mathrm{S}}\,|\,,$$

as well as ϱ and ε. Thus, the constitutive functions of a and b, also η, ζ and p, fall under the Example 4.2. So we can assume in the following, as if these functions were constant.

Now $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor, see (II4.14), and thus for each orthonormal system $\{e_1(t,x),\dots,e_n(t,x)\}$ we have

$$\operatorname{div} v = \sum_i e_i\bullet(\mathrm{D}v)^{\mathrm{S}}\,e_i\,,$$

and hence

$$(\operatorname{div} v)\circ Y = \sum_i (e_i\circ Y)\bullet((\mathrm{D}v)^{\mathrm{S}}\circ Y)(e_i\circ Y)$$
$$= \sum_i (e_i\circ Y)\bullet(Q\,(\mathrm{D}v^*)^{\mathrm{S}}\,Q^{\mathrm{T}})(e_i\circ Y)$$
$$= \sum_i (Q^{\mathrm{T}}\,e_i\circ Y)\bullet(\mathrm{D}v^*)^{\mathrm{S}}\,(Q^{\mathrm{T}}\,e_i\circ Y) = \operatorname{div} v^*\,,$$

because $\{Q(t^*)^{\mathrm{T}}\,e_1(t,x),\dots,Q(t^*)^{\mathrm{T}}\,e_n(t,x)\}$ is also an orthonormal system. Therefore, $\operatorname{div} v$ is an objective scalar.

Finally it follows that S is an objective tensor and therefore also Π. □

If one replaces a constitutive function of a physical quantity by a function, which depends on more parameters, the situation usually is changing dramatically. Therefore one has to take into account the structure of constitutive functions. We now show that under linearity conditions for Π a representation as in 4.12 is also necessary (see also III.2.5).

[6] For matrices M, N we define $M{:}N := \sum_{i,j} M_{ij}N_{ij}$, that is $|M|^2 := \sum_{i,j}|M_{ij}|^2$.

4.13 Theorem. Let $\Pi = \widehat{\Pi}(\varrho, \varepsilon, v, \mathrm{D}v)$ with a symmetric objective function $\widehat{\Pi}$, i.e. $\widehat{\Pi}_{ji} = \widehat{\Pi}_{ij}$ for $i, j = 1, \dots, n$. If $\widehat{\Pi}$ is affine linear in $\mathrm{D}v$, then Π has the representation in 4.12 where the coeffcients p, η and ζ dont depend on $(\mathrm{D}v)^{\mathrm{S}}$. *Notice:* It is assumed that $\widehat{\Pi}$ is defined for all values of $\mathrm{D}v$.

Proof [7]. For $\widehat{\Pi}$ one obtains independence of v and independence of the antisymmetric part of $\mathrm{D}v$ in the same manner as in 4.11. Therefore

$$\widehat{\Pi}(\varrho, \varepsilon, v, \mathrm{D}v) = \widehat{\Pi}(\varrho, \varepsilon, 0, (\mathrm{D}v)^{\mathrm{S}}) \,.$$

Now Π is an objective tensor, that is, $\Pi \circ Y = Q \Pi^* Q^{\mathrm{T}}$ or

$$\widehat{\Pi}(\varrho, \varepsilon, 0, (\mathrm{D}v)^{\mathrm{S}}) \circ Y = Q \widehat{\Pi}(\varrho^*, \varepsilon^*, 0, (\mathrm{D}v^*)^{\mathrm{S}}))\, Q^{\mathrm{T}} \,.$$

Inserting the transformation rules for the arguments this is

$$\widehat{\Pi}(\varrho^*, \varepsilon^*, 0, Q\,(\mathrm{D}v^*)^{\mathrm{S}}\, Q^{\mathrm{T}}) = Q \widehat{\Pi}(\varrho^*, \varepsilon^*, 0, (\mathrm{D}v^*)^{\mathrm{S}}))\, Q^{\mathrm{T}} \,,$$

where the function $\widehat{\Pi}$ is affine linear in the last argument. For $(\mathrm{D}v^*)^{\mathrm{S}} = 0$ we obtain that

$$\widehat{\Pi}(\varrho^*, \varepsilon^*, 0, 0) = Q \widehat{\Pi}(\varrho^*, \varepsilon^*, 0, 0)\, Q^{\mathrm{T}}$$

for all orthogonal matrices Q. This implies that the matrix $\widehat{\Pi}(\varrho^*, \varepsilon^*, 0, 0)$ (for fixed values of ϱ^* and ε^*) is a constant objective tensor, and therefore, by 4.14(4) for $n \geq 3$, is a multiple of the identity, that is,

$$\widehat{\Pi}(\varrho^*, \varepsilon^*, 0, 0) = \widehat{p}(\varrho^*, \varepsilon^*) \mathrm{Id} \,,$$

where physically $p = \widehat{p}(\varrho, \varepsilon)$ is the "pressure". Then $S := p\mathrm{Id} - \Pi$ is linear in the gradient, hence we have for the "stress tensor" S a representation

$$S_{ij} = \sum_{k,l=1}^{n} c_{ijkl} (\partial_k v_l + \partial_l v_k) \,, \quad c_{ijkl} = \widehat{c}_{ijkl}(\varrho, \varepsilon) \,,$$

where we can assume that $\widehat{c}_{ijkl} = \widehat{c}_{ijlk}$ for all $i, j, k, l = 1, \dots, n$. This is an objective representation of S with a scalar function $\widehat{c}_{ijkl}$ since S is like Π an objective tensor, that is,

$$S_{ij} \circ Y = \sum_{\widetilde{i}, \widetilde{j}=1}^{n} Q_{\widetilde{i}i} Q_{\widetilde{j}j} S^*_{\widetilde{i}\widetilde{j}} \,.$$

This can also be written as

$$\sum_{k,l} \big(c_{ijkl} (\partial_k v_l + \partial_l v_k) \big) \circ Y = \sum_{\widetilde{i}, \widetilde{j}} \sum_{\widetilde{k}, \widetilde{l}} c^*_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} Q_{\widetilde{i}i} Q_{\widetilde{j}j} (\partial_{\widetilde{k}} v^*_{\widetilde{l}} + \partial_{\widetilde{l}} v^*_{\widetilde{k}}) \,.$$

[7] A proof is given in [10, §15], but this proof does not match the mathematical rigour which we intend to give here, so we do the following proof.

Using the tansformation rule for $(\mathrm{D}v)^{\mathrm{S}}$, that is (II4.14) which says that it is an objective tensor, we get that on the left-hand side we can replace

$$(\partial_k v_l + \partial_l v_k)) \circ Y = \sum_{\widetilde{k},\widetilde{l}} Q_{k\widetilde{k}} Q_{l\widetilde{l}} (\partial_{\widetilde{k}} v^*_{\widetilde{l}} + \partial_{\widetilde{l}} v^*_{\widetilde{k}}) \,,$$

therefore

$$\sum_{k,l} c_{ijkl} \circ Y \sum_{\widetilde{k},\widetilde{l}} Q_{k\widetilde{k}} Q_{l\widetilde{l}} (\partial_{\widetilde{k}} v^*_{\widetilde{l}} + \partial_{\widetilde{l}} v^*_{\widetilde{k}}) = \sum_{\widetilde{i},\widetilde{j}} \sum_{\widetilde{k},\widetilde{l}} c^*_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} Q_{i\widetilde{i}} Q_{j\widetilde{j}} (\partial_{\widetilde{k}} v^*_{\widetilde{l}} + \partial_{\widetilde{l}} v^*_{\widetilde{k}}) \,.$$

Since $(\mathrm{D}v^*)^{\mathrm{S}}$ can be any symmetric matrix at a given spacetime point (this is true by assumption) the coefficients, which are symmetric in $\widetilde{k}$ and $\widetilde{l}$, have to be the same. This leads to the identity

$$\sum_{k,l=1}^{n} c_{ijkl} \circ Y \, Q_{k\widetilde{k}} Q_{l\widetilde{l}} = \sum_{\widetilde{i},\widetilde{j}=1}^{n} Q_{i\widetilde{i}} Q_{j\widetilde{j}} \, c^*_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} \quad \text{for all } i,j,\widetilde{k},\widetilde{l} \,.$$

Again we rewrite this so, that we have Q-terms only on the right-hand side, that is, we multiply the equation by $Q_{\bar{k}\widetilde{k}} Q_{\bar{l}\widetilde{l}}$, sum over $(\widetilde{k},\widetilde{l})$, and then rename $(\bar{k},\bar{l})$ as (k,l). This gives

$$c_{ijkl} \circ Y = \sum_{\widetilde{i},\widetilde{j},\widetilde{k},\widetilde{l}=1}^{n} Q_{i\widetilde{i}} Q_{j\widetilde{j}} Q_{k\widetilde{k}} Q_{l\widetilde{l}} \, c^*_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} \quad \text{for all } i,j,k,l$$

that is, $(c_{ijkl})_{i,j,k,l=1,\dots,n}$ is an objective 4-tensor. Now

$$c_{ijkl} \circ Y = \widehat{c}_{ijkl}(\varrho,\varepsilon) \circ Y = \widehat{c}_{ijkl}(\varrho^*,\varepsilon^*) = c^*_{ijkl}$$

so that

$$c^*_{ijkl} = \sum_{\widetilde{i},\widetilde{j},\widetilde{k},\widetilde{l}=1}^{n} Q_{i\widetilde{i}} Q_{j\widetilde{j}} Q_{k\widetilde{k}} Q_{l\widetilde{l}} \, c^*_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} \quad \text{for all } i,j,k,l \,.$$

This means that for fixed values of ϱ^* and ε^* the value $c^*_{ijkl} = \widehat{c}^*_{ijkl}(\varrho^*,\varepsilon^*)$ is fixed, that is, we can treat $\left(c^*_{ijkl}\right)_{i,j,k,l=1,\dots,n}$ as a constant objective 4-tensor, which is symmetric in the last two indices. Since S^* is symmetric, this implies, as shown in 4.14(6), that the symmetric part with respect to the first two indices is of the form

$$c^*_{ijkl} = a^*(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b^*\delta_{k,l}\delta_{i,j}$$

with two scalars a^*, b^*. Now use the fact, that $c_{ijkl} \circ Y = c^*_{ijkl}$ and similar $a \circ Y = a^*$, $b \circ Y = b^*$, to arrive at

$$c_{ijkl} = a(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b\delta_{k,l}\delta_{i,j} \,.$$

From this the assertion follows (see 4.14(6), (II4.32), (II4.33)). ☐

Here is the used identity for constant tensors of arbitrary size.

4.14 Lemma. We consider a constant objective m-tensor $C = (c_{i_1,\dots,i_m})_{i_1,\dots,i_m=1,\dots,n}$ (see the definition in 3.2(4)). This means that for all orthogonal matrices Q in $\mathbb{R}^n$ with $\det Q = 1$ the following identity is satisfied:

$$c_{i_1,\dots,i_m} = \sum_{\bar{i}_1,\dots,\bar{i}_m=1}^{n} Q_{i_1\bar{i}_1} \cdot \dots \cdot Q_{i_m\bar{i}_m} c_{\bar{i}_1,\dots,\bar{i}_m} \,. \tag{II4.15}$$

We assume $n \geq 2$ (for $n = 1$ there is only $Q = \mathrm{Id}$).

(1) Property (II4.15) is equivalent to, for any antisymmetric matrix A,

$$\begin{aligned} 0 = &\sum_{\bar{i}_1=1}^{n} A_{i_1\bar{i}_1} c_{\bar{i}_1,i_2,\dots,i_m} + \sum_{\bar{i}_2=1}^{n} A_{i_2\bar{i}_2} c_{i_1,\bar{i}_2,i_3,\dots,i_m} \\ &+ \dots + \sum_{\bar{i}_m=1}^{n} A_{i_m\bar{i}_m} c_{i_1,\dots,i_{m-1},\bar{i}_m} \,. \end{aligned} \tag{II4.16}$$

(2) Property (II4.15) is equivalent to

$$\begin{gathered} \delta_{i_1,r} c_{s,i_2,\dots,i_m} + \delta_{i_2,r} c_{i_1,s,i_3,\dots,i_m} + \dots + \delta_{i_m,r} c_{i_1,\dots,i_{m-1},s} \\ \text{is symmetric in } r,s \in \{1,\dots,n\} \end{gathered} \tag{II4.17}$$

for all $i_1,\dots,i_m = 1,\dots,n$.

(3) If $m = 1$ then $C = 0$.

(4) If $m = 2$ then, if $n \geq 3$, the matrix C is a multiple of the identity. For $n = 2$ it can have an additional antisymmetric part (see (II4.20)).

(5) If $m = 3$ then C satisfies (II4.15) if and only if C is antisymmetric in every pair of indices. If $n = 3$, then C satisfies for vectors $\xi \in \mathbb{R}^3$ for some $a \in \mathbb{R}$

$$C(\xi) := \big(\textstyle\sum_{k=1}^{3} c_{i,j,k}\xi_k\big)_{i,j=1,\dots,n} = a \cdot \begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} . \tag{II4.18}$$

If $n \geq 4$, then $C = 0$.

(6) If $m = 4$ we consider only the case $n \geq 3$. Then if C is symmetric in the last two arguments and C is symmetric in the first two arguments it has the form

$$C(M) := \Big(\textstyle\sum_{k,l=1}^{n} c_{i,j,k,l} M_{k,l}\Big)_{i,j=1,\dots,n} = a \cdot (M + M^{\mathrm{T}}) + b \cdot \mathrm{trace}\,(M) \cdot \mathrm{Id}. \tag{II4.19}$$

Remark: Dieses Resultat ist aus dem Anhang von [19].

Proof (1). Assume (II4.15) holds. Setting $Q = \exp\,(sA)$ with an antisymmetric matrix A, and taking the derivative with respect to s in (II4.15) at $s = 0$ one obtains (II4.16). Now assume that (II4.16) holds. Denote the right-hand side of (II4.15) by

$$F_i(Q) := \sum_{\bar{i}_1,\dots,\bar{i}_m=1}^{n} Q_{i_1\bar{i}_1} \cdot \dots \cdot Q_{i_m\bar{i}_m} c_{\bar{i}_1,\dots,\bar{i}_m} \quad \text{for } i = (i_1,\dots,i_m).$$

Consider a smooth curve $s \mapsto Q(s)$ with $Q(0) = \mathrm{Id}$. Then with $A(s) := \dot{Q}(s)\,Q^{\mathrm{T}}\,(s)$ one computes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} F_i(Q(s)) \quad &= \sum_{k=1}^{n} A_{i_1k} F_{k,i_2,\dots,i_m}(Q(s)) + \sum_{k=1}^{n} A_{i_2k} F_{i_1,k,i_3,\dots,i_m}(Q(s)) \\ &+ \dots + \sum_{k=1}^{n} A_{i_mk} F_{i_1,,\dots,i_{m-1},k}(Q(s)) \,. \end{aligned}$$

Using (II4.16), we see that the same differential equation holds for the function $s \mapsto F_i(Q(s)) - c_i$. Since $F_i(Q(0)) - c_i = 0$ for all i, we obtain $F_i(Q(s)) - c_i = 0$ for all s and i.

Since the set of orthogonal matrices with positive determinant is a connected manifold, we can reach any such matrix with a curve starting at the identity. □

Proof (2). Varying over all antisymmetric matrices one sees that (II4.16) is equivalent to the fact, that the identity (II4.17) holds. □

We consider property (II4.17) in the subsequent argumentations. We do not claim that this is the most efficient way to derive these conclusions, but at least there is a unified background.

Proof (3): *Case* $m = 1$. Then (II4.17) reads

$$\delta_{i,r} c_s = \delta_{i,s} c_r \quad \text{for all } i \text{ and } r \neq s.$$

Setting $i = r$ we get $c_s = 0$, and this for all s, hence the result follows. □

Proof (4): *Case* $m = 2$. Then (II4.17) reads

$$\delta_{i,r} c_{s,j} + \delta_{j,r} c_{i,s} = \delta_{i,s} c_{r,j} + \delta_{j,s} c_{i,r} \quad \text{for all } i, j \text{ and } r \neq s.$$

Setting $i = r$, $j = s$ we get

$$c_{s,s} = c_{r,r} \quad \text{for all } r \neq s \;,$$

hence for some number a

$$c_{i,i} = a \quad \text{for all } i \;.$$

If $n \geq 3$ set $i = r$ and let $j \neq r, s$. This gives

$$c_{s,j} = 0 \quad \text{for all } j \neq s \;.$$

Thus

$$C = a\mathrm{Id}.$$

For $n = 2$ set $i = j = r$ and obtain

$$c_{s,r} + c_{r,s} = 0 \quad \text{for } s \neq r \;,$$

hence for some number b

$$C = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{II4.20}$$

□

Proof (5): *Case* $m = 3$. Then (II4.17) reads

$$\begin{aligned} &\delta_{i,r} c_{s,j,k} + \delta_{j,r} c_{i,s,k} + \delta_{k,r} c_{i,j,s} \\ &= \delta_{i,s} c_{r,j,k} + \delta_{j,s} c_{i,r,k} + \delta_{k,s} c_{i,j,r} \quad \text{for all } i, j, k \text{ and } r \neq s. \end{aligned} \tag{II4.21}$$

We consider the case $n \geq 3$. For $r = k = j$, and three different i, k, and s this gives

$$c_{i,s,k} + c_{i,k,s} = 0 \quad \text{for all } s, k \neq i \text{ with } s \neq k.$$

For $i = j$, $r = k$, and three different i, k, and s the identity gives

$$c_{i,i,s} = 0 \quad \text{for all } s \neq i,$$

and for $j = k$, $r = i$, and different i, k, and s this gives

$$c_{s,j,j} = 0 \quad \text{for all } s \neq j.$$

For $k = i \neq j$ and $s = i$, $r = j$ this gives $c_{i,i,i} = c_{j,j,i} + c_{i,j,j}$, which is 0 by the previous results, thus

$$c_{i,i,i} = 0 \quad \text{for all } i.$$

Therefore we have seen that

$$c_{i,j,k} \quad \text{is antisymmetric in } j,k.$$

Interchanging two indices leads to a tensor, which again is objective, hence the above antisymmetry applies to the new tensor. It follows that C is antisymmetric in every pair of indices. Every such 3-tensor is objective.

For $n \geq 4$ obtain for empty set $\{r,s\} \cap \{i,j\}$

$$\delta_{k,r} c_{i,j,s} = \delta_{k,s} c_{i,j,r}$$

and then for $k = s \neq r$

$$c_{i,j,r} = 0 \quad \text{for all } r \neq i,j.$$

Together with the above antisymmetry it follows that $C = 0$. For $n = 3$ we have

$$a := c_{1,2,3} = -c_{1,3,2} = c_{3,1,2} = -c_{3,2,1} = c_{2,3,1} = -c_{2,1,3},$$

and all other components vanish. Hence for vectors $\xi \in \mathbb{R}^3$ equation (II4.18) holds. ☐

Proof (6)*: Case* $m = 4$. Let C be any constant objective 4-tensor. We consider the case $n \geq 3$. Equation (II4.17) reads

$$\begin{aligned} &\delta_{i,r} c_{s,j,k,l} + \delta_{j,r} c_{i,s,k,l} + \delta_{k,r} c_{i,j,s,l} + \delta_{l,r} c_{i,j,k,s} \\ &= \delta_{i,s} c_{r,j,k,l} + \delta_{j,s} c_{i,r,k,l} + \delta_{k,s} c_{i,j,r,l} + \delta_{l,s} c_{i,j,k,r} \quad \text{for all } i,j,k,l \text{ and } r \neq s. \end{aligned} \tag{II4.22}$$

Let us assume, that C is symmetric in the last two indices, that is

$$c_{i,j,k,l} = c_{i,j,l,k} \quad \text{for all } i,j,k,l.$$

Set $i = j$ in (II4.22). Then

$$\begin{aligned} &\delta_{i,r}(c_{s,i,k,l} + c_{i,s,k,l}) + \delta_{k,r} c_{i,i,s,l} + \delta_{l,r} c_{i,i,k,s} \\ &= \delta_{i,s}(c_{r,i,k,l} + c_{i,r,k,l}) + \delta_{k,s} c_{i,i,r,l} + \delta_{l,s} c_{i,i,k,r} \quad \text{for } r \neq s \text{ and all } i,k,l. \end{aligned} \tag{II4.23}$$

For $k,l,r,s \neq i$ with $r \neq s$ one obtains

$$\delta_{k,r} c_{i,i,s,l} + \delta_{l,r} c_{i,i,k,s} = \delta_{k,s} c_{i,i,r,l} + \delta_{l,s} c_{i,i,k,r}.$$

This is the characterization of the objective 2-tensor $(c_{i,i,k,l})_{k,l=1,\dots,n}$ in $n-1$ dimensions. Since $n-1 \geq 2$ and symmetry is assumed, the above result (4) implies with a real number b_i

$$c_{i,i,k,l} = b_i \delta_{k,l} \quad \text{for all } k,l \neq i. \tag{II4.24}$$

For $r,s \neq i$ with $r \neq s$ and $k = r$, $l = i$ one obtains

$$c_{i,i,s,i} = 0 \quad \text{for all } s \neq i. \tag{II4.25}$$

Now, in (II4.23), set $k = i$ and let $l,r,s \neq i$ with $r \neq s$. One obtains

$$\delta_{l,r} c_{i,i,i,s} = \delta_{l,s} c_{i,i,i,r} \quad \text{for all } i \text{ and } l,r,s \neq i \text{ with } r \neq s.$$

For $l = r$ this gives

$$c_{i,i,i,s} = 0 \quad \text{for all } s \neq i.$$

Together with (II4.24) this gives

$$\begin{aligned} &c_{i,i,k,l} = 0 \quad \text{for all } k \neq l, \\ &c_{i,i,k,k} = b_i \quad \text{for all } k \neq i, \\ &c_{i,i,i,i} \quad \text{so far undetermined.} \end{aligned} \tag{II4.26}$$

Now, in (II4.23), set $r = i$ and let $k, l, s \neq i$ (then $r \neq s$). One obtains

$$c_{s,i,k,l} + c_{i,s,k,l} = \delta_{k,s} c_{i,i,i,l} + \delta_{l,s} c_{i,i,k,i}.$$

The right-hand side vanishes by the first identity in (II4.26), hence

$$c_{s,i,k,l} + c_{i,s,k,l} = 0 \quad \text{for all } k, l, s \neq i,$$

or relabeled

$$c_{j,i,k,l} + c_{i,j,k,l} = 0 \quad \text{for all } i \neq j \text{ and all } k, l \neq i \text{ or } k, l \neq j.$$

Denoting the symmetrization with respect to the first two indices by

$$c'_{i,j,k,l} := \frac{1}{2}(c_{i,j,k,l} + c_{j,i,k,l}) \quad \text{for all } i, j, k, l \tag{II4.27}$$

we obtain $c'_{i,j,k,l} = c'_{j,i,k,l} = 0$ for all $i \neq j$ and all $k, l \neq i$ or $k, l \neq j$, that is

$$c'_{i,j,k,l} = 0 \quad \text{for all } i \neq j \text{ and } k, l \text{ with } \{k, l\} \neq \{i, j\}. \tag{II4.28}$$

For $\{k, l\} = \{i, j\}$ we get

$$a_{i,j} := c'_{i,j,i,j} = c'_{i,j,j,i} = c'_{j,i,i,j} = c'_{j,i,j,i} = a_{j,i}.$$

Now let $i \neq j$ in (II4.22). Then for $r = i$ and $s \neq i$ this identity becomes

$$\begin{aligned} &c_{s,j,k,l} + \delta_{k,i} c_{i,j,s,l} + \delta_{l,i} c_{i,j,k,s} \\ &= \delta_{j,s} c_{i,i,k,l} + \delta_{k,s} c_{i,j,i,l} + \delta_{l,s} c_{i,j,k,i} \quad \text{for } i \neq j, \text{ and } s \neq i, \text{ and all } k, l. \end{aligned} \tag{II4.29}$$

As first case in (II4.29) let $s = j$. Then

$$\begin{aligned} &c_{j,j,k,l} + \delta_{k,i} c_{i,j,j,l} + \delta_{l,i} c_{i,j,k,j} \\ &= c_{i,i,k,l} + \delta_{k,j} c_{i,j,i,l} + \delta_{l,j} c_{i,j,k,i} \quad \text{for all } k, l. \end{aligned}$$

For $k = l \neq i, j$ this gives

$$c_{j,j,k,k} = c_{i,i,k,k} \quad \text{for all } i \neq j \text{ and } k \neq i, j\ ,$$

thus in the second identity in (II4.26) for some number b

$$b_i = b \quad \text{for all } i.$$

For $k = l = i$ we obtain

$$c_{j,j,i,i} + c_{i,j,j,i} + c_{i,j,i,j} = c_{i,i,i,i},$$

which by definition of b becomes

$$c_{i,i,i,i} = b + c_{i,j,j,i} + c_{i,j,i,j},$$

and for $k = l = j$ we obtain

$$c_{j,j,j,j} = c_{i,i,j,j} + c_{i,j,i,j} + c_{i,j,j,i} = b + c_{i,j,i,j} + c_{i,j,j,i},$$

and interchanging i, j

$$c_{i,i,i,i} = b + c_{j,i,j,i} + c_{j,i,i,j}.$$

Adding up both equations for $c_{i,i,i,i}$ we arrive at

$$c_{i,i,i,i} = b + c'_{i,j,j,i} + c'_{i,j,i,j} = b + 2a_{i,j} \tag{II4.30}$$

by definition of $a_{i,j}$. As second case in (II4.29) let $s \neq i,j$. Then

$$\begin{aligned} &c_{s,j,k,l} + \delta_{k,i}c_{i,j,s,l} + \delta_{l,i}c_{i,j,k,s} \\ &= \delta_{k,s}c_{i,j,i,l} + \delta_{l,s}c_{i,j,k,i} \quad \text{for } i \neq j \text{ and } s \neq i,j, \text{ and all } k,l. \end{aligned}$$

For $k = s$ and $l = j$ this gives

$$c_{s,j,s,j} = c_{i,j,i,j} \quad \text{for all } i \neq j \text{ and } s \neq i,j.$$

From now on let us consider only $C' := \left(c'_{i,j,k,l}\right)_{i,j,k,l=1,\dots,n}$ given by (II4.27), that is the symmetric part of C with respect to the first two indices. Since also C' is a constant objective 4-tensor (the same for the corresponding antisymmetric part), we can apply all results also to C'. In particular, the last identity becomes

$$c'_{s,j,s,j} = c'_{i,j,i,j} \quad \text{for all } i \neq j \text{ and } s \neq i,j,$$

which by symmetry means that $c'_{\widetilde{i},\widetilde{j},\widetilde{i},\widetilde{j}} = c'_{i,j,i,j}$ for all $i \neq j$ and $\widetilde{i} \neq \widetilde{j}$. Hence for some number a

$$a_{i,j} = a \quad \text{for all } i \neq j.$$

Therefore $c'_{i,i,i,i} = b + 2a$ from (II4.30). Summing up, we have shown that C' has the following structure:

$$\begin{aligned} &c'_{i,j,k,l} = 0 \quad \text{except that} \\ &c'_{i,i,k,k} = b \quad \text{for all } k \neq i, \\ &c'_{i,i,i,i} = b + 2a \quad \text{for all } i, \\ &c'_{i,j,i,j} = c'_{i,j,j,i} = c'_{j,i,i,j} = c'_{j,i,j,i} = a \quad \text{for all } j \neq i. \end{aligned} \tag{II4.31}$$

This means that

$$c'_{i,j,k,l} = a(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b\delta_{k,l}\delta_{i,j} \quad \text{for all } i,j,k,l. \tag{II4.32}$$

Or equivalently, for all matrices $M = (M_{i,j})_{i,j=1,\dots,n}$

$$\sum_{k,l=1}^{n} c'_{i,j,k,l}M_{k,l} = a \cdot (M_{i,j} + M_{j,i}) + b \cdot \text{trace}\,(M) \cdot \delta_{i,j}\,, \tag{II4.33}$$

that is, C' satisfies (II4.19). □

5 Objectivity in reference coordinates

The objektivity can be formulated also in reference coordintes, but it should be noted that this exists only in Newtonian physics, because in relativistic physics the reference coordinates are not defined. So let

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} t^* + \mathrm{a} \\ X(t^*, x^*) \end{bmatrix}$$

be a Newtonian observer transformation. Both the (t, x)-observer and the (t^*, x^*)-observer have a transformation

$$x = \varphi(t, \underline{x}) \quad \text{und} \quad x^* = \varphi^*(t^*, \underline{x})$$

to reference coordinates. Here we assume that both observers choose the same reference configuration. (Thus we have the situation as in Fig. I23 but with $\underline{x}^* = \underline{x}$. For different reference coordinates see EX:Referenzkoordinaten.) If we define $\underline{\xi}$ by

$$x = \varphi(t, \underline{x}) \quad \Longleftrightarrow: \quad \underline{x} = \underline{\xi}(t, x)$$

(that is the inverse of $\varphi(t, \bullet)$), this implies

$$\underline{\xi}_i(t, x) = \underline{x}_i = \underline{\xi}_i^*(t^*, x^*) \quad \text{for} \quad (t, x) = Y(t^*, x^*)\,, \tag{II5.1}$$

that is, the ith component of the reference coordinates $\underline{\xi}_i$ is an objective scalar. This can be used to prove the following.

5.1 Lemma. A function $(t, \underline{x}) \mapsto \underline{p}(t, \underline{x})$ an objective scalar if

$$\underline{p}(t, \underline{x}) = \underline{p}^*(t^*, \underline{x})\,.$$

Likewise $(t, \underline{x}) \mapsto \underline{e}(t, \underline{x})$ is an objective vector if

$$\underline{e}(t, \underline{x}) = Q(t^*)\underline{e}^*(t^*, \underline{x})\,.$$

Remark: J is an objective scalar.

Proof. $\underline{p}$ is called objective scalar when this is satisfied for $p(t, x) := \underline{p}(t, \underline{\xi}(t, x))$. Hence, if $(t, x) = Y(t^*, x^*)$ is the observer transformation,

$$\underline{p}(t, \underline{x}) = \underline{p}(t, \underline{\xi}(t, x)) = \underline{p}^*(t^*, \underline{\xi}^*(t^*, x^*)) = \underline{p}^*(t^*, \underline{x})\,.$$

$\underline{e}$ is called objective vector when this is satisfied for $e(t, x) := \underline{e}(t, \underline{\xi}(t, x))$. Hence

$$\begin{aligned}\underline{e}(t, \underline{x}) = \underline{e}(t, \underline{\xi}(t, x)) = e(t, x) &= Q(t^*)e^*(t^*, x^*) \\ &= \underline{e}^*(t^*, \underline{\xi}^*(t^*, x^*)) = \underline{e}^*(t^*, \underline{x})\,,\end{aligned}$$

if $(t, x) = Y(t^*, x^*)$. □

Proof of the Remark. In 5.2 we prove $F = QF^*$. Hence the statement follows since $J = \det F$. □

It holds

5.2 Lemma. The deformation gradient F and the velocity V satisfy

$$F(t,\underline{x}) = Q(t^*)F^*(t^*,\underline{x})\,,$$
$$V(t,\underline{x}) = \dot{X}(t^*,\varphi^*(t^*,\underline{x})) + Q(t^*)V^*(t^*,\underline{x})\,,$$

for $t = t^* + \mathrm{a}$.

Proof. Since $X(t^*,x^*) = Q(t^*)x^* + \mathrm{b}(t^*)$ it follows (we remark that the two observers have the same reference configuration)

$$\varphi(t,\underline{x}) = X(t^*,\varphi^*(t^*,\underline{x})) = Q(t^*)\varphi^*(t^*,\underline{x}) + \mathrm{b}(t^*)\,, \quad t = t^* + \mathrm{a}\,.$$

If we compute the derivative with respect to $\underline{x}$ we obtain

$$F(t,x) = \mathrm{D}\varphi(t,x) = Q(t^*)\mathrm{D}\varphi^*(t^*,\underline{x}) = Q(t^*)F^*(t^*,\underline{x})\,,$$

the first assertion. If we compute the derivative with respect to t^* we obtain

$$\begin{aligned} V(t,\underline{x}) &= \partial_t\varphi(t,\underline{x}) = \partial_{t^*}\varphi(t^*+a,\underline{x}) \\ &= \dot{X}(t^*,\varphi^*(t^*,\underline{x})) + Q(t^*)\partial_{t^*}\varphi^*(t^*,\underline{x}) \\ &= \dot{X}(t^*,\varphi^*(t^*,\underline{x})) + Q(t^*)V^*(t^*,\underline{x})\,, \end{aligned}$$

the second assertion. □

For the first Piola-Kirchhoff stress tensor defined in Section I.6 we obtain the following (please, do not mix the tensor S with the tensor in III.2.4 which applies to fluids)

5.3 Piola-Kirchhoff stress tensor. Define

$S := F^{-1}\,P$ second Piola-Kirchhoff stress tensor,

$C := F^{\mathrm{T}}\,F$ right Cauchy-Green deformation tensor,

$B := F\,F^{\mathrm{T}}$ left Cauchy-Green deformation tensor.

Both, C and B, are symmetric deformation tensors. Then it holds for the tension tensor

$$-\Pi{\circ}\tau = \frac{1}{J}F\,S\,F^{\mathrm{T}} \quad \text{or} \quad S = J\,F^{-1}\,(-\Pi{\circ}\tau)\,F^{-\mathrm{T}}\,.$$

This results in the following observer dependencies

$$S(t,\underline{x}) = S^*(t^*,\underline{x})\,,$$
$$C(t,\underline{x}) = C^*(t^*,\underline{x})\,,$$

das heißt, die Werte S_{ij} und C_{ij} sind beobachterunabhängig.
Observe: Thus the tensor S is symmetric if and only if the tension tensor Π is symmetric.

Proof. By 5.2 we have for $t = t^* + \mathrm{a}$

$$F(t,\underline{x}) = Q(t^*)F^*(t^*,\underline{x})\,, \qquad \text{and} \qquad F^{*-1}(t^*,\underline{x}) = F^{-1}(t,\underline{x})Q(t^*)\,.$$

It follows, we omit the arguments,

$$C = F^{\mathrm{T}}F = (QF^*)^{\mathrm{T}}QF^* = F^{*\mathrm{T}}Q^{\mathrm{T}}QF^* = F^{*\mathrm{T}}F^* = C^*\,.$$

Since

$$J = \det F = \det(QF^*) = \det Q\cdot\det F^* = \det F^* = J^*\,,$$

and since, due to 3.7, the tensor Π is objective, i.e. $\Pi\circ Y = Q\Pi^*Q^{\mathrm{T}}$, we obtain omitting the arguments

$$\begin{aligned} FSF^{\mathrm{T}} &= -J\Pi = -J^*Q\Pi^*Q^{\mathrm{T}} \\ &= Q(F^*S^*F^{*\mathrm{T}})Q^{\mathrm{T}} = QF^*S^*(QF^*)^{\mathrm{T}} = FS^*F^{\mathrm{T}}\,, \end{aligned}$$

and from this $S = S^*$. Or explicitly with argumets, due to the objectivity $\Pi\circ Y = Q\Pi^*Q^{\mathrm{T}}$, the fact $Y^{-1}\circ\tau = \tau^*$ and $Q\circ\tau^* = Q$, we obtain

$$\begin{aligned} FSF^{\mathrm{T}} &= -J\,\Pi\circ\tau = -J\,\Pi\circ Y\circ Y^{-1}\circ\tau = -J^*Q\Pi^*\circ\tau^*Q^{\mathrm{T}} \\ &= Q(F^*S^*F^{*\mathrm{T}})Q^{\mathrm{T}} = QF^*S^*(QF^*)^{\mathrm{T}} = FS^*F^{\mathrm{T}}\,, \end{aligned}$$

and from this $S = S^*$. □

5.4 Constitutive function for *S*. If S is the second Piola-Kirchhoff stress tensor, then each constitutive relation

$$S(t,\underline{x}) = \widehat{S}(\underline{x}, C(t,\underline{x})) \tag{II5.2}$$

defines an objective function $\widehat{S}$. The same holds for

$$S(t,\underline{x}) = \widehat{S}(\underline{x}, \underline{p}(t,\underline{x}), C(t,\underline{x})) \tag{II5.3}$$

where $\underline{p}$ is an objective scalar.

Proof. In 5.3 we had shown that $C(t,\underline{x}) = C^*(t^*,\underline{x})$ and $S(t,\underline{x}) = S^*(t^*,\underline{x})$. We plug this into the constitutive relation (II5.2) and obtain $S^*(t^*,\underline{x}) = \widehat{S}(\underline{x}, C^*(t^*,\underline{x}))$. Therefore the function $\widehat{S}$ is the same for different observers. The equation (II5.3) takes the same effect, since $\underline{p}(t,\underline{x}) = \underline{p}^*(t^*,\underline{x})$. □

5.5 Example. A simple example for $\widehat{S}$ is

$$\widehat{S}(\underline{x}) = \begin{bmatrix} \lambda_1(\underline{x}) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(\underline{x}) \end{bmatrix} = \sum_{k=1}^{n} \lambda_k(\underline{x})\mathbf{e}_k {\mathbf{e}_k}^{\mathrm{T}} .$$

Then $S(t,\underline{x}) = \widehat{S}(\underline{x})$ implies

$$-\Pi\circ\tau = \frac{1}{J} F S F^{\mathrm{T}} = \frac{1}{J} F \sum_{k=1}^{n} \lambda_k \mathbf{e}_k {\mathbf{e}_k}^{\mathrm{T}} F^{\mathrm{T}} = \frac{1}{J} \sum_{k=1}^{n} \lambda_k\circ\underline{\xi}\, e_k\circ\tau\ (e_k\circ\tau)^{\mathrm{T}}$$

with $e_k\circ\tau := F\mathbf{e}_k$. It is $\{e_1(t,x),\dots,e_n(t,x)\}$ a basis in physical space. The vectors e_k are objective vectors, since if $x = \varphi(t,\underline{x})$ and $x^* = \varphi^*(t^*,\underline{x})$

$$e_k(t,x) = F(t,\underline{x})\mathbf{e}_k = Q(t^*)F^*(t^*,\underline{x})\mathbf{e}_k = Q(t^*)e_k^*(t^*,x^*)$$

See the example in 4.7 where an orthonormal basis were chosen.

6 Angular momentum

We are interested in the angular momentum of a medium with respect to a moving point

$$t \mapsto \xi(t) \in \mathbb{R}^3 ,$$

which might be the center of another observer seen from the present observer. The special case, that this point is equal to 0, is usually presented in literature, an exception one finds in Truesdell [14, I.8]. Indeed, the description with ξ is necessary, if one wants an observer independent formulation of the angular momentum. So we have the identity $(t, \xi(t)) = Y(t^*, \xi^*(t^*))$ for an observer transformation Y.

We start with the usual conservation for mass and momentum 3.7 in the special case $\mathbf{J} = 0$ and $\mathbf{r} = 0$:

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v) = 0 , \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f} . \end{aligned} \tag{II6.1}$$

For this case we define the ***angular momentum*** $\mathscr{J}$ by a matrix satisfying

Angular momentum $\mathscr{J}$:

$$\partial_t \mathscr{J} + \operatorname{div}(\mathscr{J} v^{\mathrm{T}} + \widetilde{\Sigma}) = \widetilde{G} \tag{II6.2}$$

$\mathscr{J} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ antisymmetric,

$\mathscr{J} \circ Y = \varrho^*(Q(x^* - \xi^*)) \wedge (\dot{Q}(v^* - \dot{\xi}^*)) + Q \mathscr{J}^* Q^{\mathrm{T}}$

where Y is the observer transformation

In coordinates this equation reads

$$\partial_t \mathscr{J}_{kl} + \sum_j \partial_{x_j}(\mathscr{J}_{kl} v_j + \widetilde{\Sigma}_{klj}) = \widetilde{G}_{kl} .$$

This is the form used in DeGroot & Mazur [6, CH. XII §1]. To be concrete we define the ***spin*** as a matrix satisfying

Spin $\mathscr{S}$:

$$\mathscr{J} = \mathscr{L} + \mathscr{S} \tag{II6.3}$$

$\mathscr{L} = (x - \xi) \wedge \varrho(v - \dot{\xi})$ see 6.2,

the moment of stress density,

$\mathscr{S} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ antisymmetric,

the spin, is an objective tensor

Here we use the '$\wedge$' product as in Truesdell [14, I.8], in literature usually the cross product '$\times$' is used, it is an equivalent notation:

6.1 Definition. The products '$\wedge$' and '$\times$' satisfy for $a, b \in \mathbb{R}^3$

$$a \wedge b = (a_2b_3 - b_2a_3)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3b_1 - b_3a_1)\mathbf{e}_3 \wedge \mathbf{e}_1 + (a_1b_2 - b_1a_2)\mathbf{e}_1 \wedge \mathbf{e}_2\,,$$
$$a \times b = (a_2b_3 - b_2a_3)\mathbf{e}_1 + (a_3b_1 - b_3a_1)\mathbf{e}_2 + (a_1b_2 - b_1a_2)\mathbf{e}_3\,,$$

so that they use just different basis vectors. The Grassman product '$\wedge$' can also be defined by

$$a \wedge b := a\otimes b - b\otimes a = a\,b^{\mathrm{T}} - b\,a^{\mathrm{T}}\,.$$

For an antisymmetric matrix $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ and a vector $\boldsymbol{\omega} \in \mathbb{R}^3$ the property

$$\boldsymbol{R} = \mathcal{R}(\boldsymbol{\omega}) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \iff \forall\; a \in \mathbb{R}^3 :\; \boldsymbol{\omega} \times a = \boldsymbol{R}\,a$$

is true.

We have to show the assumption on $\mathscr{L}$ in (II6.3).

6.2 Basic lemma. The following transformation formulas hold

$$(x - \xi)\circ Y = Q(x^* - \xi^*) \quad \text{with} \quad \partial_{(1,v)}(x - \xi) = v - \dot{\xi}\,,$$
$$(v - \dot{\xi})\circ Y = \dot{Q}(x^* - \xi^*) + Q(v^* - \dot{\xi}^*)\,.$$

This implies that $\mathscr{L}$ in (II6.3) satisfies the transformation rule of $\mathscr{J}$ in (II6.2).

Proof. It is

$$\xi(t) = X(t^*, \xi^*(t^*)) \quad \text{for} \quad (t, x) = Y(t^*, x^*)$$

as in (II1.3), hence

$$((x - \xi)\circ Y)(t^*, x^*) = X(t^*, x^*) - X(t^*, \xi^*(t^*)) = Q(t^*)(x^* - \xi^*(t^*))\,.$$

And we compute

$$\partial_{(1,v)}(x - \xi) = \partial_{(1,v)}x - \partial_{(1,v)}\xi$$
$$= \partial_t x + v\bullet\nabla x - \partial_t\xi - v\bullet\nabla\xi = v\bullet\nabla x - \partial_t\xi = v - \dot{\xi}\,.$$

The transformation rule for ξ is

$$(\dot{\xi}\circ Y)(t^*) = \dot{X}(t^*, \xi^*(t^*)) + Q(t^*)\dot{\xi}^*(t^*)\,, \tag{II6.4}$$

hence we obtain for $v - \dot{\xi}$

$$((v - \dot{\xi})\circ Y)(t^*, x^*) = \dot{X}(t^*, x^*) + Q(t^*)v^* - \dot{X}(t^*, \xi^*(t^*)) - Q(t^*)\dot{\xi}^*(t^*)$$
$$= \dot{Q}(t^*)(x^* - \xi^*(t^*)) + Q(t^*)(v^* - \dot{\xi}^*(t^*))\,.$$

These statements imply immediately

$$\begin{aligned}
&\mathscr{L}\circ Y = \varrho\circ Y\,((x-\xi)\wedge(v-\dot\xi))\circ Y\\
&= \varrho^*(Q(x^*-\xi^*))\wedge(\dot Q(x^*-\xi^*)+Q(v^*-\dot\xi^*))\\
&= \varrho^*(Q(x^*-\xi^*))\wedge(\dot Q(x^*-\xi^*))+\varrho^*(Q(x^*-\xi^*))\wedge(Q(v^*-\dot\xi^*))\\
&= \varrho^*(Q(x^*-\xi^*))\wedge(\dot Q(x^*-\xi^*))+Q\mathscr{L}^*Q^{\mathrm{T}}\;,
\end{aligned}$$

since for $w,z\in\mathbb{R}^3$

$$\begin{aligned}
&(Qw\wedge Qz)_{kl} = (Qw)_k(Qz)_l-(Qw)_l(Qz)_k\\
&= \sum_{\overline{k},\overline{l}} Q_{k\overline{k}}Q_{l\overline{l}}\big(w_{\overline{k}}z_{\overline{l}}-w_{\overline{l}}z_{\overline{k}}\big) = \big(Q(w\wedge z)\,Q^{\mathrm{T}}\big)_{kl}\,.
\end{aligned}\tag{II6.5}$$

Thus $\mathscr{L}$ satisfies the transformation rule of an angular momentum. □

We define the ***specific angular momentum*** by

$$\mathscr{J} = \varrho\,\mathscr{J}^{\mathrm{sp}}\,,\quad \text{similar}\quad \mathscr{L} = \varrho\mathscr{L}^{\mathrm{sp}}\,,\quad \mathscr{S} = \varrho\mathscr{S}^{\mathrm{sp}}\,.\tag{II6.6}$$

With this definition

6.3 Theorem. The system (II6.1) and the angular momentum equation (II6.2) are equivalent to[8]

$$\begin{aligned}
&\partial_{(1,v)}\varrho+\varrho\;\mathrm{div}(v-\dot\xi)=0\,,\\
&\varrho\,\partial_{(1,v)}(v-\dot\xi)+\mathrm{div}\Pi=\mathbf{f}-\varrho\ddot\xi\,,\\
&\varrho\,\partial_{(1,v)}\mathscr{J}^{\mathrm{sp}}+\mathrm{div}\widetilde\Sigma=\widetilde G\,.
\end{aligned}\tag{II6.7}$$

Proof. For the mass-momentum system we recall (II3.16) and do the following manipulation: In the mass equation we add $\mathrm{div}\,\dot\xi=0$ and in the momentum equation $\varrho\,\partial_{(1,v)}\dot\xi=\varrho\ddot\xi$. This gives the asserted equations. For the angular momentum we compute

$$\begin{aligned}
&\partial_t\mathscr{J}+\mathrm{div}(\mathscr{J}\,v^{\mathrm{T}})=\partial_t(\varrho\,\mathscr{J}^{\mathrm{sp}})+\mathrm{div}(\mathscr{J}^{\mathrm{sp}}(\varrho v)^{\mathrm{T}})\\
&=\big(\partial_t\varrho+\mathrm{div}(\varrho v)\big)\mathscr{J}^{\mathrm{sp}}+\varrho\partial_t\mathscr{J}^{\mathrm{sp}}+\varrho v\bullet\nabla\mathscr{J}^{\mathrm{sp}}=\varrho\,\partial_{(1,v)}\mathscr{J}^{\mathrm{sp}}
\end{aligned}$$

which gives the third equation. □

The third equation in (II6.7) can also be written as

[8] It is $\partial_{(1,v)}=\partial_t+v\bullet\nabla_x$ defined as partial derivative in spacetime $\mathbb{R}\times\mathbb{R}^3$. This is called "total derivative" and one denotes it also by $\partial_{(1,v)}h=\mathring{h}$ for every function h. See the footnote to the notation in 3.8.

6.4 Spin balance equation. The spin $\mathscr{S}$ satisfies

$$\boxed{\varrho\,\partial_{(1,v)}\mathscr{S}^{\mathrm{sp}} + \mathrm{div}\Sigma = 2\Pi^{\mathrm{A}} + G\,,}$$

where

$$\begin{aligned}
&\widetilde{\Sigma} = (x-\xi)\wedge\Pi + \Sigma\,,\\
&\qquad (x-\xi)\wedge\Pi \quad \text{moment of stress density,}\\
&\qquad \Sigma \quad \text{couple stress density,}\\
&\widetilde{G} = (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi}) + G\,,\\
&\qquad (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi}) \quad \text{moment of volume forces,}\\
&\qquad G \quad \text{(intrinsic) body couple density.}
\end{aligned}$$

Remark: See Hutter & Jöhnk [47, Ex. 2.4.4 with solution 2.5.4].

Proof. The third equation of (II6.7) with $\mathscr{J}^{\mathrm{sp}} = \mathscr{L}^{\mathrm{sp}} + \mathscr{S}^{\mathrm{sp}}$ and the equation of the next statement are

$$\begin{aligned}
&\varrho\,\partial_{(1,v)}(\mathscr{L}^{\mathrm{sp}} + \mathscr{S}^{\mathrm{sp}}) + \mathrm{div}\widetilde{\Sigma} = \widetilde{G}\,,\\
&\varrho\,\partial_{(1,v)}\mathscr{L}^{\mathrm{sp}} + 2\Pi^{\mathrm{A}} + \mathrm{div}\big((x-\xi)\wedge\Pi\big) = (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi})
\end{aligned}$$

for $\mathscr{L}^{\mathrm{sp}} = (x-\xi)\wedge(v-\dot{\xi})$. Taking the difference we obtain

$$\begin{aligned}
&\varrho\,\partial_{(1,v)}\mathscr{S}^{\mathrm{sp}} + \mathrm{div}\big(\widetilde{\Sigma} - (x-\xi)\wedge\Pi\big)\\
&\quad = 2\Pi^{\mathrm{A}} + \widetilde{G} - (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi})\,.
\end{aligned}$$

Hence the assertion holds. □

In the proof we have used the subsequent

6.5 Important identity. We obtain for solutions of (II6.1)

$$\varrho\,\partial_{(1,v)}\mathscr{L}^{\mathrm{sp}} + 2\Pi^{\mathrm{A}} + \mathrm{div}\big((x-\xi)\wedge\Pi\big) = (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi})$$

where $\mathscr{L} := \varrho\,(x-\xi)\wedge(v-\dot{\xi})$.

Proof. For arbitrary Π we compute

$$\begin{aligned}
&\varrho\partial_{(1,v)}\big((x-\xi)\wedge(v-\dot{\xi})\big) = \varrho\partial_{(1,v)}\Big((x-\xi)\wedge(v-\dot{\xi})\Big)\\
&= (x-\xi)\wedge\big(\varrho\partial_{(1,v)}(v-\dot{\xi})\big) + \underbrace{\varrho\big(\partial_{(1,v)}(x-\xi)\big)\wedge(v-\dot{\xi})}_{=\varrho(v-\dot{\xi})\wedge(v-\dot{\xi})=0}\\
&= (x-\xi)\wedge\big(\varrho\partial_{(1,v)}v\big) - (x-\xi)\wedge(\varrho\ddot{\xi})\\
&= (x-\xi)\wedge(\mathbf{f}-\varrho\ddot{\xi}) - (x-\xi)\wedge\mathrm{div}\Pi
\end{aligned}$$

that is

$$\varrho\,\partial_{(1,v)}\big((x-\xi)\wedge(v-\dot\xi)\big)+(x-\xi)\wedge\,\mathrm{div}\Pi=(x-\xi)\wedge(\mathbf{f}-\varrho\ddot\xi)\,.$$

Now we have to handle the term

$$\begin{aligned}(x-\xi)\wedge\,\mathrm{div}\Pi&=\sum_\gamma(x-\xi)\wedge\partial_{x_\gamma}\Pi_{\bullet\gamma}\\&=\sum_\gamma\partial_{x_\gamma}\big((x-\xi)\wedge\Pi_{\bullet\gamma}\big)-\sum_\gamma\big(\partial_{x_\gamma}(x-\xi)\big)\wedge\Pi_{\bullet\gamma}\\&=\mathrm{div}\big((x-\xi)\wedge\Pi\big)-\sum_\gamma\mathbf{e}_\gamma\wedge\Pi_{\bullet\gamma}\end{aligned}$$

where [9]

$$-\sum_\gamma\mathbf{e}_\gamma\wedge\Pi_{\bullet\gamma}=\sum_{\beta,\gamma}\Pi_{\beta\gamma}\mathbf{e}_\beta\wedge\mathbf{e}_\gamma=\sum_{\beta,\gamma}(\Pi_{\beta\gamma}-\Pi_{\gamma\beta})\mathbf{e}_\beta\otimes\mathbf{e}_\gamma=2\Pi^{\mathrm{A}}\,.$$

Hence the claimed equation. □

If Π is symmetric, then $\mathscr{L}$ satisfies the usual form of angular momentum.

6.6 Conservation of angular momentum. Assume the first two equations of 6.3 are satisfied. Then it holds: The matrix Π is symmetric if and only if

$$\varrho\,\partial_{(1,v)}\big((x-\xi)\wedge(v-\dot\xi)\big)+\mathrm{div}\big((x-\xi)\wedge\Pi\big)=(x-\xi)\wedge(\mathbf{f}-\varrho\ddot\xi)\,.$$

Hence $\mathscr{J}=\mathscr{L}=\varrho((x-\xi)\wedge(v-\dot\xi))$ by 6.2 is an angular momentum. For observers for which ξ is zero the equation reads

$$\varrho\,\partial_{(1,v)}\big(x\wedge v\big)+\mathrm{div}\big(x\wedge\Pi\big)=x\wedge\mathbf{f}\,.$$

Notice: Here for symmetric Π the spin matrix $\mathscr{S}$ is zero.

Reminder: The divergence operator works on the last index of the argument, here it is

$$\mathrm{div}(w\wedge\Pi)=\mathrm{div}\Big(\sum_j w\wedge\Pi_{\bullet j}\mathbf{e}_j\Big)=\sum_j\partial_j(w\wedge\Pi_{\bullet j})$$

for vectors w and tensors Π where $\Pi_{\bullet j}=(\Pi_{ij})_i$.

Proof. Now, the difference of the equation in 6.5, which is true for any Π, with the equation in the statement is $2\Pi^{\mathrm{A}}=0$. Therefore the assertion is proved. □

---

[9] For vectors $x\wedge y=x\otimes y-y\otimes x$ and $x\otimes y=xy^{\mathrm{T}}$, and for matrices we have $M=(M_{\alpha\beta})_{\alpha,\beta}=\sum_{\alpha,\beta}M_{\alpha\beta}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$

Hence for symmetric Π there is no spin. The only application with an unsymmetric tensor Π is in the section 17 on liquid crystals.

In order have a spin we consider a moving rigid body as they are identified in I.6.4(3) for the velocity vector $v(t,x) = A(t)x + b(t)$ with an antisymmetric matrix function A and a vector function b. Alternatively, one could define the rigid body in reference coordinates as in I.6.6

$$x = \varphi(t,\underline{x}) = x_{cen}(t) + F(t)\underline{x}\,, \quad \underline{x} \in \mathcal{B}\,,$$

with an orthogonal matrix $F(t)$. Then $B_t := \varphi(t,\mathcal{B})$ is the solid body at time t. The mass $\underline{x} \mapsto \underline{\varrho}(\underline{x})$ is given and the body $\mathcal{B}$ is translated so that the center of mass in reference coordinates $\underline{x}_{cen} = 0$. Then in physical space the mass center is $x_{cen}(t) := \varphi(t,\underline{x}_{cen})$, that is, defining ϱ as in (I6.4) by $\varrho(t,x) = \underline{\varrho}(\underline{x})$ since $J = \det F = 1$ (hence ϱ in B_t is independent of time),

$$\int_{B(t)} \varrho(t,x)(x - x_{cen}(t))\,\mathrm{d}x = F(t)\int_{\mathcal{B}} \underline{\varrho}(\underline{x})\underline{x}\,\mathrm{d}\underline{x} = 0\,.$$

For the velocity we obtain

$$v(t,x) = \partial_t\varphi(t,\underline{x}) = \dot{x}_{cen}(t) + \dot{F}(t)\underline{x} = \dot{x}_{cen}(t) + \dot{F}(t)\,F(t)^{\mathrm{T}}\,(x - x_{cen}(t))\,,$$

and with $A(t) := \dot{F}(t)\,F(t)^{\mathrm{T}}$ this is equivalent to

$$v(t,x) = v_{cen}(t) + A(t)(x - x_{cen}(t))\,, \quad v_{cen}(t) := \dot{x}_{cen}(t)\,, \tag{II6.8}$$

where $A(t)$ is an antisymmetric matrix, that is, we have derived for v the above mentioned structure. As mass-momentum equation for solid bodies we take the following system in the distributional version in $\mathbb{R}\times\mathbb{R}^3$

$$\begin{aligned} \partial_t[\varrho\mathcal{X}_B] + \operatorname{div}[\varrho v\mathcal{X}_B] &= 0\,,\\ \partial_t[\varrho v\mathcal{X}_B] + \operatorname{div}[\varrho v\,v^{\mathrm{T}}\,\mathcal{X}_B] &= [\mathbf{f}]\,,\\ \mathbf{f}\circ Y &= \varrho^*\mathcal{X}_{B^*}(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*\,, \end{aligned} \tag{II6.9}$$

where $B = \{(t,x)\,;\; x \in B_t\}$. It is clear that the following holds.

6.7 Lemma. For a rigid body in empty space the mass-momentum system (II6.9) holds, if

$$\mathbf{f} = \varrho\mathbf{f}^{\mathrm{sp}}\mathcal{X}_B\,, \quad \mathbf{f}^{\mathrm{sp}} = C(t)(x - x_{cen}(t)) + c(t)\,,$$

where

$$c = \ddot{x}_{cen} \quad \text{and} \quad C = \dot{A} + A^2\,.$$

This is compatible with the transformation rule of $\mathbf{f}$, see (II6.9).
Résumé: For a rigid body the mass-momentum equation cannot hold under a general gravitational potential, only if the potential is approximated by a linear term like on the surface of Earth.

Proof. One possible proof is: The left side of the mass equation reads for scalar test functions ζ

$$\begin{aligned}
&\langle\, \zeta\,,\, \partial_t[\varrho\mathcal{X}_B] + \mathrm{div}[\varrho v\mathcal{X}_B]\,\rangle \\
&= -\int_B (\varrho\partial_t\zeta + \varrho v\bullet\nabla\zeta)\,\mathrm{dL}^4 = -\int_B (\varrho,\varrho v)\bullet(\partial_t,\nabla)\zeta\,\mathrm{dL}^4 \\
&= -\int_{\partial B} \zeta\,(\varrho,\varrho v)\bullet\mathrm{n}_B\,\mathrm{dH}^3 + \int_B \zeta\,(\partial_t,\,\mathrm{div})(\varrho,\varrho v)\,\mathrm{dL}^4
\end{aligned}$$

where n_B is the outer normal of $B \subset \mathbb{R}\times\mathbb{R}^3$ (and ∂B is 3-dimensional). Now, if $\underline{x} \in \partial\mathcal{B}$ then for all times $(t,\varphi(t,\underline{x})) \in \partial B$ hence

$$(1, v(t,x)) = (1, \partial_t\varphi(t,\underline{x})) = \frac{\mathrm{d}}{\mathrm{d}t}(t,\varphi(t,\underline{x}))$$

lies in the tangent space of $\partial B \subset \mathbb{R}\times\mathbb{R}^3$, therefore $(1,v)\bullet\mathrm{n}_B = 0$. In the second integral, since $\mathrm{div}v = \mathrm{trace}\,A = 0$ in B,

$$(\partial_t,\,\mathrm{div})(\varrho,\varrho v) = (\partial_t + v\bullet\nabla)\varrho = \frac{\mathrm{d}}{\mathrm{d}t}\varrho(t,\varphi(t,\underline{x})) = \frac{\mathrm{d}}{\mathrm{d}t}\underline{\varrho}(\underline{x}) = 0\,.$$

We have proved the mass equation. Analogously we show for the momentum equation for vector valued test functions ζ

$$\begin{aligned}
&\Big\langle\, \zeta\,,\, \partial_t[\varrho v\mathcal{X}_B] + \mathrm{div}[\varrho v\, v^{\mathrm{T}}\,\mathcal{X}_B]\,\Big\rangle \\
&= -\sum_k \int_B (\varrho v_k\partial_t\zeta_k + \varrho v_k(v\bullet\nabla)\zeta_k)\,\mathrm{dL}^4 = -\sum_k \int_B \varrho v_k(1,v)\bullet(\partial_t,\nabla)\zeta_k\,\mathrm{dL}^4 \\
&= -\sum_k \int_{\partial B} \zeta_k\,\varrho v_k(1,v)\bullet\mathrm{n}_B\,\mathrm{dH}^3 + \sum_k \int_B \zeta_k(\partial_t,\,\mathrm{div})(\varrho v_k,\varrho v_k v)\,\mathrm{dL}^4\,.
\end{aligned}$$

The boundary term vanishes as before and, since $\partial_t\varrho + \mathrm{div}(\varrho v) = 0$ in B,

$$\begin{aligned}
&(\partial_t,\,\mathrm{div})(\varrho v_k,\varrho v_k v) = \partial_t(\varrho v_k) + \mathrm{div}(\varrho v_k v) \\
&= \varrho\partial_t v_k + \varrho v\bullet\nabla v_k = \varrho(\partial_t + v\bullet\nabla)v_k\,,
\end{aligned}$$

hence the differential equation implies

$$\begin{aligned}
&\langle\, \zeta\,,\, [\mathbf{f}]\,\rangle = \Big\langle\, \zeta\,,\, \partial_t[\varrho v\mathcal{X}_B] + \mathrm{div}[\varrho v\, v^{\mathrm{T}}\,\mathcal{X}_B]\,\Big\rangle \\
&= \langle\, \zeta\,,\, [\varrho\mathcal{X}_B(\partial_t + v\bullet\nabla)v]\,\rangle\ .
\end{aligned}$$

Therefore $\mathbf{f}$ has to be as in the assertion $\mathbf{f} = \varrho\mathbf{f}^{\mathrm{sp}}\mathcal{X}_B$ and $\mathbf{f}^{\mathrm{sp}}$ in the set B is

$$\begin{aligned}
&\mathbf{f}^{\mathrm{sp}} = (\partial_t + v\bullet\nabla)v = (\partial_t + v\bullet\nabla)(v_{cen} + A(x - x_{cen})) \\
&= \dot v_{cen} - Av_{cen} + \dot A(x - x_{cen}) + Av \\
&= \dot v_{cen} - Av_{cen} + \dot A(x - x_{cen}) + A(v_{cen} + A(x - x_{cen})) \\
&= \dot v_{cen} + (\dot A + A^2)(x - x_{cen})\,,
\end{aligned}$$

that is, c and C in the assertion are $c = \dot{v}_{cen}$ and $C = \dot{A} + A^2$. To have the standard transformation rule for $\mathbf{f}$, that is, the third equation in (II6.9), $\mathbf{f}^{\rm sp}$ has to satisfy

$$\mathbf{f}^{\rm sp}\circ Y = (\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^{\rm sp*}\,.$$

Now

$$\mathbf{f}^{\rm sp}\circ Y = C\circ Y(X(t^*,x^*) - X(t^*,x^*_{cen})) + c\circ Y = C\circ YQ(x^* - x^*_{cen}) + c\circ Y$$

and

$$\begin{aligned}&(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^{\rm sp*} = \ddot{Q}x^* + \ddot{b} + 2\dot{Q}(v^*_{cen} + A(x^* - x^*_{cen})) + Q\mathbf{f}^{\rm sp*}\\ &= \ddot{Q}((x^* - x^*_{cen}) + x^*_{cen}) + \ddot{b} + 2\dot{Q}(v^*_{cen} + A(x^* - x^*_{cen}))\\ &\qquad\qquad + Q(C^*(x^* - x^*_{cen}) + c^*)\,.\end{aligned}$$

Comparison of the coefficients of $x^* - x^*_{cen}$ gives

$$C\circ YQ = \ddot{Q} + 2\dot{Q}A + QC^*$$

and the remaining absolute term is

$$c\circ Y = \ddot{Q}x^*_{cen} + \ddot{b} + 2\dot{Q}v^*_{cen} + Qc^*\,.$$

These are the transformation rules for C and c. □

The angular momentum for the rigid body has the following distributional version.

6.8 Lemma. For a rigid body in empty space the angular momentum reads in the space of distributions $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^3)$

$$\begin{aligned}\partial_t\mathscr{J} + \operatorname{div}(v\mathscr{J}) &= [(x-\xi)\wedge(\mathbf{f}^{\rm sp} - \ddot{\xi})\varrho\mathcal{X}_B]\,,\\ \mathscr{J} &:= [(x-\xi)\wedge(v-\dot{\xi})\,\varrho\mathcal{X}_B]\,.\end{aligned}$$

Proof. First we prove that the modified momentum equation

$$\partial_t[\varrho(v-\dot{\xi})\mathcal{X}_B] + \operatorname{div}[\varrho(v-\dot{\xi})\,v^{\rm T}\,\mathcal{X}_B] = [\mathbf{f} - \varrho\ddot{\xi}\mathcal{X}_B] \qquad \text{(II6.10)}$$

holds, since for vector valued test functions ζ

$$\begin{aligned}&\Big\langle \zeta\,,\,\partial_t[\varrho(v-\dot{\xi})\mathcal{X}_B] + \operatorname{div}[\varrho(v-\dot{\xi})\,v^{\rm T}\,\mathcal{X}_B]\Big\rangle\\ &= -\int_B\Big(\partial_t\zeta\bullet\varrho(v-\dot{\xi}) + \sum_j v_j\partial_j\zeta\bullet\varrho(v-\dot{\xi})\Big)\,\mathrm{dL}^4\\ &= -\int_B\Big(\partial_t(\zeta\bullet(v-\dot{\xi}))\cdot\varrho - \varrho\zeta\bullet\partial_t(v-\dot{\xi})\\ &\quad + \sum_j\partial_j(\zeta\bullet(v-\dot{\xi}))\cdot\varrho v_j - \varrho\zeta\bullet\sum_j v_j\partial_j(v-\dot{\xi})\Big)\,\mathrm{dL}^4\end{aligned}$$

$$= -\int_B \Big(\partial_t(\zeta\bullet(v-\dot\xi))\cdot\varrho + \sum_j \partial_j(\zeta\bullet(v-\dot\xi))\cdot\varrho v_j\Big)\,\mathrm{dL}^4$$
$$+\int_B \Big(\varrho\zeta\bullet\partial_t(v-\dot\xi) + \varrho\zeta\bullet\sum_j v_j\partial_j(v-\dot\xi)\Big)\,\mathrm{dL}^4$$
$$= \Big\langle\,\zeta\bullet(v-\dot\xi)\,,\,\partial_t[\varrho\mathcal{X}_B] + \mathrm{div}[\varrho v_j\mathcal{X}_B]\,\Big\rangle$$
$$+\int_B \varrho\zeta\bullet\big(\partial_t(v-\dot\xi) + (v\bullet\nabla)(v-\dot\xi)\big)\,\mathrm{dL}^4\,.$$

The first term vanishes because of the distributional mass equation, here with test function $\zeta\bullet(v-\dot\xi)$ (we have to continue v somehow outside B). In the second term we have $(\partial_t + (v\bullet\nabla))\dot\xi = \ddot\xi$, and $\partial_t v + (v\bullet\nabla)v = \mathbf{f}^{\mathrm{sp}}$ for the momentum equation in B. Therefore

$$\Big\langle\,\zeta\,,\,\partial_t[\varrho(v-\dot\xi)\mathcal{X}_B] + \mathrm{div}[\varrho(v-\dot\xi)\,v^{\mathrm{T}}\,\mathcal{X}_B]\,\Big\rangle$$
$$= \int_B \zeta\bullet\big(\varrho\mathbf{f}^{\mathrm{sp}} - \varrho\ddot\xi\big)\,\mathrm{dL}^4 = \Big\langle\,\zeta\,,\,[\mathbf{f} - \varrho\ddot\xi\mathcal{X}_B]\,\Big\rangle$$

so that (II6.10) is proved.

In the main part of the proof we choose in the weak equation of (II6.10)

$$0 = \sum_k \int_B \big(\partial_t\zeta_k\cdot\varrho(v-\dot\xi)_k + \nabla\zeta_k\cdot(\varrho(v-\dot\xi)_k v) + \zeta_k\varrho(\mathbf{f}^{\mathrm{sp}}-\ddot\xi)_k\big)\,\mathrm{dL}^4$$
$$= \sum_k \int_B \big((\partial_t + v\bullet\nabla)\zeta_k\cdot\varrho(v-\dot\xi)_k + \zeta_k\varrho(\mathbf{f}^{\mathrm{sp}}-\ddot\xi)_k\big)\,\mathrm{dL}^4$$

the vector valued test functions ζ as

$$\zeta_k(t,x) = \sum_l \zeta_{kl}(t,x)\cdot(x-\xi)_l \quad\text{with}\quad \zeta_{kl}+\zeta_{lk} = 0\,,$$

that is, we multiply the k-th component of (II6.10) with $(x-\xi)_l$ and subtract the l-th component of the multiple by $(x-\xi)_k$. We obtain

$$0 = \sum_{k,l}\int_B \Big((\partial_t + v\bullet\nabla)\big(\zeta_{kl}(x-\xi)_l\big)\cdot\varrho(v-\dot\xi)_k$$
$$+\zeta_{kl}\,(x-\xi)_l\varrho(\mathbf{f}^{\mathrm{sp}}-\ddot\xi)_k\Big)\,\mathrm{dL}^4$$
$$= \sum_{k,l}\int_B \Big((\partial_t + v\bullet\nabla)\zeta_{kl}\cdot(x-\xi)_l\varrho(v-\dot\xi)_k$$
$$+\zeta_{kl}\,\underbrace{(\partial_t + v\bullet\nabla)(x-\xi)_l}_{=(v-\dot\xi)_l}\cdot\varrho(v-\dot\xi)_k + \zeta_{kl}\,(x-\xi)_l\varrho(\mathbf{f}^{\mathrm{sp}}-\ddot\xi)_k\Big)\,\mathrm{dL}^4$$

The second term vanishes since $(k,l) \mapsto \zeta_{kl}$ is antisymmetric, and therefore with $\mathscr{J}_{kl} := [(x-\xi)_l(v-\dot{\xi})_k\varrho\mathcal{X}_B]$

$$\begin{aligned}0 &= \sum_{k,l}\int_B \Big((\partial_t\zeta_{kl} + v\bullet\nabla\zeta_{kl})\cdot(x-\xi)_l\varrho(v-\dot{\xi})_k \\ &\qquad\qquad + \zeta_{kl}\,(x-\xi)_l\varrho(\mathbf{f}^{\mathrm{sp}} - \ddot{\xi})_k\Big)\,\mathrm{dL}^4 \\ &= \sum_{k,l}\Big\langle\, \zeta_{kl}\,,\, \partial_t\mathscr{J}_{kl} + \operatorname{div}(\mathscr{J}_{kl}v) + [(x-\xi)_l\varrho(\mathbf{f}^{\mathrm{sp}} - \ddot{\xi})_k\mathcal{X}_B]\,\Big\rangle\,,\end{aligned}$$

the assertion. □

Now we let B_t a moving rod with $x_{cen}(t)$ as center of mass

$$B_t := \Big\{\,(t, x_{cen}(t) + s\mathrm{n}(t) + y)\ ;\ -\frac{\ell}{2} \leq s \leq \frac{\ell}{2},\ y\bullet\mathrm{n}(t) = 0,\ |y| < \varepsilon\,\Big\}$$

where ℓ is the length and $\mathrm{n}(t)$ the direction of the rod. The mass density in B is assumed to be constant and equal to

$$\varrho := \frac{m}{\mathrm{L}^3(B_t)} = \frac{m}{\ell\pi\varepsilon^2}\,, \quad m > 0\,,$$

and v is the velocity in (II6.8). We perform the limit $\varepsilon \to 0$, that is, we consider the moving rod in the limit as moving stick of length ℓ,

$$\begin{gathered}[\varrho\mathcal{X}_B] \longrightarrow m\mu \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3) \text{ as } \varepsilon \to 0\,, \\ \langle\,\zeta\,,\,\mu\,\rangle := \int_{\mathbb{R}} \frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \zeta(t, x_{cen}(t) + s\mathrm{n}(t))\,\mathrm{d}s\,\mathrm{d}t\,, \\ t \mapsto A(t) \text{ has a limit as } \varepsilon \to 0,\end{gathered} \tag{II6.11}$$

and therefore also v in (II6.8). For the angular momentum $\mathscr{J}$ this means

$$\begin{aligned}&\mathscr{J} = [(x-\xi)\wedge(v-\dot{\xi})\,\varrho\mathcal{X}_B] \\ &= [(x-\xi)\wedge(v_{cen}-\dot{\xi})\,\varrho\mathcal{X}_B] + [(x-\xi)\wedge(A(x-x_{cen}))\,\varrho\mathcal{X}_B] \\ &\longrightarrow (x-\xi)\wedge(v_{cen}-\dot{\xi})\,m\mu + (x-\xi)\wedge(A(x-x_{cen}))\,m\mu\,.\end{aligned} \tag{II6.12}$$

We obtain

6.9 Theorem. If for the stick μ we define

$$\begin{aligned}\mathscr{J} &:= \mathscr{L} + \mathscr{S}\,, \\ \mathscr{L} &:= (x-\xi)\wedge(v_{cen}-\dot{\xi})\,m\mu\,, \\ \mathscr{S} &:= (x-\xi)\wedge(A(x-x_{cen}))\,m\mu\,,\end{aligned}$$

then the differential equations become

$$\begin{aligned}
\partial_t(m\mu) + \operatorname{div}(vm\mu) &= 0\,,\\
v &= v_{cen} + A(x - x_{cen})\,,\\
\partial_t(vm\mu) + \operatorname{div}(v\,v^{\mathrm{T}}\,m\mu) &= \mathbf{f}\,,\\
\mathbf{f}\circ Y &= (\ddot{X} + 2\dot{Q}v^*)m\mu + Q\mathbf{f}^*\,,\\
\partial_t\mathscr{J} + \operatorname{div}(v\mathscr{J}) &= (x-\xi)\wedge(\mathbf{f}^{\mathrm{sp}} - \ddot{\xi})m\mu\,,
\end{aligned}$$

where $\mathbf{f} = \mathbf{f}^{\mathrm{sp}}m\mu$ and $\mathbf{f}^{\mathrm{sp}}$ as in 6.7.

Proof. That $\mathscr{J} = \mathscr{L} + \mathscr{S}$ is true in the limit follows from (II6.12). Since the mass and momentum equation follows (II6.9), where $\mathbf{f}^{\mathrm{sp}}$ has a limit and

$$[\mathbf{f}] = \mathbf{f}^{\mathrm{sp}}[\varrho\mathcal{X}_B] \longrightarrow \mathbf{f}^{\mathrm{sp}}m\mu$$

and we define in the limit $\mathbf{f} := \mathbf{f}^{\mathrm{sp}}m\mu$. In the angular momentum we obtain with the notation of the previous proof

$$\begin{aligned}
&\partial_t\mathscr{J}_{kl} + \operatorname{div}(v\mathscr{J}_{kl}) + [(x-\xi)_l\varrho(\mathbf{f}^{\mathrm{sp}} - \ddot{\xi})_k\mathcal{X}_B]\\
&\longrightarrow \partial_t\mathscr{J}_{kl} + \operatorname{div}(v\mathscr{J}_{kl}) + (x-\xi)_l(\mathbf{f}^{\mathrm{sp}} - \ddot{\xi})_k m\mu\,.
\end{aligned}$$

Thus the result is shown. □

Hence for the stick we have an angular momentum $\mathscr{J} = \mathscr{L} + \mathscr{S}$, where $\mathscr{S}$ is the spin. If the observer $t \mapsto \xi(t)$ is located at the center $t \mapsto x_{cen}(t)$ of the stick this spin has a familiar stucture.

6.10 Lemma. If $\xi = x_{cen}$ then for the spin $\mathscr{S}$ in 6.9

$$\mathscr{S} = \mathrm{n}\wedge A\mathrm{n}\Big(\frac{m\ell^2}{12}\boldsymbol{\mu}_{x_{cen}} + m\ell^3\boldsymbol{\lambda}_\ell\Big) \quad \text{in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3).$$

Here the measure $\boldsymbol{\mu}_{x_{cen}}$ is defined as in I.2.8 and $\boldsymbol{\lambda}_\ell$ is "bounded distribution" in ℓ as $\ell \to 0$. In detail

$$\begin{aligned}
\langle\,\zeta\,,\boldsymbol{\mu}_{x_{cen}}\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} &:= \int_{\mathbb{R}} \zeta(t, x_{cen}(t))\,\mathrm{d}t\,,\\
\langle\,\zeta\,,\boldsymbol{\lambda}_\ell\,\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} &:= \int_{\mathbb{R}} \frac{1}{\ell^4}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \big(\zeta(t, x_{cen}(t) + s\mathrm{n}(t)) - \zeta(t, x_{cen}(t))\big)s^2\,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Proof that $\boldsymbol{\lambda}_\ell$ is bounded. For real values test functions $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$, if $\operatorname{supp}\zeta \subset [-R, R]\times\mathbb{R}^3$,

$$\begin{aligned}
&\Big|\langle\,\zeta\,,\boldsymbol{\lambda}_\ell\,\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)}\Big|\\
&\le \int_{\mathbb{R}} \frac{1}{\ell^4}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |\zeta(t, x_{cen}(t) + s\mathrm{n}(t)) - \zeta(t, x_{cen}(t))|\,s^2\,\mathrm{d}s\,\mathrm{d}t\\
&\le \|\mathrm{D}\zeta\|_{C^0}\int_{-R}^{+R}\frac{1}{\ell^4}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |s|^3\,\mathrm{d}s\,\mathrm{d}t \le \frac{R}{16}\|\mathrm{D}\zeta\|_{C^0}\,,
\end{aligned}$$

which says that $\boldsymbol{\lambda}_\ell$ are locally bounded distributions in ℓ. □

Proof. For every test function ζ in $C_c^\infty(\mathbb{R}^{3\times3})$, with $\zeta(t,x)$ being antisymmetric for all (t,x),

$$\begin{aligned}\langle \zeta\,,\,\mathscr{S}\,\rangle &= \langle \zeta\,,\,(x-\xi)\wedge(A(x-x_{cen}))\,m\mu\,\rangle \\ &= \langle \zeta\,,\,(x-x_{cen})\wedge(A(x-x_{cen}))\,m\mu\,\rangle \qquad \big(x-x_{cen}=s\mathrm{n}\big) \\ &= \int_{\mathbb{R}}\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} m\zeta(t,x_{cen}(t)+s\mathrm{n}(t)) \,\boldsymbol{:}\, (\mathrm{n}(t)\wedge A(t)\mathrm{n}(t))s^2\,\mathrm{d}s\,\mathrm{d}t\,,\end{aligned}$$

and by writing

$$\widetilde{\zeta}(t,x) := \zeta(t,x)\,\boldsymbol{:}\,(\mathrm{n}(t)\wedge A(t)\mathrm{n}(t))$$

this is

$$= \int_{\mathbb{R}}\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} m\widetilde{\zeta}(t,x_{cen}(t)+s\mathrm{n}(t))s^2\,\mathrm{d}s\,\mathrm{d}t\,. \tag{II6.13}$$

We split this into two expressions by writing

$$\begin{aligned}&\widetilde{\zeta}(t,x_{cen}(t)+s\mathrm{n}(t)) \\ &= \widetilde{\zeta}(t,x_{cen}(t)) + \big(\widetilde{\zeta}(t,x_{cen}(t)+s\mathrm{n}(t)) - \widetilde{\zeta}(t,x_{cen}(t))\big)\end{aligned}$$

The first term in (II6.13) gives the integral

$$\begin{aligned}&= \int_{\mathbb{R}}\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} m\widetilde{\zeta}(t,x_{cen}(t))s^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\widetilde{\zeta}(t,x_{cen}(t))\underbrace{\left(\frac{m}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} s^2\,\mathrm{d}s\right)}_{=\frac{m\ell^2}{12}}\,\mathrm{d}t = \left\langle\,\widetilde{\zeta}\,,\,\frac{m\ell^2}{12}\,\mu_{x_{cen}}\right\rangle\,.\end{aligned}$$

The second term in (II6.13) gives the integral

$$= \int_{\mathbb{R}}\frac{1}{\ell}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} m\big(\widetilde{\zeta}(t,x_{cen}(t)+s\mathrm{n}(t)) - \widetilde{\zeta}(t,x_{cen}(t))\big)s^2\,\mathrm{d}s\,\mathrm{d}t = \left\langle\,\widetilde{\zeta}\,,\,m\ell^3\,\boldsymbol{\lambda}_\ell\,\right\rangle\,.$$

This is the result. □

7 Exercises

7.1 Gruppeneigenschaft. Zeige: Eine Menge $\mathcal{G}$ von Matrizen hat die Gruppeneigenschaft bzgl. der Matrixmultiplikation genau dann, wenn die zugehörige Menge der Transformationen $\mathcal{T}$ (wie in 1.1) die Gruppeneigenschaft bzgl. der Hintereinanderschaltung hat.

7.2 Antisymmetrie. Sei Q eine zeitabhängige orthogonale Transformation, also $QQ^{\mathrm{T}} = \mathrm{Id}$. Zeige, dass dann $\dot{Q}Q^{\mathrm{T}}$ antisymmetrisch ist.

Lorentz transformations

7.3 Lorentz transformation. Zeige, dass die Lorentz Transformation $\mathbf{L}_c(V,Q)$ die Determinante 1 hat.

Solution. Bezüglich der orthonormalen Vektoren

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ und } \begin{bmatrix} 0 \\ \frac{V}{|V|} \end{bmatrix}$$

hat die Lorentz Transformation nach (II2.4) die Matrix

$$\begin{bmatrix} \gamma & \gamma|V| \\ \frac{1}{|V|}\cdot\left(\gamma - \frac{1}{\gamma}\right) & \gamma \end{bmatrix},$$

deren Determinante gleich 1 ist. Auf dem zu den beiden Vektoren senkrechten Raum ist die Transformation die Identität (siehe (II2.4)). ☐

7.4 Einstein's addition theorem. In 2.2(3) zeige $Q^{\mathrm{T}}V$ in die Richtung von $\pm v^*$. Dann gilt mit $\overline{v} := Qv^*$

$$v = \frac{V + \overline{v}}{1 + \frac{1}{c^2} V \bullet \overline{v}}\,.$$

Solution. Es ist $\mathbf{B}_c(V)Qv^* = Qv^* + (\gamma - 1)\widehat{V}\bullet Qv^*\,\widehat{V} = \gamma Qv^*$, wobei $\widehat{V} = \frac{V}{|V|}$. ☐

Objectivity of differential equations

7.5 Exercise. In the momentum equation of (II3.16) the term $\varrho\overset{\circ}{v} - \mathbf{f}$ is an objective vector.

Solution. The second equation of (II3.16) is $\varrho\overset{\circ}{v} - \mathbf{f} = -\,\mathrm{div}\Pi$. Here $\mathbf{f}$ is a classical force with the rule (II3.18)

$$\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*\,.$$

And $v\circ Y = \dot{X} + Qv^*$ implies, see the computation below,

$$\overset{\circ}{v}\circ Y = \big(\ddot{X} + 2\dot{Q}v^*\big) + Q\overset{\circ}{v^*}\,, \tag{II7.1}$$

so that

$$\big(\varrho\overset{\circ}{v} - \mathbf{f}\big)\circ Y = Q\big(\varrho^*\overset{\circ}{v^*} - \mathbf{f}^*\big)\,,$$

a transfomation rule like $\mathrm{div}\Pi$ has. To derive (II7.1) we note that

$$\overset{\circ}{v} = (\partial_t + v\bullet\nabla_x)v = \partial_t v + (\mathrm{D}_x v)v\,.$$

Taking the derivatives of the transformation rule for $v\circ Y$ (see above) gives

$$\begin{aligned} &(\partial_t v)\circ Y + ((\mathrm{D}_x v)\circ Y)\dot{X} = \partial_{t^*}(v\circ Y) \\ &= \ddot{X} + \dot{Q}v^* + Q\partial_{t^*}v^*\,. \end{aligned}$$

Since $\dot{X} = v\circ Y - Qv^*$ (see above) we obtain

$$\begin{aligned}&\ddot{X} + \dot{Q}v^* + Q\partial_{t^*}v^* = (\partial_t v)\circ Y + ((\mathrm{D}_x v)\circ Y)(v\circ Y - Qv^*)\\&= (\partial_t v + v\bullet\nabla_x v)\circ Y - ((\mathrm{D}_x v)\circ Y)Qv^*\\&= \overset{\circ}{v}\circ Y - (\dot{Q} + Q\mathrm{D}_{x^*}v^*)v^*\,,\end{aligned}$$

since $((\mathrm{D}_x v)\circ Y)Q = \dot{Q} + Q\mathrm{D}_{x^*}v^*$ (see the derivation of (II4.12) in 4.11), therefore

$$\overset{\circ}{v}\circ Y = \ddot{X} + 2\dot{Q}v^* + Q(\partial_{t^*} + \mathrm{D}_{x^*}v^*)v^* = \ddot{X} + 2\dot{Q}v^* + Q\overset{\circ}{v^*}\,,$$

which is (II7.1). □

Objectivity of functions

7.6 Constitutive relations. Sei u ein objektiver Skalar sowie a und b objektive Vektoren. Zeige:

(1) Die Beziehung $u = \widehat{u}(a,b)$ ist objektiv, d.h. $\widehat{u}$ ist dieselbe Funktion für verschiedene Beobachter, falls

$$\widehat{u}(Qa, Qb) = \widehat{u}(a,b)$$

für alle Werte von a und b und alle orthogonalen Matrizen Q mit positiver Determinante.

(2) Dies ist erfüllt, falls es eine Funktion $\widetilde{u}$ gibt mit

$$\widehat{u}(a,b) = \widetilde{u}(a\bullet b)\,.$$

7.7 Objective inequality and equality. Es seien a und b objektive Vektoren. Dann ist

$$a\bullet b \geq 0$$

eine objektive Ungleichung. Ebenso die Gleichung

$$a\bullet b = 1.$$

Solution. Ist $(t,x) = Y(t^*,x^*)$ eine Newton'sche Beobachtertransformation, so gilt $a\circ Y = Qa^*$ und $b\circ Y = Qb^*$, also

$$a\bullet b = (Qa^*)\bullet(Qb^*) = a^*\bullet b^*.$$

Also ist $u := a\bullet b$ ein objektiver Skalar, somit $u \geq 0$ dieselbe Ungleichung für alle Beobachter und $u = 1$ dieselbe Gleichung für alle Beobachter. □

7.8 Material derivative. Ist v eine Geschwindigkeit und g ein objektiver Skalar, so ist auch

$$\overset{\circ}{g} := \partial_t g + v\bullet\nabla g$$

ein objektiver Skalar.

7.9 Transformation formula. Es sei w ein objektiver Vektor und $\overset{\circ}{w}$ definiert wie in 7.8 mit einer Geschwindigkeit v. Zeige: Dann gilt die Transformationsformel

$$\overset{\circ}{w}\circ Y = \dot{Q}w^* + Q\overset{\circ}{w^*}$$

Beachte: $\overset{\circ}{w}$ benötigt v zu Definition, während $\overset{\circ}{w^*}$ mit v^* definiert ist.

Solution. Es gilt nach der Kettenregel

$$\partial_{t^*}(w\circ Y) = (\partial_t w)\circ Y + \sum_i \dot X_i (\partial_{x_i} w)\circ Y\,,$$
$$\partial_{x_j^*}(w\circ Y) = \sum_i Q_{ij}(\partial_{x_i} w)\circ Y\,.$$

Wegen $w\circ Y = Qw^*$ gilt

$$\partial_{x_j^*}(w\circ Y) = Q\partial_{x_j^*} w^*\,,$$
$$\partial_{t^*}(w\circ Y) = \dot Q w^* + Q\partial_{t^*} w^*\,,$$

also wegen $v\circ Y = \dot X + Qv^*$

$$\begin{aligned}
\overset{\circ}{w}\circ Y &= (\partial_t w + v\bullet\nabla w)\circ Y = (\partial_t w)\circ Y + v\circ Y\bullet(\nabla w)\circ Y \\
&= (\partial_t w)\circ Y + \sum_i (\dot X_i + \sum_j Q_{ij} v_j^*)(\partial_{x_i} w)\circ Y \\
&= \partial_{t^*}(w\circ Y) + \sum_j v_j^* \partial_{x_j^*}(w\circ Y) \\
&= \dot Q w^* + Q\partial_{t^*} w^* + \sum_j v_j^* Q\partial_{x_j^*} w^* = \dot Q w^* + Q\overset{\circ}{w}{}^*\,.
\end{aligned}$$

□

7.10 Objective vector. Seien J und q objektive Vektoren mit der konstitutiven Gleichung für beliebige q

$$J = \widehat{J}(q)\,.$$

Zeige: Ist $\widehat{J}$ objektiv, so folgt

$$\widehat{J}(Qq) = Q\widehat{J}(q)$$

für alle $q \in \mathbb{R}^n$ und alle orthogonalen Transformationen $Q \in \mathbb{R}^{n\times n}$ mit Determinante 1.

Solution. Let Y be an observer transformation, then it holds

$$J\circ Y = QJ^*\,,\quad q\circ Y = Qq^*\,.$$

Since $\widehat{J}$ is objective, it follows for the two observers

$$J^*(t^*,x^*) = \widehat{J}(q^*(t^*,x^*))\,,\quad J(t,x) = \widehat{J}(q(t,x))\,,$$

and hence

$$\begin{aligned}
Q\widehat{J}(q^*) &= QJ^* = J\circ Y \quad \text{(since J is an objective vector)} \\
&= \widehat{J}(q\circ Y) = \widehat{J}(Qq^*) \quad \text{(since q is an objective vector)}\,.
\end{aligned}$$

Hence $\widehat{J}(Qq^*) = Q\widehat{J}(q^*)$ where $q^*(t^*,x^*)$ is an arbitrary vector in $\mathbb{R}^n$. □

7.11 Objective tensor. Sei $n \geq 3$ and Π ein objektiver Tensor, der nur von objektiven Skalaren abhängt. Dann gibt es ein $p \in \mathbb{R}$, so dass

$$\Pi = p\mathrm{Id}\,,$$

und p hängt von jenen objektiven Skalaren ab.

7.12 Objective tensor. Sei S ein objektiver Tensor, q ein objektiver Vektor. Zeige: Die Gleichung

$$S = \widehat{S}(q) \text{ mit } \widehat{S}(\widetilde{q}) := \widetilde{q}\widetilde{q}^{\mathrm{T}} \text{ für alle } \widetilde{q} \in \mathbb{R}^n$$

definiert eine objektive Funktion $\widehat{S}$, d.h. sie ist dieselbe Funktion für alle Beobachter.

7.13 Objective tensor. Sei S ein Tensor. Es gelte, dass Sn für jeden objektiven Vektor n einen objektiven Vektor ergibt. Zeige, dass dann S ein objektiver Tensor ist. *Hinweis:* n kann jeden Vektor annehmen.

7.14 Representation with respect to a basis. Es seien $D = (d_{ij})_{ij}$ und $E = (e_{ij})_{ij}$ Matrizen mit

$$d_{ij} = \sum_k \lambda_k e_{ki} e_{kj}\,, \quad e_k = (e_{ki})_i\,,$$

wobei $\lambda_k \in \mathbb{R}$ Zahlen seien. Zeige:

(1) Es gilt

$$D = \sum_{k=1}^{n} \lambda_k e_k \, {e_k}^{\mathrm{T}} = E \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} E^{\mathrm{T}}\,.$$

(2) Sind λ_k objektive Skalare und e_k objektive Vektoren, also $\lambda_k{\circ}Y = \lambda_k^*$ und $e_k{\circ}Y = Qe_k^*$, so gelten die Transformationsformeln

$$E{\circ}Y = QE^*\,, \quad D{\circ}Y = QD^*Q^{\mathrm{T}}\,.$$

7.15 Entropy. Es sei η ein objektiver Skalar und es seien (ϱ, v, e) Größen, die sich wie die Dichte, die Geschwindigkeit und die Energie verhalten. Es gelte mit einer objektiven Funktion $\widehat{\eta}$

$$\eta = \widehat{\eta}(\varrho, v, e)$$

Dann folgt mit einer neuen konstitutiven Funktion $\widetilde{\eta}$

$$\eta = \widetilde{\eta}(\varrho, \varepsilon)\,, \quad \varepsilon := e - \frac{\varrho}{2}|v|^2\,.$$

Bemerkung: Es ist also $\widehat{\eta}(\varrho, v, e) = \widetilde{\eta}\big(\varrho, e - \frac{\varrho}{2}|v|^2\big)$.

7.16 Deformation. Sei η eine objektiver Skalar und F transformiere bei Beobachterwechsel wie der Deformationsgradient, d.h.

$$F{\circ}Y = QF^* \text{ für alle orthonormalen } Q \text{ mit Determinante } 1.$$

Es sei

$$\eta = \widehat{\eta}(F)$$

mit einer objektiven Funktion $\widehat{\eta}$. Zeige dann:

(1) Es ist $\eta'_F \, F^{\mathrm{T}}$ symmetrisch.

(2) Dies ist zum Beispiel der Fall, wenn $\eta = \widetilde{\eta}(F^{\mathrm{T}} F)$ mit einer Funktion $\widetilde{\eta}$.

Beachte: Es ist $\eta'_F \, F^{\mathrm{T}} = \big(\sum_k \eta'_{F_{ik}} F_{jk}\big)_{ij}$.

III Energy and entropy

Entropy principle in the course of time

The entropy principle is considered as an essential contribution to continuum physics. Let us give a short historical overview. The origin of thermodynamics are seen in the paper [99] von Sadi Carnot 1824. Later Émile Clapeyron 1834 in [102] and Rudolf Clausius 1850 in [103] (see the contributions in [100]) have revisited and extended it, also William Thomson (Lord Kelvin) in the paper 1848 [121].

The entropy principle was formulated in the 19th century by Rudolf Clausius and was used in the development and optimization of machines, that is, in the transfer of heat to motion, see [http:www.animatedengines.com]. The heat was produced in coal-burning stoves and later partly replaced by liquid fossil fuels. These fundamentals of thermodynamics are also now formulated in an axiomatic way and taught to students as the laws of thermodynamics. Here is an example:

[MIT, Lecture on Thermodynamics (Spakovszky, Fall 2008)]:

Zeroth Law. There exists for every thermodynamic system in equilibrium a property called temperature. Equality of temperature is a necessary and sufficient condition for thermal equilibrium.

First Law. There exists for every thermodynamic system a property called the energy. The change of energy of a system is equal to the mechanical work done on the system in an adiabatic process. In a non-adiabatic process, the change in energy is equal to the heat added to the system minus the mechanical work done by the system.

Second Law. There exists for every thermodynamic system in equilibrium an extensive scalar property called the entropy, S, such that in an infinitesimal reversible change of state of the system, $dS = dQ/T$, where T is the absolute temperature and dQ is the amount of heat received by the system. The entropy of a thermally insulated system cannot decrease and is constant if and only if all processes are reversible.

It is important to understand the historical concepts since only this way it is possible to recognize the achievements of the entropy principle.

References: The history is contained in I.Müller [11, Grundzüge der Thermodynamik] and Hutter [8, Fluid- und Thermodynamik], in particular in the section [8, 6.1 Grundsätzliches sowie geschichtliche Bemerkungen]. Also see [Wikipedia: Second Law of Thermodynamics] or as a somewhat older version [133]. It is suggested for interested readers also to look at Truesdell [96] which is a comprehensive study of the history.

The "Zeroth Law of Thermodynamics" states that there exists a temperature, and this temperature is important for the physical behavior of the system and it is related to the energy and entropy. The entropy we treat in section 1 and the energy in section 2. But do not forget that this absolute temperature is connected with the temperature as measurable quantity, that is, there exist "thermometers", see 6.1 and 6.2. This temperature is the historical basis for the entropy principle.

The "First Law of Thermodynamics" postulates that the system of conservation laws, that is the mass and momentum balance, has to be completed by a conservation equation for the total energy

$$\boxed{\partial_t e + \mathrm{div}\widetilde{q} = \widetilde{g}}$$

(where we discuss here mostly the case of smooth functions only). Here e is the ***(total) energy density*** of the system, the vector field $\widetilde{q}$ is the corresponding ***energy flux*** and $\widetilde{g}$ is the ***energy production***. All these variables are functions of (t, x) and they are studied in section 2. The energy flux $\widetilde{q}$ contains the heat that is supplied to the system. The total energy density e contains the kinetic energy, we studied it already in II.3.13.

The "Second Law of Thermodynamics" postulates the existence of a quantity which is called entropy. The property that "the entropy of a thermally insulated system cannot decrease", is not quantified in the text above. Therefore we refer to the properties of the entropy in the book of De Groot & Mazur [6, Chapter III] (see also Prigogine & Defay [91, Chapter III]):

[De Groot & Mazur (1962), Chapter III page 20]:
"According to the principles of thermodynamics one can introduce for any macroscopic system a state function S, the entropy of the system, which has the following properties."
"The variation of the entropy $\mathrm{d}S$ may be written as the sum of two terms"

$$\begin{aligned} &(1) \quad \mathrm{d}S = \mathrm{d}_e S + \mathrm{d}_i S, \\ &(2) \quad \mathrm{d}_i S \geq 0 \quad (\text{"The Second Law of Thermodynamics"}), \\ &(4) \quad \mathrm{d}_e S = \frac{\mathrm{d}Q}{T} \quad (\text{"Theorem of Carnot-Clausius"}), \end{aligned} \tag{III0.1}$$

"where $\mathrm{d}_e S$ is the entropy supplied to the system by its surroundings, and $\mathrm{d}_i S$ the entropy produced inside the system."

After a remark, with reference to the next pages of the book, it is said:

[De Groot & Mazur (1962), Chapter III page 21]:
"We may remark at this point that thermodynamics in the customary sense is concerned with the study of the reversible transformations for which the equality (2) holds. In thermodynamics of irreversible processes, however, one of the important objektives is to relate the quantity $\mathrm{d}_i S$, the entropy production, to the various irreversible phenomena which may occur inside the system. Before calculating the entropy production in terms of quantities which characterize the irreversible phenomena, we shall rewrite (1) and (2) in a form which is more suitable for the description of systems in which the densities of the extensive properties (such as mass and energy ....) are continuous functions of space coordinates. Let us write"

$$(6) \quad S = \int_V \varrho s \, \mathrm{dL}^n \,, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \varrho s \, \mathrm{dL}^n \,,$$

$$(7) \quad \frac{\mathrm{d}_e S}{\mathrm{d}t} = - \int_{\partial V} J_{s,tot} \bullet \nu_V \, \mathrm{dH}^{n-1} \,,$$

$$(8) \quad \frac{\mathrm{d}_i S}{\mathrm{d}t} = \int_V \sigma \, \mathrm{dL}^n \,,$$

"where s is the entropy per unit mass, $J_{s,tot}$ the total energy flow per unit area and unit time, and σ the entropy source strength or entropy production per unit volume and unit time." (The notations of measures are changed compared with the original and one has to set $n = 3$.)

Here V is the test volume and ν_V the outer unit normal on ∂V. It is meant, that the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \varrho s \, \mathrm{dL}^n = - \int_{\partial V} J_{s,tot} \bullet \nu \, \mathrm{dH}^{n-1} + \int_V \sigma \, \mathrm{dL}^n$$

holds. The text continuous as follows:

[De Groot & Mazur (1962), Chapter III page 22]:
"With (6), (7) and (8), formula (1) may be rewritten, using also Gauss' theorem, in the form"

$$\int_V \Big(\frac{\partial}{\partial t} (\varrho s) + \mathrm{div} J_{s,tot} - \sigma \Big) \, \mathrm{dL}^n = 0 \,.$$

"From this relation it follows, since (1) and (2) must hold for an arbitrary volume V, that"

$$(10) \quad \frac{\partial}{\partial t} (\varrho s) + \mathrm{div} J_{s,tot} = \sigma \,,$$

$$(11) \quad \sigma \geq 0 \,.$$

"These two formulae are the local forms of (1) and (2), i.e. the local mathematical expression for the second law of thermodynamics."

So fare the historical overview. Therefore it is common that nowadays one assumes that the "Second law of thermodynamics" is satisfied in this local version

$$\boxed{\partial_t \eta + \mathrm{div}\psi = \sigma \geq 0\,,} \tag{III0.2}$$

where we set η for the ***entropy*** (in older versions it is $\eta = \varrho s$ where s denotes the "specific entropy", i.e. the entropy per unit mass) and ψ for the ***entropy flux*** (in the above notation $J_{s,tot}$). For the ***entropy production*** we use the notation σ of DeGroot & Mazur. Hence it is clear how the entropy principle becomes a manifest in a differential inequality. At this point we remark that the quantities "entropy" and "entropy flux" are not given a priori, as for example by the theorem of Carnot-Clausius (siehe dazu (III0.1)), rather they are determined by the context in which they arise. They result in the special physical situation, that is in the differential equations and in the constitutive equations.

Mathematical equivalent versions of the entropy principle

Now we show that the inverse conclusion also holds, that is, the formulation with test volumes is equivalent to the differential inequality. If $V \subset \mathbb{R}^n$ is a test volume then we define the entropy of the material in V

$$S(t) := \int_V \eta(t,x)\,\mathrm{dL}^3(x)\,.$$

Then from (III0.2) it follows with Gauß' theorem, since V is a fixed domain,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(t) &= \int_V \partial_t\eta(t,x)\,\mathrm{dL}^3(x) = \int_V \big(-\,\mathrm{div}\psi(t,x) + \sigma(t,x)\big)\,\mathrm{dL}^3(x)\\ &= -\int_{\partial V}\psi(t,x)\bullet\nu_V(x)\,\mathrm{dH}^2(x) + \int_V \sigma(t,x)\,\mathrm{dL}^3(x),\end{aligned}$$

hence

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}S(t) + \int_{\partial V}\psi(t,x)\bullet\nu_V(x)\,\mathrm{dH}^2(x) = \int_V \sigma(t,x)\,\mathrm{dL}^3(x) \geq 0\,,} \tag{III0.3}$$

or for mathematicians

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(-\,S(t)\big) + \int_V \underbrace{\sigma(t,x)}_{\geq 0}\,\mathrm{dL}^3(x) = \int_{\partial V}\psi(t,x)\bullet\nu_V(x)\,\mathrm{dH}^2(x)\,.$$

Since this is true for all V, this statement is equivalent to (III0.2). That means, that the time derivative of S plus what of ψ is moving out of V (or, minus the amount of ψ wich enters V) is positiv (nonnegative).

References: The entropy principle one finds, here only for example, in DeGroot & Mazur [6, Chapter III Entropy law and entropy balance], in I.Müller [87, 5 Entropy principles], and in Wilmanski [15, 6 Entropy principle]. Further we recommend from the recent literature Hutter & Wang [9, 17 Thermodynamics-Fundamentals, 18 Thermodynamics-Field Formulation]. We mention in the mathematical literature Feireisl [39, 1.3.3 The second law of thermodynamics] and also Feireisl & Novotný [40, Chap. 2 Weak Solutions, A Priori Estimates].

The entropy inequality is just the differential inequality (III0.2). This inequality is always true, as long as one does not leave the basic principles. So, the entropy inequality (III0.2) is undisputed.

Entropy in limit situations

However the entropy inequality can be of different form if one considers a certain limit. It always has to be justified why (III0.2) is no longer the exact way to formulate the entropy principle. There can be several reasons for this:

- If the limit is a distributional problem then, in the simplest case, one can asume that (III0.2) stays with the same form $\partial_t H + \mathrm{div}_x \Psi \geq 0$, but H und Ψ are distributions in the space $\mathcal{D}'(\mathcal{U})$. This e.g. covers problems with surfaces or moving Dirac distributions (see section 6).
- The entropy principle splits into several inequalities, e.g. if a thin layer becomes a hypersurface. This way a separate entropy inequality is formed on the interface, and this inequality is separated from the entropy principle in the surrounding bulk. This is the case in Alt & Witterstein [20] where an isothermal phase field model is performed.
- A thermal radiation leads to an additional entropy production term as in Greve [5, 6 Entropieprinzip, S.222] and Hutter & Jöhnk [47, 2.3.5 Entropy Balance]. If one starts in the relativistic case with an inequality like (III0.2) such entropy production term will arise if one considers the nonrelativistic limit $\mathbf{c} \to \infty$. But this production term has an extra justification.

This are just three important reasons.

1 Entropy inequality

What is entropy? The entropy η is a quantity for which the "entropy principle" holds, that is, it has a nonnegative production term. And it is an objective scalar, that is, it is transformed for two observers by $\eta \circ Y = \eta^*$ where Y is the coordinate transformation.

1.1 Entropy principle. Let $\mathcal{P}$ be a class of physical processes. Then there exist for each process in $\mathcal{P}$ variables η, the ***entropy***, and ψ, the ***entropy flux***, such that

$$\boxed{\sigma := \partial_t \eta + \mathrm{div}\psi \geq 0} \tag{III1.1}$$

in the domain $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ in which the process is defined. Here σ is the ***entropy production***.[1] The differential equation $\partial_t \eta + \mathrm{div}\psi = \sigma$ is supposed to be a scalar equation. It is called entropy identity. See also Fig. 1.
Important: Of course, the entropy on the whole (η, ψ) depends on the process $\mathcal{P}$, but how is not said here. It is not unique, if the processes described by $\mathcal{P}$ are not detailed enough.

> ***1.3.1.1 Entropy principle*** **The entropy principle reflects the experience that thermodynamic processes are dissipative and irreversible. In later chapters it will be shown how this experience leads to the concept of entropy and to the entropy inequality. Here, the entropy principle is stated in four parts in an axiomatic manner:**
>
> **(a) there exists an additive objective scalar quantity, the entropy;**
> **(b) the specific entropy and the entropy flux are given by constitutive equations;**
> **(c) the entropy production is non-negative for all thermodynamic processes;**
> **(d) the temperature is continuous across an ideal wall.**

Fig. 1: From I.Müller [87, Sec. 1.3], according '(d)' see 6.1

The property that the entropy identity is a scalar equation is motivated by the entropy inequality $\sigma \geq 0$. The reason is, that this inequality must hold for all observers (see the next statement and II.4.4).

1.2 Property. The equation (III1.1) is a scalar one if according to II.3.1

$$\begin{aligned} \eta \circ Y &= \eta^* \,, \\ \psi \circ Y &= \eta^* \dot{X} + Q\psi^* \,, \\ \sigma \circ Y &= \sigma^* \,. \end{aligned}$$

This says that η is an objective scalar. Since σ is an objective scalar, the inequality $\sigma \geq 0$ is objective.

---

[1] The notation σ is adopted from DeGroot & Mazur [6], which here does not collide with the notation S for the stress tensor.

Remark: We mention that the fact that the entropy is mathematically non-trivial (i.e. not zero), is not stated in the definition. But it is clear that the entropy is the physical entropy with their physical properties. It is the subject of the following sections to explain what these properties are.

The entropy equality is objective, if its weak version

$$\int_{\mathcal{U}} (\partial_t\zeta\eta + \nabla\zeta\bullet\psi + \zeta\sigma)\,\mathrm{dL}^{n+1} = 0 \text{ for } \zeta \in C_0^\infty(\mathcal{U};\mathbb{R}) \tag{III1.2}$$

holds for test functions ζ which transform with $\zeta\circ Y = \zeta^*$ (see section 6 for the distributioal entropy). Also the entropy inequality is equivalent to

$$\int_{\mathcal{U}} \zeta\sigma\,\mathrm{dL}^{n+1} \geq 0 \text{ for } \zeta \in C_0^\infty(\mathcal{U};\mathbb{R}) \text{ with } \zeta \geq 0. \tag{III1.3}$$

Here Y is the observer transformation. We have the freedom to choose the constitutive equations of entropy and entropy flux, but finally the entropy inequality must hold as a scalar differential inequality. On the other hand it is obvious, that an entropy inequality for a class $\mathcal{P}$ can be true only if the proof of this inequality requires knowledge on the differential equations and constitutive relations, that characterize $\mathcal{P}$. Often it is neccessary in order to understand the situation to go to the distributional formulation in section 6.

Entropy in gas theory

It is the task to calculate in concrete examples the entropy production and to ensure its positivity by requirements on the constitutive equations. This means that for concrete materials the constitutive relations must be chosen in such away that the entropy principle holds, wherein the entropy inequality $\sigma \geq 0$ does require special knowledge of the entropy and the entropy flux. Thus the choice of the entropy is part of describing a concrete material, the entropy is included in the constitutive functions. Here we compute constitutive functions in a special case, namely we take the fundamental example of gas theory.

1.3 Example from gas theory. The set of physical processes $\mathcal{P}$ is determined by the solutions of the equations

$$\begin{aligned} &\partial_t\varrho + \mathrm{div}(\varrho v) = 0\,, \\ &\partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) = -p\;\mathrm{div}v + g\,, \end{aligned} \tag{III1.4}$$

where ϱ, ε, p and g are objective scalars, v a velocity, and q an objective vector. Further, for $\mathcal{P}$ we assume conditions which are related to the entropy principle

$$\sigma = \partial_t\eta + \mathrm{div}\psi \geq 0\,.$$

Assertion: This principle is fulfilled for $\mathcal{P}$, if:

$$\eta = \widehat{\eta}(\varrho,\varepsilon)\,, \quad p = \widehat{p}(\varrho,\varepsilon)\,,$$
$$\eta = \varrho\eta'_{\varrho} + (\varepsilon + p)\eta'_{\varepsilon} \quad \text{(Gibbs relation)},$$
$$\psi = \eta v + \eta'_{\varepsilon} q \quad \text{(Clausius-Duhem flux)},$$
$$\sigma = \nabla\eta'_{\varepsilon}\bullet q + \eta'_{\varepsilon} g \geq 0 \quad \text{(Residual inequality)}.$$

The representation of (η,ψ) has the required objective property (see 1.2). *Differential equations for a gas:* The equations (III1.4) are the conservation of mass and the equation for the internal energy. This applies to ideal gases, that means for fluids without viscosity (see II.3.13 and (III2.5)), where one sets $g = 0$ as realization of the fact that the energy is conserved.

b

Proof. It follows with $\eta = \widehat{\eta}(\varrho,\varepsilon)$ and $\psi = \eta v + \psi_0$ that

$$\begin{aligned}\sigma &= \partial_t\eta + \operatorname{div}\psi = \partial_t\eta + v\bullet\nabla\eta + \eta\operatorname{div}v + \operatorname{div}\psi_0\\ &= \overset{\circ}{\eta} + \eta\operatorname{div}v + \operatorname{div}\psi_0\,,\end{aligned}$$

if $\overset{\circ}{h} := (\partial_t + v\bullet\nabla)h$ [2] for every funcktion h. Since $\eta = \widehat{\eta}(\varrho,\varepsilon)$, it follows

$$\overset{\circ}{\eta} = \eta'_{\varrho}\overset{\circ}{\varrho} + \eta'_{\varepsilon}\overset{\circ}{\varepsilon}\,.$$

Since the differential equations for ϱ and ε imply

$$\begin{aligned}&\overset{\circ}{\varrho} + \varrho\operatorname{div}v = 0\,,\\ &\overset{\circ}{\varepsilon} + (\varepsilon + p)\operatorname{div}v + \operatorname{div}q = g\,,\end{aligned}$$

we obtain

$$\begin{aligned}\sigma &= \overset{\circ}{\eta} + \eta\operatorname{div}v + \operatorname{div}\psi_0 = \eta'_{\varrho}\overset{\circ}{\varrho} + \eta'_{\varepsilon}\overset{\circ}{\varepsilon} + \eta\operatorname{div}v + \operatorname{div}\psi_0\\ &= (\eta - \eta'_{\varrho}\varrho - \eta'_{\varepsilon}(\varepsilon + p))\operatorname{div}v + \eta'_{\varepsilon}g - \eta'_{\varepsilon}\operatorname{div}q + \operatorname{div}\psi_0\\ &= (\eta - \eta'_{\varrho}\varrho - \eta'_{\varepsilon}(\varepsilon + p))\operatorname{div}v + \eta'_{\varepsilon}g + \nabla\eta'_{\varepsilon}\bullet q + \operatorname{div}(\psi_0 - \eta'_{\varepsilon}q)\,.\end{aligned}$$

The last term vanishes because of the choice of ψ. The first div$v$-term vanishes if the pressure p is selected according to the Gibbs relation. Altogether the assertion follows. □

Conseqences of the entropy principle

[2] We prefer $\overset{\circ}{h} := \partial_{(1,v)}h = \partial_t h + v\bullet\nabla h$ as abbreviation. See the footnote to the notation in II.3.8.

When is the entropy principle in this particular case fulfilled? It is fulfilled, if the pressure p is given by the Gibbs relation and if for the heat flux q the second law is satisfied in the form $\nabla\eta'_\varepsilon\bullet q \geq 0$. In addition $g = 0$, that is, we assume conservation of energy. This means, that the two variables p and q are related by constitutive equations to the new function η. In detail this means, we assume $\eta'_\varepsilon \neq 0$, that the pressure p fulfills

$$p = \frac{1}{\eta'_\varepsilon}(\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon) \tag{III1.5}$$

and that the heat flux q is given for example by Fourier's law

$$\begin{aligned} q = \widehat{q}(\varrho,\varepsilon,\nabla\eta'_\varepsilon) = \widehat{c}(\varrho,\varepsilon,|\nabla\eta'_\varepsilon|)\nabla\eta'_\varepsilon \\ \text{with a scalar } c = \widehat{c}(\varrho,\varepsilon,|\nabla\eta'_\varepsilon|) \geq 0\,, \end{aligned} \tag{III1.6}$$

which Joseph Fourier postulated in the early 19th century, see [Wikipedia: Joseph Fourier] and [106] (we mention (III1.12) where Fourier's law is written with the help of temperature). Then it follows

$$\sigma = \nabla\eta'_\varepsilon\bullet q = c|\nabla\eta'_\varepsilon|^2 \geq 0\,.$$

We mention that this inequality also applies to non-isotropic material, for example, with a term $q = M\nabla\eta'_\varepsilon$ where M is a positive semidefinite matrix. This leads to the entropy inequality

$$\sigma = \nabla\eta'_\varepsilon\bullet q = \nabla\eta'_\varepsilon\bullet M\nabla\eta'_\varepsilon \geq 0\,.$$

If M is an objective matrix then q remains an objective vector (see II.4.9).

Therefore the entropy principle leads to equations, here the Clausius-Duhem flux and the Gibbs relation, and to a residual inequality. Here this inequality is $\sigma = \nabla\eta'_\varepsilon\bullet q \geq 0$ which we call residual inequality. These conditions on the process $\mathcal{P}$ involve the entropy function η. There are also theorems that say how the functions Π and q have to look if the entropy inequality is satisfied. Such sufficient conditions one finds for example in [112] and [19, Section 15].

Empirical temperature

Die gemessene Temperatur (siehe z.B. Fig. 3 and 6.1) war historisch weit vor der theoretisch fundierten Temperatur in Gebrauch. Im 19. Jahrhundert hat William Thomson (Lord Kelvin) den Zusammenhang mit der absoluten Temperatur hergestellt. Heutzutage wird die zu messende Temperatur mit der theoretischen Temperatur, siehe die Gleichung (III1.7) bzw. 1.6, identifiziert. Mit dem Zusammenhang von empirischer Temperatur und theoretischer Temperatur haben sich Physiker allgemein auseinandergesetzt, wobei besonders zu empfehlen sei Hutter & Wang [9, 17.2.5 Empirical Temperature, Gas Temperature and Temperature Scales]. Weiter siehe auch I.Müller [87, 1.3.2.4 Determination of Λ^ε], sowie Truesdell [96, 11H. Critique: Empirical and Absolute Temperatures], ...

Temperature θ and internal energy ε as differential forms

We now provide the relationship with the somewhat older literature, in which the entropy is based on differential forms. This has only a limited validity, since it describes the entropy principle only in special situations, such as above in the description of gases in 1.3. We now are not interested in ideal gases, we treat the Gibbs relation separately.

1.4 Gibbs relation. Let $p = \widehat{p}(\varrho, \varepsilon)$ and $\eta = \widehat{\eta}(\varrho, \varepsilon)$ with $\eta'_\varepsilon > 0$. Define

$$\boxed{\begin{aligned} &\theta = \frac{1}{\eta'_\varepsilon} \quad \text{the } \textbf{\textit{absolute temperature}}, \\ &f = \varepsilon - \theta\eta \text{ the } \textbf{\textit{Helmholtz energy}} \\ &\qquad\qquad (\textbf{\textit{inner free energy}}). \end{aligned}} \tag{III1.7}$$

Further, we denote with upper index "sp" the corresponding specific quantities, that is, in a region where $\varrho > 0$ we define

$$\varepsilon^{\mathrm{sp}} := \frac{\varepsilon}{\varrho}, \quad p^{\mathrm{sp}} := \frac{p}{\varrho}, \quad \eta^{\mathrm{sp}} := \frac{\eta}{\varrho}, \quad f^{\mathrm{sp}} := \frac{f}{\varrho},$$
$$\mathrm{v}^{\mathrm{sp}} := \frac{1}{\varrho} \text{ the specific volume.}$$

Then the following statements are equivalent:

(1) $\eta = \varrho\eta'_\varrho + (\varepsilon + p)\eta'_\varepsilon$ (Gibbs relation as in 1.3).

(2) $\eta^{\mathrm{sp}}{}'_\varrho + (\varepsilon^{\mathrm{sp}} + p^{\mathrm{sp}})\eta^{\mathrm{sp}}{}'_\varepsilon = 0$.

(3) $\theta\,\mathrm{d}\eta^{\mathrm{sp}} = \mathrm{d}\varepsilon^{\mathrm{sp}} + p\,\mathrm{d}\mathrm{v}^{\mathrm{sp}}$ (see the introduction to this chapter).

(4) $\mathrm{d}f^{\mathrm{sp}} = -\eta^{\mathrm{sp}}\,\mathrm{d}\theta - p\,\mathrm{d}\mathrm{v}^{\mathrm{sp}}$.

(5) $\mathrm{d}(\varepsilon^{\mathrm{sp}} + p^{\mathrm{sp}}) = \theta\,\mathrm{d}\eta^{\mathrm{sp}} + \mathrm{v}^{\mathrm{sp}}\,\mathrm{d}p$ ($\varepsilon^{\mathrm{sp}} + p^{\mathrm{sp}}$ is the ***enthalpy***).

(6) $\mathrm{d}(f^{\mathrm{sp}} + p^{\mathrm{sp}}) = -\eta^{\mathrm{sp}}\,\mathrm{d}\theta + \mathrm{v}^{\mathrm{sp}}\,\mathrm{d}p$ ($f^{\mathrm{sp}} + p^{\mathrm{sp}}$ is the ***free enthalpy***).

The free energy is also called ***Gibbs energy***.

The statements 1.4(3)–1.4(6) are identities in the space of differential forms with (ϱ, ε) as the independent variables. If $\eta'_{\varepsilon\varepsilon}$ has a sign, physically negative, then (ϱ, θ) can also be chosen as independent variables (see 1.5 below).

With $f = \widehat{f}(\varrho, \theta)$ it then follows from 1.4(4)[3] (see also the last statement in 1.6 below)

$$\frac{\partial}{\partial\theta} f^{\mathrm{sp}} = -\eta^{\mathrm{sp}}\,, \quad \frac{\partial}{\partial\varrho} f^{\mathrm{sp}} = -p\frac{\partial}{\partial\varrho} v^{\mathrm{sp}} = \frac{p}{\varrho^2}\,. \tag{III1.8}$$

and expressed in terms of f

$$\boxed{\frac{\partial}{\partial\theta} f = -\eta\,, \quad \varrho^2 \frac{\partial}{\partial\varrho}\Big(\frac{f}{\varrho}\Big) = p\,.} \tag{III1.9}$$

Proof (1)⇔(2). It is

$$\frac{\partial}{\partial\varrho}\eta^{\mathrm{sp}} + (\varepsilon^{\mathrm{sp}} + p^{\mathrm{sp}})\frac{\partial}{\partial\varepsilon}\eta^{\mathrm{sp}} = \frac{1}{\varrho^2}\Big(\varrho\frac{\partial}{\partial\varrho}\eta - \eta + (\varepsilon + p)\frac{\partial}{\partial\varepsilon}\eta\Big)$$

and therefore (2) is equivalent to (1). □

Proof (3)⇔(1). Statement (3) can be written as

$$\mathrm{d}\eta^{\mathrm{sp}} - \eta_{\prime\varepsilon}\,\mathrm{d}\varepsilon^{\mathrm{sp}} - \eta_{\prime\varepsilon} p\,\mathrm{d}\Big(\frac{1}{\varrho}\Big) = 0\,.$$

For each function $h = \widehat{h}(\varrho, \varepsilon)$ we know

$$\mathrm{d}h = h_{\prime\varrho}\,\mathrm{d}\varrho + h_{\prime\varepsilon}\,\mathrm{d}\varepsilon\,.$$

We sort in the above expression the ϱ and ε derivative. The ε-derivative is equal to

$$\frac{\partial}{\partial\varepsilon}\eta^{\mathrm{sp}} - \eta_{\prime\varepsilon}\frac{\partial}{\partial\varepsilon}\varepsilon^{\mathrm{sp}} = 0\,,$$

which is in general true, and the ϱ-derivative is

$$\begin{aligned}\frac{\partial}{\partial\varrho}\eta^{\mathrm{sp}} - \eta_{\prime\varepsilon}\frac{\partial}{\partial\varrho}\varepsilon^{\mathrm{sp}} - \eta_{\prime\varepsilon} p\frac{\partial}{\partial\varrho}\frac{1}{\varrho} &= \frac{\partial}{\partial\varrho}\frac{\eta}{\varrho} + \eta_{\prime\varepsilon}\frac{\varepsilon + p}{\varrho^2} \\ &= \frac{1}{\varrho^2}\Big(\varrho\frac{\partial}{\partial\varrho}\eta - \eta + \eta_{\prime\varepsilon}(\varepsilon + p)\Big)\,,\end{aligned}$$

which again is the left-hand side of (1). Therefore (3) is equivalent to (1). □

Proof (4)⇔(3). Because $f^{\mathrm{sp}} = \varepsilon^{\mathrm{sp}} - \theta\eta^{\mathrm{sp}}$, we obtain

$$\mathrm{d}f^{\mathrm{sp}} = \mathrm{d}\varepsilon^{\mathrm{sp}} - \mathrm{d}(\theta\eta^{\mathrm{sp}}) = \mathrm{d}\varepsilon^{\mathrm{sp}} - \theta\,\mathrm{d}\eta^{\mathrm{sp}} - \eta^{\mathrm{sp}}\,\mathrm{d}\theta\,,$$

and therefore with statement (4) we get

$$\mathrm{d}\varepsilon^{\mathrm{sp}} - \theta\,\mathrm{d}\eta^{\mathrm{sp}} = \mathrm{d}f^{\mathrm{sp}} + \eta^{\mathrm{sp}}\,\mathrm{d}\theta = -p\,\mathrm{d}v^{\mathrm{sp}}\,,$$

which is exactly statement (3). □

[3] The formula is the same for the variables (ϱ, ε) and the variables (ϱ, θ)

Proof (3)⇔(5). Assertion (5) is

$$\theta\, \mathrm{d}\eta^{\mathrm{sp}} = \mathrm{d}(\varepsilon^{\mathrm{sp}} + p^{\mathrm{sp}}) - \mathrm{v}^{\mathrm{sp}}\, \mathrm{d}p = \mathrm{d}\varepsilon^{\mathrm{sp}} + \mathrm{d}(p\mathrm{v}^{\mathrm{sp}}) - \mathrm{v}^{\mathrm{sp}}\, \mathrm{d}p = \mathrm{d}\varepsilon^{\mathrm{sp}} + p\, \mathrm{d}\mathrm{v}^{\mathrm{sp}}\,,$$

which is (3). □

Proof (4)⇔(6). Assertion (6) is

$$\begin{aligned}\mathrm{d}f^{\mathrm{sp}} &= -\,\mathrm{d}p^{\mathrm{sp}} - \eta^{\mathrm{sp}}\, \mathrm{d}\theta + \mathrm{v}^{\mathrm{sp}}\, \mathrm{d}p \\ &= -\,\mathrm{d}(\mathrm{v}^{\mathrm{sp}}p) - \eta^{\mathrm{sp}}\, \mathrm{d}\theta + \mathrm{v}^{\mathrm{sp}}\, \mathrm{d}p = -\eta^{\mathrm{sp}}\, \mathrm{d}\theta - p\, \mathrm{d}\mathrm{v}^{\mathrm{sp}}\,,\end{aligned}$$

which is (4). □

The statements of this theorem change if the parameters change on which the entropy depends. In particular, this is true if the entropy depends on the gradients of these variables as treated in section IV.11. Therefore we make no use of differential forms. The definition of the absolute temperature θ in (III1.7) only used that beside its dependence on ϱ the entropy is given as a function of the internal energy, where it is assumed that $\eta'_{\varepsilon} > 0$.

References: For the dual variables in connection with Gibbs equation see Hutter [8, 6.9 Zustandsgleichungen]. For the inverse absolute temperature see the mathematical treatment of the Legendre-Fenchel transform.

Temperature θ as dual variable to ε

We now introduce the inverse absolute temperature $\beta = \frac{1}{\theta}$ as the dual variable of the internal energy ε. The dual function of η we denote by φ. Here we do not write down the dependence on ϱ.

1.5 Energy and inverse absolute temperature. Let η be an entropy, which is a function of ε and other variables (these are not written here, but taken as parameters). It is assumed that $\eta'_{\varepsilon\varepsilon} \neq 0$ (physically $\eta'_{\varepsilon\varepsilon} < 0$). Then, instead of ε one can also take

$$\boxed{\beta := \eta'_{\varepsilon}(\varepsilon) \text{ the } \textit{\textbf{inverse absolute temperature}}}$$

as independent variable (an ε-interval is mapped to a β-interval). If we define φ as ***dual function*** to η, that is,

$$\boxed{\eta(\varepsilon) + \varphi(\beta) = \beta\varepsilon \quad \text{for } \beta = \eta'_{\varepsilon}(\varepsilon),} \tag{III1.10}$$

then

$$\begin{gathered}\beta = \eta'_{\varepsilon}(\varepsilon)\,, \quad \varepsilon = \varphi'_{\beta}(\beta)\,, \\ \varphi'_{\beta} = (\eta'_{\varepsilon})^{-1}\,, \quad \eta'_{\varepsilon} = (\varphi'_{\beta})^{-1}\,, \\ 1 = \eta'_{\varepsilon\varepsilon}(\varepsilon)\varphi'_{\beta\beta}(\beta) \text{ for } \varepsilon \text{ and } \beta \text{ as above,}\end{gathered}$$

and

η given:	$\varphi(\beta) = \varepsilon\eta'_{\varepsilon}(\varepsilon) - \eta(\varepsilon)$ for $\beta = \eta'_{\varepsilon}(\varepsilon)\,,$
φ given:	$\eta(\varepsilon) = \beta\varphi'_{\beta}(\beta) - \varphi(\beta)$ for $\varepsilon = \varphi'_{\beta}(\beta)\,.$

Note: The functions usually depend on other variables, but in a way that the consideration does not change. This is true if the variables are

$$(\varepsilon, u_1, \ldots, u_N) \quad \text{and} \quad (\beta, u_1, \ldots, u_N)$$

and the transformations are given by

$$\beta = \widehat{\beta}(\varepsilon, u_1, \ldots, u_N) \quad \text{and} \quad \varepsilon = \widehat{\varepsilon}(\beta, u_1, \ldots, u_N).$$

The dual variables ε and β also allow φ to be calculated from the entropy η and vice versa, the entropy η from φ.

Proof. The equation (III1.10), written in the variable ε, is

$$\eta(\varepsilon) + \varphi(\eta'_{\varepsilon}(\varepsilon)) = \eta'_{\varepsilon}(\varepsilon)\varepsilon\,.$$

The derivative with respect to ε of this identity is

$$\eta'_{\varepsilon}(\varepsilon) + \varphi'_{\beta}(\eta'_{\varepsilon}(\varepsilon))\eta'_{\varepsilon\varepsilon}(\varepsilon) = \eta'_{\varepsilon\varepsilon}(\varepsilon)\varepsilon + \eta'_{\varepsilon}(\varepsilon)\,,$$

which due to the condition $\eta'_{\varepsilon\varepsilon}(\varepsilon) \neq 0$ is equivalent to

$$\varphi'_{\beta}(\eta'_{\varepsilon}(\varepsilon)) = \varepsilon\,.$$

Thus

$$\varphi'_{\beta} = (\eta'_{\varepsilon})^{-1}$$

is proved. The other statements are direct consequences and the last assertion follows, by differentiating $\beta = \eta'_{\varepsilon}(\varphi'_{\beta}(\beta))$ with respect to β and obtaining $1 = \eta'_{\varepsilon\varepsilon}(\varphi'_{\beta}(\beta))\varphi'_{\beta\beta}(\beta)$. □

By this result we could be tempted to use the inverse temperature rather than the temperature as variable, but traditionally the temperature is the variable which is considered, in fact, the temperature can be measured (see 6.1 and Fig. 3). The inverse temperature is, however, a Lagrangian parameter as one can see in (III4.4). We will now introduce the absolute temperature θ as an independent variable. Then the internal free energy f plays the role of φ. Hence f can be calculated from the entropy η, and vice versa, the entropy η can be calculated from the inner free energy f.

1.6 Absolute temperature. Let in 1.5

$$\varepsilon \in]\varepsilon_{min}, \varepsilon_{max}[\text{ be mapped to } \beta \in]0,\infty[$$

(it is $\eta'_{\varepsilon} > 0$ and η'_{ε} maps $]\varepsilon_{min}, \varepsilon_{max}[$ to $]0,\infty[$, and it is $\eta'_{\varepsilon\varepsilon} < 0$, so the mapping is monotonically decreasing). Then we define, with φ as in 1.5,

$$\theta := \frac{1}{\beta} = \frac{1}{\eta'_{\varepsilon}(\varepsilon)} \text{ the } \textit{\textbf{absolute temperature}},$$
$$f(\theta) := \theta\varphi\Big(\frac{1}{\theta}\Big) \text{ the } \textit{\textbf{inner free energy}}.$$

Hence it follows from (III1.10)

$$f = \varepsilon - \theta\eta\,.$$

It follows

$$0 < \frac{1}{\theta^3} = f'_{\theta\theta}(\theta)\eta'_{\varepsilon\varepsilon}(\varepsilon) \quad \text{for } \theta = \frac{1}{\eta'_{\varepsilon}(\varepsilon)}$$

and

$$\eta \text{ given:} \quad f(\theta) = \varepsilon - \theta\eta(\varepsilon) \quad \text{für } \theta = \frac{1}{\eta'_{\varepsilon}(\varepsilon)}\,,$$
$$f \text{ given:} \quad \eta(\varepsilon) = -f'_{\theta}(\theta) \quad \text{for } \varepsilon = f(\theta) - \theta f'_{\theta}(\theta) = -\theta^2\Big(\frac{f}{\theta}\Big)'_{\theta}\,.$$

Remark: In this statement the Gibbs relation is not used.

Proof. All assertions follow from the identity

$$\eta + \varphi = \beta\varepsilon \qquad \text{(III1.11)}$$

in 1.5. From (III1.11) it follows

$$\eta + \frac{1}{\theta}f = \frac{1}{\theta}\varepsilon$$

and hence

$$\theta\eta + f = \varepsilon\,,$$

which was to be proven. Further it holds

$$\theta = \frac{1}{\beta} = \frac{1}{\eta'_{\varepsilon}}$$

and

$$\varepsilon = \varphi'_{\beta} = \Big(\frac{f}{\theta}\Big)'_{\theta}\theta'_{\beta} = -\theta^2\Big(\frac{f}{\theta}\Big)'_{\theta} = f - \theta f'_{\theta}\,.$$

Consequently with (III1.11)

$$\eta = \beta\varepsilon - \varphi = \frac{1}{\theta}(f - \theta f{\prime}_\theta) - \frac{f}{\theta} = -f{\prime}_\theta\,.$$

The statement for the second derivatives follows by taking the derivate of

$$\eta(\varepsilon) = -f{\prime}_\theta\big(\frac{1}{\eta{\prime}_\varepsilon(\varepsilon)}\big)$$

with respect to ε

$$\eta{\prime}_\varepsilon(\varepsilon) = -f{\prime}_{\theta\theta}(\theta)\big(\frac{1}{\eta{\prime}_\varepsilon(\varepsilon)}\big){\prime}_\varepsilon = f{\prime}_{\theta\theta}(\theta)\frac{\eta{\prime}_{\varepsilon\varepsilon}(\varepsilon)}{\eta{\prime}_\varepsilon(\varepsilon)^2}\,,$$

which gives the stated formula. Alternatively, we take the derivative of the equation

$$\varphi(\beta) = \beta f\big(\frac{1}{\beta}\big)$$

with respect to β

$$\varphi{\prime}_\beta(\beta) = f(\theta) + \beta f{\prime}_\theta(\theta)\theta{\prime}_\beta = f(\theta) - \theta f{\prime}_\theta(\theta)$$

and get after a further derivation

$$\varphi{\prime}_{\beta\beta}(\beta) = (f - \theta f{\prime}_\theta){\prime}_\theta\theta{\prime}_\beta = -\theta f{\prime}_{\theta\theta}(\theta)\cdot(-\theta^2) = \theta^3 f{\prime}_{\theta\theta}(\theta)\,,$$

what is plugged into the equation $1 = \eta{\prime}_{\varepsilon\varepsilon}\cdot\varphi{\prime}_{\beta\beta}$. □

1.7 Example. As an example we consider the entropy function

$$\eta(\varepsilon) = c\log\varepsilon + d \quad \text{for } \varepsilon > 0\,,$$

where c and d may depend on ϱ. Then one concludes

(1) $\beta = \eta{\prime}_\varepsilon(\varepsilon) = \frac{c}{\varepsilon}\,,\quad \varphi(\beta) = \beta\,\varepsilon - \eta = c - c\log\varepsilon - d = c\big(1 + \log\frac{\beta}{c}\big) - d.$

(2) $\theta = \frac{1}{\beta} = \frac{\varepsilon}{c}\,,\quad f(\theta) = \varepsilon - \theta\eta = \theta\,(c - \eta) = \theta\,(c - d - c\cdot\log\,(c\,\theta)).$

Thus $\varepsilon = c\,\theta$. *Hint:* In IV.2.5 we will consider c and d depending on ϱ.

In 1.6 we use $\varepsilon_{min} = 0$ and $\varepsilon_{max} = \infty$. The constitutive equations are usually given in the variables (ϱ, θ). Then Fourier's law (cf. (III1.6)) for the heat flux q reads

$$\begin{aligned} &q = \widehat{q}(\varrho, \theta, \nabla\theta) = -\widehat{a}(\varrho, \theta, |\nabla\theta|)\nabla\theta \\ &\text{with a scalar } a = \widehat{a}(\varrho, \theta, |\nabla\theta|) \geq 0, \end{aligned} \tag{III1.12}$$

where the sign is due to the entropy inequality.

2 Energy equation

What is energy? In the introduction of this chapter we have made clear, that the existence of energy is the content of the "First Law of Thermodynamics". The total energy e, which we already introduced in II.3, has different aspects: It is a variable

- which contains the kinetic energy, that is, $e = e_{\text{int}} + e_{\text{kin}}$, where for a single fluid with velocity v the kinetic energy is $e_{\text{kin}} = \frac{\varrho}{2}|v|^2$ and the internal energy $e_{\text{int}} = \varepsilon$ (see (III2.6)).
- which depends on the absolute temperature θ, and so it describes the interaction between the kinetic energy and temperature (see 1.4).
- whose transformation behavior is that of e_{kin} (see (III2.3)).
- which with its equation closes the existing system of mass and momentum equation as shown in (III2.5).

The energy identity is the last equation of the

2.1 Energy system (Definition). In Section II.3 we have already learned about the energy, and we have considered in II.3.12 the general system

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}\widetilde{J} = \mathbf{r}\,,\\
&\partial_t(\varrho v) + \operatorname{div}\widetilde{\Pi} = \widetilde{\mathbf{f}}\,,\\
&\partial_t e + \operatorname{div}\widetilde{q} = \widetilde{g}\,,
\end{aligned} \tag{III2.1}$$

which was defined as ***mass-momentum-energy system***. This definition says that the system is transformed from one observer to another observer with the matrix (we use the notation in (II1.3))

$$Z := \begin{bmatrix} 1 & 0 & 0 \\ \dot{X} & Q & 0 \\ \frac{1}{2}|\dot{X}|^2 & \dot{X}^{\mathrm{T}} Q & 1 \end{bmatrix}. \tag{III2.2}$$

Here e is the ***(total) energy***, the vector field $\widetilde{q}$ the corresponding ***energy flux*** and $\widetilde{g}$ is the total ***energy production***.

In particular, the energy is transformed by the equation

$$e \circ Y = \tfrac{1}{2}|\dot{X}|^2 \varrho^* + \varrho^* \dot{X} \bullet (Q v^*) + e^*\,, \tag{III2.3}$$

which is the same transformation rule as for the kinetic energy $e_{\text{kin}} := \frac{\varrho}{2}|v|^2$ in (II3.24). The definition means that to the energy belongs a mass density

ϱ and a velocity v. We then had shown in II.3.12 that with

$$\begin{aligned}\widetilde{J} &= \varrho v + \mathbf{J}\,,\\ \widetilde{\Pi} &= \varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi\,,\\ \widetilde{q} &= ev + \frac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}}\, v + q\,,\end{aligned} \tag{III2.4}$$

the general mass-momentum-energy system can be written as

Mass-momentum-energy system:

$$\begin{aligned}\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) &= \mathbf{r}\,,\\ \partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) &= \widetilde{\mathbf{f}}\,,\\ \partial_t e + \operatorname{div}(ev + \tfrac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}}\, v + q) &= \widetilde{g}\,,\end{aligned} \tag{III2.5}$$

with transformation rules as in (III2.6)

and representations for $\widetilde{\mathbf{f}}$ and $\widetilde{g}$ as in (III2.7).

Here, see (II3.31),

(III2.6)
- $e = \varepsilon + \frac{\varrho}{2}|v|^2$, ε internal energy,
- ϱ, ε, $\mathbf{r}$ are objective scalars,
- v is a velocity, i.e. $v\circ Y = \dot{X} + Qv^*$
- $\mathbf{J}$, q are objective vectors, Π is an objective tensor,

and for the right-hand sides in the momentum and energy equation, see (II3.32) and (II3.31),

(III2.7)
- $\widetilde{\mathbf{f}} = (\mathbf{r} + \mathbf{J}\bullet\nabla)v + \mathbf{f}\,,$
  - $\mathbf{f}$ a classical force, i.e. $\mathbf{f}\circ Y = \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*\,,$
- $\widetilde{g} = \dfrac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + v\bullet \mathrm{D}v\mathbf{J} + (\mathrm{D}v)^{\mathrm{A}} : \Pi + g\,,$
  - g an objective scalar.

Of course, $(\mathrm{D}v)^{\mathrm{A}} : \Pi = 0$ if Π is symmetric (see II.3.14). For the definition of the internal energy ε it is important to deduce the differential equation for the kinetic energy (analogous to (I3.32)).

2.2 Lemma. From the mass and momentum balance in (III2.5) it follows

$$\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \operatorname{div}\Big(\frac{1}{2}|v|^2(\varrho v + \mathbf{J}) + \Pi^{\mathrm{T}}\, v\Big)$$
$$= v\bullet\mathbf{f} + \frac{1}{2}|v|^2\mathbf{r} + v\bullet(\mathrm{D}v\mathbf{J}) + \mathrm{D}v : \Pi\,.$$

Proof. With $\overset{\circ}{h} := (\partial_t + v\bullet\nabla)h = \partial_{(1,v)}h$ it follows from the first equation of (III2.5)

$$\overset{\circ}{\varrho} + \varrho\,\mathrm{div}v = \partial_t\varrho + v\bullet\nabla\varrho + \varrho\,\mathrm{div}v = \partial_t\varrho + \mathrm{div}(\varrho v) = \mathbf{r} - \mathrm{div}\mathbf{J}\,.$$

From there we conclude with the product rule

$$\begin{aligned}&\partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}})\\ &= \underbrace{(\,\partial_t\varrho + \mathrm{div}(\varrho v)\,)}_{=\,\mathbf{r}\,-\,\mathrm{div}\mathbf{J}}v + \varrho\underbrace{(\,\partial_t v + v\bullet\nabla v\,)}_{=\,\overset{\circ}{v}}\,,\end{aligned}$$

thus, the second equation in (III2.5) becomes[4]

$$\begin{aligned}&\varrho\overset{\circ}{v} + \mathrm{div}\Pi = \varrho(\partial_t v + v\bullet\nabla v) + \mathrm{div}\Pi\\ &= -(\mathbf{r} - \mathrm{div}\mathbf{J})v + \partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + \Pi)\\ &= -\mathbf{r}v - \mathrm{D}v\mathbf{J} + \partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi)\\ &= -\mathbf{r}v - \mathrm{D}v\mathbf{J} + \widetilde{\mathbf{f}} = \mathbf{f}\,.\end{aligned}$$

(Hence the second equation in 2.3 is shown.) For the kinetic energy we now compute

$$\begin{aligned}&\Big(\frac{\varrho}{2}|v|^2\Big)^{\circ} = \frac{1}{2}|v|^2\overset{\circ}{\varrho} + \frac{1}{2}\varrho\big(|v|^2\big)^{\circ} = \frac{1}{2}|v|^2\overset{\circ}{\varrho} + v\bullet(\varrho\overset{\circ}{v})\\ &= \frac{1}{2}|v|^2\big(\mathbf{r} - \mathrm{div}\mathbf{J} - \varrho\,\mathrm{div}v\big) + v\bullet\big(\mathbf{f} - \mathrm{div}\Pi\big)\\ &= -\Big(\frac{\varrho}{2}|v|^2\Big)\,\mathrm{div}v + \frac{1}{2}|v|^2(\mathbf{r} - \mathrm{div}\mathbf{J}) + v\bullet\mathbf{f} - v\bullet\mathrm{div}\Pi\\ &= -\Big(\frac{\varrho}{2}|v|^2\Big)\,\mathrm{div}v - \mathrm{div}\Big(\Pi^{\mathrm{T}}v + \frac{1}{2}|v|^2\mathbf{J}\Big)\\ &\qquad\qquad + v\bullet\mathbf{f} + \frac{1}{2}|v|^2\mathbf{r} + \sum_j v_j\nabla v_j\bullet\mathbf{J} + \mathrm{D}v : \Pi\,,\end{aligned}$$

where we used

$$\begin{aligned}\mathrm{div}(\Pi^{\mathrm{T}}v) &= v\bullet\mathrm{div}\Pi + \mathrm{D}v : \Pi\,,\\ \mathrm{div}\Big(\frac{1}{2}|v|^2\mathbf{J}\Big) &= \frac{1}{2}|v|^2\,\mathrm{div}\mathbf{J} + \sum_j v_j\nabla v_j\bullet\mathbf{J}\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \mathrm{div}\Big(\frac{1}{2}|v|^2(\varrho v + \mathbf{J}) + \Pi^{\mathrm{T}}v\Big)\\ &\qquad = v\bullet\mathbf{f} + \frac{1}{2}|v|^2\mathbf{r} + v\bullet(\mathrm{D}v\mathbf{J}) + \mathrm{D}v : \Pi\,,\end{aligned}$$

that is the assertion. □

[4] It is $v\bullet\nabla v = (v\bullet\nabla)v = \sum_j v_j\partial_{x_j}v = (\mathrm{D}v)v$

We now derive an often used representation of the conservation laws (III2.5) for the variabels $(\varrho, v, \varepsilon)$.

2.3 Lemma. The general mass-momentum-energy system (III2.5) is equivalent to [5]

$$\begin{aligned} &\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \\ &\varrho(\partial_t v + v\bullet\nabla v) + \operatorname{div}\Pi = \mathbf{f}\,, \\ &\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) = -\,(\mathrm{D}v)^{\mathrm{S}} : \Pi + g\,. \end{aligned} \tag{III2.8}$$

This system one can also write in the form (III2.10).

Reminder: The assumption $g = 0$ is known as ***energy conservation*** and this is possible since g is an objective scalar. Also all terms of the energy equation in (III2.10) are objectiv scalars including $(\mathrm{D}v)^{\mathrm{S}} : \Pi$. The right-hand side of the momentum equation is the classical force $\mathbf{f}$ which includes the fictitious forces $\mathbf{f}^{fic}$. Hence $\mathbf{f} = \mathbf{f}^{fic} + \mathbf{f}^{obj}$, where $\mathbf{f}^{obj}$ is an objective vector and contains e.g. the gravitational force. The assumption that $\mathbf{f}^{fic}$ is zero is known as ***inertial system***. This can be true only for certain observers, since it is not an objective term, see (III2.7).

Proof. The second equation of the claim was shown in the previous proof. Subtracting the equation for the kinetic energy 2.2 from the energy equation, we obtain, due to $e = \frac{\varrho}{2}|v|^2 + \varepsilon$,

$$\begin{aligned} &\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) \\ &= \widetilde{g} - \Big(v\bullet\mathbf{f} + \frac{1}{2}|v|^2\mathbf{r} + v\bullet(\mathrm{D}v\mathbf{J}) + \mathrm{D}v : \Pi\Big) = g - (\mathrm{D}v)^{\mathrm{S}} : \Pi\,, \end{aligned}$$

where we used (III2.7) for $\widetilde{g}$. The equations (III2.7) contain also the transformation rule of $\mathbf{f}$ and g, that is, $\mathbf{f}$ is a classical force and g an objective scalar. ☐

We mention that the first and third equation we used in 1.3 for the special case $\mathbf{J} = 0$, $\mathbf{r} = 0$, $\Pi = p\mathrm{Id}$, $g = 0$. The energy equation for the internal energy ε in (III2.8), also called the "thermal energy equation", is often used, this is why we have derived this version of the system. We require now for the original system (III2.5), what is equivalent to (III2.8), that the entropy principle is satisfied. The following is true

2.4 Theorem. For solutions of (III2.5) the entropy principle is satisfied with

$$\eta = \widehat{\eta}(\varrho, \varepsilon)\,, \quad \psi = \eta v + \eta'_{\varrho}\mathbf{J} + \eta'_{\varepsilon}q\,,$$

if in the differential equations (let $\eta'_{\varepsilon} > 0$)

$$\Pi = p\mathrm{Id} - S\,, \quad \eta = \varrho\eta'_{\varrho} - (\varepsilon + p)\eta'_{\varepsilon}\,,$$

[5] The inner product for matrices is $M : N := \sum_{ij} M_{ij}N_{ij} = M\bullet N$ and the inner product of vectors $v\bullet w := \sum_i v_i w_i$.

and if the following residual inequality

$$\sigma = \eta'_\varepsilon (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} S + \eta'_\varrho \mathbf{r} + \nabla\eta'_\varrho \bullet \mathbf{J} + \eta'_\varepsilon g + \nabla\eta'_\varepsilon \bullet q \geq 0 \qquad \text{(III2.9)}$$

holds. *Remark:* By Gibbs relation p is a function of (ϱ, ε).

Here the tensor S is defined as $S := p\mathrm{Id} - \Pi$, please, do not mix it with the tensor S in elasticity theory where in II.5.3 the connection with Π is written down.

Proof. The differential equations (III2.5) are equivalent to the system in 2.3, which is written as (define $\mathring{h} = \partial_t h + v \bullet \nabla h = \partial_{(1,v)} h$ for each function h)

$$\begin{aligned} &\mathring{\varrho} + \varrho\,\mathrm{div}v + \mathrm{div}\mathbf{J} = \mathbf{r}\,, \\ &\varrho\,\mathring{v} + \mathrm{div}\Pi = \mathbf{f}\,, \\ &\mathring{\varepsilon} + \varepsilon\,\mathrm{div}v + \mathrm{div}q = g - (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} \Pi\,. \end{aligned} \qquad \text{(III2.10)}$$

Now because $\eta = \widehat{\eta}(\varrho, \varepsilon)$

$$\mathring{\eta} = \eta'_\varrho \mathring{\varrho} + \eta'_\varepsilon \mathring{\varepsilon}$$

and hence

$$\begin{aligned} 0 \leq \sigma &= \partial_t \eta + \mathrm{div}\psi = \partial_t \eta + v \bullet \nabla\eta + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\ &= \mathring{\eta} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\ &= \eta'_\varrho \mathring{\varrho} + \eta'_\varepsilon \mathring{\varepsilon} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\ &= \eta'_\varrho(-\varrho\,\mathrm{div}v - \mathrm{div}\mathbf{J} + \mathbf{r}) + \eta'_\varepsilon(-\varepsilon\,\mathrm{div}v - \mathrm{div}q - (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} \Pi + g) \\ &\quad + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\ &= (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} \big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon \Pi\big) \\ &\quad + \eta'_\varrho(-\mathrm{div}\mathbf{J} + \mathbf{r}) + \eta'_\varepsilon(-\mathrm{div}q + g) + \mathrm{div}(\psi - \eta v) \\ &= (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} \big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon \Pi\big) \\ &\quad + \eta'_\varrho \mathbf{r} + \nabla\eta'_\varrho \bullet \mathbf{J} + \eta'_\varepsilon g + \nabla\eta'_\varepsilon \bullet q + \mathrm{div}(\psi - \eta v - \eta'_\varrho \mathbf{J} - \eta'_\varepsilon q)\,. \end{aligned}$$

The choice of ψ makes the last term to 0 and the Gibbs relation the first one. From this the statement follows. ☐

The residual inequality in (III2.9) contains five terms. In the easiest case all single terms are assumed to be greater or equal 0, e.g.

$$\sigma = \underbrace{\eta'_\varepsilon (\mathrm{D}v)^{\mathrm{S}} \boldsymbol{:} S}_{\geq 0} + \underbrace{\eta'_\varrho \mathbf{r}}_{=0} + \underbrace{\nabla\eta'_\varrho \bullet \mathbf{J}}_{\geq 0} + \underbrace{\eta'_\varepsilon g}_{=0} + \underbrace{\nabla\eta'_\varepsilon \bullet q}_{\geq 0} \geq 0\,.$$

For a standard equation of a gas or of a fluid there is no mass source or sink, that is $\mathbf{r} = 0$. Moreover we have the general inequality $\frac{1}{\theta} = \eta'_\varepsilon > 0$

and $\eta = \widehat{\eta}(\varrho, \varepsilon)$ is a concave function in (ϱ, ε), which is quite standard (see IV.2.5 and Müller & Ruggeri [57, Chap.6 1.2 Universal Principles of the Constitutive Theory]). Then the system (III2.10) with $\Pi = p\mathrm{Id} - S$ and $g = 0$ can be written as

$$\begin{aligned}
&\overset{\circ}{\varrho} + \mathrm{div}\mathbf{J} = -\varrho\,\mathrm{div}v \quad \text{with} \quad \nabla\eta'_{\varrho}\bullet\mathbf{J} \geq 0\,,\\
&\varrho\,\overset{\circ}{v} + \mathrm{div}(p\mathrm{Id} - S) = \mathbf{f} \quad \text{with} \quad (\mathrm{D}v)^{\mathrm{S}}\,\mathbf{:}S \geq 0\,,\\
&\overset{\circ}{\varepsilon} + \mathrm{div}q = (\mathrm{D}v)^{\mathrm{S}}\,\mathbf{:}S - (\varepsilon + p)\,\mathrm{div}v \quad \text{with} \quad \nabla\eta'_{\varepsilon}\bullet q \geq 0\,.
\end{aligned}$$

This shows that in principle the system is a solvable system of differential equations.

In chapter IV we will get to know different versions of constitutive equations for the free energy and entropy. In all cases we will use the entropy inequality, how it is developed in this chapter. In many cases we have to deal with

$$\sigma = \eta'_{\varepsilon}\,(\mathrm{D}v)^{\mathrm{S}}\,\mathbf{:}S + \nabla\eta'_{\varepsilon}\bullet q \geq 0\,. \tag{III2.11}$$

Now a statement about the constitutive equations of terms satisfying the residual inequality in (III2.11). Here we now take (ϱ, v, θ) as independent variables.

2.5 Proposition. Consider solutions of (III2.5) assuming the standard case $\mathbf{r} = 0$, $\mathbf{J} = 0$, and $g = 0$. Suppose that Π and q are given by constitutive functions depending on $(\varrho, \theta, \nabla\varrho, \nabla\theta, (\mathrm{D}v)^{\mathrm{S}})$ and that Π is symmetric.
Then it holds: If $n \geq 3$ and Π and q are (affine) linear in the variables representing derivatives, then objectivity and the entropy principle imply that they are of the form

$$\begin{aligned}
&\Pi = p\mathrm{Id} - S, \quad p = \widehat{p}(\varrho, \theta),\\
&S = 2\widehat{a}(\varrho, \theta)\,(\mathrm{D}v)^{\mathrm{S}} + \widehat{b}(\varrho, \theta)\,\mathrm{div}(v)\mathrm{Id},\\
&q = -\widehat{c}(\varrho, \theta)\nabla\theta
\end{aligned}$$

with objective scalars p, a, b, and c, satisfying

$$a \geq 0\,, \quad b + \frac{2a}{n} \geq 0\,, \quad c \geq 0\,.$$

This shows that the stress tensor S and the heat flux q have no cross terms, if both satisfy the linearity assumption for derivatives. This statement is intended to complete the statement in II.4.13 (see also [19, Proposition 11.5]).

Proof (Objectivity of q). Besides the constitutive equation for q in the formulation of the statement let us assume sthe more general constititve relation

$$q = \widehat{q}(\varrho, v, \theta, \nabla\varrho, \mathrm{D}v, \nabla\theta)\,.$$

Then, since the function $\widehat{q}$ is objective, it follows for another observeer

$$q^* = \widehat{q}(\varrho^*, v^*, \theta^*, \nabla\varrho^*, \mathrm{D}v^*, \nabla\theta^*)\,.$$

Since $q\circ Y = Qq^*$, we obtain, applying known transformation rules (the rules for ϱ and v and their space derivatives in (II4.5) and (II4.13)),

$$\begin{aligned}&\widehat{q}(\varrho^*, \dot{X} + Qv^*, \theta^*, Q\nabla\varrho^*, \dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^* Q^{\mathrm{T}}, Q\nabla\theta^*)\\ &= Q\widehat{q}(\varrho^*, v^*, \theta^*, \nabla\varrho^*, \mathrm{D}v^*, \nabla\theta^*)\,.\end{aligned}$$

For a given (t^*, x^*) we can choose an observer transformation (the following is analogue to the argumentation in the proof of II.4.11), such that at this given point $Q(t^*) = \mathrm{Id}$, and such that $\dot{X}(t^*)$ is a given vector and $\dot{Q}(t^*)Q^{\mathrm{T}}(t^*)$ a given antisymmetric matrix. This implies that $\widehat{q}$ has to be independent of the v-variables and the antisymmetric part of the $\mathrm{D}v$-variable. Thus with a new constitutive function

$$q = \widehat{q}(\varrho, \theta, \nabla\varrho, (\mathrm{D}v)^{\mathrm{S}}, \nabla\theta)\,,$$

and since $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor (see the transformation rule in (II4.14)), the above identity now becomes

$$\begin{aligned}&\widehat{q}(\varrho^*, \theta^*, Q\nabla\varrho^*, Q\,(\mathrm{D}v^*)^{\mathrm{S}} Q^{\mathrm{T}}, Q\nabla\theta^*)\\ &= Q\widehat{q}(\varrho^*, \theta^*, \nabla\varrho^*, (\mathrm{D}v^*)^{\mathrm{S}}, \nabla\theta^*)\,.\end{aligned}$$

For zero value of the derivatives, that is for vanishing value of $\nabla\varrho^*(t^*, x^*)$, $(\mathrm{D}v^*)^{\mathrm{S}}(t^*, x^*)$, $\nabla\theta^*(t^*, x^*)$ (for example choose a constant solution) we obtain

$$\widehat{q}(\varrho^*, \theta^*, 0, 0, 0) = Q\widehat{q}(\varrho^*, \theta^*, 0, 0, 0)\,.$$

Since $Q(t^*)$ can be any orthogonal matrix independent of the values of (ϱ^*, θ^*), this implies

$$\widehat{q}(\varrho^*, \theta^*, 0, 0, 0) = 0\,. \tag{III2.12}$$

Using (III2.12) and since $\widehat{q}$ is (affine) linear in the variables $\partial_j\varrho$, $\partial_j\theta$, and $\partial_j v_k + \partial_k v_j$, we have a representation

$$\begin{aligned}q_i = \;&\sum_{j=1}^{n} \widehat{a}_{ij}(\varrho, \theta)\partial_j\theta + \sum_{j=1}^{n} \widehat{b}_{ij}(\varrho, \theta)\partial_j\varrho\\ &+ \sum_{k,l=1}^{n} \widehat{c}_{ikl}(\varrho, \theta)(\partial_k v_l + \partial_l v_k)\end{aligned}$$

with coefficients a_{ij}, b_{ij}, and c_{ijk}, where we can assume that $c_{ikl} = c_{ilk}$ for all $i, k, l = 1, \ldots, n$. We know that ϱ, θ are objective scalars, that q, $\nabla\varrho$, $\nabla\theta$ are objective vectors, and that $(\mathrm{D}v)^{\mathrm{S}}$ is an objective tensor. This information and the identity $q\circ Y = Qq^*$, that is

$$q_i\circ Y = \sum_{\widetilde{i}=1}^{n} Q_{i\widetilde{i}}q^*_{\widetilde{i}}\,,$$

implies

$$\begin{aligned}&\sum_{j,\widetilde{j}=1}^{n} \widehat{a}_{ij}(\varrho^*, \theta^*)Q_{j\widetilde{j}}\partial_{\widetilde{j}}\theta^* + \sum_{j,\widetilde{j}=1}^{n} \widehat{b}_{ij}(\varrho^*, \theta^*)Q_{j\widetilde{j}}\partial_{\widetilde{j}}\varrho^*\\ &\quad+ \sum_{k,l,\widetilde{k},\widetilde{l}=1}^{n} \widehat{c}_{ikl}(\varrho^*, \theta^*)Q_{k\widetilde{k}}Q_{l\widetilde{l}}(\partial_{\widetilde{k}}v^*_{\widetilde{l}} + \partial_{\widetilde{l}}v^*_{\widetilde{k}})\\ &= \sum_{\widetilde{i},\widetilde{j}=1}^{n} Q_{i\widetilde{i}}\widehat{a}_{\widetilde{i}\widetilde{j}}(\varrho^*, \theta^*)\partial_{\widetilde{j}}\theta^* + \sum_{\widetilde{i},\widetilde{j}=1}^{n} Q_{i\widetilde{i}}\widehat{b}_{\widetilde{i}\widetilde{j}}(\varrho^*, \theta^*)\partial_{\widetilde{j}}\varrho^*\\ &\quad+ \sum_{\widetilde{i},\widetilde{k},\widetilde{l}=1}^{n} Q_{i\widetilde{i}}\widehat{c}_{\widetilde{i}\widetilde{k}\widetilde{l}}(\varrho^*, \theta^*)\big(\partial_{\widetilde{k}}v^*_{\widetilde{l}} + \partial_{\widetilde{l}}v^*_{\widetilde{k}}\big).\end{aligned}$$

Now fix (t^*, x^*). There is a process $(\varrho^*, v^*, \theta^*)$ with given values and space derivatives at (t^*, x^*). Thus fixing $\varrho^*(t^*, x^*)$ and $\theta^*(t^*, x^*)$, varying over all spatial derivatives at (t^*_0, x^*_0), we see that the following identities have to be satisfied at (t^*_0, x^*_0):

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n}\widehat{a}_{ij}(\varrho^*,\theta^*)Q_{j\widetilde{j}} = \sum_{\widetilde{i}=1}^{n}Q_{i\widetilde{i}}\widehat{a}_{\widetilde{i}\widetilde{j}}(\varrho^*,\theta^*) && \text{for all } i,\widetilde{j},\\
&\textstyle\sum_{j=1}^{n}\widehat{b}_{ij}(\varrho^*,\theta^*)Q_{j\widetilde{j}} = \sum_{\widetilde{i}=1}^{n}Q_{i\widetilde{i}}\widehat{b}_{\widetilde{i}\widetilde{j}}(\varrho^*,\theta^*) && \text{for all } i,\widetilde{j},\\
&\textstyle\sum_{k,l=1}^{n}\widehat{c}_{ikl}(\varrho^*,\theta^*)Q_{k\widetilde{k}}Q_{l\widetilde{l}} = \sum_{\widetilde{i}=1}^{n}Q_{i\widetilde{i}}\widehat{c}_{\widetilde{i}\widetilde{k}\widetilde{l}}(\varrho^*,\theta^*) && \text{for all } i,\widetilde{k},\widetilde{l}.
\end{aligned}$$

Note, that for the last identity we have used the symmetry of c_{ikl} in k and l. The first identity is equivalent to

$$\widehat{a}_{ij}(\varrho^*,\theta^*) = \textstyle\sum_{\widetilde{i},\widetilde{j}=1}^{n}Q_{i\widetilde{i}}Q_{j\widetilde{j}}\widehat{a}_{\widetilde{i}\widetilde{j}}(\varrho^*,\theta^*) \quad \text{for all } i,j$$

and all orthogonal matrices Q with positive determinant. This says, that (for fixed values of $\varrho^*(t^*,x^*)$ and $\theta^*(t^*,x^*)$) the tensor $(\widehat{a}_{ij}(\varrho^*,\theta^*))_{i,j=1,\dots,n}$ behaves like a constant objective tensor, which implies that it is a multiple of the identity (see II.4.14(4)). The same follows for the b-term. The third identity is equivalent to

$$\widehat{c}_{ikl}(\varrho^*,\theta^*) = \textstyle\sum_{\widetilde{i},\widetilde{k},\widetilde{l}=1}^{n}Q_{i\widetilde{i}}Q_{k\widetilde{k}}Q_{l\widetilde{l}}\widehat{c}_{\widetilde{i}\widetilde{k}\widetilde{l}}(\varrho^*,\theta^*) \quad \text{for all } i,k,l,$$

and all orthogonal matrices Q with positive determinant. This says, that (for fixed values of $\varrho^*(t^*,x^*)$ and $\theta^*(t^*,x^*)$) the 3-tensor $(\widehat{c}_{ikl}(\varrho^*,\theta^*))_{i,k,l=1,\dots,n}$ behaves like a constant objective 3-tensor, which is symmetric in the last two indices. This implies that it has to vanish (see II.4.14(5)). Thus it follows that

$$q_i = \alpha\,\partial_i\theta + \beta\,\partial_i\varrho \tag{III2.13}$$

with two objektive scalars α and β depending on (ϱ,θ). □

Proof (Objectivity of Π*).* For $\widehat{\Pi}$ one obtains independence of v and the antisymmetric part of $\mathrm{D}v$ in the same manner as for q. Then

$$\widehat{\Pi}(\varrho^*,\theta^*,0,0,0) = Q\widehat{\Pi}(\varrho^*,\theta^*,0,0,0)\,Q^{\mathrm{T}}$$

for all orthogonal matrices Q. This implies that $\widehat{\Pi}(\varrho^*,\theta^*,0,0,0)$ (for fixed values of ϱ^* and θ^*) is a constant objective tensor, and therefore, for $n\geq 3$, is a multiple of the identity, that is,

$$\widehat{\Pi}(\varrho^*,\theta^*,0,0,0) = \widehat{p}(\varrho^*,\theta^*)\mathrm{Id}\,.$$

Then $\Pi = p\mathrm{Id} - S$ and S has a representation

$$\begin{aligned}
S_{ij} \;&= \sum_{k=1}^{n}\widehat{a}_{ijk}(\varrho,\theta)\partial_k\theta + \sum_{k=1}^{n}\widehat{b}_{ijk}(\varrho,\theta)\partial_k\varrho\\
&\quad + \sum_{k,l=1}^{n}\widehat{c}_{ijkl}(\varrho,\theta)(\partial_k v_l + \partial_l v_k),
\end{aligned}$$

where we can assume that $c_{ijkl} = c_{ijlk}$ for all $i,j,k,l = 1,\dots,n$. Now Π and then also S is an objective tensor (see II.4.14), that is

$$S_{ij}\circ Y = \sum_{\widetilde{i},\widetilde{j}=1}^{n} Q_{i\widetilde{i}}Q_{j\widetilde{j}}S^*_{\widetilde{i},\widetilde{j}}.$$

This leads, with the above notation, to the identities

$$\begin{aligned}
&\textstyle\sum_{k=1}^{n}a_{ijk}Q_{k\widetilde{k}} = \sum_{\widetilde{i},\widetilde{j}=1}^{n}Q_{i\widetilde{i}}Q_{j\widetilde{j}}a_{\widetilde{i}\widetilde{j}\widetilde{k}} && \text{for all } i,j,\widetilde{k},\\
&\textstyle\sum_{k=1}^{n}b_{ijk}Q_{k\widetilde{k}} = \sum_{\widetilde{i},\widetilde{j}=1}^{n}Q_{i\widetilde{i}}Q_{j\widetilde{j}}b_{\widetilde{i}\widetilde{j}\widetilde{k}} && \text{for all } i,j,\widetilde{k},\\
&\textstyle\sum_{k,l=1}^{n}c_{ijkl}Q_{k\widetilde{k}}Q_{l\widetilde{l}} = \sum_{\widetilde{i},\widetilde{j}=1}^{n}Q_{i\widetilde{i}}Q_{j\widetilde{j}}c_{\widetilde{i}\widetilde{j}\widetilde{k}\widetilde{l}} && \text{for all } i,j,\widetilde{k},\widetilde{l}.
\end{aligned}$$

Again we rewrite this so that we have Q-terms only on the right-hand side. This gives, that $(a_{ijk})_{i,j,k=1,\dots,n}$ behaves like a constant objective 3-tensor. This implies that it is antisymmetric in each pair of indices (for $n=3$, for $n\geq 4$ it follows that $a_{ijk}=0$), hence $a_{ijk}+a_{jik}=0$. Therefore this term gives no contribution to the symmetric part of S. The same follows for the b-term. The third identity gives that $(c_{ijkl})_{i,j,k,l=1,\dots,n}$ is a constant objective 4-tensor, which is symmetric in the last two indices. Since S is symmetric, which is assumed, this implies that the symmetric part with respect to the first two indices is of the form

$$c_{ijkl} = a(\delta_{k,i}\delta_{l,j} + \delta_{l,i}\delta_{k,j}) + b\delta_{k,l}\delta_{i,j}$$

with two scalars a, b (see II.4.14(6) for this statement). Thus it follows that

$$S_{ij} = a(\partial_i v_j + \partial_j v_i) + b \operatorname{div} v \cdot \delta_{i,j} \tag{III2.14}$$

with two objektive scalars a and b depending on (ϱ, θ). □

Proof (Entropy inequality). If $\mathbf{r}=0$, $\mathbf{J}=0$, and $g=0$ then we have for the entropy production

$$\begin{aligned} 0 \leq \theta\sigma &= (\mathrm{D}v)^{\mathrm{S}} : S + \theta\nabla\Big(\frac{1}{\theta}\Big)\bullet q = (\mathrm{D}v)^{\mathrm{S}} : S - \frac{1}{\theta}\nabla\theta\bullet q \\ &= 2a|\,(\mathrm{D}v)^{\mathrm{S}}\,|^2 + b(\operatorname{div} v)^2 - \frac{\alpha}{\theta}|\nabla\theta|^2 - \frac{\beta}{\theta}\nabla\theta\bullet\nabla\varrho \end{aligned}$$

using the special representations (III2.13) and (III2.14), and exploiting the symmetry of S. We consider this as a quadratic equation in $((\mathrm{D}v)^{\mathrm{S}}, \nabla\theta, \nabla\varrho)$. For this one shows that there are local solutions where at a point (t_0, x_0) the gradients $(\mathrm{D}v)^{\mathrm{S}}(t_0,x_0)$, $\nabla\theta(t_0,x_0)$, and $\nabla\varrho(t_0,x_0)$ have arbitrary given values. Consequently the quadratic inequality is non-negative if and only if [6]

$$a \geq 0\,, \quad b + \frac{2}{n}a \geq 0\,, \quad \alpha \leq 0\,, \quad \beta = 0\,.$$

□

[6] For each $n\times n$-matrix $|M|^2 = |M - \frac{1}{n}(\operatorname{trace} M)\mathrm{Id}|^2 + \frac{1}{n}(\operatorname{trace} M)^2$.

3 Mixtures

Wir betrachten eine Mischung von verschiedenen Spezies, so dass, im Allgemeinen, die Anziehungskräfte der Moleküle derselben Spezies sehr verschieden sind von den Anziehungskräften der Moleküle verschiedener Spezies. Unter anderem hängt das davon ab, wie die Moleküle aussehen. Wir betrachten ein Kontinuumsmodel, d.h. wir sind hier an den räumlich weitreichenden Konsequenzen interessiert. Gegeben ist also ein System von Massen, wobei α der Index der Spezies mit der Massendichte $\varrho_\alpha \geq 0$ sei, für die wir die Massenbilanz

$$\partial_t \varrho_\alpha + \operatorname{div} \widetilde{J}_\alpha = \mathbf{r}_\alpha \text{ for all } \alpha \tag{III3.1}$$

fordern. Dies heißt, dass die einzelnen Massen miteinander mit der Rate $\mathbf{r}_\alpha$ reagieren, und dass sie sich räumlich mit dem Fluss $\widetilde{J}_\alpha$ fortbewegen. Bei jeder dieser Gleichungen handelt es sich wie in II.3.1 um eine skalare Gleichung, d.h. ϱ_α und $\mathbf{r}_\alpha$ sind objektive Skalare und es gilt die Transformationsregel

$$\widetilde{J}_\alpha \circ Y = \varrho_\alpha^* \dot{X} + Q \widetilde{J}_\alpha^* \,,$$

wobei Y wie immer die Beobachtertransformation sei. Dies ist erfüllt, wenn wir bezüglich $\widetilde{J}_\alpha$ wie in II.3.4 die Aussage machen, dass

$$\widetilde{J}_\alpha = \varrho_\alpha v_\alpha + \mathbf{J}_\alpha \,,$$

wobei v_α eine Geschwindigkeit und $\mathbf{J}_\alpha$ ein objektiver Vektor ist.

Da sich die Komponenten der Mischung gegenseitig beeinflussen, also gegenseitig Kräfte ausüben, erzwingt dies weitere Aussagen über die Dynamik. So wird die gesamte Flüssigkeit mit einer Geschwindigkeit v transportiert, und die Frage ist, wie ist v zu verstehen? Im Allgemeinen haben wir je nach Materialkomposition die folgenden Optionen.

3.1 Mean velocity. With objective scalars a_α one defines a velocity

$$v = \sum_\alpha a_\alpha v_\alpha \quad \text{if} \quad \sum_\alpha a_\alpha = 1 \,. \tag{III3.2}$$

(1) ***Barycentric velocity.*** Define the mass fractions

$$c_\alpha := \frac{\varrho_\alpha}{\varrho} \,, \text{ where } \varrho := \sum_\beta \varrho_\beta > 0 \text{ the total mass.}$$

Then the corresponding mean of velocity is

$$v = \sum_\alpha c_\alpha v_\alpha \,.$$

(2) ***Mean molar velocity.*** Define the concentrations, or molar fractions,

$$n_\alpha := \frac{N_\alpha}{N} \text{ with } N := \sum_\beta N_\beta > 0 \,, \;\; \varrho_\alpha = M_\alpha N_\alpha \,,$$

where $M_\alpha = \text{const}$ is the molar mass and N_α the molar density. Then the velocity mean is

$$v = \sum_\alpha n_\alpha v_\alpha \,.$$

Remark: These variables are used for solids in Fig. **IV**18.
Explanation: In general one denotes by ***Mol*** the amount of substance in chemical reactions, i.e. the amount of individual particles (atoms, molecules, ions, electrons, photons), for information see [Wikipedia: Mol].
Comparison with IV.2.4*:* It is $N_\alpha = \frac{\mathrm{n}_\alpha}{V}$ the molar density where (realize the different notation) n_α is the amount of substance.

(3) ***Dominant species.*** Let β be the dominant component and

$$v = \sum_\alpha \delta_{\alpha,\beta} v_\alpha = v_\beta \,.$$

Remark: This one obtains in the limit $\dfrac{\varrho_\alpha}{\varrho_\beta} \to 0$ für $\alpha : \alpha \neq \beta$.

We shall use various reference velocities, which can all be written as weighted averages of the component velocities $\boldsymbol{v}_k$ in the following way

$$\boldsymbol{v}^a = \sum_{i=1}^n a_i \boldsymbol{v}_i \,, \quad \left(\sum_{i=1}^n a_i = 1 \right), \tag{24}$$

where $a_1, a_2, \ldots, a_n$ are the (normalized) weights. In the following table we list four amongst the most useful choices of weights and corresponding reference velocities and diffusion flows

weights a_i	reference velocity $\boldsymbol{v}^a = \Sigma_i a_i \boldsymbol{v}_i$	diffusion flow $\boldsymbol{J}^a = \rho_i (\boldsymbol{v}_i - \boldsymbol{v}^a)$
mass fractions c_i	barycentric velocity $\boldsymbol{v} = \Sigma_i c_i \boldsymbol{v}_i$	$\boldsymbol{J}_i = \rho_i (\boldsymbol{v}_i - \boldsymbol{v})$
molar fractions n_i	mean molar velocity $\boldsymbol{v}^m = \Sigma_i n_i \boldsymbol{v}_i$	$\boldsymbol{J}_i^m = \rho_i (\boldsymbol{v}_i - \boldsymbol{v}^m)$
$\rho_i v_i$	mean volume velocity $\boldsymbol{v}^0 = \Sigma_i \rho_i v_i \boldsymbol{v}_i$	$\boldsymbol{J}_i^0 = \rho_i (\boldsymbol{v}_i - \boldsymbol{v}^0)$
δ_{in}	n^{th} component velocity $\boldsymbol{v}_n = \Sigma_i \delta_{in} \boldsymbol{v}_i$	$\boldsymbol{J}_i^r = \rho_i (\boldsymbol{v}_i - \boldsymbol{v}_n)$

Fig. 2: From DeGroot & Mazur [6, Chap. XI §2]

In a single fluid often dependencies on the velocity drop out by objectivity, see e.g. II.4.11. For mixtures the situation is quite different, since differences $v_{\alpha_1} - v_{\alpha_2}$ of two velocities v_{α_1} and v_{α_2} are objective vectors, as for example the relative velocities $u_\alpha = v_\alpha - v$ in (III3.4) below.

Proof. The property (III3.2) says

$$v\circ Y = \sum_\alpha (a_\alpha\circ Y) v_\alpha\circ Y\,.$$

Since a_α are objective scalars

$$a_\alpha\circ Y = a_\alpha^* \tag{III3.3}$$

we obtain since v_α are velocities

$$v\circ Y = \sum_\alpha a_\alpha^*(\dot X + Q v_\alpha^*) = \sum_\alpha a_\alpha^* \dot X + Q\sum_\alpha a_\alpha^* v_\alpha^* = \dot X + Q v^*$$

since the sum of the a_α^* equals 1 (this is true in all examples). Hence v is indeed a velocity. □

For a given velocity mean v the ***relative velocity of species*** $\boldsymbol{\alpha}$ is defined as

$$\boxed{u_\alpha := v_\alpha - v\,.} \tag{III3.4}$$

Because of

$$u_\alpha\circ Y = v_\alpha\circ Y - v\circ Y = (\dot X + Q v_\alpha^*) - (\dot X + Q v^*) = Q(v_\alpha^* - v^*) = Q u_\alpha^*$$

the quantity u_α has the transformation rule of an objective vector, therefore it may occur in constitutive functions.

References: As introduction we recommend I.Müller [11, 7 Mischungen und Mischphasen] and Müller & Müller [13, 8 Mixtures, solutions, and alloys]. The following cassification of mixtures you find in Hutter & Jöhnk [47, 7 Theory of Mixtures].

It depends a concrete material which velocity mean one has to consider. If you consider a mixture of Class I the velocity mean is not prescibed. If you consider a mixture of Class II you have to choose a barycentrical mean, then there are the following fundamental identities

$$\boxed{\sum_\alpha \varrho_\alpha u_\alpha = 0 \quad\text{and}\quad \sum_\alpha \varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} = \varrho v v^{\mathrm{T}} + \sum_\alpha \varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}}\,.} \tag{III3.5}$$

The left equation is treated in (III3.8) and the right is subject of IV.9.1. In the following we denote the different classes of mixtures by I, II, III.

Mixture Case I

The equations in this Case I are the individual mass conservations (III3.1) with $u_\alpha = 0$ but arbitrary $\mathbf{J}_\alpha$, the momentum balance for the common velocity v, and at last equation the energy balance for e, that is,

Mixtures Case I:

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \ldots, m\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\, \mathbf{J}^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,,$$
$$\partial_t e + \operatorname{div}(e v + \frac{1}{2}|v|^2 \mathbf{J} + \Pi^{\mathrm{T}}\, v + q) = \widetilde{g}\,, \tag{III3.6}$$

$\varrho := \sum_\alpha \varrho_\alpha$ total mass, $\quad v$ velocity,

$\mathbf{J} := \sum_\alpha \mathbf{J}_\alpha\,, \quad \mathbf{r} := \sum_\alpha \mathbf{r}_\alpha\,,$

$e = \varepsilon + \frac{\varrho}{2}|v|^2$ energy, $\quad \widetilde{\mathbf{f}}, \widetilde{g}$ see (II3.32).

The individual mass equations are scalar laws as in II.3.1 and therefore also the equation for the total mass ϱ

$$\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,.$$

Hence this equation together with the momentum and energy equations in (III3.6) are supposed to be a mass-momentum-energy system as in II.3.12. The additional entropy equality is a scalar one and does not belong to the equations of the system. However, it gives restrictions on the quantities of these equations, for example constitutive relations. We will consider mixtures of Case I in section IV.6 where we apply this to the growth of bones (IV6.1), and we will use it in section IV.11 for modelling of the components of reaction-diffusion systems (IV11.2). There we deal with a solid body (IV11.8) which we take as a dominant component, and further we consider chemical reactions in (IV11.21).

Mixture Case II

Besides the individual mass equations (III3.1) with velocities v_α we assume individual momentum balances

$$\partial_t(\varrho_\alpha v_\alpha) + \operatorname{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \widetilde{\mathbf{f}}_\alpha \,. \tag{III3.7}$$

The joint velocity v is defined as barycentric mean so that with u_α defined as in (III3.4)

$$\boxed{\varrho v = \sum_\alpha \varrho_\alpha v_\alpha \,, \quad \text{equivalent to} \quad \sum_\alpha \varrho_\alpha u_\alpha = 0 \,,} \tag{III3.8}$$

which follows from the definitions of ϱ and v

$$\sum_\alpha \varrho_\alpha u_\alpha = \sum_\alpha \varrho_\alpha (v_\alpha - v) = \sum_\alpha \varrho_\alpha v_\alpha - \big(\sum_\alpha \varrho_\alpha\big) v = 0 \,.$$

In addition we postulate a single energy equation so that altogether

Mixtures Case II :

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v_\alpha + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \,,$$

$$\partial_t(\varrho_\alpha v_\alpha) + \operatorname{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \widetilde{\mathbf{f}}_\alpha \quad \text{für } \alpha = 1, \dots, m,$$

$$\partial_t e + \operatorname{div} \widetilde{q} = \widetilde{g} \,, \tag{III3.9}$$

$\varrho := \sum_\alpha \varrho_\alpha$ total mass density,

v velocity as barycentric mean,

$e = \varepsilon_{mix} + \frac{\varrho}{2}|v|^2$ energy, $\widetilde{q}$, $\widetilde{g}$ see examples

$\widetilde{\mathbf{f}}_\alpha$ as in (II3.32) for each α.

The mass and momentum equation are for each α by definition a mass-momentum system as defined in II.3.6. In IV.9.1 we will perform the mass and the momentum equation for the total mass ϱ and the velocity mean v. This together with the energy conservation will be a mass-momentum-energy system in II.3.12. The entropy equality is an additional scalar equation and it provides with additional constitutive equations and inequalities. As example we consider the fractionation in section IV.9 and the arrangement of space objects in layers, see section IV.16.

Mixtures of Case III

In this case one has also energy equations for the single species. But the entropy equality still is a scalar equation and serves all species.

The above classification of mixtures one finds in Hutter & Jöhnk [47, 7 Theory of Mixtures]. In [47, 7.1, p.255] it says: "The three just described classes of mixture theories may also occur in a mixed form; all the more, such mixed forms are often applied in practice. For example, the dispersion of a pollutant in the groundwater is formulated by a model which contains elements of classes I and II. The pollutant and the water form together a mixture of class I; the polluted water together with the soil a mixture of class II. All combinations are thinkable, and it lies in the talent and depth of physical understanding of the scientist who develops a model to make the choice appropriate to a given situation."

Hier we consider only mixtures of Class I and Class II.

4 Lagrange multipliers

Als Ergänzung zum Entropieprinzip sei noch folgende allgemeine Prozedur genannt, die sich auf das Masse-Impuls-Energie System anwenden lässt. Wir betrachten eine Menge $\mathcal{P}'$, die aus Größen (u^k, q^k, g^k) für $k = 1, \ldots, N$ sowie (η, ψ, σ) besteht zwischen denen konstitutive Beziehungen gelten. Es sei eine Klasse von physikalischen Prozessen $\mathcal{P}$ definiert durch Elemente in $\mathcal{P}'$, die den Erhaltungsgleichungen

$$\partial_t u^k + \mathrm{div} q^k = g^k \text{ for } k = 1, \ldots, N \tag{III4.1}$$

genügen und ggf. noch weitere Bedingungen erfüllen. Liu & Müller machen für die größere Klasse $\mathcal{P}'$ die folgende Annahme.

4.1 Lagrange multipliers. Folgendes wird angenommen: Es gibt Multiplikatoren Λ_k, $k = 1, \ldots, N$, so dass für die Größen in $\mathcal{P}'$ gilt

$$\partial_t \eta + \mathrm{div}\psi - \sigma = \sum_{k=1}^{N} \Lambda_k \cdot (\partial_t u^k + \mathrm{div} q^k - g^k)\,. \tag{III4.2}$$

Bemerkung: Λ_k sind in der Regel Funktionen der Größen in $\mathcal{P}'$.

4.2 Theorem. Es erfülle $\mathcal{P}'$ die Gleichung in 4.1. Dann ist für physikalische Prozesse in $\mathcal{P}$ die Entropieungleichung für (η, ψ) erfüllt, falls

$$\sigma \geq 0\,.$$

Remark: Also gilt das Entropieprinzip für $\mathcal{P}$, falls in $\mathcal{P}'$ gilt

$$\sigma := \partial_t \eta + \mathrm{div}\psi - \sum_{k=1}^{N} \Lambda_k \cdot (\partial_t u^k + \mathrm{div} q^k - g^k) \geq 0\,.$$

References: Siehe die Arbeiten I. Müller [117], I-Shih Liu [112], und auch im Buch von I. Müller [87, 5.4.3 Rational Thermodynamics with Lagrange multipliers] sowie im Buch von I-Shih Liu [86, 7.3 Method of Lagrange Multipliers]. Ebenfalls sei Wilmanski [15, Theorem in 6.2. Entropy Inequality]) erwähnt. Ich habe in dem Paper [18] von der Lagrange Methode Gebrauch gemacht.

Ingo Müller sagt in seinem Buch [87, 5.4.3.3 Lagrange multipliers]:

> "Thus we may think of the field equations as constraints:
> The fields that satisfy the entropy inequality are constrained by
> the requirement that they must be solutions of the field equations."

Die klassische Bedeutung von Lagrange-Multiplikatoren ergibt sich aus dem Beitrag [Wikipedia: Lagrange-Multiplikator], worin unter anderem gesagt wird:

> "Die Bedeutung der Lagrange-Multiplikatoren in der Physik
> wird bei der Anwendung in der klassischen Mechanik sichtbar.
> Hierfür wurden sie von Lagrange eingeführt. ...
> Eine physikalische Zwangsbedingung, die die Bewegung
> einschränkt, erscheint als Nebenbedingung des Extremums. ... "

Allerdings lässt sich diese Prozedur nicht anwenden, wenn man distributionelle Erhaltungsgleichungen hat. Aus diesem Grund verzichten wir hier auf diese sehr effiziente Methode. Wir werden sie aber in Situationen, wo sie gebraucht wird, explizit benutzen, so zum Beispiel

- in der Astrophysik, wo die Energiegleichung durch die Entropiegleichung ersetzt wird, siehe IV.16.12.
- bei Phasenübergängen, wobei hier nur die Arbeit von Niezgodka & Sprekels [58, 3.2 Thermomechanical model of dynamical phase transitions] genannt sei.
- beim Beweis des Entropieprinzips für höhere Momente, siehe REF-multiply...

References: Als mathematische Literatur verweisen wir auf das Buch [40] von Feireisl & Novotny. Siehe z.B. [40, 4.1], oder [40, 3.5.5], wo das Masse-Impuls-Entropie System gelöst wird, wobei ein **J**-Term in der Massengleichung proportional zu $-\nabla\varrho$ verwendet wird. In [40, 3.6.6] wird dasselbe System ohne den **J**-Term genannt.

Um zu zeigen, wie effektiv diese Methode von Liu & Müller funktioniert, geben wir hier die Version für das Masse-Impuls-Energie System für Flüssigkeiten (III2.5) mit $\mathbf{J} = 0$ an:

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v) = \mathbf{r}\,,\\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,,\\
&\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}}\, v + q) = \widetilde{g}\,.
\end{aligned} \tag{III4.3}$$

Es sei nun $\mathcal{P}'$ die Menge der Funktionen, für die $e = \varepsilon + \frac{\varrho}{2}|v|^2$ und $\eta = \widehat{\eta}(\varrho, \varepsilon)$ ist. Dann folgt

$$\begin{aligned}
\mathring{\eta} &= \eta'_{\varrho}\mathring{\varrho} + \eta'_{\varepsilon}\mathring{\varepsilon}\\
&= \big(\eta'_{\varrho} + \eta'_{\varepsilon}\frac{|v|^2}{2}\big)\mathring{\varrho} - \eta'_{\varepsilon} v\bullet\mathring{(\varrho v)} + \eta'_{\varepsilon}\mathring{e}\,,
\end{aligned}$$

und daher vermuten wir (es ist $\eta'_{\varepsilon} = \frac{1}{\theta}$ nach 1.6)

$$\Lambda_{\varrho} = \eta'_{\varrho} + \eta'_{\varepsilon}\frac{|v|^2}{2}\,, \quad \Lambda_v = -\eta'_{\varepsilon} v\,, \quad \Lambda_e = \eta'_{\varepsilon}\,. \tag{III4.4}$$

Hierbei seien Λ_{ϱ}, $\Lambda_v := (\Lambda_{v_k})_{k=1,\dots,n}$ und Λ_e die Multiplikatoren. Es gilt mit diesen Multiplikatoren 4.1, was im folgenden gezeigt wird.

4.3 Theorem. Mit den Multiplikatoren in (III4.4) gilt für das System (III4.3) für alle Funktionen in $\mathcal{P}'$

$$\begin{aligned}&\partial_t\eta + \operatorname{div}(\eta v + \eta'_\varepsilon q) - \sigma\\ &= \Lambda_\varrho \cdot (\partial_t \varrho + \operatorname{div}(\varrho v) - \mathbf{r})\\ &+\Lambda_v \cdot (\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) - \widetilde{\mathbf{f}}))\\ &+\Lambda_e \cdot (\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}}\, v + q) - \widetilde{g})\end{aligned} \tag{III4.5}$$

wenn

$$\begin{aligned}&\sigma := \eta'_\varrho \mathbf{r} + \eta'_\varepsilon\big(\frac{|v|^2}{2}\mathbf{r} - v\bullet\widetilde{\mathbf{f}} + \widetilde{g}\big)\\ &+\mathrm{D}v\,\boldsymbol{:}\big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi\big) + \nabla\eta'_\varepsilon\bullet q\,.\end{aligned} \tag{III4.6}$$

Proof. Man mache unter Benutzung von (III4.4) ähnliche Manipulationen wie im Beweis von 2.4. □

Es sei bemerkt, dass das Entropieprinzip nur die Positivität von σ als Ganzes verlangt. Mit den Definitionen in (II3.32) ergibt sich $\frac{|v|^2}{2}\mathbf{r} - v\bullet\widetilde{\mathbf{f}} + \widetilde{g} = g$ und das Verschwinden des Dv-Terms ergibt die wohlbekannte Darstellung des Drucktensors

$$\Pi = p\mathrm{Id} - S\,, \quad 0 = \eta - \varrho\eta'_\varrho - (\varepsilon + p)\eta'_\varepsilon$$

mittels der Gibbs' Relation. Wir sehen also, dass das Entropieprinzip einerseits wohlbekannte konstitutive Gleichungen im System beinhaltet und andererseits in der Residualungleichung

$$\sigma = \eta'_\varrho \mathbf{r} + \eta'_\varepsilon g + \eta'_\varepsilon \mathrm{D}v\boldsymbol{:}S + \nabla\eta'_\varepsilon\bullet q \geq 0$$

mündet. Diese Ungleichung wird in der Regel dadurch verifiziert, dass alle Terme einzeln größer oder gleich 0 sind. So wird $\mathbf{r} = 0$ gesetzt, da es sich um die Gesamtmasse handelt, und die Energieerhaltung sagt $g = 0$. Der Stresstensor wird mit $\mathrm{D}v\boldsymbol{:}S \geq 0$ und der Wärmefluss mit $\nabla\eta'_\varepsilon\bullet q \geq 0$ gewählt. Es sei jedoch betont, dass nur die gesamte Entropieproduktion $\sigma \geq 0$ sein muss (siehe z.B. IV.6.3).

Entropy equation versus energy equation

Es sei noch folgende nichttriviale Anwendung genannt, die manchmal benutzt wird. Für Lösungen des Systems (III4.3) sei die Entropieungleichung $\sigma \geq 0$ erfüllt, wobei σ wie in (III4.6) definiert ist. Da der Multiplikator

$$\Lambda_e = \eta'_\varepsilon = \frac{1}{\theta} > 0$$

in (III4.4) überall nichtnull ist, folgt daraus, dass die Energieidentität eine Linearkombination der übrigen Differentialgleichungen von (III4.3) und der Entropiegleichung ist. Also ist (III4.3) äquivalent zum System

$$\begin{aligned}&\partial_t\varrho + \operatorname{div}(\varrho v) = \mathbf{r}\,,\\&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{r}v + \mathbf{f}\,,\\&\partial_t\eta + \operatorname{div}(\eta v + \eta_{'\varepsilon} q) = \sigma \quad \text{(gegeben in (III4.6)).}\end{aligned}\tag{III4.7}$$

Voraussetzung dafür ist, dass σ wie in (III4.6) in den Termen des Systems (III4.3) ausgedrückt werden kann. Also kann das Masse-Impuls-Energie System auch geschrieben werden als Masse-Impuls-Entropie System. Man findet die Schreibweise mit Entropieidentität gelegentlich in der Elastizitätstheorie und häufig in der Astrophysik, wobei wir als Beispiel IV.16.12 angeben.

5 Dissipation inequality

What is free energy? The internal free energy f is a variable which is calculated from the internal energy and the entropy by the formula

$$\boxed{f = \varepsilon - \theta\eta\,.} \tag{III5.1}$$

We have two interpretations of this formula, one is more mathematically and the other more physically:

- The variable $\frac{1}{\theta}$ is the dual variable to ε and $\varphi_f := \frac{1}{\theta}f$ is the dual function to the entropy η. That is $\eta + \varphi_f = \frac{1}{\theta}\varepsilon$ (see 7.3 and 1.5).
- The Gibbs relation says that the pressure on one hand is expressed by the inner free energy, $p = \varrho f'_{\varrho} - f$, and on the other hand by the entropy, $p = \theta(\eta - \varrho\eta'_{\varrho}) - \varepsilon$ (see 1.4(1)).

The total internal energy f^{tot} contains still the kinetic term like the total energy, that is, we have

$$\boxed{f^{tot} := f + \frac{\varrho}{2}|v|^2 = \varepsilon + \frac{\varrho}{2}|v|^2 - \theta\eta = e - \theta\eta\,.} \tag{III5.2}$$

We only consider the case of a smooth variable f in a mixture of materials, that means, here the case of distributions is not considered. Equation (III5.1) we have met in the introduction of temperature in 1.6.

Here at the end of the section we will present a principle in the isothermal case, which goes back to the dissipation inequality. We start from the general mass-momentum-energy system

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,,\\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,,\\
&\partial_t e + \operatorname{div}(ev + \tfrac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}}\,v + q) = \widetilde{g}\,,\\
&\widetilde{\mathbf{f}} = (\mathbf{r} + \mathbf{J}\bullet\nabla)v + \mathbf{f}\,,\\
&\widetilde{g} = \tfrac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + v\bullet \mathrm{D}v\mathbf{J} + (\mathrm{D}v)^{\mathrm{A}} : \Pi + g\,.
\end{aligned} \tag{III5.3}$$

The functions $\widetilde{\mathbf{f}}$ and $\widetilde{g}$ are as in (II3.32), i.e. $\mathbf{f}$ is a classical force and g an objective scalar. The system (III5.3) by 2.3 is equivalent to

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,,\\
&\varrho(\partial_t v + v\bullet\nabla v) + \operatorname{div}\Pi = \mathbf{f}\,,\\
&\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) = -(\mathrm{D}v)^{\mathrm{S}} : \Pi + g\,.
\end{aligned} \tag{III5.4}$$

We show

5.1 Dissipation inequality. Let the entropy inequality in 1.1 be satisfied for the entropy η and the entropy flux

$$\psi = \underbrace{\eta v + \frac{q}{\theta}}_{\text{Clausius-Duhem}} + \widetilde{\psi}\,.$$

Then the free energy f in (III5.1) satisfies

$$\boxed{\partial_t f + \operatorname{div}(fv - \theta\widetilde{\psi}) + (\mathrm{D}v)^{\mathrm{S}} \mathbin{\bullet\!\!:} \Pi - g + R_\theta = -\theta\sigma \le 0} \tag{III5.5}$$

where R_θ is defined by

$$R_\theta := \eta(\partial_t + v\bullet\nabla)\theta + \nabla\theta\bullet\Big(\frac{1}{\theta}q + \widetilde{\psi}\Big)\,. \tag{III5.6}$$

Proof. The two relevant equations are the energy equation (III5.4) and the entropy identity in 1.1

$$\begin{aligned}&\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) + (\mathrm{D}v)^{\mathrm{S}} \mathbin{\bullet\!\!:} \Pi - g = 0\,,\\ &\partial_t\eta + \operatorname{div}\psi = \sigma \ge 0\,,\end{aligned}$$

which is with (III2.10) and the decomposition $\psi - \eta v = \frac{1}{\theta}q + \widetilde{\psi}$

$$\begin{aligned}&\overset{\circ}{\varepsilon} + \varepsilon\operatorname{div}v + \operatorname{div}q + (\mathrm{D}v)^{\mathrm{S}} \mathbin{\bullet\!\!:} \Pi - g = 0\,,\\ &\overset{\circ}{\eta} + \eta\operatorname{div}v + \operatorname{div}\big(\frac{1}{\theta}q + \widetilde{\psi}\big) = \sigma \ge 0\,.\end{aligned}$$

Indem wir die zweite Gleichung mit $\theta \ge 0$ multiplizieren, schreibt sich dies als

$$\begin{aligned}&\overset{\circ}{\varepsilon} + \varepsilon\operatorname{div}v + \operatorname{div}q + (\mathrm{D}v)^{\mathrm{S}} \mathbin{\bullet\!\!:} \Pi - g = 0\,,\\ &\theta\overset{\circ}{\eta} + \theta\eta\operatorname{div}v + \underbrace{\theta\operatorname{div}\big(\tfrac{1}{\theta}q\big)}_{=\ \operatorname{div}q + \theta\nabla(\frac{1}{\theta})\bullet q} + \theta\operatorname{div}\widetilde{\psi} = \theta\sigma \ge 0\,.\end{aligned} \tag{III5.7}$$

Bei der Differenz dieser Gleichungen fällt nun $\operatorname{div}q$ weg, und wir erhalten

$$\begin{aligned}0 \ge -\theta\sigma = {}&\overset{\circ}{\varepsilon} - \theta\overset{\circ}{\eta} + (\varepsilon - \theta\eta)\operatorname{div}v\\ &+ (\mathrm{D}v)^{\mathrm{S}} \mathbin{\bullet\!\!:} \Pi - g - \theta\nabla\big(\frac{1}{\theta}\big)\bullet q - \theta\operatorname{div}\widetilde{\psi}\,.\end{aligned}$$

Mit der freien Energie f ist nun wegen $(\theta\eta)^\circ = \theta\overset{\circ}{\eta} + \eta(\partial_t\theta + v\bullet\nabla\theta)$

$$\begin{aligned}&\overset{\circ}{\varepsilon} - \theta\overset{\circ}{\eta} + (\varepsilon - \theta\eta)\operatorname{div}v\\ &= \overset{\circ}{f} + \eta(\partial_t\theta + v\bullet\nabla\theta) + f\operatorname{div}v\\ &= \partial_t f + \operatorname{div}(fv) + \eta(\partial_t\theta + v\bullet\nabla\theta)\,,\end{aligned}$$

also folgt wegen $-\theta\,\mathrm{div}\widetilde{\psi} = \mathrm{div}(-\theta\widetilde{\psi}) + \nabla\theta\bullet\widetilde{\psi}$

$$0 \geq -\theta\sigma = \partial_t f + \mathrm{div}(fv - \theta\widetilde{\psi}) + (\mathrm{D}v)^{\mathrm{S}} : \Pi - g + R_\theta\,,$$

wenn $R_\theta := \eta(\partial_t + v\bullet\nabla)\theta - \theta\nabla\big(\frac{1}{\theta}\big)\bullet q + \nabla\theta\bullet\widetilde{\psi}$. □

The quantity R_θ contains terms with spacetime derivatives of θ.

References: See e.g. [50, 4.4.3 The Dissipation Inequality] for the dissipation inequality. The entropy inequality can be found there in [50, (4.4.15)]. See also [Wikipedia: Clausius-Duhem inequality], where also the dissipation is quoted.

Adding up now equation 2.2 for the kinetic energy,

$$\begin{aligned}\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \mathrm{div}\Big(\frac{1}{2}|v|^2(\varrho v + \mathbf{J}) + \Pi^{\mathrm{T}} v\Big)\\ -\frac{1}{2}|v|^2\mathbf{r} - v\bullet\mathbf{f} - \mathrm{D}v:(\Pi + v\,\mathbf{J}^{\mathrm{T}}) = 0\,,\end{aligned} \tag{III5.8}$$

to the equation in (III5.5), one gets the

5.2 Free energy inequality ($\boldsymbol{\theta}$ variable). Let

$$f = \varepsilon - \theta\eta\,, \quad \widetilde{\psi} = \psi - \eta v - \frac{1}{\theta}q\,.$$

If the entropy principle is fulfilled for (η, ψ), then it holds

$$\boxed{\begin{aligned}\partial_t f^{tot} + \mathrm{div}\Big(f^{tot}v + \frac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}} v - \theta\widetilde{\psi}\Big) - \widetilde{g}\\ +R_\theta = -\theta\sigma \leq 0\,.\end{aligned}}$$

Here f^{tot} is the (total) free energy defined in (III5.2). The term $\widetilde{g}$ can be found in (III5.3) and R_θ in (III5.6).

This identity is equivalent to the entropy principle, i.e. to the entropy inequality. This equivalence can be seen easily if the above conclusions are carried out in the reversed direction. We now consider the isothermal limit.

5.3 Comment. Passing with $\theta \to$ const it is expected that $R_\theta \to 0$. To show this is not at all obvious. If the limit $\theta \to$ const is considered, one really takes a sequence of solutions with $q \to \infty$. The entropy principle shows that $\nabla\big(\frac{1}{\theta}\big)\bullet q$ is in L^1 in spacetime, which is a natural bound on the solutions. But for the sequence it is not clear that this implies that R_θ really converges to 0, since it contains the term $-\theta\nabla\big(\frac{1}{\theta}\big)\bullet q$. This difficulty can be avoided if one assumes $-\theta\sigma - \frac{1}{\theta}\nabla\theta\bullet q \leq 0$, which is usually fulfilled.

In this limit, motivated by 5.2, one considers

$$\boxed{\begin{aligned} f^{tot} = f + \frac{\varrho}{2}|v|^2\,, \quad \varphi^{tot} = f^{tot}v + \frac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}}\,v + \varphi \\ g^{tot} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + v\bullet\mathrm{D}v\mathbf{J} + (\mathrm{D}v)^{\mathrm{A}}\mathbin{:}\Pi + g\,. \end{aligned}} \tag{III5.9}$$

Here f^{tot} is called ***(total) free energy*** and φ^{tot} the ***free energy flux***. The quantity g^{tot} is not zero as it is the case of the entropy princple, quite the contrary, if the data have e.g. a force $\mathbf{f}$ so g^{tot} has to contain the term $v\bullet\mathbf{f}$. Only a part of g^{tot} (in the above formula g) is an objective scalar and only an objective quantity can be zero. The right-hand side of the differential equation in 5.2 is $\sigma_f := -\theta\sigma$ and because of this one formulates the

5.4 Free energy inequality ($\boldsymbol{\theta}$ = const). For all physical processes in $\mathcal{P}$, which are isothermal, there is a (total) free energy f^{tot} and an associated flux φ^{tot}, such that

$$\boxed{\sigma_f := \partial_t f^{tot} + \mathrm{div}\varphi^{tot} - g^{tot} \leq 0\,.}$$

Here, the function g^{tot} on the left-hand side one has to choose in such a way that the left side of this inequality is an objective scalar. Only then the inequality is observer-independent.

It is now clear that the free energy inequality is a consequence of the entropy principle in the isothermal case, i.e. if θ = const > 0, and if no heat flux is present in the equations describing the material. Besides this there are similar comments for the free energy inequality as for the entropy principle. Usually, if ϱ is the total mass and v is the velocity, the free energy is given by the formula in (III5.9), that is, by the inner free energy.

5.5 Conclusion. Let the mass and momentum equation

$$\begin{aligned} &\partial_t\varrho + \mathrm{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \\ &\varrho(\partial_t v + v\bullet\nabla v) + \mathrm{div}\Pi = \mathbf{f}\,, \end{aligned}$$

belong to the set $\mathcal{P}$ and let f^{tot}, φ^{tot}, g^{tot} be given by (III5.9). Then the free energy inequality 5.4 is equivalent to

$$\boxed{\sigma_f = \partial_t f + \mathrm{div}\varphi + (\mathrm{D}v)^{\mathrm{S}}\mathbin{:}\Pi - g \leq 0\,.}$$

Proof. As a consequence of the two equations, which belong to $\mathcal{P}$, we derive (III5.8). Subtracting this from the energy equation we obtain

$$
\begin{aligned}
0 \geq \sigma_f &= \partial_t\Big(f^{tot} - \frac{\varrho}{2}|v|^2\Big) + \mathrm{div}\Big(\varphi^{tot} - \frac{1}{2}|v|^2(\varrho v + \mathbf{J}) - \Pi^{\mathrm{T}}\, v\Big) \\
&-g^{tot} + \frac{1}{2}|v|^2\mathbf{r} + v\bullet\mathbf{f} + \mathrm{D}v\mathbin{:}(\Pi + v\,\mathbf{J}^{\mathrm{T}}) \\
&= \partial_t f + \mathrm{div}\varphi + \mathrm{D}v\mathbin{:}\Pi - g\,,
\end{aligned}
$$

the assertion. □

The inner free energy $f = \varepsilon - \theta\eta$ is an objective scalar like the inner energy ε, therefore the (total) free energy f^{tot} has the same transformation rule as the energy e. If one decides to accept a constant temperature $\theta = \mathrm{const}$, one looses the energy equation and has a free energy inequality instead of the entropy inequality. This is the necessary consequence of the fact that the temperature is a constant. Often the free energy inequality is applied without knowledge about the temperature, i.e. one does not realize how terms depend on the constant temperature. Therefore the free energy inequality is adequate for such problems. But in general θ is variable and one has to use the energy equation for e, concerning this matter see the note IV.3.4. In the next chapter we will get to know examples where we use the free energy and examples where we use the entropy.

6 Distributional entropy

It is not a big deal to generalise the entropy principle to the case of distributions: If $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ (physically is $n = 3$) is the domain of consideration, there exists an ***entropy*** $H \in \mathcal{D}'(\mathcal{U})$ and an ***entropy flux*** $\Psi \in \mathcal{D}'(\mathcal{U}; \mathbb{R}^n)$ such that

$$\boxed{\Sigma := \partial_t H + \mathrm{div}_x \Psi \geq 0 \text{ in } \mathcal{D}'(\mathcal{U})\,,} \tag{III6.1}$$

which means that

$$\langle\, \zeta\,,\, \partial_t H + \mathrm{div}_x \Psi \,\rangle_{\mathcal{D}(\mathcal{U})} \geq 0 \text{ for all test functions } \zeta \geq 0\,.$$ [7]

These formulations apply, for example, to the case

- where the entropy and entropyflux is given by a function η, that is $H = [\eta]$, and a vector field ψ, that is $\Psi = [\psi]$, which in general are not continuous. For example, on the boundary between two media, these functions may have jumps.
- where the mass is concentrated on certain moving points, hence one has to deal with 1-dimensional distributions in spacetime $\mathbb{R} \times \mathbb{R}^n$. Therefore also the entropy is concentrated on these moving points.

In the first case $H = [\eta]$ and $\Psi = [\psi]$ with a function η in $\mathcal{U}$ and a vector field ψ in $\mathcal{U}$. Therefore the entropy principle (III6.1) is equivalent to

$$\Sigma := \partial_t[\eta] + \mathrm{div}_x[\psi] \geq 0 \text{ in } \mathcal{D}'(\mathcal{U})\,, \tag{III6.2}$$

which means that

$$\int_{\mathcal{U}} (\partial_t \zeta \cdot \eta + \nabla\zeta \bullet \psi)\, \mathrm{dL}^{n+1} \leq 0 \text{ for all test functions } \zeta \geq 0\,,$$

where we assume that the underlying mass-momentum-energy system is written in the distributional space $\mathcal{D}'(\mathcal{U})$

$$\begin{aligned} \partial_t[\varrho] + \mathrm{div}[\widetilde{\mathbf{J}}] &= [\mathbf{r}]\,, \\ \partial_t[\varrho v] + \mathrm{div}[\widetilde{\Pi}] &= [\widetilde{\mathbf{f}}]\,, \\ \partial_t[e] + \mathrm{div}[\widetilde{q}] &= [\widetilde{g}]\,. \end{aligned} \tag{III6.3}$$

As elementary example we want to measure the temperature. Since $\eta'_\varepsilon = \frac{1}{\theta}$ for the temperature θ we see that the inverse temperature is a multiplier (in this connection see section 4). This indicates, that the temperature at an

[7] Es ist $\langle\, \zeta\,,\, \Sigma\,\rangle := \langle\, \zeta\,,\, \partial_t H + \mathrm{div}_x \Psi\,\rangle = -\langle\, \partial_t \zeta\,,\, H\,\rangle - \langle\, \nabla\zeta\,,\, \Psi\,\rangle$. Die Aussage lautet mit Quantoren $\forall\, \zeta \in \mathscr{D}(\mathcal{U}) : \big(\, \zeta \geq 0 \Rightarrow \langle\, \zeta\,,\, \Sigma\,\rangle \geq 0\,\big)$.

interface is continuous (see Fig. 1(d)), and in fact, in the following example this will be the main assumption. In 6.1 we consider the case of a solution with velocity $v = 0$ (at least for one observer), and that there is no reaction or diffusion in the mass conservation, hence $\widetilde{J} = 0$ and $\mathbf{r} = 0$. Consequently $\widetilde{\mathbf{f}} = \mathbf{f}$, $\widetilde{\Pi} = \Pi$, $e = \varepsilon$ [8], $\widetilde{q} = q$, and $\widetilde{g} = g$. Thus (III6.3) becomes

$$\begin{aligned} &\partial_t \varrho = 0\,, \\ &\mathrm{div}[\Pi] = [\mathbf{f}]\,, \\ &\partial_t \varepsilon + \mathrm{div}[q] = [g]\,. \end{aligned} \tag{III6.4}$$

We show

6.1 Thermometer. Let $\mathcal{U} = \mathbb{R} \times (D_o \cup \boldsymbol{S} \cup D_m)$, where $\mathbb{R} \times D_o$ is the to be measured area and $\mathbb{R} \times D_m$ is the position of the measuring instrument. We will give a situation in which no velocity occurs, i.e. $v = 0$. Under these assumptions the momentum and the energy equations in $\mathscr{D}'(\mathcal{U})$ are [9]

$$\begin{aligned} &[\Pi] = p_o \,\mathrm{Id}\, \mu_{\mathbb{R}\times D_o} + p_m \mathrm{Id}\, \mu_{\mathbb{R}\times D_m}\,, && [\mathbf{f}] = \mathbf{f}_o\, \mu_{\mathbb{R}\times D_o} + \mathbf{f}_m\, \mu_{\mathbb{R}\times D_m}\,, \\ &[\varepsilon] = \varepsilon_o\, \mu_{\mathbb{R}\times D_o} + \varepsilon_m\, \mu_{\mathbb{R}\times D_m}\,, && \\ &[q] = q_o\, \mu_{\mathbb{R}\times D_o} + q_m\, \mu_{\mathbb{R}\times D_m}\,, && [g] = g_o\, \mu_{\mathbb{R}\times D_o} + g_m\, \mu_{\mathbb{R}\times D_m}\,. \end{aligned}$$

$$[\psi] = \frac{q_o}{\theta_o}\mu_{\mathbb{R}\times D_o} + \frac{q_m}{\theta_m}\mu_{\mathbb{R}\times D_m}$$

Under the assumption that the external forces are 0, and that the entropy flux has the Clausiuc-Duhem form, it follows that

$$\theta_o = \theta_m \text{ on } \mathbb{R} \times \boldsymbol{S}\,.$$

This is under the condition that on the boundary no entropy or entropy flux is present. *Remark:* This implies that the absolute temperature in the contact zone $\boldsymbol{S}$ between the region and the measuring stick is the same. If the temperature in the region is higher than in the measuring stick then $q_o \bullet \nu_D > 0$. Thus, the measurement affects the temperature in the region. If we wait long enough, so θ_m will be approximately the same throughout the whole measuring stick and then this is taken as a measurement of the temperature in the region. *Remark:* The representation of $[\psi]$ is the assertion that the entropy principle in each domain is the classical one of Clausis-Duhem.

Proof. For $H = [\eta]$ and $\Psi = [\psi]$ the entropy inequality

$$\Sigma := \partial_t H + \mathrm{div}\Psi \geq 0 \text{ in } \mathscr{D}'(\mathcal{U}) \tag{III6.5}$$

[8] Since $v = 0$ there is no kinetic energy in $[e]$.
[9] Es ist $\mu_A(E) := \mathrm{L}^4(A \cap E)$ für $E \subset \mathbb{R} \times \mathbb{R}^3$.

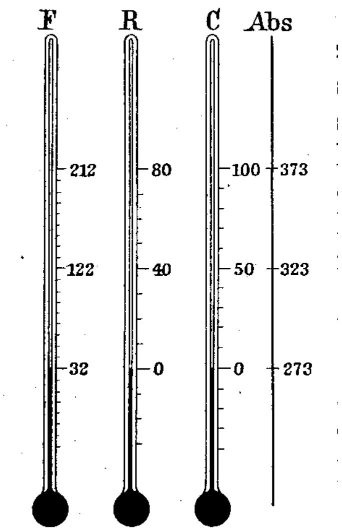


Abb. 485. Die gebräuchlichen Quecksilberthermometer mit ihren Fundamentalpunkten.

Fig. 3: "Die gebräuchlichen Quecksilberthermometer mit ihren Fundamentalpunkten" from "Grimsehl's Lehrbuch der Physik". It shows the temperature in Fahrenheit [℉], Réaumur [°Ré], Celsius [℃] and Kelvin [K] (Absolute temperature). It is 0℃ = 273.15 K, see [Wikipedia: Kelvin].

is valid. The assumption that the Clausius-Duhem identity holds outside $\mathbb{R} \times \boldsymbol{S}$, means that the entropy identities

$$\partial_t \eta + \operatorname{div}\Big(\frac{q_o}{\theta}\Big) = \sigma_0 \geq 0 \text{ in } \mathbb{R} \times D_0\,,$$
$$\partial_t \eta_m + \operatorname{div}\Big(\frac{q_m}{\theta_m}\Big) = \sigma_m \geq 0 \text{ in } \mathbb{R} \times D_m$$

hold. The fact that no entropy occurs on $\mathbb{R} \times \boldsymbol{S}$ means that in (III6.5) we have

$$H = \eta_o \mu_{\mathbb{R}\times D_o} + \eta_m \mu_{\mathbb{R}\times D_m}\,,$$
$$\Psi = \frac{q_o}{\theta_o}\mu_{\mathbb{R}\times D_o} + \frac{q_m}{\theta_m}\mu_{\mathbb{R}\times D_m}\,,$$
$$\Sigma = \sigma_o \mu_{\mathbb{R}\times D_o} + \sigma_m \mu_{\mathbb{R}\times D_m}\,.$$

Thus the distributional differential identities and the entropy inequality

(III6.5) mean for test functions $\zeta \in \mathscr{D}(\mathcal{U})$ that for $j = 1, \dots, 3$

$$\int_{\mathbb{R}} \Big(\int_{D_o} (\partial_{x_j}\zeta \cdot p_o + \zeta \mathbf{f}_{oj})\, \mathrm{dL}^3 + \int_{D_m} (\partial_{x_j}\zeta \cdot p_m + \zeta \mathbf{f}_{mj})\, \mathrm{dL}^3 \Big)\, \mathrm{dL}^1 = 0\,,$$
$$\int_{\mathbb{R}} \Big(\int_{D_o} (\partial_t\zeta \cdot \varepsilon_o + \nabla\zeta \bullet q_o)\, \mathrm{dL}^3 + \int_{D_m} (\partial_t\zeta \cdot \varepsilon_m + \nabla\zeta \bullet q_m)\, \mathrm{dL}^3 \Big)\, \mathrm{dL}^1 = 0\,,$$
$$\int_{\mathbb{R}} \Big(\int_{D_o} (\partial_t\zeta \cdot \eta + \nabla\zeta \bullet (\frac{1}{\theta_o} q_o) + \zeta\sigma)\, \mathrm{dL}^3$$
$$+ \int_{D_m} (\partial_t\zeta \cdot \eta_m + \nabla\zeta \bullet (\frac{1}{\theta_m} q_m) + \zeta\sigma_m)\, \mathrm{dL}^3 \Big)\, \mathrm{dL}^1 = 0\,.$$

This leads to the following equations on the time independent boundary $\mathbb{R} \times \Gamma$, with $\mathrm{n} = \nu_{D_o} = -\nu_{D_m}$,

$$p_m = p_o\,,$$
$$q_m \bullet \mathrm{n} = q_o \bullet \mathrm{n}\,,$$
$$(\frac{1}{\theta_o} q_o - \frac{1}{\theta_m} q_m) \bullet \mathrm{n} = 0\,.$$

Since during the measurement $q_o \bullet \mathrm{n} \neq 0$, from this it follows

$$\theta_m = \theta_o \text{ on } \mathbb{R} \times \boldsymbol{S}\,.$$

Hence the measurement requires an arbitrary $q_o \bullet \mathrm{n}$. □

This result gives rise to the following remark.

6.2 Empirical temperature. In Hutter & Wang [9, 17.2.5 Empirical Temperature, Gas Temperature and Temperature Scales] it is said: "It is, however, not possible to define the temperature in a direct manner. It must be defined indirectly via the notion of thermodynamic equilibrium. One says: Two systems have the same temperature, if they are in thermal equilibrium with one another." The daily perceived temperature is based on this effect.

In the second case we apply the distributional mass-momentum-energy system to colliding particles. In addition to the distributional mass-momentum system in I.3.1 we prove an equation for the kinetic energy (see also [19, 7.11]).

6.3 Kinetic energy of a mass point. It follows from I.3.1 for the trajectory $t \to \xi(t)$ with mass $t \to m(t)$ in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$

$$\partial_t \big(\frac{m}{2}|v|^2 \boldsymbol{\mu}_\xi\big) + \mathrm{div}\big(\frac{m}{2}|v|^2 v \boldsymbol{\mu}_\xi\big) = \big(\frac{\mathbf{r}}{2}|v|^2 + v \bullet \mathbf{f}\big)\boldsymbol{\mu}_\xi\,. \tag{III6.6}$$

Proof. Let $t \to \xi(t)$ be the movement of a mass point with mass $t \to m(t)$ as in I.2.9. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{m}{2}|\dot\xi|^2\Big) = \frac{\dot m}{2}|\dot\xi|^2 + m\dot\xi\bullet\ddot\xi = \frac{\mathbf{r}}{2}|\dot\xi|^2 + \dot\xi\bullet\mathbf{f}$$

and therefore, since $\dot\xi(t) = v(t,\xi(t))$,

$$\begin{aligned}
&\Big\langle \zeta\,,\,\partial_t\big(\frac{m}{2}|\dot\xi|^2\boldsymbol{\mu}_\xi\big) + \mathrm{div}\big(\frac{m}{2}|\dot\xi|^2 v\boldsymbol{\mu}_\xi\big)\Big\rangle_{\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)}\\
&= -\Big\langle \partial_t\zeta\,,\,\frac{m}{2}|\dot\xi|^2\boldsymbol{\mu}_\xi\Big\rangle_{\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)} - \Big\langle \nabla\zeta\,,\,\frac{m}{2}|\dot\xi|^2 v\boldsymbol{\mu}_\xi\Big\rangle_{\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)}\\
&= -\int_{\mathbb{R}}\big(\partial_t\zeta(t,\xi(t)) + \dot\xi(t)\bullet\nabla\zeta(t,\xi(t))\big)\frac{m(t)}{2}|\dot\xi(t)|^2\,\mathrm{d}t\\
&= -\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}t}\big(\zeta(t,\xi(t))\big)\frac{m(t)}{2}|\dot\xi(t)|^2\,\mathrm{d}t = \int_{\mathbb{R}}\zeta(t,\xi(t))\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{m(t)}{2}|\dot\xi(t)|^2\big)\,\mathrm{d}t\\
&= \int_{\mathbb{R}}\zeta(t,\xi(t))\big(\frac{\mathbf{r}(t)}{2}|\dot\xi(t)|^2 + \dot\xi(t)\bullet\mathbf{f}(t)\big)\,\mathrm{d}t\\
&= \Big\langle \zeta\,,\,\big(\frac{\mathbf{r}}{2}|\dot\xi|^2 + \dot\xi\bullet\mathbf{f}\big)\boldsymbol{\mu}_\xi\Big\rangle_{\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)} = \Big\langle \zeta\,,\,\big(\frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f}\big)\boldsymbol{\mu}_\xi\Big\rangle_{\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)}\,.
\end{aligned}$$

□

We obtain

6.4 Mass-momentum-energy equation for a mass point. If the mass point is given as in 6.3 then there are equivalent:

(1) The distributional equations

$$\begin{aligned}
&\partial_t(m\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\boldsymbol{\mu}_\xi) = \mathbf{r}\boldsymbol{\mu}_\xi\,,\\
&\partial_t(mv\boldsymbol{\mu}_\xi) + \mathrm{div}\,(mv\,v^{\mathrm{T}}\,\boldsymbol{\mu}_\xi) = (\mathbf{r}v + \mathbf{f})\boldsymbol{\mu}_\xi\,,\\
&\partial_t\big((\varepsilon + \tfrac{m}{2}|v|^2)\boldsymbol{\mu}_\xi\big) + \mathrm{div}\big((\varepsilon + \tfrac{m}{2}|v|^2)v\boldsymbol{\mu}_\xi\big) = \big(\tfrac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f}\big)\boldsymbol{\mu}_\xi
\end{aligned}\tag{III6.7}$$

in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ are fulfilled.

(2) It is $v(t,\xi(t)) = \dot\xi(t)$ the velocity, and the ordinary differential equations

$$\dot m = \mathbf{r}\,,\quad m\ddot\xi = \mathbf{f}\,,\quad \varepsilon = \mathrm{const}\tag{III6.8}$$

are satisfied.

Proof. Wir nehmen zuerst die Aussage in I.3.1. Dann gehen wir zur Energiegleichung und ziehen die Formel (III6.6) für die kinetische Energie ab. Es bleibt $\partial_t(\varepsilon\boldsymbol{\mu}_\xi) + \mathrm{div}(\varepsilon v\boldsymbol{\mu}_\xi) = 0$. Wie bei der Massenerhaltung in I.2.9 ist dies äquivalent zu $\dot\varepsilon = 0$. □

Now we are able to formulate the collision problem where for example two particles collide and produce a whole family of particles.

6.5 Collision of mass points. Let $t \in]-\infty, t_0[\mapsto \xi_-^\alpha(t)$ the trajectories before the collision and $t \in]t_0, \infty[\mapsto \xi_+^\beta(t)$ the trajectories afterwards where $\alpha = 1, \dots, \alpha_{max}$ and $\beta = 1, \dots, \beta_{max}$. The collision takes place at the spacetime point (t_0, x_0). Let the distributional conservation laws for mass, momentum, and energy

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,, \\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}}) = \mathbf{f}\,, \\
&\partial_t e + \operatorname{div}(e v) = v \bullet \mathbf{f}
\end{aligned}$$

be satisfied in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$[10], where

$$\begin{aligned}
\varrho &:= \sum_\alpha m_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta m_+^\beta \boldsymbol{\mu}_{\xi_+^\beta} \\
\mathbf{f} &:= \sum_\alpha \mathbf{f}_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta \mathbf{f}_+^\beta \boldsymbol{\mu}_{\xi_+^\beta} \\
e &:= \sum_\alpha (\varepsilon_-^\alpha + \frac{m_-^\alpha}{2}|v|^2) \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta (\varepsilon_+^\beta + \frac{m_+^\beta}{2}|v|^2) \boldsymbol{\mu}_{\xi_+^\beta}\,.
\end{aligned}$$

Conclusion: In the time intervals $]-\infty, t_0[$ and $]t_0, \infty[$ for all particles the equations in 6.4(2) hold and the energies are constant. And at the collision time t_0

$$\begin{aligned}
\sum_\alpha m_-^\alpha &= \sum_\beta m_+^\beta\,, \\
\sum_\alpha m_-^\alpha \dot\xi_-^\alpha &= \sum_\beta m_+^\beta \dot\xi_+^\beta\,, \\
\sum_\alpha \big(\varepsilon_-^\alpha + \frac{m_-^\alpha}{2}|\dot\xi_-^\alpha|^2\big) &= \sum_\beta \big(\varepsilon_+^\beta + \frac{m_+^\beta}{2}|\dot\xi_+^\beta|^2\big)\,.
\end{aligned}$$

Remark: The velocity v is used only in the points (t, x) with $x = \xi_-^\alpha(t)$ for $t < t_0$ und $x = \xi_+^\beta(t)$ for $t > t_0$. The multiplication with v is relevant only in these points. In the proof there is $v(t, \xi_-^\alpha(t)) = \dot\xi_-^\alpha(t)$ for $t < t_0$ and $v(t, \xi_+^\beta(t)) = \dot\xi_+^\beta(t)$ for $t > t_0$.

Proof. Wenn, wie im Beweis von I.3.2,

$$\partial_t\Big(\sum_\alpha g_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta g_+^\beta \boldsymbol{\mu}_{\xi_+^\beta}\Big) + \operatorname{div}\Big(\sum_\alpha g_-^\alpha v \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta g_+^\beta v \boldsymbol{\mu}_{\xi_+^\beta}\Big) = \sum_\alpha r_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta r_+^\beta \boldsymbol{\mu}_{\xi_+^\beta}\,,$$

[10] Attention: ϱ is a distribution and ϱv is the multiplication of this distribution with a bounded vector field v.

one obtains for test functions $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$, as in the proof of I.3.2,

$$\begin{aligned}
0 &= \left\langle \partial_t\zeta\,,\,\Big(\sum_\alpha g_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta g_+^\beta \boldsymbol{\mu}_{\xi_+^\beta}\Big)\right\rangle \\
&+ \left\langle \nabla\zeta\,,\,\Big(\sum_\alpha g_-^\alpha v\boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta g_+^\beta v\boldsymbol{\mu}_{\xi_+^\beta}\Big)\right\rangle + \left\langle \zeta\,,\,\sum_\alpha r_-^\alpha \boldsymbol{\mu}_{\xi_-^\alpha} + \sum_\beta r_+^\beta \boldsymbol{\mu}_{\xi_+^\beta}\right\rangle \\
&= \sum_\alpha \int_{-\infty}^{t_0} \zeta(t,\xi_-^\alpha(t))\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\big(g_-^\alpha(t,\xi_-^\alpha(t))\big) + r_-^\alpha(t,\xi_-^\alpha(t))\Big) \\
&+ \sum_\beta \int_{t_0}^{\infty} \zeta(t,\xi_+^\beta(t))\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\big(g_+^\beta(t,\xi_+^\beta(t))\big) + r_+^\beta(t,\xi_+^\beta(t))\Big) \\
&+ \zeta(t,x_0)\Big(\sum_\alpha g_-^\alpha(t,x_0) - \sum_\beta g_+^\beta(t,x_0)\Big)\,,
\end{aligned}$$

which gives the ODE equations and

$$\sum_\alpha g_-^\alpha(t,x_0) = \sum_\beta g_+^\beta(t,x_0)\,,$$

the identity in the collision point (t_0,x_0). □

6.6 Examples. We define $v_-^\alpha := \dot\xi_-^\alpha(t_0)$ and $v_+^\beta := \dot\xi_+^\beta(t_0)$.

(1) ***Elastic collision.*** $\beta_{max} = \alpha_{max} = 2$: This collision we treated already in I.3.2. If we choose $\varepsilon_-^\alpha = \varepsilon_+^\beta = 0$ the collision laws are

$$\begin{aligned}
m_-^1 + m_-^2 &= m_+^1 + m_+^2\,,\\
m_-^1 v_-^1 + m_-^2 v_-^2 &= m_+^1 v_+^1 + m_+^2 v_+^2\,,\\
\frac{m_-^1}{2}|v_-^1|^2 + \frac{m_-^2}{2}|v_-^2|^2 &= \frac{m_+^1}{2}|v_+^1|^2 + \frac{m_+^2}{2}|v_+^2|^2\,.
\end{aligned}$$

An elastic collision preserves the masses, that is, $m_+^1 = m_-^1$ and $m_+^2 = m_-^2$. There is a twodimensional degree of freedom for this collision.

(2) ***Explosion.*** $\alpha_{max} = 1$, $\beta_{max} > 1$: If we define $v_+^\beta = v_- + u_+^\beta$ we have the following laws

$$m_- = \sum_\beta m_+^\beta\,,\quad 0 = \sum_\beta m_+^\beta u_+^\beta\,,$$

$$\varepsilon_- = \sum_\beta \big(\varepsilon_+^\beta + \frac{m_+^\beta}{2}|u_+^\beta|^2\big)\,.$$

If the inner energies ε_+^β are nonnegative the body must be hot enough to explode.

(3) ***Coalescence.*** $\alpha_{max} = 2$, $\beta_{max} = 1$: Two bodies collide and as a result one body is formed. The mass conservation and the conservation of momentum says that

$$m_+ = m_-^1 + m_-^1\,, \quad v_+ = \frac{m_-^1}{m_+} v_-^1 + \frac{m_-^2}{m_+} v_-^2\,.$$

Hence the initial conditions (m_+, v_+) for the combined body is prescribed. The definitions $u_-^1 := v_-^1 - v_+$, $u_-^2 := v_-^2 - v_+$ lead to

$$\varepsilon_+ = \sum_\alpha \big(\varepsilon_-^\alpha + \frac{m_-^\alpha}{2}|u_-^\alpha|^2\big)\,.$$

This gives the temperature of the resulting body. The temperature is mainly determined by the relative velocity by which the bodies are colliding.

In the general case, hence also in all three examples, the definitions

$$\begin{aligned}
&m := \sum_\alpha m_-^\alpha = \sum_\beta m_+^\beta \quad \text{(assume } m > 0\text{)},\\
&v := \sum_\alpha \frac{m_-^\alpha}{m} v_-^\alpha = \sum_\beta \frac{m_+^\beta}{m} v_+^\beta \quad \text{(barycentric mean)},\\
&u_-^\alpha = v_-^\alpha - v\,, \quad u_+^\beta = v_+^\beta - v\,,
\end{aligned}$$

at the collision time t_0 lead to

$$\sum_\alpha m_-^\alpha u_-^\alpha = 0\,, \quad \sum_\beta m_+^\beta u_+^\beta = 0\,,$$

$$\sum_\alpha \big(\varepsilon_-^\alpha + \frac{m_-^\alpha}{2}|u_-^\alpha|^2\big) = \sum_\beta \big(\varepsilon_+^\beta + \frac{m_+^\beta}{2}|u_+^\beta|^2\big)\,.$$

7 Exercises

Entropy

7.1 Übung. Der Druck sei als Funktion $p = \widetilde{p}(\varrho, \theta)$ gegeben, ebenso die innere Energie $\varepsilon = \widetilde{\varepsilon}(\varrho, \theta)$. Es gelte $\varrho\varepsilon{}'_\varrho + \theta p{}'_\theta = \varepsilon + p$. Berechne daraus die Entropie, wenn die Gibbs Relation erfüllt ist. Mache die Probe.
Erinnerung: Die Temperatur θ hängt von der Entropie wie in (III1.7) ab.
Zeige: Als Funktion $\eta := \widetilde{\eta}(\varrho, \theta)$ lautet die Entropie

$$\widetilde{\eta}^{\mathrm{sp}}(\varrho, \theta) = \int_{\theta_0}^{\theta} \frac{\widetilde{\varepsilon}^{\mathrm{sp}}{}'_\theta(\varrho_0, \widetilde{\theta})}{\widetilde{\theta}} \, \mathrm{d}\widetilde{\theta} - \int_{\varrho_0}^{\varrho} \frac{\widetilde{p}^{\mathrm{sp}}{}'_\theta(\widetilde{\varrho}, \theta)}{\widetilde{\varrho}} \, \mathrm{d}\widetilde{\varrho} + C \, ,$$

wobei η^{sp}, $\varepsilon^{\mathrm{sp}}$, p^{sp} die spezifischen Größen (wie in 1.4) bezeichnen, und ϱ_0, θ_0, C sind Konstanten.

Die Gibbs-Relation, eine Folgerung aus dem Enttropieprinzip, ist eine differentielle Bedingung an (p, ε, η). Sie drückt eine Größe durch die anderen aus. In dieser Aufgabe ist (p, ε) gegeben und η wird dadurch ausgedrückt.

Part 1: Notwendige Bedingung für die Existenz von η. Mit $\eta = \widehat{\eta}(\varrho, \varepsilon)$ sind die beiden gegebenen Gleichungen

$$\theta \, \eta{}'_\varepsilon = 1 \, , \quad \eta - \varrho\eta{}'_\varrho - (\varepsilon + p)\eta{}'_\varepsilon = 0 \, , \tag{III7.1}$$

wobei $\theta = \widehat{\theta}(\varrho, \varepsilon)$ und $p = \widehat{p}(\varrho, \varepsilon)$. Mit der spezifischen Entropie

$$\eta^{\mathrm{sp}} := \frac{1}{\varrho}\eta$$

lauten die beiden Gleichungen

$$\eta^{\mathrm{sp}}{}'_\varepsilon = \frac{1}{\varrho}\eta{}'_\varepsilon = \frac{1}{\varrho\theta} \, , \quad \eta^{\mathrm{sp}}{}'_\varrho = \Big(\frac{\eta}{\varrho}\Big){}'_\varrho = -\frac{\varepsilon + p}{\varrho^2\theta} \, .$$

Also gibt es η^{sp} genau dann, wenn

$$\Big(\frac{1}{\varrho\theta}\Big){}'_\varrho = (\eta^{\mathrm{sp}}{}'_\varepsilon){}'_\varrho = (\eta^{\mathrm{sp}}{}'_\varrho){}'_\varepsilon = -\Big(\frac{\varepsilon + p}{\varrho^2\theta}\Big){}'_\varepsilon \tag{III7.2}$$

Nun schreiben wir η als Funktion von (ϱ, θ), also

$$\eta = \widehat{\eta}(\varrho, \widetilde{\varepsilon}(\varrho, \theta)) =: \widetilde{\eta}(\varrho, \theta) \, .$$

Wenn wir für beliebige Funktionen h dasselbe tun, also $h = \widehat{h}(\varrho, \widetilde{\varepsilon}(\varrho, \theta)) =: \widetilde{h}(\varrho, \theta)$, so gelten die Kettenregeln

$$\widetilde{h}{}'_\theta = \widehat{h}{}'_\varepsilon \widetilde{\varepsilon}{}'_\theta \, , \quad \widetilde{h}{}'_\varrho = \widehat{h}{}'_\varrho + \widehat{h}{}'_\varepsilon \widetilde{\varepsilon}{}'_\varrho$$

oder

$$\widehat{h}{}'_\varepsilon = \widetilde{h}{}'_\theta \frac{1}{\widetilde{\varepsilon}{}'_\theta} \, , \quad \widehat{h}{}'_\varrho = \widetilde{h}{}'_\varrho - \widetilde{h}{}'_\theta \frac{\widetilde{\varepsilon}{}'_\varrho}{\widetilde{\varepsilon}{}'_\theta} \, , \tag{III7.3}$$

was ergibt

$$\widehat{h} := -\frac{\varepsilon + p}{\varrho^2\theta} \quad \Rightarrow \quad -\Big(\frac{\varepsilon + \widehat{p}}{\varrho^2\widehat{\theta}}\Big){}'_\varepsilon = -\Big(\frac{\widetilde{\varepsilon} + \widetilde{p}}{\varrho^2\theta}\Big){}'_\theta \frac{1}{\widetilde{\varepsilon}{}'_\theta} = -\frac{\widetilde{p}{}'_\theta\theta - (\widetilde{\varepsilon} + \widetilde{p})}{\varrho^2\theta^2\widetilde{\varepsilon}{}'_\theta} - \frac{1}{\varrho^2\theta} \, ,$$

$$\widehat{h} := \frac{1}{\varrho\theta} \quad \Rightarrow \quad \Big(\frac{1}{\varrho\widehat{\theta}}\Big){}'_\varrho = \Big(\frac{1}{\varrho\theta}\Big){}'_\varrho - \Big(\frac{1}{\varrho\theta}\Big){}'_\theta \frac{\widetilde{\varepsilon}{}'_\varrho}{\widetilde{\varepsilon}{}'_\theta} = -\frac{1}{\varrho^2\theta} + \frac{1}{\theta^2\varrho} \frac{\widetilde{\varepsilon}{}'_\varrho}{\widetilde{\varepsilon}{}'_\theta} \, .$$

Dann wird aus (III7.2)

$$\frac{1}{\theta^2\varrho}\widetilde{\varepsilon}_{\prime\varrho} = -\frac{\widetilde{p}_{\prime\theta}\theta - (\widetilde{\varepsilon}+\widetilde{p})}{\varrho^2\theta^2}$$

das heißt

$$\boxed{\varrho\widetilde{\varepsilon}_{\prime\varrho} + \theta\widetilde{p}_{\prime\theta} = \varepsilon + p\,.} \qquad \text{(III7.4)}$$

Also, wenn diese notwendige Bedingung für ε und p erfüllt ist, existiert η^{sp} und damit η, was zu beweisen war. □

Part 2: Gibbs Darstellung in (ϱ,θ)*.* Die Gibbs Relation in (ϱ,ε), siehe (III7.1), ist

$$\eta - \varrho\eta_{\prime\varrho} = (\varepsilon+p)\eta_{\prime\varepsilon}\,,$$

wobei die rechte Seite wegen der notwendigen Bedingung (III7.4)

$$= (\varrho\widetilde{\varepsilon}_{\prime\varrho} + \theta\widetilde{p}_{\prime\theta})\eta_{\prime\varepsilon} = \varrho\widetilde{\varepsilon}_{\prime\varrho}\eta_{\prime\varepsilon} + \widetilde{p}_{\prime\theta}$$

ist, und die linke Seite, wenn wir (III7.3) für η benutzen,

$$= \widetilde{\eta} - \varrho\Big(\widetilde{\eta}_{\prime\varrho} - \underbrace{\frac{\widetilde{\eta}_{\prime\theta}}{\widetilde{\varepsilon}_{\prime\theta}}}_{=\widehat{\eta}_{\prime\varepsilon}}\,\widetilde{\varepsilon}_{\prime\varrho}\Big) = \widetilde{\eta} - \varrho\widetilde{\eta}_{\prime\varrho} + \varrho\eta_{\prime\varepsilon}\widetilde{\varepsilon}_{\prime\varrho}\,.$$

Also ist die Gibbs relation äquivalent zu

$$\boxed{\widetilde{\eta} - \varrho\widetilde{\eta}_{\prime\varrho} = \widetilde{p}_{\prime\theta}\,,}$$

oder in der Schreibweise für die spezifischen Größen

$$\varrho^2\widetilde{\eta}^{\mathrm{sp}}_{\prime\varrho} + \widetilde{p}^{\mathrm{sp}}_{\prime\theta} = 0\,, \qquad \text{(III7.5)}$$

eine Darstellung, die wir jetzt benutzen werden. □

Part 3: Darstellung von η*.* Indem wir (III7.5) durch ϱ^2 dividieren und in ϱ aufintegrieren, erhalten wir

$$\widetilde{\eta}^{\mathrm{sp}}(\varrho,\theta) = -\int_{\varrho_0}^{\varrho}\frac{\widetilde{p}_{\prime\theta}(\widetilde{\varrho},\theta)}{\widetilde{\varrho}^2}\,\mathrm{d}\widetilde{\varrho} + C(\theta)$$

mit einer Konstanten ϱ_0 und einer zu betimmenden Funktion $\theta \mapsto C(\theta)$. Indem wir diese Identität nach θ differenzieren, erhalten wir

$$\begin{aligned}
&-\int_{\varrho_0}^{\varrho}\frac{\widetilde{p}_{\prime\theta\theta}(\widetilde{\varrho},\theta)}{\widetilde{\varrho}^2}\,\mathrm{d}\widetilde{\varrho} + C_{\prime\theta}(\theta) = \widetilde{\eta}^{\mathrm{sp}}_{\prime\theta}(\varrho,\theta)\\
&= \frac{1}{\varrho}\widetilde{\eta}_{\prime\theta}(\varrho,\theta) = \frac{1}{\varrho}\widehat{\eta}_{\prime\varepsilon}(\varrho,\varepsilon)\widetilde{\varepsilon}_{\prime\theta}(\varrho,\theta) = \frac{1}{\varrho\theta}\widetilde{\varepsilon}_{\prime\theta}(\varrho,\theta) = \frac{1}{\theta}\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta}(\varrho,\theta)
\end{aligned}$$

nach (III7.3) für η. Indem wir die notwendige Bedingung (III7.4)

$$\theta\widetilde{p}_{\prime\theta} - p = \varepsilon - \varrho\widetilde{\varepsilon}_{\prime\varrho} = -\varrho^2\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\varrho}$$

nach θ differenzieren, erhalten wir $\theta\widetilde{p}_{\prime\theta\theta} = -\varrho^2\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\varrho\theta} = -\varrho^2(\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta})_{\prime\varrho}$, also ist

$$C_{\prime\theta}(\theta) = -\int_{\varrho_0}^{\varrho}\frac{(\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta})_{\prime\varrho}(\widetilde{\varrho},\theta)}{\theta}\,\mathrm{d}\widetilde{\varrho} + \frac{1}{\theta}\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta}(\varrho,\theta) = \frac{1}{\theta}\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta}(\varrho_0,\theta)$$

und somit

$$C(\theta) = C(\theta_0) + \int_{\theta_0}^{\theta}\frac{\widetilde{\varepsilon}^{\mathrm{sp}}_{\prime\theta}(\varrho_0,\widetilde{\theta})}{\widetilde{\theta}}\,\mathrm{d}\widetilde{\theta}\,.$$

Indem man $C(\theta)$ in obige Formel einsetzt, folgt die Behauptung. □

7.2 Ideales Gas. Wende 7.1 an auf

$$\varepsilon = c_v \varrho\theta\,, \quad p + \varepsilon = c_p \varrho\theta\,,$$

mit $c_p > c_v$. Zeige:

$$\eta = c_v \varrho \log \frac{p}{\varrho^\gamma} + \text{const} \cdot \varrho\,, \quad \gamma := \frac{c_p}{c_v}\,.$$

7.3 Lemma. Gegeben sei eine Entropie der Gestalt $\eta = \widehat{\eta}(\varepsilon, u_1, \dots, u_N)$ mit beliebigen Größen $u_1,\dots,u_n$. Es sei $\eta'_\varepsilon > 0$ und $\eta'_{\varepsilon\varepsilon} \neq 0$. Sei

$$\theta(\varepsilon, u_1, \dots, u_N) := \frac{1}{\eta'_\varepsilon(\varepsilon, u_1, \dots, u_N)}\,,$$
$$f(\theta, u_1, \dots, u_N) := \varepsilon - \theta\eta(\varepsilon, u_1, \dots, u_N) \text{ für } \theta = \theta(\varepsilon, u_1, \dots, u_N)\,.$$

Dann ist

$$\boxed{f'_\theta = -\eta\,, \quad f'_{u_i} = -\theta\eta'_{u_i}}$$

für $i = 1, \dots, N$ und außerdem gilt

$$f'_{\theta\theta}\eta'_{\varepsilon\varepsilon} = \theta^{-3} > 0\,.$$

Solution. Aus $f = \varepsilon - \theta\eta$ folgt, wenn man bezüglich ε ableitet,

$$f'_\theta \theta'_\varepsilon = 1 - \theta\eta'_\varepsilon - \theta'_\varepsilon \eta = -\theta'_\varepsilon \eta\,,$$

also

$$f'_\theta = -\eta\,.$$

Die Ableitung bezüglich u_i ergibt

$$f'_\theta \theta'_{u_i} + f'_{u_i} = -\theta\eta'_{u_i} - \theta'_{u_i}\eta\,,$$

also wegen $f'_\theta = -\eta$

$$f'_{u_i} = -\theta\eta'_{u_i}\,.$$

□

Energy

7.4 Energieerhaltung. Sei $\boldsymbol{\mu}_\xi$ wie in I.3.1 Lösung der Impulsgleichung wie dort. Zeige, dass dann für die kinetische Energie die Gleichung

$$\partial_t(\frac{m}{2}|v|^2\boldsymbol{\mu}_\xi) + \text{div}(\frac{m}{2}|v|^2 v\boldsymbol{\mu}_\xi) = v\bullet\mathbf{f}\boldsymbol{\mu}_\xi$$

im Raum der Distributionen gilt.

IV Various applications

After we have introduced the basic principles, we will now give some examples in which the objectivity and the entropy principle resp. the free energy inequality are used. We will also get to know some explicit solutions, which give an insight into the various applications.

1 Tidal period

We study the effect of gravity on the system of Earth and Moon. We look at this problem from the center of mass which can be considerd as inertial frame. This is an important usage of the notion “inertial frame”, which we defined in II.3.9, where we have already pointed out the fact, that inertial frames can be used only approximately. Because of the importance of this notion in physics we will come back to this fact in 1.1 and ask the question: What can the obsever do to exclude fictitious forces? In this example we use only the mass-momentum system of Earth and Moon, and in addition the equation of gravity.

So, we as observer with coordinates (t^*, x^*) see the two celestial bodies (*de:* Himmelskörper) from a certain position, let's say, $B_{\mathrm{Earth}}(t^*) \subset \mathbb{R}^3$ is the domain occupied by the Earth and $B_{\mathrm{Moon}}(t^*) \subset \mathbb{R}^3$ the one occupied by the Moon. The mass densities are $(t^*, x^*) \mapsto \varrho^*_{\mathrm{Earth}}(t^*, x^*)$, which is positive in $B_{\mathrm{Earth}}(t^*)$ and 0 outside $B_{\mathrm{Earth}}(t^*)$, and $(t^*, x^*) \mapsto \varrho^*_{\mathrm{Moon}}(t^*, x^*)$, which is positive in $B_{\mathrm{Moon}}(t^*)$ and 0 outside $B_{\mathrm{Moon}}(t^*)$. The total masses of Earth and Moon are given by

$$M_{\mathrm{Earth}}(t^*) := \int_{\mathbb{R}^3} \varrho^*_{\mathrm{Earth}}(t^*, x^*)\,\mathrm{d}x^*\,, \quad M_{\mathrm{Moon}}(t^*) := \int_{\mathbb{R}^3} \varrho^*_{\mathrm{Moon}}(t^*, x^*)\,\mathrm{d}x^*\,.$$

(Later in 1.1 we will show that the conservation of mass implies that M_{Earth} and M_{Moon} are constants.) Then

$$(t^*, x^*) \mapsto \varrho^*_{\mathrm{Earth}}(t^*, x^*) + \varrho^*_{\mathrm{Moon}}(t^*, x^*)$$

is the total mass distribution of Earth and Moon. The center of mass of the system (*de:* Schwerpunkt des Systems) we denote by x^*_0, defined by

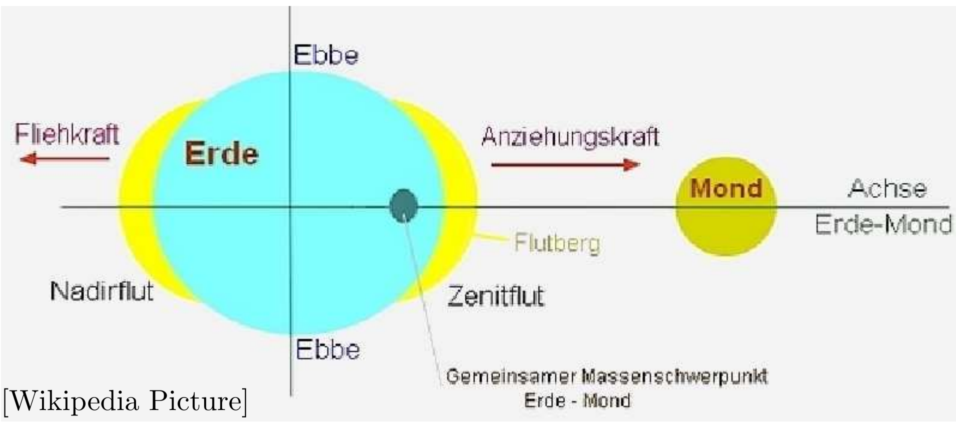
$$0 = \int_{\mathbb{R}^3} \Big(\varrho^*_{\mathrm{Earth}}(t^*, x^*) + \varrho^*_{\mathrm{Moon}}(t^*, x^*)\Big)(x^* - x^*_0)\,\mathrm{d}x^*\,. \tag{IV1.1}$$

For this system the mass and momentum equations are in the sense of distributions (we use the formulation for L^∞-functions)

$$\begin{aligned}
&\partial_{t^*}[\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}}] + \operatorname{div}_{x^*}[(\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}})v^*] = 0\,,\\
&\partial_{t^*}[(\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}})v^*] + \operatorname{div}_{x^*}[(\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}})v^*\,v^{*\mathrm{T}} + \Pi^*]\\
&\qquad = [\mathfrak{g}(\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}})\nabla\boldsymbol{\phi}^*]\,,\\
&\operatorname{div}_{x^*}(-[\nabla_{x^*}\boldsymbol{\phi}^*]) = [\varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}}]\,,
\end{aligned} \tag{IV1.2}$$

where the last equation is the equation for the gravity field $\boldsymbol{\phi}^*$. That the gravity of Moon and Earth is the only force term $\mathbf{f}^* := \mathfrak{g}(\varrho^*_{\text{Earth}}+\varrho^*_{\text{Moon}})\nabla\boldsymbol{\phi}^*$ means that the coordinate system (t^*, x^*) is supposed to be an inertial one (see the discussion in II.3.9 and the one following 1.1). It then follows (see Lemma 1.1) that the center of mass satisfies $\ddot{x}^*_0 = 0$, and we will then normalize the inertial observer by an uniform motion (*de:* gleichförmige Bewegung) so that $x^*_0 = 0$, that is, we let the observer be located at the center of mass.

Other effects like the gravity of the Sun are neglected (this has an effect mainly when Earth, Moon, and Sun are positioned on a line, which gives rise to a spring tide (*de:* Springflut)). Therefore besides the gravity of the two celestial objects there is no other source which influences the system essentially. Havng done so it follows that the main effects of this system are a combination of the centrifugal force (*de:* Fliehkraft) and the attractive force (*de:* Anziehungskraft). (To this see Fig. 1, [Wikipedia: Tide], [Wikipedia: Marée], and [136].) Both forces are in fact related to each other, which is in some sense obvious since in the equations (IV1.2) there is only the $\mathfrak{g}$-term. We mention that we do not consider any varying temperature, the movement we treat is purely mechanical.



[Wikipedia Picture]

Fig. 1: Tide (Observe that the axis of the Earth is inclined)

To come out with argumentations which are simpler, that is, they contain only the main effects, we make some special assumptions. We assume that Earth and Moon are both approximated by balls $B_{\text{Earth}}(t^*) = \mathrm{B}_{R_{\text{Earth}}}(x^*_{\text{Earth}}(t^*))$ and $B_{\text{Moon}}(t^*) = \mathrm{B}_{R_{\text{Moon}}}(x^*_{\text{Moon}}(t^*))$, where the mass densities $\varrho^*_{\text{Earth}}(t^*,\bullet)$ and $\varrho^*_{\text{Moon}}(t^*,\bullet)$ are constant in $B_{\text{Earth}}(t^*)$ resp. $B_{\text{Moon}}(t^*)$. Further it is assumed that v^* and Π^* is located in $B_{\text{Earth}}(t^*)$ resp. $B_{\text{Moon}}(t^*)$, where the bodies are rotating around themselves. It is shown in 16.5 that a rotation changes the shape of a celestial body. These rotations results for the Earth in a small change by around 3% of the ball, so we do not consider it here (or, which is not realistic, we assume that there is no rotation). Also we assume that the distance between Earth and Moon is constant. Indeed, the movement of them is like a Kepler ellipse (as in I.3.3), the distance varies between 356.400 km (perigee, *de:* Perigäum, Erdnähe) and 406.700 km (apogee, *de:* Apogäum, Erdferne), see [Wikipedia: Moon], but this difference in distance is only about 1.3%. To make things concrete, here is a list of constants:

$$\begin{aligned}
R_{\text{Earth}} &= 6371.0\,km \text{ (this is the mean radius)},\\
M_{\text{Earth}} &= 5.97219\cdot 10^{24}\,kg\,,\\
R_{\text{Moon}} &= 1737.10\,km = 0.273\,R_{\text{Earth}}\,,\\
M_{\text{Moon}} &= 7.3457\cdot 10^{22}\,kg = 0.012300\,M_{\text{Earth}}\,,\\
|x^*_{\text{Earth}} - x^*_{\text{Moon}}| &= 384.402\cdot 10^{3}\,km \text{ (mean semi-major axis)}.
\end{aligned}$$

The potential $\boldsymbol{\phi}^*$ can be split into $\boldsymbol{\phi}^* = \boldsymbol{\phi}^*_{\text{Earth}} + \boldsymbol{\phi}^*_{\text{Moon}}$ with

$$\operatorname{div}(-[\nabla\boldsymbol{\phi}^*_{\text{Earth}}]) = [\varrho^*_{\text{Earth}}]\,,\quad \operatorname{div}(-[\nabla\boldsymbol{\phi}^*_{\text{Moon}}]) = [\varrho^*_{\text{Moon}}]\,. \tag{IV1.3}$$

First we show

1.1 Lemma. Under the assumptions above the equations (IV1.2) imply that the centers x^*_{Earth} and x^*_{Moon} fulfill the differential equations

$$\begin{aligned}
\ddot{x}^*_{\text{Earth}} &= \mathfrak{g}\nabla\boldsymbol{\phi}^*_{\text{Moon}}(\bullet, x^*_{\text{Earth}})\,,\\
\ddot{x}^*_{\text{Moon}} &= \mathfrak{g}\nabla\boldsymbol{\phi}^*_{\text{Earth}}(\bullet, x^*_{\text{Moon}})\,.
\end{aligned}$$

The observer can be chosen so that $x^*_0 = 0$ and then (IV1.1) reads

$$0 = M_{\text{Earth}}x^*_{\text{Earth}}(t^*) + M_{\text{Moon}}x^*_{\text{Moon}}(t^*)\,, \tag{IV1.4}$$

where M_{Earth} and M_{Moon} turn out to be constants.

Hint: This choice of the observer makes his coordinates to an "inertial system". How he manages this: 1. Position: Center of mass of Earth and Moon. 2. Rotation: Orientation on stellar objects (not planets and not the sun). Then approximately no other forces are present.

Proof. Let $\varrho^* := \varrho^*_{\text{Earth}} + \varrho^*_{\text{Moon}}$ be the total mass. The mass and the momentum equation of (IV1.2) are in the weak sense for $\eta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^3;\mathbb{R})$

$$0 = \int_{\mathbb{R}^4} (\partial_{t^*}\eta + v^*\bullet\nabla\eta)\varrho^*\,\mathrm{d}(t^*,x^*) \tag{IV1.5}$$

and for $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^4} (\partial_{t^*}\zeta\bullet(\varrho^* v^*) + \mathrm{D}\zeta{:}(\varrho^* v^* v^{*\mathrm{T}} + \Pi^*))\,\mathrm{d}(t^*,x^*) \\ &+ \int_{\mathbb{R}^4} \zeta\bullet(\mathfrak{g}\varrho^*\nabla\boldsymbol{\phi}^*)\,\mathrm{d}(t^*,x^*)\,. \end{aligned} \tag{IV1.6}$$

First we treat the mass equation. Let us look at the Earth, that is, we only consider test functions η with support in a neighbourhood of $B^*_{\text{Earth}}(t^*)$. Translating this to the center of the Earth

$$\begin{aligned} &\mathrm{B}_{R_{\text{Earth}}}(0) = \{y\in\mathbb{R}^3;\ y + x^*_{\text{Earth}}(t^*) \in B^*_{\text{Earth}}(t^*)\}\,, \\ &\widetilde{\eta}(t^*,y) := \eta(t^*, y + x^*_{\text{Earth}}(t^*))\,, \\ &\widetilde{\varrho}^*(t^*,y) := \varrho^*(t^*, y + x^*_{\text{Earth}}(t^*))\,, \\ &\widetilde{v}^*(t^*,y) := v^*(t^*, y + x^*_{\text{Earth}}(t^*))\,, \end{aligned}$$

where $\widetilde{\eta}$ has support in a neighbourhood of $\overline{\mathrm{B}_{R_{\text{Earth}}}(0)}$. Since $\nabla_y\widetilde{\eta} = \nabla_{x^*}\eta$ and $\partial_{t^*}\widetilde{\eta} = \partial_{t^*}\eta + \dot{x}^*_{\text{Earth}}(t^*)\bullet\nabla_{x^*}\eta$ the transformed mass equation reads

$$0 = \int_{\mathbb{R}^4} \widetilde{\varrho}^*\big(\partial_{t^*}\widetilde{\eta} + (\widetilde{v}^* - \dot{x}^*_{\text{Earth}}(t^*))\bullet\nabla\widetilde{\eta}\big)\,\mathrm{d}(t^*,y)\,.$$

We now choose $\widetilde{\eta}(t^*,y) = b(t^*)\varphi(y)$ where $b\in C^\infty_0(\mathbb{R};\mathbb{R})$ and $\varphi\in C^\infty(\mathbb{R}^3;\mathbb{R})$ with $\varphi(y) = 1$ for $y \in \mathrm{B}_{R_{\text{Earth}}}(0)$ and support in a neighbourhood of $\mathrm{B}_{R_{\text{Earth}}}(0)$. With this the integral becomes since $\widetilde{\varrho}^*$ vanishes outside $\mathrm{B}_{R_{\text{Earth}}}(0)$

$$0 = \int_{\mathbb{R}^4} \widetilde{\varrho}^*\partial_{t^*}\widetilde{\eta}\,\mathrm{d}(t^*,y) = \int_{\mathbb{R}} b'(t^*) \underbrace{\int_{\mathrm{B}_{R_{\text{Earth}}}(0)} \varrho^*(t^*, y + x^*_{\text{Earth}}(t^*))\,\mathrm{d}y}_{=M_{\text{Earth}}(t^*)}\ \mathrm{d}t^*\,.$$

Since b is an arbitrary test function, it follows that M_{Earth} is constant. Next we choose $\widetilde{\eta}(t^*,y) = a(t^*)\bullet\psi(y)$ with $a\in C^\infty_0(\mathbb{R};\mathbb{R}^3)$ and $\psi(y) = y$ for $y\in\mathrm{B}_{R_{\text{Earth}}}(0)$ and support in a neighbourhood of $\mathrm{B}_{R_{\text{Earth}}}(0)$. Then the integral becomes

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^4} \widetilde{\varrho}^* \big(\partial_{t^*}\widetilde{\eta} + (\widetilde{v}^* - \dot{x}^*_{\text{Earth}})\bullet\nabla\widetilde{\eta}\big)\,\mathrm{d}(t^*,y) \\
&= \int_{\mathbb{R}}\int_{\mathrm{B}_{R_{\text{Earth}}}(0)} \varrho^*_{\text{Earth}}\big(a'(t^*)\bullet y + (\widetilde{v}^*(t^*,y) - \dot{x}^*_{\text{Earth}}(t^*))\bullet a(t^*)\big)\,\mathrm{d}y\,\mathrm{d}t^* \\
&= \int_{\mathbb{R}} a'(t^*)\bullet \underbrace{\int_{\mathrm{B}_{R_{\text{Earth}}}(0)} \varrho^*_{\text{Earth}}\, y\,\mathrm{d}y}_{=\,0}\;\mathrm{d}t^* \\
&\qquad\qquad + \int_{\mathbb{R}} a(t^*)\bullet\Big(\int_{\mathrm{B}_{R_{\text{Earth}}}(0)} \varrho^*_{\text{Earth}}(\widetilde{v}^*(t^*,y) - \dot{x}^*_{\text{Earth}}(t^*))\,\mathrm{d}y\Big)\,\mathrm{d}t^* \\
&= \int_{\mathbb{R}} a(t^*)\bullet\Big(\frac{M_{\text{Earth}}}{\mathrm{L}^3(\mathrm{B}_{R_{\text{Earth}}}(0))}\int_{\mathrm{B}_{R_{\text{Earth}}}(0)} \big(\widetilde{v}^*(t^*,y) - \dot{x}^*_{\text{Earth}}(t^*)\big)\,\mathrm{d}y\Big)\,\mathrm{d}t^* \\
&= M_{\text{Earth}}\int_{\mathbb{R}} a(t^*)\bullet\Big(\frac{3}{4\pi R^3_{\text{Earth}}}\int_{B_{\text{Earth}}(t^*)} v^*(t^*,x^*)\,\mathrm{d}x^* - \dot{x}^*_{\text{Earth}}(t^*)\Big)\,\mathrm{d}t^*\,.
\end{aligned}
$$

Since a is arbitrary we conclude

$$
\dot{x}^*_{\text{Earth}}(t^*) = v^*_{\text{Earth}}(t^*) := \frac{3}{4\pi R^3_{\text{Earth}}}\int_{B_{\text{Earth}}(t^*)} v^*(t^*,x^*)\,\mathrm{d}x^*\,. \tag{IV1.7}
$$

Similar we obtain by looking at the Moon that M_{Moon} is constant and

$$
\dot{x}^*_{\text{Moon}}(t^*) = v^*_{\text{Moon}}(t^*) := \frac{3}{4\pi R^3_{\text{Moon}}}\int_{B_{\text{Moon}}(t^*)} v^*(t^*,x^*)\,\mathrm{d}x^*\,.
$$

In these integrals an arbitrary rotation of Earth and Moon is included, because we assume that these bodies are spherically symmetric (but on the other hand a rotation leads to an ellipsoidal shape as shown by Newton, see 16.5).

We now come to the second and third equation of (IV1.2), the conservation of momentum paired with the gravitational law. According to the support of the Earth we write

$$
\begin{aligned}
\widetilde{\zeta}(t^*,y) &:= \zeta(t^*, y + x^*_{\text{Earth}}(t^*))\,,\\
\widetilde{\boldsymbol{\phi}}^*(t^*,y) &:= \boldsymbol{\phi}^*(t^*, y + x^*_{\text{Earth}}(t^*))\,,\\
\widetilde{v}^*(t^*,y) &:= v^*(t^*, y + x^*_{\text{Earth}}(t^*))\,,\\
\widetilde{\Pi}^*(t^*,y) &:= \Pi^*(t^*, y + x^*_{\text{Earth}}(t^*))\,,
\end{aligned}
$$

where $\widetilde{\zeta}$ has support in a neighbourhood of $\mathrm{B}_{R_{\text{Earth}}}(0)$. Since $\Pi^* = \Pi^*_{\text{Earth}} + \Pi^*_{\text{Moon}}$, where Π^*_{Earth} concentrates on the Earth and Π^*_{Moon} on the Moon, we obtain for the momentum equation

$$0 = \int_{\mathbb{R}\times \mathrm{B}_{R_{\mathrm{Earth}}}(0)} \Big(\partial_{t^*}\widetilde{\zeta}\bullet(\varrho^*_{\mathrm{Earth}}\widetilde{v}^*) \\ + \mathrm{D}\widetilde{\zeta}\,\boldsymbol{:}\Big(\varrho^*_{\mathrm{Earth}}\widetilde{v}^*\,(\widetilde{v}^* - \dot{x}^*_{\mathrm{Earth}})^{\mathrm{T}} + \widetilde{\Pi}^*_{\mathrm{Earth}}\Big)\Big)\,\mathrm{d}(t^*,y) \\ + \int_{\mathbb{R}\times \mathrm{B}_{R_{\mathrm{Earth}}}(0)} \widetilde{\zeta}\bullet(\mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\widetilde{\boldsymbol{\phi}}^*)\,\mathrm{d}(t^*,y)\,.$$

Choose now $\widetilde{\zeta}(t^*,y) := \psi(y)b(t^*)$ with $b \in C^\infty_0(\mathbb{R};\mathbb{R}^3)$ and $\psi \in C^\infty_0(\mathbb{R}^3;\mathbb{R})$ with support in a neighbourhood of $\mathrm{B}_{R_{\mathrm{Earth}}}(0)$ and $\psi(y) = 1$ for y in the closure of $\mathrm{B}_{R_{\mathrm{Earth}}}(0)$. With this the integral becomes (the term with $\mathrm{D}\widetilde{\zeta}$ vanishes)

$$0 = \int_{\mathbb{R}} b'(t^*)\bullet\int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \varrho^*_{\mathrm{Earth}}\widetilde{v}^*\,\mathrm{d}y\,\mathrm{d}t^* + \int_{\mathbb{R}} b(t^*)\bullet\int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\widetilde{\boldsymbol{\phi}}^*\,\mathrm{d}y\,\mathrm{d}t^*$$

(now do partial integration in t^*)

$$= \int_{\mathbb{R}} b(t^*)\bullet\Big(-\frac{\mathrm{d}}{\mathrm{d}t^*}\Big(\int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \varrho^*_{\mathrm{Earth}}\widetilde{v}^*\,\mathrm{d}y\Big) + \int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\widetilde{\boldsymbol{\phi}}^*\,\mathrm{d}y\Big)\,\mathrm{d}t^*\,.$$

Since b is an arbitrary test function we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t^*}\Big(\int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \varrho^*_{\mathrm{Earth}}\widetilde{v}^*\,\mathrm{d}y\Big) = \int_{\mathrm{B}_{R_{\mathrm{Earth}}}(0)} \mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\widetilde{\boldsymbol{\phi}}^*\,\mathrm{d}y$$

or in the original coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t^*}\Big(\int_{B_{\mathrm{Earth}}(t^*)} \varrho^*_{\mathrm{Earth}}v^*\,\mathrm{d}x1^*\Big) = \int_{B_{\mathrm{Earth}}(t^*)} \mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\boldsymbol{\phi}^*\,\mathrm{d}x^*\,.$$

Using (IV1.7) we remark that

$$\int_{B_{\mathrm{Earth}}(t^*)} \varrho^*_{\mathrm{Earth}}v^*\,\mathrm{d}x^* = \frac{3M_{\mathrm{Earth}}}{4\pi R^3_{\mathrm{Earth}}}\int_{B_{\mathrm{Earth}}(t^*)} v^*\,\mathrm{d}x^* = M_{\mathrm{Earth}}\dot{x}^*_{\mathrm{Earth}}\,,$$

and we obtain

$$(M_{\mathrm{Earth}}\dot{x}^*_{\mathrm{Earth}})^{\boldsymbol{\cdot}} = \int_{B_{\mathrm{Earth}}(t^*)} \mathfrak{g}\varrho^*_{\mathrm{Earth}}\nabla\boldsymbol{\phi}^*\,\mathrm{d}x^* \\ = \frac{3\mathfrak{g}M_{\mathrm{Earth}}}{4\pi R^3_{\mathrm{Earth}}}\Big(\underbrace{\int_{B_{\mathrm{Earth}}(t^*)} \nabla\boldsymbol{\phi}^*_{\mathrm{Earth}}\,\mathrm{d}x^*}_{=\,0\text{ (see the text)}} + \int_{B_{\mathrm{Earth}}(t^*)} \nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}\,\mathrm{d}x^*\Big)\,.$$

Since $\nabla\boldsymbol{\phi}^*_{\mathrm{Earth}}(t^*,x^*) = C(x^*_{\mathrm{Earth}}(t^*) - x^*)$ with a constant C by (I2.21), the first integral vanishes. Since $x^* \mapsto \boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*,x^*)$ is harmonic in $B_{\mathrm{Earth}}(t^*)$

hence also the components of $x^* \mapsto \nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*, x^*)$ are harmonic, we conclude from the mean value property of harmonic functions

$$\frac{3}{4\pi R^3_{\mathrm{Earth}}} \int_{B_{\mathrm{Earth}}(t^*)} \nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*, x^*)\,\mathrm{d}x^* = \nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*, x^*_{\mathrm{Earth}}(t^*))\,.$$

Therefore

$$(M_{\mathrm{Earth}}\dot{x}^*_{\mathrm{Earth}})^{\cdot}\,(t^*) = \mathfrak{g}M_{\mathrm{Earth}}\nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*, x^*_{\mathrm{Earth}}(t^*))\,.$$

In the same way it follows near the Moon

$$(M_{\mathrm{Moon}}\dot{x}^*_{\mathrm{Moon}})^{\cdot}\,(t^*) = \mathfrak{g}M_{\mathrm{Moon}}\nabla\boldsymbol{\phi}^*_{\mathrm{Earth}}(t^*, x^*_{\mathrm{Moon}}(t^*))\,,$$

and the sum of both terms is

$$\begin{aligned}&\mathfrak{g}\big(M_{\mathrm{Earth}}\nabla\boldsymbol{\phi}^*_{\mathrm{Moon}}(t^*, x^*_{\mathrm{Earth}}(t^*)) + M_{\mathrm{Moon}}\nabla\boldsymbol{\phi}^*_{\mathrm{Earth}}(t^*, x^*_{\mathrm{Moon}}(t^*))\big)\\ &= \mathfrak{g}M_{\mathrm{Earth}}M_{\mathrm{Moon}}\Big(\frac{x^*_{\mathrm{Moon}}(t^*) - x^*_{\mathrm{Earth}}(t^*)}{4\pi D^3} + \frac{x^*_{\mathrm{Earth}}(t^*) - x^*_{\mathrm{Moon}}(t^*)}{4\pi D^3}\Big) = 0\,,\end{aligned}$$

thus $\ddot{x}^*_0 = 0$. Hence without producing a fictitious force the observer can be chosen so that $x^*_0 = 0$. □

Thus the two centers perform a movement in a two dimensional plane, which we called to be the (x^*_1, x^*_2)-plane. Since we assume a constant distance between Moon and Earth, $x^*_{\mathrm{Moon}}(t^*) - x^*_{\mathrm{Earth}}(t^*)$ moves on a circular line in the (x^*_1, x^*_2)-plane. This means that the eccentricity 0.0549 of the Moon is set to be zero.

1.2 Lemma. For a circular movement of $x^*_{\mathrm{Moon}}(t^*) - x^*_{\mathrm{Earth}}(t^*)$ in the (x^*_1, x^*_2)-plane we consider different variables with the transformation

$$\begin{aligned}&t = t^*\,, \quad x = Q(t^*)x^*\,,\\ &Q(t^*) = \begin{bmatrix}\cos(\omega t^*) & \sin(\omega t^*) & 0\\ -\sin(\omega t^*) & \cos(\omega t^*) & 0\\ 0 & 0 & 1\end{bmatrix},\\ &\boxed{\omega^2 = \frac{GM}{D^3}\,, \quad M := M_{\mathrm{Earth}} + M_{\mathrm{Moon}}\,,}\end{aligned} \tag{IV1.8}$$

where D is the (constant) distance between Moon and Earth. Then

$$x_{\mathrm{Earth}} := Q(t^*)x^*_{\mathrm{Earth}}(t^*)\,, \quad x_{\mathrm{Moon}} := Q(t^*)x^*_{\mathrm{Moon}}(t^*)$$

are constant vectors with

$$0 = M_{\mathrm{Earth}}x_{\mathrm{Earth}} + M_{\mathrm{Moon}}x_{\mathrm{Moon}}\,, \quad D := |x_{\mathrm{Moon}} - x_{\mathrm{Earth}}|\,.$$

We can choose the coordinates so, that both points x_{Earth} and x_{Moon} lie on the x_1-axis with negative value of x_{Earth} and positive value of x_{Moon}.

Proof. From (IV1.4) it follows that

$$\frac{M}{M_{\text{Earth}}} x^*_{\text{Moon}}(t^*) = x^*_{\text{Moon}}(t^*) - x^*_{\text{Earth}}(t^*) = -\frac{M}{M_{\text{Moon}}} x^*_{\text{Earth}}(t^*) \,. \tag{IV1.9}$$

The differential equations for the motions of Earth and Moon are

$$\begin{aligned}\ddot{x}^*_{\text{Earth}} &= \mathfrak{g}\nabla\phi^*_{\text{Moon}}(\bullet, x^*_{\text{Earth}}) = GM_{\text{Moon}} \frac{x^*_{\text{Moon}} - x^*_{\text{Earth}}}{|x^*_{\text{Moon}} - x^*_{\text{Earth}}|^3} \\ &= \frac{GM_{\text{Moon}}}{D^3}(x^*_{\text{Moon}} - x^*_{\text{Earth}}) = -\frac{GM_{\text{Moon}}}{D^3}\frac{M}{M_{\text{Moon}}} x^*_{\text{Earth}} \,,\end{aligned}$$

hence

$$\ddot{x}^*_{\text{Earth}} = -\frac{GM}{D^3} x^*_{\text{Earth}} \,.$$

Similarly

$$\begin{aligned}\ddot{x}^*_{\text{Moon}} &= \mathfrak{g}\nabla\phi^*_{\text{Earth}}(\bullet, x^*_{\text{Moon}}) = GM_{\text{Moon}} \frac{x^*_{\text{Earth}} - x^*_{\text{Moon}}}{|x^*_{\text{Earth}} - x^*_{\text{Moon}}|^3} \\ &= \frac{GM_{\text{Moon}}}{D^3}(x^*_{\text{Earth}} - x^*_{\text{Moon}}) = -\frac{GM_{\text{Earth}}}{D^3}\frac{M}{M_{\text{Earth}}} x^*_{\text{Moon}} \,,\end{aligned}$$

hence

$$\ddot{x}^*_{\text{Moon}} = -\frac{GM}{D^3} x^*_{\text{Moon}} \,.$$

Since $D = |x^*_{\text{Earth}} - x^*_{\text{Moon}}|$ by assumption is constant,

$$\omega := \sqrt{\frac{GM}{D^3}}$$

defines a constant ω and the differential equations are satisfied for

$$\begin{aligned} &x^*_{\text{Earth}}(t^*) = Q^* x_{\text{Earth}} \,, \quad x^*_{\text{Moon}}(t^*) = Q^* x_{\text{Moon}} \,, \\ &Q^* = \begin{bmatrix} \cos(\omega t^*) & -\sin(\omega t^*) & 0 \\ \sin(\omega t^*) & \cos(\omega t^*) & 0 \\ 0 & 0 & 1 \end{bmatrix} \,, \end{aligned}$$

with constant vectors x_{Earth} and x_{Moon} if the coordinate x^*_3 is chosen appropriately. In fact, $\ddot{x}^*_{\text{Earth}} = -\omega^2 x^*_{\text{Earth}}$ and $\ddot{x}^*_{\text{Moon}} = -\omega^2 x^*_{\text{Moon}}$. If we let

$$Q(t^*) := Q^{*-1}$$

we have proved the assertion and $|x^*_{\text{Earth}} - x^*_{\text{Moon}}| = |x_{\text{Earth}} - x_{\text{Moon}}|$. □

Thus in the new coordinates we turn with the same speed as the Moon surrounds the Earth, it is a period of approximately 27.3 days,

$$\omega_{\text{Moon}} = \frac{2\pi}{P_{\text{Moon}}} \,, \quad P_{\text{Moon}} = 27.321582\ d \quad (\text{“sidereal period”}) \,.$$

However our ω is defined in a different way,

$$\omega = \frac{(GM)^{\frac{1}{2}}}{D^{\frac{3}{2}}} \text{ with } G = 6.67384 \cdot 10^{-11} \frac{m^3}{kg\, s^2}\,,$$
$$M = M_{\text{Earth}} + M_{\text{Moon}} = 5.97219 \cdot 10^{24} \cdot (1 + 0.0123)\, kg\,,$$
$$D := |x_{\text{Moon}} - x_{\text{Earth}}| = 384399\, km = 3.84399 \cdot 10^8\, m\,,$$

so that

$$\omega = \frac{\big(6.67384 \cdot 10^{-11} \cdot 5.97219 \cdot 10^{24} \cdot (1 + 0.0123)\big)^{\frac{1}{2}}}{\big(3.84399 \cdot 10^8\big)^{\frac{3}{2}}} \frac{1}{s}$$
$$= \sqrt{\frac{6.67384 \cdot 5.97219 \cdot 1.0123}{(3.84399)^3 \cdot 10^{11}}}\frac{1}{s} = \frac{1}{3.7520 \cdot 10^5\, s} = \frac{2\pi}{27.285\, d} \approx \omega_{\text{Moon}}\,.$$

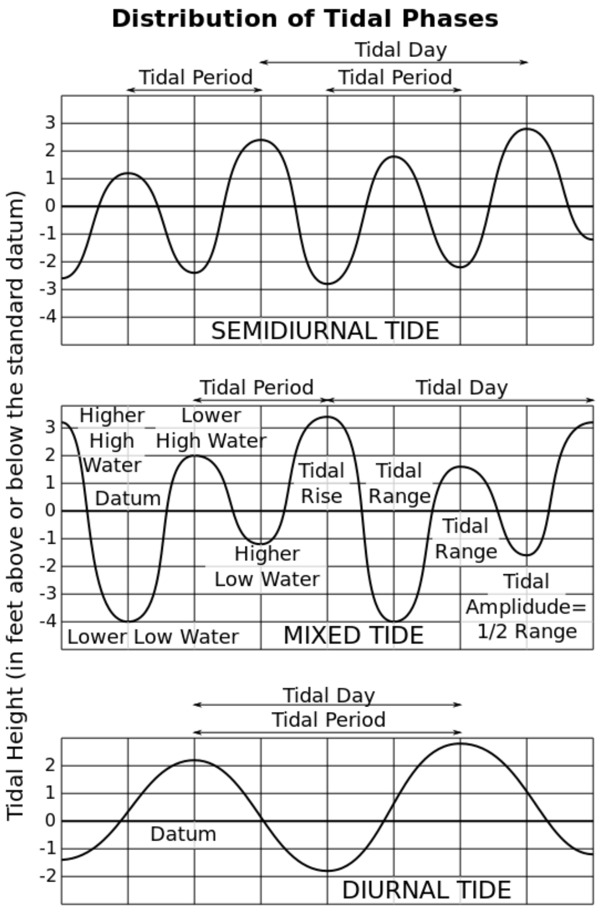


Fig. 2: Temporal variation of ebb and high tide (from Wikipedia).

Let us now focus again on the momentum equation near the Earth, which tells us more about the tidal period. In particular, it answers the question of two high tides during a tidal day, which is due to a combination of centrifugal force and force of attraction, both induced by gravity.

1.3 Theorem. Let $d_{\text{Earth}} < 0 < d_{\text{Moon}}$, where $x_{\text{Earth}} = d_{\text{Earth}}\mathbf{e}_1$ and $x_{\text{Moon}} = d_{\text{Moon}}\mathbf{e}_1$, hence $M_{\text{Earth}}d_{\text{Earth}} + M_{\text{Moon}}d_{\text{Moon}} = 0$. Then the momentum equation in (IV1.2) near the Earth $\mathrm{B}_{R_{\text{Earth}}}(x_{\text{Earth}})$ becomes

$$\begin{aligned}
&\partial_t(\varrho_{\text{Earth}}v) + \operatorname{div}_x(\varrho_{\text{Earth}}v\,v^{\mathrm{T}} + \Pi_{\text{Earth}}) = \mathbf{f}\,,\\
&\mathbf{f} = \mathbf{f}_{tide} + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Earth}} + 2\omega\varrho_{\text{Earth}}(-v_2, v_1, 0)\,,\\
&\mathbf{f}_{tide} = \omega^2\varrho_{\text{Earth}}(x_1, x_2, 0) + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Moon}}\,,\\
&\omega^2 = \frac{GM}{D^3}\,,\quad M = M_{\text{Earth}} + M_{\text{Moon}}\,,\quad D = d_{\text{Moon}} - d_{\text{Earth}}\,,
\end{aligned}$$

where $\mathbf{f}_{tide}$ is the force induced by the gravity between Moon and Earth. *Remark:* The explicit form of $\mathbf{f}_{tide}$ will be shown in 1.4.

Proof. The momentum equation in (IV1.2) in a neighbourhood of the Earth reads

$$\partial_{t^*}(\varrho^*_{\text{Earth}}v^*) + \operatorname{div}_{x^*}(\varrho^*_{\text{Earth}}v^*\,v^{*\mathrm{T}} + \Pi^*_{\text{Earth}}) = \mathbf{f}^* := \mathfrak{g}\varrho^*_{\text{Earth}}\nabla\boldsymbol{\phi}^*\,,$$

where $\boldsymbol{\phi}^* = \boldsymbol{\phi}^*_{\text{Earth}} + \boldsymbol{\phi}^*_{\text{Moon}}$ is the gravity potential of Earth and Moon. We transform the coordinates to $(t,x) = Y(t^*,x^*) = (t^*, Q(t^*)x^*)$, where $Q(t^*)$ is defined in 1.2. Then the transformed momentum equation is

$$\partial_t(\varrho_{\text{Earth}}v) + \operatorname{div}_x(\varrho_{\text{Earth}}v\,v^{\mathrm{T}} + \Pi_{\text{Earth}}) = \mathbf{f}\,,$$

where the quantities are given by

$$\begin{aligned}
&\varrho\circ Y = \varrho^*\,,\quad \boldsymbol{\phi}\circ Y = \boldsymbol{\phi}^*\,,\\
&v\circ Y = \dot{X} + Qv^*\,,\quad \Pi\circ Y = Q\Pi^*Q^{\mathrm{T}}\,,
\end{aligned}$$

and where $\mathbf{f}$ is given by (not writing arguments)

$$\begin{aligned}
\mathbf{f}\circ Y &= \varrho^*(\ddot{X} + 2\dot{Q}v^*) + Q\mathbf{f}^*\,,\\
\varrho^*(\ddot{X} + 2\dot{Q}v^*) &= \varrho^*(\ddot{X} + 2\dot{Q}Q^{\mathrm{T}}(v - \dot{X}))\\
&= \varrho((\ddot{Q} - 2\dot{Q}Q^{\mathrm{T}}\dot{Q})x^* + 2\dot{Q}Q^{\mathrm{T}}v)\\
&= \varrho((\ddot{Q}Q^{\mathrm{T}} - 2(\dot{Q}Q^{\mathrm{T}})^2)x + 2\dot{Q}Q^{\mathrm{T}}v)\,,\\
Q\mathbf{f}^* &= \mathfrak{g}\varrho^*_{\text{Earth}}Q\nabla\boldsymbol{\phi}^* = \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}\\
&= \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Earth}} + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Moon}}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{f} &= \varrho_{\text{Earth}}((\ddot{Q}Q^{\mathrm{T}} - 2(\dot{Q}Q^{\mathrm{T}})^2)x + 2\dot{Q}Q^{\mathrm{T}}v)\\
&\quad + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Earth}} + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Moon}}\\
&= \mathbf{f}_{tide} + 2\varrho_{\text{Earth}}\dot{Q}Q^{\mathrm{T}}v + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Earth}}\,,
\end{aligned}$$

where

$$\mathbf{f}_{tide} := \varrho_{\text{Earth}}(\ddot{Q}Q^{\mathrm{T}} - 2(\dot{Q}Q^{\mathrm{T}})^2)x + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Moon}}\,.$$

Since $t^* \mapsto Q(t^*)$ is given, we compute

$$\ddot{Q}Q^{\mathrm{T}} - 2(\dot{Q}Q^{\mathrm{T}})^2 = \omega^2 I\,, \quad 2\dot{Q}Q^{\mathrm{T}} = 2\omega A\,,$$
$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$Ix = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \quad Av = \begin{bmatrix} -v_2 \\ v_1 \\ 0 \end{bmatrix}.$$

Therefore we obtain

$$\mathbf{f} = \mathbf{f}_{tide} + 2\omega\varrho_{\text{Earth}}(-v_2, v_1, 0) + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Earth}}\,,$$
$$\mathbf{f}_{tide} = \omega^2\varrho_{\text{Earth}}(x_1, x_2, 0) + \mathfrak{g}\varrho_{\text{Earth}}\nabla\boldsymbol{\phi}_{\text{Moon}}\,.$$

□

On the surface of the Earth $\mathbf{f}_{tide}$ has besides a term which shows in the direction of the normal two terms, which are directed towards the two vectors $(R_{\text{Earth}}, 0, 0)$ and $(-R_{\text{Earth}}, 0, 0)$ in an almost symmetric way.

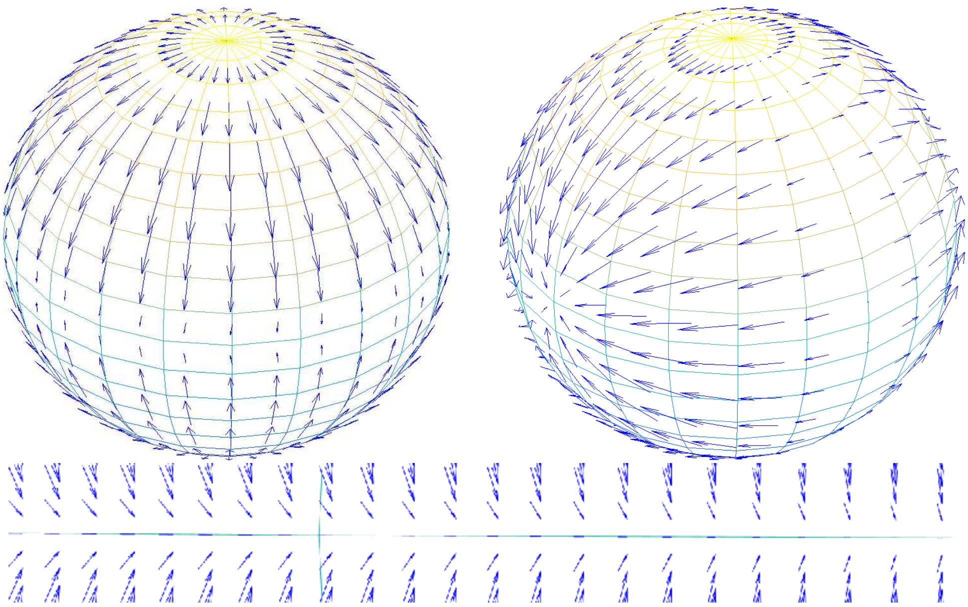


Fig. 3: Tides on the surface of the Earth (plotted are tangential components). *Left:* $\mathbf{a}_{tide}$ all terms. *Below:* Detail near the plain of the surrounding Moon. *Right:* First summand of $\mathbf{a}_{tide}$.

1.4 Lemma. With $x = x_{\text{Earth}} + R_{\text{Earth}}\xi$ with $|\xi| = 1$ the tidal acceleration $\mathbf{a}_{tide}$ defined by

$$\mathbf{f}_{tide} = \varrho_{\text{Earth}}\mathbf{a}_{tide}$$

satisfies the following identity:

$$\mathbf{a}_{tide}(x) = \Big(\frac{GM_{\text{Moon}}}{|x_{\text{Moon}} - x|^3} - \frac{GM_{\text{Moon}}}{|x_{\text{Moon}} - x_{\text{Earth}}|^3}\Big)\begin{bmatrix} |x_{\text{Moon}} - x_{\text{Earth}}| \\ 0 \\ 0 \end{bmatrix}$$

$$-\frac{GMR_{\text{Earth}}}{|x_{\text{Moon}} - x_{\text{Earth}}|^3}\begin{bmatrix} 0 \\ 0 \\ \xi_3 \end{bmatrix} + R_{\text{Earth}}\Big(\frac{GM}{|x_{\text{Moon}} - x_{\text{Earth}}|^3} - \frac{GM_{\text{Moon}}}{|x_{\text{Moon}} - x|^3}\Big)\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}.$$

Proof. Writing $x = x_{\text{Earth}} + R_{\text{Earth}}\xi$ with $|\xi| \le 1$ one obtains

$$\begin{aligned}\mathbf{a}_{tide} &= \frac{\mathbf{f}_{tide}}{\varrho_{\text{Earth}}} = \omega^2(x_1, x_2, 0) + \mathfrak{g}\nabla\phi_{\text{Moon}} \\ &= \frac{GM}{|x_{\text{Moon}} - x_{\text{Earth}}|^3}(x_1, x_2, 0) + \frac{GM_{\text{Moon}}}{|x_{\text{Moon}} - x|^3}(x_{\text{Moon}} - x),\end{aligned}$$

hence

$$\frac{\mathbf{a}_{tide}}{GM_{\text{Moon}}} = \frac{M}{M_{\text{Moon}}|x_{\text{Moon}} - x_{\text{Earth}}|^3}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \frac{1}{|x_{\text{Moon}} - x|^3}\begin{bmatrix} d_{\text{Moon}} - x_1 \\ -x_2 \\ -x_3 \end{bmatrix}.$$

Since

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_{\text{Earth}} + R_{\text{Earth}}\xi_1 \\ R_{\text{Earth}}\xi_2 \\ R_{\text{Earth}}\xi_3 \end{bmatrix} \quad \text{and} \quad -\frac{M}{M_{\text{Moon}}}d_{\text{Earth}} = d_{\text{Moon}} - d_{\text{Earth}}$$

by (IV1.9), this is

$$\begin{aligned} &= \frac{M}{M_{\text{Moon}}|x_{\text{Moon}} - x_{\text{Earth}}|^3}\begin{bmatrix} d_{\text{Earth}} + R_{\text{Earth}}\xi_1 \\ R_{\text{Earth}}\xi_2 \\ 0 \end{bmatrix} \\ &\quad + \frac{1}{|x_{\text{Moon}} - x|^3}\begin{bmatrix} d_{\text{Moon}} - d_{\text{Earth}} - R_{\text{Earth}}\xi_1 \\ -R_{\text{Earth}}\xi_2 \\ -R_{\text{Earth}}\xi_3 \end{bmatrix} \\ &= \Big(\frac{1}{|x_{\text{Moon}} - x|^3} - \frac{1}{|x_{\text{Moon}} - x_{\text{Earth}}|^3}\Big)\begin{bmatrix} d_{\text{Moon}} - d_{\text{Earth}} \\ 0 \\ 0 \end{bmatrix} \\ &\quad - \frac{M}{M_{\text{Moon}}|x_{\text{Moon}} - x_{\text{Earth}}|^3}\begin{bmatrix} 0 \\ 0 \\ R_{\text{Earth}}\xi_3 \end{bmatrix} \\ &\quad + \Big(\frac{M}{M_{\text{Moon}}|x_{\text{Moon}} - x_{\text{Earth}}|^3} - \frac{1}{|x_{\text{Moon}} - x|^3}\Big)\begin{bmatrix} R_{\text{Earth}}\xi_1 \\ R_{\text{Earth}}\xi_2 \\ R_{\text{Earth}}\xi_3 \end{bmatrix}. \end{aligned}$$

□

In Abbildung Fig. 3 ist die tangentiale Kraft gezeigt. Der Anteil der normal zur Oberfläche wirkenden Kraft ist auf der Erdobefläche nicht vom Gravitationsfeld der Erde zu unterscheiden, er ist vielmehr von dieser abzuziehen (was eine Störung der Erdanziehung von der Größe $\leq 10^{-4}\frac{m}{s^2}$ bedeutet, siehe hierzu das Ende von Abschnitt I.4).

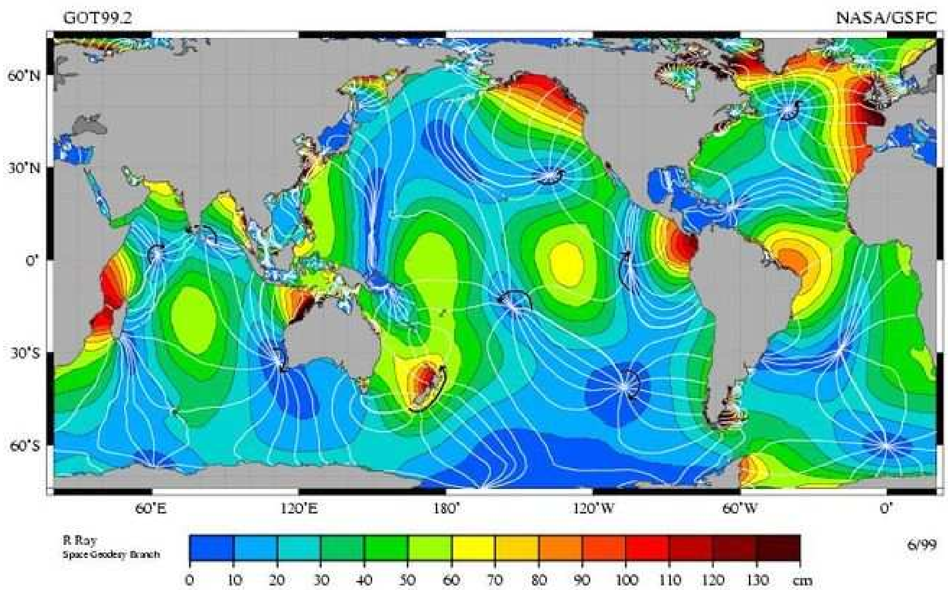


Fig. 4: Tide actually on Earth.

Bei der Auswirkung der Gezeiten auf den Meeresspiegel ist zu berücksichtigen, dass die Erde pro Tag eine Drehung um die eigene Achse macht, wobei die Erdachse nicht senkrecht auf der Verbindung Erde-Mond (in Fig. 1 gezeigt) steht, sondern gegenüber der Bahn des Mondes geneigt ist. Deshalb ist der Einfluß nicht einfach zu berechnen, zumal auch die Landverteilung in der Realität zur Höhe des Meeresspiegels beiträgt (siehe Fig. 4). Die Erdrotation hat auch zur Folge, dass in jedem auch inneren Punkt der Erde der Abstand vom Mond innerhalb eines Tages variiert und somit die Kraft auf den betreffenden Punkt nicht konstant ist. Das hat wiederum zur Konsequenz, dass die Erde permanent etwas "durchgeknetet" wird und dadurch Wärme entsteht.

2 Fluids and gases

In this section, the equations for a liquid or a gas are treated, that means the equations for mass, momentum and energy. We have already considered these equations in (III2.5) and for

$$\mathbf{J} = 0, \quad \mathbf{r} = 0, \quad \Pi \text{ symmetric}, \quad g = 0 \tag{IV2.1}$$

(these specifications are possible due to the objectivity of the quantities) they have the following form:

Mass-momentum-energy system:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}(e v + \Pi^{\mathrm{T}}\, v + q) = v \bullet \mathbf{f} \tag{IV2.2}$$

ϱ mass, v velocity, ϱv momentum,

ε internal energy, $e = \varepsilon + \frac{\varrho}{2}|v|^2$ total energy,

Π pressure tensor, q heat flux, $\mathbf{f}$ force.

The variables are $(\varrho, v, \varepsilon)$ are the quantities we are searching for, and for Π and q we use constitutive equations (the symmetry of Π implies $\Pi^{\mathrm{T}} = \Pi$), which are subject to restrictions due to the entropy principle. This has been shown in III.2.4 in a general setting. It has been shown that the entropie inequality implies certain equalities as well as a residual inequalilty. And there is the possibility of using the temperature (see III.1.4 or III.1.6)

$$\theta = \frac{1}{\eta'_{\varepsilon}(\varrho, \varepsilon)}\,, \quad \eta'_{\varepsilon}(\varrho, \varepsilon) > 0$$

instead of ε. This means that now the variables (ϱ, v, θ) are the unknown variables instead of $(\varrho, v, \varepsilon)$. We assume that this transformation is bijective. Furthermore (see also III.1.4 or III.1.6), the internal free energy f in the new variables (ϱ, θ) is given by

$$f = \varepsilon - \theta\eta\,, \tag{IV2.3}$$

in detail, since $f = \widehat{f}(\varrho, \theta)$,

$$\widehat{f}(\varrho, \theta) = \varepsilon - \theta\widehat{\eta}(\varrho, \varepsilon) \quad \text{for} \quad \theta\widehat{\eta}'_{\varepsilon}(\varrho, \varepsilon) = 1\,. \tag{IV2.4}$$

We will show in 2.1 below that $\eta(\varrho, \varepsilon) = -f'_{\theta}(\varrho, \theta)$. Therefore we can consider ε in the new variables

$$\varepsilon = \widetilde{\varepsilon}(\varrho, \theta) = \widehat{f}(\varrho, \theta) - \theta\widehat{f}'_{\theta}(\varrho, \theta)\,,$$

and for the Gibbs relation, a consequence of the entropy principle, also in the new variables

$$p = \widehat{p}(\varrho, \theta) = \varrho \widehat{f}{}'_{\varrho}(\varrho, \theta) - \widehat{f}(\varrho, \theta) \,.$$

We prove that for the Gibbs relation the following lemma holds.

2.1 Gibbs relation. Let $\eta = \widehat{\eta}(\varrho, \varepsilon)$ and $f = \widehat{f}(\varrho, \theta)$ with $\theta > 0$. Assume (IV2.3) holds, where the variables depend on each other. Then the equations

$$\boxed{\eta - \varrho\eta'_{\varrho} - (\varepsilon + p)\eta'_{\varepsilon} = 0 \,, \quad \theta\, \eta'_{\varepsilon} = 1} \tag{IV2.5}$$

are equivalent to

$$\boxed{p = \varrho f'_{\varrho} - f \,, \quad \varepsilon = f - \theta f'_{\theta} \,.} \tag{IV2.6}$$

Here the following is satisfied

$$f'_{\theta} = -\eta \,, \quad f'_{\varrho} = -\theta\eta'_{\varrho} \,.$$

Explanation: The assumption on ε and θ means $\theta = \widehat{\theta}(\varrho, \varepsilon)$ and $\varepsilon = \widetilde{\varepsilon}(\varrho, \theta)$ where $\theta \mapsto \widetilde{\varepsilon}(\varrho, \theta)$ is the inverse of $\varepsilon \mapsto \widehat{\theta}(\varrho, \varepsilon)$.

Proof (IV2.5)⇒(IV2.6). We first write the equation (IV2.5) for η, due to $\eta'_{\varepsilon} = \frac{1}{\theta}$, into

$$\theta(\eta - \varrho\eta'_{\varrho}) - (\varepsilon + p) = 0$$

or

$$p = \theta(\eta - \varrho\eta'_{\varrho}) - \varepsilon = -\varrho\theta\eta'_{\varrho} - f$$

where we have used the definition of f, from this definition also

$$\varepsilon = f + \theta\eta \tag{IV2.7}$$

follows. In the equations for p and ε we have only to plug in

$$\eta = -f'_{\theta} \,, \quad \theta\eta'_{\varrho} = -f'_{\varrho} \tag{IV2.8}$$

to get the result. These two equations we obtain when we use the definition (IV2.7) of f

$$\widehat{f}(\varrho, \theta) = \widetilde{\varepsilon}(\varrho, \theta) - \theta\widehat{\eta}(\varrho, \widetilde{\varepsilon}(\varrho, \theta))$$

and compute its derivatives. Because then

$$\begin{aligned} f'_{\theta} &= \varepsilon'_{\theta} - \theta\eta'_{\varepsilon}\varepsilon'_{\theta} - \eta = (1 - \theta\eta'_{\varepsilon})\varepsilon'_{\theta} - \eta = -\eta \,, \\ f'_{\varrho} &= \varepsilon'_{\varrho} - \theta\eta'_{\varrho} - \theta\eta'_{\varepsilon}\varepsilon'_{\varrho} = (1 - \theta\eta'_{\varepsilon})\varepsilon'_{\varrho} - \theta\eta'_{\varrho} = -\theta\eta'_{\varrho} \,, \end{aligned}$$

the two equations in (IV2.8) as stated. □

Proof (IV2.6)⇒(IV2.5). We use the equation for the free energy and the equation for ε

$$\theta\eta = \varepsilon - f = -\theta f_{'\theta}$$

in order to obtain $\eta = -f_{'\theta}$. Then we write the equation for the free energy in the form

$$f(\varrho, \widehat{\theta}(\varrho,\varepsilon)) = \varepsilon - \widehat{\theta}(\varrho,\varepsilon)\eta(\varrho,\varepsilon)$$

and calculate the derivative with respect to ε

$$f_{'\theta}\theta_{'\varepsilon} = 1 - \theta_{'\varepsilon}\eta - \theta\eta_{'\varepsilon}\,,$$

what is, due to $\eta = -f_{'\theta}$, the identity $1 = \theta\eta_{'\varepsilon}$. Next we do the derivative with respect to ϱ

$$f_{'\varrho} + f_{'\theta}\theta_{'\varrho} = -\theta_{'\varrho}\eta - \theta\eta_{'\varrho}$$

and obtain $f_{'\varrho} = -\theta\eta_{'\varrho}$. The representation of p results therefore again using the equation for the free energy

$$p = \varrho f_{'\varrho} - f = \varrho f_{'\varrho} + \theta\eta - \varepsilon = \theta(\eta - \varrho\eta_{'\varrho}) - \varepsilon\,.$$

Thus we have shown

$$\theta\eta_{'\varepsilon} = 1\,, \quad \theta(\eta - \varrho\eta_{'\varrho}) - (p+\varepsilon) = 0\,,$$

quod erat demonstrandum. □

A version of III.2.4 is the following

2.2 Entropy theorem. For the system (IV2.2) the entropy inequality

$$\sigma := \partial_t\eta + \mathrm{div}\psi \geq 0$$

is satisfied, if $f = \widehat{f}(\varrho,\theta)$ with $\theta > 0$ and

$$\Pi = p\mathrm{Id} - S\,, \quad S \text{ symmetric,}$$
$$p = \varrho f_{'\varrho} - f\,, \quad \varepsilon = f - \theta f_{'\theta}\,,$$

and if for the entropy including flux and production

$$\eta = -f_{'\theta}\,, \quad \psi = \eta v + \frac{1}{\theta}q\,,$$

$$\sigma = \frac{1}{\theta}\mathrm{D}v\,\mathbf{:}\,S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0\,. \tag{IV2.9}$$

Remarks: Here we assume again the relation (IV2.3). Because of the assumption (IV2.1) the stress tensor S is symmetric.

Es sei bemerkt, dass die Darstellung $\Pi = p\mathrm{Id} - S$ sozusagen nur aus dem Entropieprinzip folgert. Der Spannungstensor S ist definiert in der Residualungleichung für σ, siehe (IV2.10), und hat daher nur die Ungleichung (IV2.9) zu erfüllen. Bei dieser Prozedur fiel die Definition des Druckes p in der Gibbs Relation (IV2.5) ab.

Proof (with ε-variable). It is (as in the proof of III.1.3)

$$\begin{aligned}\sigma &= \partial_t\eta + \mathrm{div}\psi \qquad (\text{because of } \psi = \eta v + \psi_0)\\ &= \partial_t\eta + v\bullet\nabla\eta + \eta\,\mathrm{div}v + \mathrm{div}\psi_0\\ &= \overset{\circ}{\eta} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 \qquad (\text{since } \overset{\circ}{h} := (\partial_t + v\bullet\nabla)h).\end{aligned}$$

We assume (see also III.1.6) that (ϱ,ε) are independent variables and that $\eta = \widehat{\eta}(\varrho,\varepsilon)$. Then one computes

$$\overset{\circ}{\eta} = \eta'_{\varrho}\overset{\circ}{\varrho} + \eta'_{\varepsilon}\overset{\circ}{\varepsilon}$$

and since it follows from the differential equations for ϱ and ε (see III.2.3)

$$\begin{aligned}&\overset{\circ}{\varrho} + \varrho\,\mathrm{div}v = 0\,,\\ &\overset{\circ}{\varepsilon} + \varepsilon\,\mathrm{div}v + \mathrm{div}q = -\mathrm{D}v : \Pi\,,\end{aligned}$$

we get

$$\begin{aligned}\sigma &= \overset{\circ}{\eta} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 = \eta'_{\varrho}\overset{\circ}{\varrho} + \eta'_{\varepsilon}\overset{\circ}{\varepsilon} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0\\ &= \mathrm{D}v : \Big((\eta - \eta'_{\varrho}\varrho - \eta'_{\varepsilon}\varepsilon)\mathrm{Id} - \eta'_{\varepsilon}\Pi\Big) - \eta'_{\varepsilon}\,\mathrm{div}q + \mathrm{div}\psi_0\\ &= \mathrm{D}v : \widetilde{S} + \nabla\eta'_{\varepsilon}\bullet q + \mathrm{div}(\psi_0 - \eta'_{\varepsilon}q)\,,\end{aligned}$$

if

$$\widetilde{S} := (\eta - \eta'_{\varrho}\varrho - \eta'_{\varepsilon}\varepsilon)\mathrm{Id} - \eta'_{\varepsilon}\Pi\,, \tag{IV2.10}$$

and if $\psi_0 = \eta'_{\varepsilon}q$. Thus, if

$$\eta = \widehat{\eta}(\varrho,\varepsilon)\,, \quad \psi = \eta v + \eta'_{\varepsilon}q\,,$$

then the entropy principle $\sigma \geq 0$ is fulfilled if

$$\sigma = \mathrm{D}v : \widetilde{S} + \nabla\eta'_{\varepsilon}\bullet q \geq 0\,.$$

From 2.1 and the of Π it follows that

$$\theta = \frac{1}{\eta'_{\varepsilon}}\,, \quad \eta'_{\varepsilon} > 0\,, \quad \eta - \varrho\eta'_{\varrho} - (\varepsilon + p)\eta'_{\varepsilon} = 0\,,$$

so dass also

$$\widetilde{S} := \eta'_{\varepsilon}(p\mathrm{Id} - \Pi) = \eta'_{\varepsilon}S = \frac{1}{\theta}S\,,$$

quod erat demonstrandum. □

The following proof only deals with the variable θ.

Proof (with θ-variable). Es gilt zunächst wie im ersten Beweis

$$\begin{aligned}\sigma &= \partial_t\eta + \mathrm{div}\psi \quad (\text{wenn } \psi = \eta v + \psi_0 \text{ ist})\\ &= \partial_t\eta + v\bullet\nabla\eta + \eta\,\mathrm{div}v + \mathrm{div}\psi_0\\ &= \mathring{\eta} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 \quad (\text{wenn } \mathring{h} := (\partial_t + v\bullet\nabla)h \text{ ist}).\end{aligned}$$

Die Darstellung $\eta = -f_{\prime\theta}$ bedeutet $\eta = \widetilde{\eta}(\varrho,\theta) := -f_{\prime\theta}(\varrho,\theta)$, weshalb nun

$$\mathring{\eta} = \eta_{\prime\varrho}\mathring{\varrho} + \eta_{\prime\theta}\mathring{\theta}\,.$$

Aus den Differentialgleichungen für ϱ und ε folgt wieder

$$\begin{aligned}&\mathring{\varrho} + \varrho\,\mathrm{div}v = 0\,,\\ &\mathring{\varepsilon} + \varepsilon\,\mathrm{div}v + \mathrm{div}q = -\mathrm{D}v\mathbin{\vdots}\Pi\,.\end{aligned}$$

Da $\varepsilon = \widetilde{\varepsilon}(\varrho,\theta) := f(\varrho,\theta) - \theta f_{\prime\theta}(\varrho,\theta)$ ist jetzt

$$\mathring{\varepsilon} = \varepsilon_{\prime\varrho}\mathring{\varrho} + \varepsilon_{\prime\theta}\mathring{\theta}$$

und wegen $\eta = \widetilde{\eta}(\varrho,\theta) := -f_{\prime\theta}(\varrho,\theta)$ ist daher

$$\begin{aligned}\mathring{\eta} &= \eta_{\prime\varrho}\mathring{\varrho} + \frac{\eta_{\prime\theta}}{\varepsilon_{\prime\theta}}(\mathring{\varepsilon} - \varepsilon_{\prime\varrho}\mathring{\varrho})\\ &= \Big(\eta_{\prime\varrho} - \varepsilon_{\prime\varrho}\frac{\eta_{\prime\theta}}{\varepsilon_{\prime\theta}}\Big)\mathring{\varrho} + \frac{\eta_{\prime\theta}}{\varepsilon_{\prime\theta}}\mathring{\varepsilon}\,.\end{aligned}$$

Nun ist

$$\begin{aligned}&\eta_{\prime\varrho} = -f_{\prime\theta\varrho}\,, \quad \eta_{\prime\theta} = -f_{\prime\theta\theta}\,,\\ &\varepsilon_{\prime\varrho} = f_{\prime\varrho} - \theta f_{\prime\theta\varrho}\,, \quad \varepsilon_{\prime\theta} = -\theta f_{\prime\theta\theta}\,.\end{aligned}$$

Also gilt

$$\mathring{\eta} = -\frac{f_{\prime\varrho}}{\theta}\mathring{\varrho} + \frac{1}{\theta}\mathring{\varepsilon}$$

und es folgt wie im obigen Beweis

$$\begin{aligned}\sigma &= \mathring{\eta} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 = -\frac{f_{\prime\varrho}}{\theta}\mathring{\varrho} + \frac{1}{\theta}\mathring{\varepsilon} + \eta\,\mathrm{div}v + \mathrm{div}\psi_0\\ &= \mathrm{D}v\mathbin{\vdots}\Big((\eta + \frac{f_{\prime\varrho}\varrho}{\theta} - \frac{1}{\theta}\varepsilon)\mathrm{Id} - \frac{1}{\theta}\Pi\Big) - \frac{1}{\theta}\,\mathrm{div}q + \mathrm{div}\psi_0\\ &= \frac{1}{\theta}\mathrm{D}v\mathbin{\vdots}S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q + \mathrm{div}(\psi_0 - \frac{1}{\theta}q)\\ &= \frac{1}{\theta}\mathrm{D}v\mathbin{\vdots}S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q\,,\end{aligned}$$

if

$$\begin{aligned}&\Pi = p\mathrm{Id} - S\,, \quad \psi_0 = \frac{1}{\theta}q\,,\\ &\frac{p}{\theta} = \eta + \frac{f_{\prime\varrho}\varrho}{\theta} - \frac{1}{\theta}\varepsilon\,,\end{aligned}$$

wobei die letzte Identität gleich

$$p = \theta\eta + f_{\prime\varrho}\varrho - \varepsilon = f_{\prime\varrho}\varrho - f$$

ist. ***Explanation:*** Es sei bemerkt, dass wir verschiedene Abhängigkeiten in der Entropie

$$\eta = \widehat{\eta}(\varrho,\varepsilon) = \widetilde{\eta}(\varrho,\theta)$$

hatten, also ist $\eta'_\varrho = \widehat{\eta}'_\varrho(\varrho,\varepsilon)$ von $\eta'_\varrho = \widetilde{\eta}'_\varrho(\varrho,\theta)$ verschieden (eigentlich müssten diese Ableitungen mit $\widehat{\eta}'_\varrho$ und $\widetilde{\eta}'_\varrho$ bezeichnet werden), obwohl wir beides in verschiedenem Zusammenhang gleich bezeichnet hatten. In der Physikliteratur sorgt die folgende Definition für eine eindeutige Interpretation:

$$\left(\frac{\partial\eta}{\partial\varrho}\right)_\varepsilon := \widehat{\eta}'_\varrho \text{ und } \left(\frac{\partial\eta}{\partial\varrho}\right)_\theta := \widetilde{\eta}'_\varrho \,.$$

Es werden als Index die Variablen genannt, die konstant gehalten werden. Dies definiert in eindeutiger Weise das Koordinatensystem (das sind diese Variablen plus die Variable, bezüglich der die Ableitung genommen wird). □

Let us summarize the consequences of the entropy theorem. It is sufficient that in the differential equations (IV2.2)

$$\Pi = p\mathrm{Id} - S\,, \quad p = \varrho f'_\varrho - f\,, \quad \varepsilon = f - \theta f'_\theta \tag{IV2.11}$$

hold. Moreover, the stress tensor S and the heat flux q satisfy the entropy inequality

$$\sigma = \frac{1}{\theta}\mathrm{D}v\mathbin{:}S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0\,. \tag{IV2.12}$$

Thus, it remains to choose in (IV2.12) the stress tensor S and the heat flux q such that $\sigma \geq 0$. This is satisfied if, for example,

$$\mathrm{D}v\mathbin{:}S \geq 0\,, \quad \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0\,. \tag{IV2.13}$$

The simplest case is, see III.2.5, that

Linear stress tensor:

$$S = 2\widehat{a}(\varrho,\theta)\,(\mathrm{D}v)^{\mathrm{S}} + \widehat{b}(\varrho,\theta)\,\mathrm{div}(v)\mathrm{Id}\,,$$

$$a \geq 0\,, \quad b + \frac{2a}{n} \geq 0$$

Linear heat conduction:

$$q = -\widehat{c}(\varrho,\theta)\nabla\theta\,, \quad c \geq 0 \tag{IV2.14}$$

---

a shear viscosity,

$b + \dfrac{2}{n}a$ bulk viscosity,

c heat conductivity.

With this choice of S and q the differential equation read

Fluid equations:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}((e+p)v - Sv + q) = v\bullet\mathbf{f} \qquad \text{(IV2.15)}$$

$$p = \varrho f'_{\varrho} - f\,, \quad e = \varepsilon + \frac{\varrho}{2}|v|^2\,, \quad \varepsilon = f - \theta f'_{\theta}\,,$$

$f = \widehat{f}(\varrho,\theta)\,,$ S and q satisfy (IV2.14).

Or written differently, with $\overset{\circ}{h} = \partial_t h + v\bullet\nabla h$ for each function h and under usage of III.2.2 for the kinetic energy,

Fluid equations:

$$\overset{\circ}{\varrho} + \varrho\operatorname{div}v = 0\,,$$
$$\varrho\overset{\circ}{v} + \nabla p - \operatorname{div}S = \mathbf{f}\,,$$
$$\overset{\circ}{\varepsilon} + (\varepsilon + p)\operatorname{div}v + \operatorname{div}q = \mathrm{D}v{:}S \qquad \text{(IV2.16)}$$

$p = \varrho f'_{\varrho} - f$ pressure, $\varepsilon = f - \theta f'_{\theta}$ internal energy,

$f = \widehat{f}(\varrho,\theta)$ internal free energy,

S and q satisfy (IV2.14).

Note: $\overset{\circ}{v} = \partial_t v + v\bullet\nabla v$ is a quadratic term in v.

2.3 Lemma. Prove the statements (IV2.15) and (IV2.16).

Proof (IV2.15). See the above text. □

Proof (IV2.16). See III.2.3. For your completeness: For the kinetic energy we have, see III.2.2, for solutions of the mass and momentum conservation

$$\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \operatorname{div}\Big(\frac{\varrho}{2}|v|^2 v + \Pi^{\mathrm{T}}\,v\Big) = v\bullet\mathbf{f} + \mathrm{D}v{:}\Pi\,. \qquad \text{(IV2.17)}$$

Subtracting this from the energy equation for $e = \varepsilon + \frac{\varrho}{2}|v|^2$, we get

$$\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) = -\,(\mathrm{D}v)^{\mathrm{S}}{:}\Pi$$

and since $\overset{\circ}{\varepsilon} = \partial_t\varepsilon + v\bullet\nabla\varepsilon$

$$\overset{\circ}{\varepsilon} + \varepsilon\operatorname{div}v + \operatorname{div}q = -\,(\mathrm{D}v)^{\mathrm{S}}{:}\Pi\,.$$

Since $\Pi = p\mathrm{Id} - S$ the equation becomes

$$\overset{\circ}{\varepsilon} + \varepsilon\,\mathrm{div}v + \mathrm{div}q = -p\,\mathrm{div}v + (\mathrm{D}v)^{\mathrm{S}} \mathbin{\vdots} S\,.$$

And the symmetry of S gives $(\mathrm{D}v)^{\mathrm{S}} \mathbin{\vdots} S = \mathrm{D}v \mathbin{\vdots} S$. This is the assertion. □

In the general case the stress tensor S can depend also nonlinear on $(\mathrm{D}v)^{\mathrm{S}}$, and still satisfy the entropy principle. As example we take Rheology, see [Wikipedia: Rheology] (*de:* [Wikipedia: Rheologie]), and the Dirichlet problem in the next section 3.

Ideal gases

The choice of the entropy η and the free energy f is crucial for the type of liquid or gas, which we want to consider. The simplest case is the "Ideal gas law" (see [Wikipedia: Ideal gas])

$$\boxed{\begin{array}{ll} PV = \mathrm{m}RT\,, & p = P\,,\ \varrho = \dfrac{\mathrm{m}}{V}\,,\ \theta = T\,, \\ U = c_V T\,, & U = \varepsilon^{\mathrm{sp}} = \dfrac{\varepsilon}{\varrho}\,,\ R = c_P - c_V > 0\,, \end{array}} \tag{IV2.18}$$

where c_V and c_P are positive constants. The constant R is the specific gas constant.

2.4 Gas constant. The ***universal gas constant*** $\mathcal{R}$ is, see [125],

$$\mathcal{R} = 8.314472\,\frac{J}{K \cdot mol} \qquad [J = Pa \cdot m^3,\ Pa = \frac{kg}{m \cdot s^2}]\,.$$

The ***specific gas constant*** R ist

$$\begin{aligned} R &= \frac{\mathcal{R}}{M}\,[\frac{m^2}{K \cdot s^2}], && M \text{ molar mass (molecular weight)}, \\ M &= \frac{\mathrm{m}}{\mathrm{n}}, && \mathrm{m}\,[kg] \text{ mass, } \mathrm{n}\,[mol] \text{ amount of substance.} \end{aligned}$$

From [Wikipedia: Molar mass], see also [Wikipedia: Molecular mass]: "In chemistry, the molar mass ... is a physical property. It is defined as the mass of a given substance (chemical element or chemical compound) divided by its amount of substance."

For example, the molar mass of water is

$$M_{H_2O} = 2 \cdot M_H + M_O = 2 \cdot 1.00794\,\frac{g}{mol} + 15.9994\,\frac{g}{mol} = 18.01528\,\frac{g}{mol}\,,$$

and the one of air

$$\begin{aligned} &M_{N_2} = 28.014\,\frac{g}{mol}\,, \quad M_{O_2} = 31.998\,\frac{g}{mol}\,, \quad M_{Ar} = 39.948\,\frac{g}{mol}\,, \\ &M_{dryair} = 78\%\,M_{N_2} + 21\%\,M_{O_2} + 1\%\,M_{Ar} \\ &= (0.78 \cdot 28.014 + 0.21 \cdot 31.998 + 0.01 \cdot 39.948)\frac{g}{mol} = 28.97\frac{g}{mol}\,. \end{aligned}$$

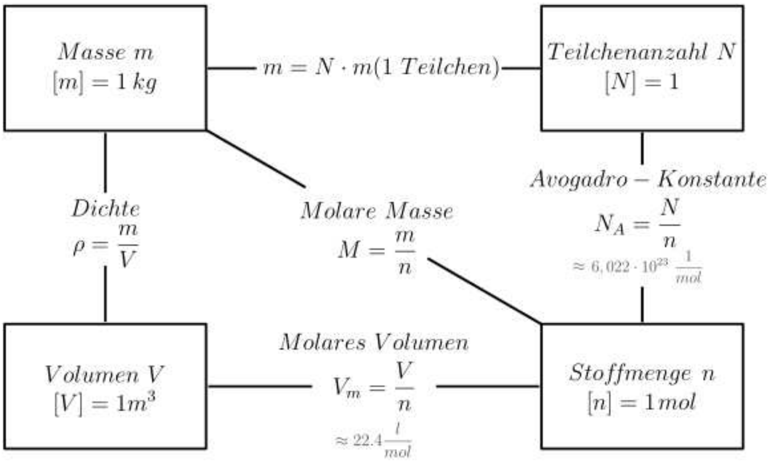


Fig. 5: "Der Zusammenhang zwischen Masse, Stoffmenge, Volumen und Teilchenanzahl" by Johannes Schneider from [Wikipedia: Molare Masse]

We prove the following

2.5 Lemma. Let c_V and c_P be positive constants and $R = c_P - c_V > 0$. Let the Gibbs relation be satisfied, more exactly (IV2.5) or (IV2.6), and for the internal energy ε let

$$\varepsilon = \widehat{\varepsilon}(\varrho, \theta) = c_V \theta \varrho \,.$$

Then the following statements are equivalent.

(1) $PV = \mathrm{m}RT$.

(2) $p = \widehat{p}(\varrho, \theta) = R\theta\varrho$.

(3) $\eta = \widehat{\eta}(\varrho, \varepsilon) = c_V \varrho \log \varepsilon - c_P \varrho \log \varrho + c\varrho$ for some $c \in \mathbb{R}$.

(4) $f = \widehat{f}(\varrho, \theta) = -c_V \theta \varrho \log \theta + R\theta\varrho \log \varrho + d\varrho\theta$ for some $d \in \mathbb{R}$.

Es sei noch erwähnt, dass $(\varrho, \varepsilon) \mapsto \widehat{\eta}(\varrho, \varepsilon)$ eine konkave Funktion ist.
Remark: This lemma can also be formulated with the help of specific quantities $\varepsilon^{\mathrm{sp}}$, η^{sp} and f^{sp}.

Proof (1)⇔(2). Follows directly by the definitions in (IV2.18). □

Proof (2)⇒(3). We compute

$$\eta'_{\varepsilon} = \frac{1}{\theta} = \frac{c_V \varrho}{\varepsilon} \,.$$

By integration we obtain with a function $\varrho \mapsto d(\varrho)$

$$\eta = c_V \varrho \log \varepsilon + d(\varrho) \,.$$

The Gibbs relation yields

$$0 = \varrho\eta_{'\varrho} - \eta + (\varepsilon + p)\eta_{'\varepsilon} = \varrho d_{'\varrho} - d + (c_V + R)\varrho = \varrho^2 \left(\frac{d}{\varrho}\right)_{'\varrho} + c_P\varrho$$

due to $c_V + R = c_P$. Hence, for some constant c

$$\frac{d}{\varrho} = -c_P \log\varrho + c\,.$$

From this it follows (3). □

Proof (2)⇒(4). The equation (IV2.3) is $\varepsilon = f + \theta\eta = f - \theta f_{'\theta}$, hence

$$\left(\frac{f}{\theta}\right)_{'\theta} = \frac{\theta f_{'\theta} - f}{\theta^2} = -\frac{\varepsilon}{\theta^2} = -\frac{c_V\varrho}{\theta}\,.$$

In addition, it follows from the Gibbs relation

$$\left(\frac{f}{\varrho}\right)_{'\varrho} = \frac{\varrho f_{'\varrho} - f}{\varrho^2} = \frac{p}{\varrho^2} = \frac{R\theta}{\varrho}\,.$$

Thus it is

$$\left(\frac{f}{\theta}\right)_{'\theta} = -\frac{c_V\varrho}{\theta}\,, \quad \left(\frac{f}{\varrho}\right)_{'\varrho} = \frac{R\theta}{\varrho}\,,$$

from which it follows

$$\frac{f}{\theta} = -c_V\varrho\log\theta + d_1(\varrho)\,, \quad \frac{f}{\varrho} = R\theta\log\varrho + d_2(\theta)\,,$$

that is

$$f = R\theta\varrho\log\varrho + d_2(\theta)\varrho = -c_V\theta\varrho\log\theta + d_1(\varrho)\theta\,.$$

These two equations for f we write as

$$\frac{d_2(\theta)}{\theta} + c_V\log\theta = \frac{d_1(\varrho)}{\varrho} - R\log\varrho = d\,,$$

where d, of course, has to be a constant. From this we get

$$d_1(\varrho) = R\varrho\log\varrho + d\varrho\,, \quad d_2(\theta) = -c_V\theta\log\theta + d\theta\,,$$

and therefore

$$f = -c_V\theta\varrho\log\theta + R\theta\varrho\log\varrho + d\varrho\theta\,.$$

□

Proof (4)⇒(3). Since $\eta = -f'_\theta$ one computes

$$\begin{aligned}\eta &= c_V\varrho(\log\theta + 1) - (c_P - c_V)\varrho\log\varrho - d\varrho\\ &= c_V\varrho(\log(\theta\varrho) + 1) - c_P\varrho\log\varrho - d\varrho\\ &= c_V\varrho\log\varepsilon - c_P\varrho\log\varrho + c_V\varrho(-\log c_V + 1) - d\varrho\,,\end{aligned}$$

hence on has to choose $c = c_V(1 - \log c_V) - d$. □

Proof (3)⇒(4). It is

$$\begin{aligned}f &= \varepsilon - \theta\eta = \varepsilon - c_V\theta\varrho\log\varepsilon + c_P\theta\varrho\log\varrho - c\theta\varrho\\ &= -c_V\theta\varrho\log\theta + (c_P - c_V)\theta\varrho\log\varrho + c_V\theta\varrho(1 - \log c_V) - c\theta\varrho\end{aligned}$$

hence on has to choose $d = c_V(1 - \log c_V) - cs$. □

Proof (4)⇒(2). By (IV2.6) we have $p = \varrho f'_\varrho - f$. Then using f from (4) gives (2). □

Proof of the concavity. It is $\eta = c_V\varrho\log\varepsilon - c_P\varrho\log\varrho + c\varrho$, hence

$$\begin{aligned}\eta'_\varrho = c_V\log\varepsilon - c_P(\log\varrho + 1) + c\,,\quad \eta'_\varepsilon = c_V\frac{\varrho}{\varepsilon}\,,\\ \eta'_{\varrho\varrho} = -\frac{c_P}{\varrho}\,,\quad \eta'_{\varrho\varepsilon} = \frac{c_V}{\varepsilon}\,,\quad \eta'_{\varepsilon\varepsilon} = -\frac{c_V\varrho}{\varepsilon^2}\,.\end{aligned}$$

It follows that

$$\mathrm{D}^2\eta = -\begin{bmatrix}\frac{c_P}{\varrho} & -\frac{c_V}{\varepsilon}\\ -\frac{c_V}{\varepsilon} & \frac{c_V\varrho}{\varepsilon^2}\end{bmatrix}$$

is negative definit because

$$\begin{aligned}\begin{bmatrix}z_1\\ z_2\end{bmatrix}\bullet\begin{bmatrix}\frac{c_P}{\varrho} & -\frac{c_V}{\varepsilon}\\ -\frac{c_V}{\varepsilon} & \frac{c_V\varrho}{\varepsilon^2}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} &= \frac{c_P}{\varrho}z_1^2 - \frac{2c_V}{\varepsilon}z_1z_2 + \frac{c_V\varrho}{\varepsilon^2}z_2^2\\ &= \frac{1}{\varrho\varepsilon^2}\big(c_P\varepsilon^2z_1^2 - 2c_V\varrho\varepsilon z_1z_2 + c_V\varrho^2z_2^2\big)\\ &= \frac{1}{\varrho\varepsilon^2}\big(R\varepsilon^2z_1^2 + c_V(\varrho z_1 - \varepsilon z_2)^2\big) \geq 0\,,\end{aligned}$$

since $R = c_P - c_V > 0$. □

Under these classical equations of ideal gas law holds the following theorem.

2.6 Theorem. For the ideal gas law we have to take system (IV2.15) with

$$\Pi = p\mathrm{Id} - S\,,\quad p = R\theta\varrho\,,\quad e = c_V\theta\varrho + \frac{\varrho}{2}|v|^2\,. \tag{IV2.19}$$

And for S and q we take in general (IV2.12) (which is satisfied by the ansatz (IV2.13) and (IV2.14)). Under these assumptions an equivalent system to (IV2.15) is

Ideal fluid:

$$\partial_t \varrho + \mathrm{div}(\varrho v) = 0\,,$$
$$\varrho(\partial_t v + v\bullet\nabla v) + \mathrm{div}\Pi = \mathbf{f}\,,$$
$$c_V\,\varrho(\partial_t \theta + v\bullet\nabla\theta) + \mathrm{div} q = -\mathrm{D}v \mathbin{:} \Pi$$

Independent variables: (IV2.20)

ϱ density, v velocity, θ temperature.

Entropy production:

$$\sigma = \frac{1}{\theta}\mathrm{D}v \mathbin{:} S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0 \text{ see (IV2.12)},$$

$\Pi = p\mathrm{Id} - S\,,\quad p$ see (IV2.19).

The last equation is the equation of ***heat conduction***, and the first two equations are the Navier-Stokes system. *Remark:* The system does not depend on the c and d terms in 2.5.

Proof. We define the specific variables by

$$e^{\mathrm{sp}} = \frac{e}{\varrho}\,, \quad \varepsilon^{\mathrm{sp}} = \frac{\varepsilon}{\varrho}\,,$$

such that it holds

$$e^{\mathrm{sp}} = c_V\theta + \frac{|v|^2}{2}\,, \quad \varepsilon^{\mathrm{sp}} = c_V\theta\,.$$

If we subtract from the energy equation the equation for the kinetic energy (see (IV2.17)), we obtain

$$\partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) = -\mathrm{D}v \mathbin{:} \Pi\,.$$

Now, because of the mass conservation, we have

$$\partial_t\varepsilon + \mathrm{div}(\varepsilon v) = \varrho(\partial_t\varepsilon^{\mathrm{sp}} + v\bullet\nabla\varepsilon^{\mathrm{sp}})\,.$$

□

Often is occurs in fluid problems that the temperature is nearly constant, at least physically, i.e. $\theta \approx$ const, see for example section 3. Mathematically in such situation one sets $\theta =$ const. However sometimes it is not known where the constant temperature has to show up or there is no temperature at all. In such a situation it is necessary to consider the free energy inequality III.5.4 instead of the entropy principle III.1.1. But, please, dont forget that the entropy principle is the true real basis.

3 Navier-Stokes equation

In this section we consider viscous liquids. In physics they are provided in the compressible case with the general name “Navier-Stokes equation”. Later in this section we will derive from the compressible case the “Incompressible Navier-Stokes equation”. In this incompressible case there cannot be any heat variation, see the remark 3.4. Therefore we will perform this derivation in the isothermal situation, that is $\theta = \text{const}$. If one makes instead on θ an assumption on q, i.e. $q = 0$, one considers adiabatic processis like in section on sound waves 7 and in section on self-gravitation 16.

Compared with the general theory we make in this section the following assumptions:

$$\mathbf{J} = 0, \quad \mathbf{r} = 0, \quad \Pi \text{ symmetrisch}, \quad \theta = \text{const}.$$

So we are dealing with the ***mass-momentum conservation*** for a compressible fluid

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,, \end{aligned} \tag{IV3.1}$$

where in this section we consider only the isothermal situation that the temperature $\theta = \text{const}$ is constant. Consequently, instead of the entropy inequality, we assume that the ***free energy inequality***, see 3.1,

$$\sigma_f := \partial_t f^{tot} + \operatorname{div}\varphi^{tot} - g^{tot} \leq 0 \tag{IV3.2}$$

is satisfied. This is the version of the entropy principle in the isothermal case (see III.5.4) and it determines the structure of the tensor Π in the system. In other words: Since $\theta = \text{const}$ is assumed, there is no heat flux and therefore also no energy equation. It follows that the inner free energy f satisfies an inequality and determines the given equations (IV3.1).

3.1 Free energy inequality. For system (IV3.1) the free energy inequality (IV3.2) is satisfied for

$$f^{tot} = f + \frac{\varrho}{2}|v|^2\,, \quad \varphi^{tot} = f^{tot} v + \Pi^{\mathrm{T}}\, v\,, \quad g^{tot} = v \bullet \mathbf{f}\,,$$

if in system (IV3.1)

$$\Pi = p\mathrm{Id} - S\,, \quad p = \varrho f'_{\varrho} - f\,, \quad f = \widehat{f}(\varrho)\,,$$

and if the residual inequality

$$\sigma_f = -\mathrm{D}v \colon S \leq 0$$

is satisfied. *Note:* It is $\sigma_f = -\theta\sigma$ where σ is the entropy production, see section III.5. This writing assumes the knowledge of θ.

Proof. It follows from (IV3.1) the equation for the kinetic energy (see III.2.2)

$$\partial_t\Big(\frac{\varrho}{2}|v|^2\Big) + \operatorname{div}\Big(\frac{\varrho}{2}|v|^2 v + \Pi^{\mathrm{T}} v\Big) = v\bullet\mathbf{f} + \mathrm{D}v{:}\Pi\,. \tag{IV3.3}$$

Subtracting this equation from (IV3.2) we obtain, see III.5.5,

$$\begin{aligned}
0 \ge \sigma_f &:= \partial_t f^{tot} + \operatorname{div}\varphi^{tot} - g^{tot}\\
&= \partial_t\Big(f^{tot} - \frac{\varrho}{2}|v|^2\Big) + \operatorname{div}\Big(\varphi^{tot} - \frac{\varrho}{2}|v|^2 v - \Pi^{\mathrm{T}} v\Big) - g^{tot} + v\bullet\mathbf{f} + \mathrm{D}v{:}\Pi\\
&= \partial_t f + \operatorname{div}(fv) + \mathrm{D}v{:}\Pi\\
&= (\partial_t + v\bullet\nabla)f + \mathrm{D}v{:}(f\mathrm{Id} + \Pi)\\
&= f'_\varrho\,\underbrace{(\partial_t + v\bullet\nabla)\varrho}_{=-\varrho\,\operatorname{div}v} + \mathrm{D}v{:}(f\mathrm{Id} + \Pi)\\
&= \mathrm{D}v{:}((f - \varrho f'_\varrho)\mathrm{Id} + \Pi) = -\mathrm{D}v{:}S\,,
\end{aligned}$$

if $S := -(f - \varrho f'_\varrho)\mathrm{Id} - \Pi$, or $p := \varrho f'_\varrho - f$ and $\Pi = p\mathrm{Id} - S$. ☐

Therefore we have the following system

(Compressible) Navier-Stokes equation:

$$\partial_t\varrho + \operatorname{div}(\varrho v) = 0$$

$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f} \tag{IV3.4}$$

$p = \varrho f'_\varrho - f$ pressure, S objective stress tensor

$f = \widehat{f}(\varrho)$ free energy $(\theta = \text{const})$

$-\sigma_f = \mathrm{D}v{:}S \ge 0$ the residual inequality $(-\sigma_f = \theta\sigma)$

It is $p + f = \varrho f'_\varrho$ the Gibbs relation (see 2.1 or III.1.4, here in the isothermal case). This equation has the same structure as (III1.10) for the temperature, just here ϱ is the variable instead of ε. So we can determine the dual variable to ϱ which is the chemical potential μ.

3.2 Pressure as a function of the chemical potential. Let $f = \widehat{f}(\varrho)$. Consider the Gibbs equation

$$p + f = \varrho f'_\varrho$$

and define the ***chemical potential*** by

$$\boxed{\mu := f'_\varrho(\varrho)\,.} \tag{IV3.5}$$

If we assume that the free energy f is a convex function of ϱ, i.e.

$$f'_{\varrho\varrho} > 0\,,$$

then the ϱ-interval $]0,\infty[$ is mapped with the definition (IV3.5) on a μ-interval $]\mu_-,\mu_+[$, and we can define

$$p = \widehat{p}(\mu) := \varrho f'_\varrho(\varrho) - f(\varrho) \text{ for } \mu = f'_\varrho(\varrho)\,.$$

It follows

$$\boxed{p'_\mu(\mu) = \varrho \text{ for } \mu = f'_\varrho(\varrho)\,.}$$

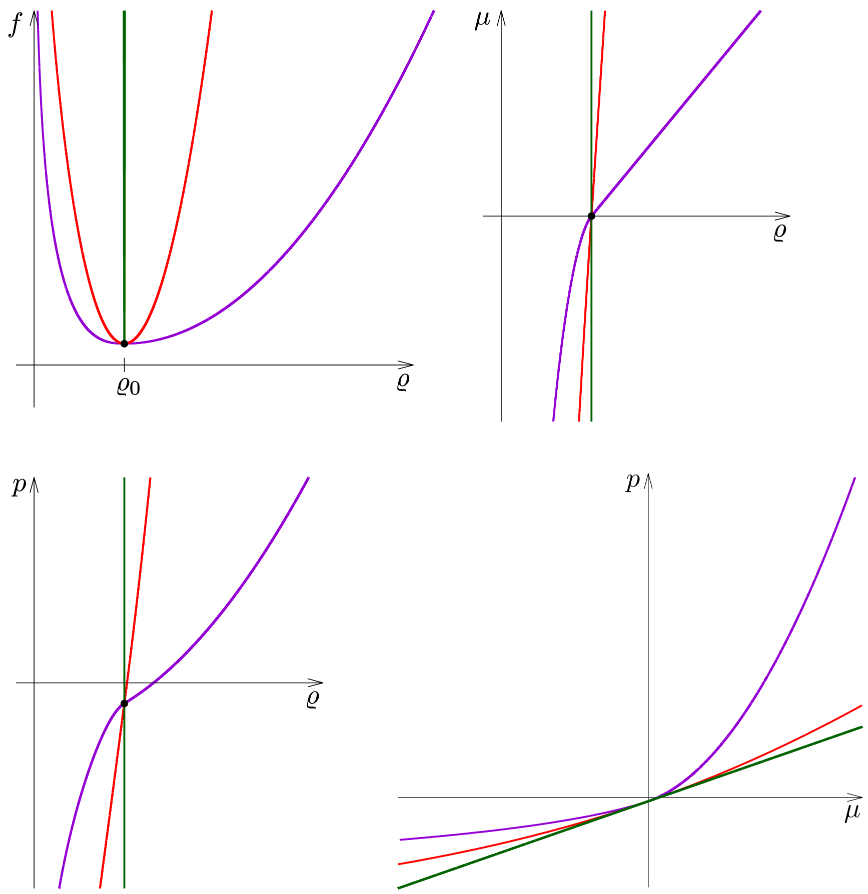


Fig. 6: The pressure p and chemical potential μ.

Proof. (Compare the Legendre-Fenchel transformation.) Since $\mu = f'_\varrho(\varrho)$ we can write the Gibbs relation (see 2.1) as

$$p(f'_\varrho(\varrho)) = p(\mu) = \varrho f'_\varrho(\varrho) - f(\varrho)$$

and therefore the derivative with respect to ϱ

$$p'_\mu(f'_\varrho(\varrho))f'_{\varrho\varrho}(\varrho) = \varrho f'_{\varrho\varrho}(\varrho)$$

and by cancelling the factor $f'_{\varrho\varrho}(\varrho) > 0$ it follows $p'_{\mu}(\mu) = \varrho$ für $\mu = f'_{\varrho}(\varrho)$. □

See section 11 for more about chemical potentials.

3.3 Example. If $\varrho_0 > 0$ is given and the free energy is

$$f = \widehat{f}(\varrho) := \frac{1}{2\delta}|\varrho - \varrho_0|^2 , \quad \delta > 0 ,$$

then

$$\begin{aligned} \mu &= f'_{\varrho} = \frac{\varrho - \varrho_0}{\delta} , \\ p &= \varrho f'_{\varrho} - f = \varrho\mu - \frac{1}{2\delta}|\varrho - \varrho_0|^2 \\ &= \varrho\mu - \frac{1}{2}(\varrho - \varrho_0)\mu = \frac{1}{2}(\varrho + \varrho_0)\mu , \end{aligned}$$

hence

$$p = \widehat{p}(\mu) := \frac{1}{2}(\delta\mu + 2\varrho_0)\mu = \Big(\frac{\delta\mu}{2} + \varrho_0\Big)\mu .$$

Now consider the limit $\delta \to 0$ (or consider the limit $\varrho \to \varrho_0$) one obtains

$$\varrho = \varrho_0 , \quad p = \varrho_0\mu$$

in the limit. Now μ can be any function, wheras $\varrho = \varrho_0$ is a constant.
Remark: This example is physically applicable only if for the solution the mass density ϱ varies in a neigbourhood of ϱ_0, see Fig. 6.

3.4 Remark. For the incompressible limit I.Müller [87, Incompressibility] writes: “It follows that a vanishing compressibility implies that there is no thermal expansion”. If a physical process contains different temperatures at different positions, the material must be compressible and the energy equality has to be used.

If we let $\varrho \to \varrho_0 = \text{const} > 0$, then in the limit p as a function of μ is an independent function. This allows us to consider the case $\varrho = \varrho_0 = \text{const}$ and to use μ in the differential equations. If p is a strict monotone function of μ, the function p itself, instead of μ, can be considered as the independent variable. Thus we obtain

Incompressible Navier-Stokes equation:

$$\mathrm{div} v = 0\,,$$
$$\partial_t(\varrho_0 v) + \mathrm{div}(\varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f}\,.$$

$S = \widehat{S}(p,(\mathrm{D}v)^{\mathrm{S}})$ objective tensor,

$S : (\mathrm{D}v)^{\mathrm{S}} \geq 0$ Residual inequality,

$\varrho = \varrho_0 = \mathrm{const}$,

v and p independent variables. (IV3.6)

It is usually assumed that (see (I3.1))

$$S = 2a\,(\mathrm{D}v)^{\mathrm{S}} = a(\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}})\,, \quad a = \widehat{a}(p,(\mathrm{D}v)^{\mathrm{S}})\,,$$

i.e.

$$S_{ij} = a(\partial_j v_i + \partial_i v_j) \text{ für } i,j = 1,\dots,n.$$

If the scalar coefficient is constant, i.e. $a = \mathrm{const}$ (see the statement in II.4.12 about the objectivity), then it follows

$$\mathrm{div} S = a\Delta v + a\sum_i (\partial_i\, \mathrm{div} v)\mathbf{e}_i = a\Delta v\,.$$

Since $\mathrm{div}(p\mathrm{Id}) = \nabla p$ we have the following system

Special Navier-Stokes equation:

$$\mathrm{div} v = 0\,,$$
$$\varrho_0(\partial_t v + (v\bullet\nabla)v) + \nabla p - a\Delta v = \mathbf{f} \qquad \text{(IV3.7)}$$

with $\varrho_0 = \mathrm{const} > 0$ and $a = \mathrm{const} > 0$

Here $\mathbf{f}$ is the (classical) force and it follows from the entropy principle (i.e. from the free energy inequality) that $a \geq 0$ (often μ, ν, or η is written for a). Normally, the incompressible Navier-Stokes equation is solved for (v,p), if $\varrho_0 > 0$ is given and under the assumption that $\mathbf{f} = \mathbf{f}(t,x)$ is a given function. However, from the frame-indifference (see (II3.18)) it follows that $\mathbf{f}$ contains a linear function of ϱ_0 and $\varrho_0 v$ (if fictitious forces are present).

Im konkreten Fall gehen wir von einem beschränkten Gebiet aus, was sich mit der Zeit verändern kann und in dem sich die Flüssigkeit befindet. Allgemein betrachten wir die Massen- und Impulserhaltung in der gesamten Raumzeit $\mathbb{R}\times\mathbb{R}^n$, wobei die Größen der Impulserhaltung in der Flüssigkeit die bekannten Terme der Navier-Stokes Gleichung sind. Das Außengebiet kann aus folgendem Material bestehen, wobei wir annehmen, dass keinen Massenaustausch mit der Flüssigkeit stattfindet:

- ***Solid body.*** Wir wählen einen Beobachter im Außengebiet, d.h. die Flüssigkeit befindet sich in einem Gebiet $\Omega = \mathbb{R}\times D$, wobei $D \subset \mathbb{R}^n$ fest gewählt sei. Der Außenraum $\mathbb{R}^n \backslash D$ ist der feste Körper. (Für einen beliebigen Beobachter ist dann die Beobachtertransformation anzuwenden.) Die Massenerhaltung für die Flüssigkeit $\mathrm{div}[v\mathcal{X}_D] = 0$ impliziert $v\bullet\nu_D = 0$. Aufgrund der Viskosität ist die Tangentialkomponente der Geschwindigkeit gleich der Tangentialkomponente der Geschwindigkeit des Außenraumes, also gleich 0. Damit gilt $v = 0$ auf ∂D, eine Dirichlet-Bedingung wie in 3.5.

- ***Vacuum.*** In diesem Fall sind die Massen- und Impulserhaltung in 3.7 im distributionellen Sinne zu lösen, wobei $\Omega_t \subset \mathbb{R}^n$ die Menge ist, in der sich die Flüssigkeit zur Zeit t befindet, d.h. es ist
$$\Omega = \{(t,x)\,;\ x \in \Omega_t\}\,.$$
Hierbei sind die Bedingungen an dem Rand von Ω aus dem distributionellen Masse-Impuls System zu nehmen, z.B. folgt für die Geschwindigkeit $(v - v_{\partial\Omega_t})\bullet\nu_\Omega = 0$, siehe 3.8. Die Terme im Komplement von Ω, also im Vakuum, sind dabei auf 0 zu setzen. Es ist klar, dass dies für alle Beobachter formuliert ist, nur in der Anfangsbedingung geht der gewählte Beobachter ein.

Dirichlet problem

The domain $D \subset \mathbb{R}^n$ is time independent and it is $v = 0$ on ∂D as Dirichlet condition. This condition on the boundary $\partial D \subset \mathbb{R}^n$ is, as described above, the right boundary condition for the solid body $\mathbb{R}^n \setminus D$.

Mathematics: It should be remarked that the regularity of the solution for $n = 3$ is a borderline case for the theory (for $n = 2$ and $n = 1$ regularity can be shown). As an alternative one considers the equation
$$\partial_t(\varrho_0 v) + \mathrm{div}(\varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f}$$
with an objective tensor $S = \widehat{S}((\mathrm{D}v)^{\mathrm{S}})$, where (see [36] and the physical argumentation in this paper) following Ladyzhenskaya one takes
$$S = 2a\,(\mathrm{D}v)^{\mathrm{S}}\ ,\quad a = \nu_0 + \nu_1|\,(\mathrm{D}v)^{\mathrm{S}}\,|^r \text{ with } r > 0$$
(in [36] the assumption $r \geq \frac{1}{5}$ is made) with $\nu_0 > 0$ and $\nu_1 > 0$. With respect to this tensor S there are a-priori estimates, and with this tensor S the free energy inequality is satisfied.

3.5 Problem. Let $D \subset \mathbb{R}^n$ be open and bounded. Then the ***Dirichlet problem*** reads

$$\begin{aligned} &v = 0 \text{ on } \mathbb{R} \times \partial D\,, \\ &\mathrm{div} v = 0 \text{ in } \mathbb{R} \times D\,, \\ &\partial_t(\varrho_0 v) + \mathrm{div}(\varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f} \text{ in } \mathbb{R} \times D\,. \end{aligned}$$

Remark: The time interval is for simplicity the entire real axis.

This problem is solved in the space of divergence free fields.

3.6 Theorem. Das Problem in 3.5 ist äquivalent zu

$$\begin{aligned} &\mathrm{div} v = 0 \text{ in } \mathbb{R} \times D \text{ und } v = 0 \text{ auf } \mathbb{R} \times \partial D\,, \\ &\int_{\mathbb{R}} \int_D \sum_k \big(\partial_t \zeta_k \varrho_0 v_k + \sum_i \partial_{x_i} \zeta_k (\varrho_0 v_k v_i - S_{ki}) + \zeta_k \mathbf{f}_k\big)\, \mathrm{d}x\, \mathrm{d}t = 0 \\ &\text{für alle } \zeta \in C_0^\infty(\mathbb{R} \times D; \mathbb{R}^n) \text{ mit } \mathrm{div}\zeta = 0 \text{ in } \mathbb{R} \times D, \end{aligned}$$

und zu der Tatsache, dass daraus die Existenz des Druckes p folgt, d.h.

$$\partial_k[p] = \mathbf{f}_k^\star := [\mathbf{f}_k] - \partial_t[\varrho_0 v_k] - \sum_i \partial_{x_i}[\varrho_0 v_k v_i - S_{ki}]$$

im Raum der Distributionen $\mathcal{D}'(\mathbb{R} \times D; \mathbb{R}^n)$ gilt.

Proof that p existss. The existence of v in the assertion implies that (realize that $\mathbf{f}_k^\star$ are distributions)

$$\sum_k \langle\, \zeta_k\,,\, \mathbf{f}_k^\star \,\rangle_{\mathscr{D}(\mathbb{R}\times D)} = 0$$

for all $\zeta \in C_0^\infty(\mathbb{R}\times D; \mathbb{R}^n) = \mathcal{D}(\mathbb{R}\times D; \mathbb{R}^n)$ with $\mathrm{div}\zeta = 0$. Hence $\partial_l \mathbf{f}_k^\star = \partial_k \mathbf{f}_l^\star$ in $\mathcal{D}'(\Omega; \mathbb{R})$ for all k and l, by putting $\zeta = \partial_l \xi \mathbf{e}_k - \partial_k \xi \mathbf{e}_l$ with a real valued $\xi \in C_0^\infty(\mathbb{R}\times\Omega; \mathbb{R})$. This also makes it possible to define p (what is quite nontrivial, see [Roger Temam, Navier-Stokes Equations, North-Holland 1977, Proposition I.1.1 and I.1.2], which uses results of Deny & Lions [J. Deny and J.L. Lions, Les espaces du type de BEPPO LEVI, Ann. Inst. Fourier 5, pp.305-370, 1954] and Nečas [J. Nečas, Equation aux dérivées partielles, Presses de lUniversité de Montréal, 1965].). Both results together give 3.5. If on the other hand 3.5 is satisfied then

$$\int_{\mathbb{R}} \int_\Omega \sum_k \big(\partial_t \zeta_k \varrho_0 v_k + \sum_i \partial_{x_i} \zeta_k (\varrho_0 v_k v_i + p\delta_{k,i} - S_{ki}) + \zeta_k \mathbf{f}_k\big)\, \mathrm{d}x\, \mathrm{d}t = 0$$

for all ζ with $\zeta = 0$ on $\mathbb{R} \times \partial D$. The assertion follows. □

So far the representation of the Dirichlet problem.

Neumann problem

Physically more interesting is the problem with variable fluid domain

$$\Omega = \{(t,x)\,;\ x \in \Omega_t\} \subset \mathbb{R}\times\mathbb{R}^n\,,\qquad \Gamma := \partial\Omega \subset \mathbb{R}\times\mathbb{R}^n$$

where the unknown domains Ω_t are bounded in $\mathbb{R}^n$. In the distributionally version this means the following.

3.7 Problem. Bestimme bei Anfangsbedingungen die Menge $\Omega \subset \mathbb{R}\times\mathbb{R}^n$ und die Geschwindigkeit $v\colon\overline{\Omega} \to \mathbb{R}^n$ sowie den Druck $p\colon\overline{\Omega} \to \mathbb{R}$ so, dass

$$\begin{aligned}
&\partial_t[\varrho_0\mathcal{X}_\Omega] + \sum_i \partial_{x_i}[\varrho_0 v_i \mathcal{X}_\Omega] = 0\,,\\
&\partial_t[\varrho_0 v_k\mathcal{X}_\Omega] + \sum_i \partial_{x_i}[(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki})\mathcal{X}_\Omega] + [\mathbf{f}_k\mathcal{X}_\Omega] = 0
\end{aligned}\tag{IV3.8}$$

für $k = 1,\dots,n$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^n)$. *Bemerkung:* Hier wird der Einfachheit halber der ganze Raum $\mathbb{R}^n$ benutzt. Und obwohl Anfangsbedingungen gebraucht werden, ist auch die Zeit auf ganz $\mathbb{R}$ ausgedehnt.

Wenn wir diese Formulierung ausschreiben, erhalten wir

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\Omega_t} \big(\partial_t\xi\varrho_0 + \nabla\xi\bullet(\varrho_0 v)\big)\,\mathrm{d}x\,\mathrm{d}t = 0 \text{ für } \xi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})\,,\\
&\int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t\zeta_k\varrho_0 v_k + \sum_i \partial_{x_i}\zeta_k(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki})\\
&\qquad\qquad + \zeta_k\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \text{ für } \zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)\,.
\end{aligned}\tag{IV3.9}$$

Zu dieser schwachen Version können wir natürlich auch die starke Version aufschreiben, die dann Neumann-Bedingungen auf dem Rande von Ω_t enthalten.

3.8 The strong version. Das Problem in 3.7 lautet in der starken Version:

$$\begin{aligned}
&(v - v_\Gamma)\bullet\nu_\Omega = 0 \text{ auf } \partial\Omega_t\,,\\
&\mathrm{div} v = 0 \text{ in } \Omega_t\,,\\
&\partial_t(\varrho_0 v) + \mathrm{div}(\varrho_0 v v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f} \text{ in } \Omega_t\,,\\
&\tau\bullet S\nu_\Omega = 0 \text{ für alle Tangentialfelder } \tau \text{ an } \partial\Omega_t,\\
&p = \nu_\Omega\bullet S\nu_\Omega \text{ auf } \partial\Omega_t.
\end{aligned}$$

Definition: Dabei ist $\nu_\Omega \in \mathbb{R}^n$ die äußere Normale an $\Omega_t \subset \mathbb{R}^n$, und $v_\Gamma \in \mathrm{span}\,\{\nu_\Omega\}$ die Geschwindigkeit, mit der sich $t \mapsto \Omega_t$ in Normalenrichtung span $\{\nu_\Omega\}$ ausbreitet, siehe I.4.1. Die äußere Normale $n_\Omega \in \mathbb{R}\times\mathbb{R}^n$ an die Raumzeitmenge $\Omega \subset \mathbb{R}\times\mathbb{R}^n$ ist mit ν_Ω verknüpft durch[1]

$$n_\Omega = \frac{(-v_\Gamma\bullet\nu_\Omega, \nu_\Omega)}{\sqrt{1+|v_\Gamma|^2}} \quad \text{mit } \Gamma = \partial\Omega\,. \tag{IV3.10}$$

Proof. Die distributionelle Massenerhaltung

$$\int_\Omega \underbrace{\big(\,\partial_t\xi\varrho_0 + \nabla_x\xi\bullet(\varrho_0 v)\,\big)}_{=\nabla_{(t,x)}\xi\bullet(\varrho_0,\varrho_0 v)}\,\mathrm{d}(t,x) = 0 \text{ für } \xi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$$

ergibt

$$0 = \operatorname{div}_{(t,x)}(\varrho_0,\varrho_0 v) = \partial_t\varrho_0 + \operatorname{div}_x(\varrho_0 v) = \varrho_0\operatorname{div}_x v \text{ in } \Omega\,,$$
$$0 = n_\Omega\bullet(\varrho_0,\varrho_0 v) \text{ auf } \Gamma = \partial\Omega\,,$$

wobei die Gleichung auf Γ wegen (IV3.10) bedeutet

$$(v - v_\Gamma)\bullet\nu_\Omega = 0\,,$$

d.h. das ist die Tatsache, dass v tangential (in Zeit und Raum) auf $\partial\Omega \subset \mathbb{R}\times\mathbb{R}^n$ ist. Entsprechend folgt für die Impulserhaltung

$$\sum_k \int_\Omega \Big(\underbrace{\partial_t\zeta_k\varrho_0 v_k + \nabla_x\zeta_k\bullet\big(\varrho_0 v_k v + (p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k\big)}_{=\nabla_{(t,x)}\zeta_k\bullet(\varrho_0 v_k,\,\varrho_0 v_k v + (p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k)} + \zeta_k\mathbf{f}_k\Big)\,\mathrm{d}(t,x) = 0$$
$$\text{für } \zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)\,.$$

Dies ergibt für $k = 1,\dots,n$

$$\operatorname{div}_{(t,x)}\bullet(\varrho_0 v_k, \varrho_0 v_k v + (p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k) - \mathbf{f}_k = 0 \text{ in } \Omega\,,$$
$$0 = n_\Omega\bullet(\varrho_0 v_k, \varrho_0 v_k v + (p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k) \text{ auf } \Gamma = \partial\Omega\,.$$

Die erste Identität ist die Differentialgleichung

$$\partial_t(\varrho_0 v_k) + \operatorname{div}_x(\varrho_0 v_k v + (p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k) = \mathbf{f}_k$$

für $k = 1,\dots,n$, und die zweite Identität ist unter der Berücksichtigung, dass $n_\Omega\bullet(1,v) = 0$ aus der Massenerhaltung ist, $0 = n_\Omega\bullet(0,(p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k)$, und damit

$$0 = \nu_\Omega\bullet((p\mathrm{Id}-S)^{\mathrm{T}}\,\mathbf{e}_k) = \big((p\mathrm{Id}-S)\nu_\Omega\big)\bullet\mathbf{e}_k\,,$$

d.h. die Behauptung. □

[1] Siehe dazu auch das Skript [22]

Mathematics: There is a pioneering work of Solonnikov, see the paper of Shibata & Shimizu [62]. It is based on the identity

$$L^2(\Omega;\mathbb{R}^n) = J_0(\Omega) \oplus G(\Omega)\,.$$

This is stated in a paper of Ladyzhenskaya & Solonnikov [111, (1.8)]. Another identity, mentioned in [62], is given in (IV3.11) and reads

$$L^2(\Omega;\mathbb{R}^n) = J(\Omega) \oplus G_0(\Omega)\,.$$

This allows us to write the problem in the space of solenoidal vector fields.

3.9 Lemma. The distributional equations (IV3.8) are equivalent to

$$\begin{aligned}
&\mathrm{div} v = 0 \text{ in } \Omega,\\
&0 = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t\overline{\zeta}_k\varrho_0 v_k + \sum_i \partial_{x_i}\overline{\zeta}_k(\varrho_0 v_k v_i - S_{ki}) + \overline{\zeta}_k\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\text{for } \overline{\zeta} \text{ with } \mathrm{div}\overline{\zeta} = 0 \text{ in } \Omega,\\
&p\nu_\Omega = S\nu_\Omega \text{ on } \partial\Omega_t\,,\\
&\int_{\mathbb{R}}\int_{\Omega_t}\Big(\nabla\xi\bullet(-\nabla p + \mathrm{div}S) + \xi\varrho_0\sum_{ki}\partial_{x_k}v_i\partial_{x_i}v_k\Big)\,\mathrm{d}x\,\mathrm{d}t = 0\\
&\text{for } \xi \text{ with } \xi = 0 \text{ on } \partial\Omega_t.
\end{aligned}$$

These statements are not independent. The two weak equations for v and p are of variational structure. *Remark:* According the $\mathbf{f}$ terms see 3.10. The term $\mathrm{div}\,\mathrm{div}S$ is zero for the linear case of S.

Proof. The problem in (IV3.9) consists of the mass equation

$$\int_{\mathbb{R}}\int_{\Omega_t}\big(\partial_t\xi\varrho_0 + \nabla\xi\bullet(\varrho_0 v)\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for $\xi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ and the momentum equation

$$\int_{\mathbb{R}}\int_{\Omega_t}\sum_k\big(\partial_t\zeta_k\varrho_0 v_k + \sum_i \partial_{x_i}\zeta_k(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki}) + \zeta_k\mathbf{f}_k\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$. The mass equation is equivalent to

$$\mathrm{div} v = 0 \text{ in } \Omega$$

and $v\bullet\nu_\Omega = v_\Gamma\bullet\nu_\Omega$, where $\Gamma_t := \partial\Omega_t$. That is, the domain occupied by the fluid is defined by the velocity v. Thus v has no constraint on Γ. We therefore use the identity

$$\begin{aligned} L^2(\Omega_t;\mathbb{R}^n) &= J(\Omega_t) \oplus G_0(\Omega_t)\,, \\ J(\Omega_t) &:= \{\zeta\,;\;\; \mathrm{div}\zeta = 0\}\,, \\ G_0(\Omega_t) &:= \{\nabla\xi\,;\; \xi \in W_0^{1,2}(\Omega_t)\} \end{aligned} \tag{IV3.11}$$

to obtain variational formulations for v and p. (This is not the decomposition considered by Ladyzhenskaya & Solonnikov, the decomposition used here is mentioned in Shibata & Shimizu.) In the momentum equation we replace $\zeta(t,x)$ by

$$\begin{aligned} \zeta(t,x) = \overline{\zeta}(t,x) + \nabla\xi(t,x)\,, \\ \overline{\zeta}(t,\bullet) \in J(\Omega_t)\,, \quad \nabla\xi(t,\bullet) \in G_0(\Omega_t)\,, \end{aligned}$$

and obtain

$$\begin{aligned} 0 = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t(\overline{\zeta}_k + \partial_{x_k}\xi)\varrho_0 v_k \\ + \sum_i \partial_{x_i}(\overline{\zeta}_k + \partial_{x_k}\xi)(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki}) \\ + (\overline{\zeta}_k + \partial_{x_k}\xi)\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

Since we can choose $\xi = 0$ while $\overline{\zeta}$ is arbitrarily, and $\overline{\zeta} = 0$ while ξ is arbitrarily, the momentum equation results in two identities

$$\begin{aligned} 0 = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t\overline{\zeta}_k\varrho_0 v_k + \sum_i \partial_{x_i}\overline{\zeta}_k(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki}) \\ + \overline{\zeta}_k\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{IV3.12}$$

for $\overline{\zeta}(t,\bullet) \in J(\Omega_t)$ and

$$\begin{aligned} 0 = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t\partial_{x_k}\xi\varrho_0 v_k + \sum_i \partial_{x_i}\partial_{x_k}\xi(\varrho_0 v_k v_i + p\delta_{ki} - S_{ki}) \\ + \partial_{x_k}\xi\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{IV3.13}$$

for ξ with $\nabla\xi(t,\bullet) \in G_0(\Omega_t)$. In equation (IV3.12) we have

$$\begin{aligned} \int_{\mathbb{R}}\int_{\Omega_t}\sum_{ki}\partial_{x_i}\overline{\zeta}_k p\delta_{ki}\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \partial_{x_k}\overline{\zeta}_k p\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\mathbb{R}}\int_{\Omega_t}\mathrm{div}\overline{\zeta}\cdot p\,\mathrm{d}x\,\mathrm{d}t = 0 \end{aligned}$$

therefore this equation reduces to

$$0 = \int_{\mathbb{R}}\int_{\Omega_t}\sum_k \Big(\partial_t\overline{\zeta}_k\varrho_0 v_k + \sum_i \partial_{x_i}\overline{\zeta}_k(\varrho_0 v_k v_i - S_{ki}) + \overline{\zeta}_k\mathbf{f}_k\Big)\,\mathrm{d}x\,\mathrm{d}t\,.$$

In the equation (IV3.13) we compute

$$\int_{\mathbb{R}}\int_{\Omega_t}\sum_{ki}\partial_{x_i}\partial_{x_k}\xi\cdot p\delta_{ki}\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}}\int_{\Omega_t}\Delta\xi\cdot p\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}}\int_{\Omega_t}\nabla\xi\bullet\nabla p\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}\int_{\Gamma_t}\nabla\xi\bullet(p\nu_\Omega)\,\mathrm{d}\mathrm{H}^{n-1}(x)\,\mathrm{d}t\,,$$

and

$$-\int_{\mathbb{R}}\int_{\Omega_t}\sum_{ki}\partial_{x_i}\partial_{x_k}\xi\cdot S_{ki}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathbb{R}}\int_{\Omega_t}\sum_{ki}\partial_{x_k}\xi\partial_{x_i}S_{ki}\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}}\int_{\Gamma_t}\sum_k\partial_{x_k}\xi\sum_i S_{ki}\nu_\Omega\bullet\mathbf{e}_i\,\mathrm{d}\mathrm{H}^{n-1}(x)\,\mathrm{d}t$$
$$= \int_{\mathbb{R}}\int_{\Omega_t}\nabla\xi\bullet\,\mathrm{div}S\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}}\int_{\Gamma_t}\nabla\xi\bullet(S\nu_\Omega)\,\mathrm{d}\mathrm{H}^{n-1}x\,\mathrm{d}t\,,$$

and since $(v - v_\Gamma)\bullet\nu_\Omega = 0$ (siehe 3.8)

$$\int_{\mathbb{R}}\int_{\Omega_t}\sum_k\big(\partial_t\partial_{x_k}\xi(\varrho_0 v_k) + \sum_i\partial_{x_i}\partial_{x_k}\xi(\varrho_0 v_k v_i)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}}\int_{\Omega_t}\sum_k\partial_{x_k}\xi\big(\partial_t(\varrho_0 v_k) + \sum_i\partial_{x_i}(\varrho_0 v_k v_i)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}}\int_{\Omega_t}\sum_k\varrho_0\partial_{x_k}\xi\big(\partial_t v_k + \sum_i v_i\partial_{x_i}v_k + v_k\underbrace{\mathrm{div}v}_{=0}\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathbb{R}}\int_{\Omega_t}\varrho_0\xi\sum_{ki}\partial_{x_k}v_i\partial_{x_i}v_k\,\mathrm{d}x\,\mathrm{d}t\,.$$

Here we have used that $\xi = 0$ on $\partial\Omega$. Altogether equation (IV3.13) becomes

$$\int_{\mathbb{R}}\int_{\Omega_t}\Big(\nabla\xi\bullet(-\nabla p + \,\mathrm{div}S) + \xi\varrho_0\sum_{ki}\partial_{x_k}v_i\partial_{x_i}v_k\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\mathbb{R}}\int_{\Gamma_t}\nabla\xi\bullet(p\nu_\Omega - S\nu_\Omega)\,\mathrm{d}\mathrm{H}^{n-1}(x)\,\mathrm{d}t = 0\,.$$

This is equivalent to the boundary condition

$$p\nu_\Omega = S\nu_\Omega$$

and

$$\int_{\mathbb{R}}\int_{\Omega_t}\Big(\nabla\xi\bullet(-\nabla p + \,\mathrm{div}S) + \xi\varrho_0\sum_{ki}\partial_{x_k}v_i\partial_{x_i}v_k\Big)\,\mathrm{d}x\,\mathrm{d}t = 0\,.$$

□

The force term has the following property.

3.10 Remark. If we write $\mathbf{f}$ with respect to one of the identities

$$L^2(\Omega_t;\mathbb{R}^n) = J_0(\Omega_t) \perp G(\Omega_t)\,,$$
$$L^2(\Omega_t;\mathbb{R}^n) = J(\Omega_t) \perp G_0(\Omega_t)$$

as

$$\mathbf{f} = \overline{\mathbf{f}} + \nabla\varphi$$

and in the same way

$$\zeta = \overline{\zeta} + \nabla\xi$$

we compute in 3.6 resp. 3.9

$$\int_{\Omega_t} \zeta \bullet \mathbf{f}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_t} \overline{\zeta} \bullet \overline{\mathbf{f}}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_t} \nabla\xi \bullet \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

Please keep in mind that $\overline{\mathbf{f}}$ as well as $\nabla\varphi$ depends on the decomposition, therefore are different functions in each case.

If the force field is given by rotation and gravity there is a wide field of application, see for example the existence theorem proved in Solonnikov [64]. His result in the non-viscous case is an old theorem shown by Newton, see 16.5.

4 Euler's equation

The temperature dependent fluid equation (IV2.15) with zero viscosity, that is $S=0$, and with zero thermal conductivity, that is $q=0$, are the equations in this section. That is, we have a small viscosity or a large Reynolds number, see [Wikipedia: Inviscid flow], or mathematically we let the stress tensor $S\to 0$ in the momentum equation. Therefore we have the following assumptions with respect to the general equation in (III2.5)

$$\mathbf{J}=0, \quad \mathbf{r}=0, \quad \Pi = p\mathrm{Id}, \quad q=0, \quad g=0.$$

Thus the resulting compressible or incompressible Euler equations are limits of the main fluid equations in (IV2.2), and we end up with the

Euler equations:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + p\mathrm{Id}) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}((e+p)v) = v\bullet\mathbf{f} \tag{IV4.1}$$

ϱ mass, v velocity, p pressure,

ε internal energy, $e = \varepsilon + \frac{\varrho}{2}|v|^2$ total energy,

$\mathbf{f}$ (classical) force.

You can read [Wikipedia: Euler equations] [Euler_equations(Wikipedia).pdf] as an introduction, it is said there: "Historically, only the incompressible equations have been derived by Euler".

References: In the **incompressible** case we mainly use the classical book of Acheson [1]. Also a recent publication by Hutter & Wang [8] has an extensive chapter [8, 6 Function-Theoretical Methods Applied to Plane Potential Flows]. For the numerics in the **compressible** case we recommend Kröner [51]. See also the references below.

Incompressible case

Let us consider the incompressible case that is $\varrho = \varrho_0 = \text{const}$. With this assumption we should keep 3.4 in mind, that is, we have constant temperature $\theta = \text{const}$ (usually in the incompressible equations there is no reference to temperature). And it means that the system has no energy equation, that is, instead of the entropy inequality we require the free energy inequality (for its derivation see section III.5). Also now the pressure p is an independent variable (see 3.2 for details), whereas before it was a function of ϱ and θ (see

(IV4.16) in the compressible case). Hence there is no restriction at all for the pressure. We obtain

Incompressible Euler equations:
$\operatorname{div} v = 0\,,$
$\varrho_0(\partial_t v + v \bullet \nabla v) + \nabla p = \mathbf{f}$
p pressure, v velocity,
$\varrho_0 > 0$ constant mass, $\mathbf{f}$ force.

(IV4.2)

Or equivalently, one can say that this is the (incompressible) Navier-Stokes equation (see section 3) in the limit $S \to 0$.

References: See Acheson [1, 1.3 Equation of motion of an ideal fluid], [1, 1.4 Vorticity: irrotational flow], [1, 1.5 The vorticity equation], and see Rill [61, Reibungsfreie, inkompressible Strömung], and Adams [3, 2 Potentialströmungen].

Let us define the

4.1 Vorticity. Let $n = 3$. The ***vorticity*** of a velocity v is defined as

$$\begin{aligned}\boldsymbol{\omega} &:= \operatorname{rot} v := (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)\\ &= \sum_{(i,j,k)\text{ cyclic}} (\partial_j v_k - \partial_k v_j)\mathbf{e}_i = \sum_{j=1,2,3} \mathbf{e}_j \times \partial_j v\,,\end{aligned}$$

where (i,j,k) cyclic means that $(i,j,k) = (1,2,3), (2,3,1), (3,1,2)$. The curl of v can also be written for any orthonormal system $\{e_1, e_2, e_3\}$, where (e_1, e_2, e_3) has the same orientation as the standard system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. This version reads [2]

$$\operatorname{rot} v = \sum_{j=1,2,3} e_j \times \partial_{e_j} v$$

which is a generalization of the definition. In the world of the nabla operator we write

$$\operatorname{rot} v = \nabla \times v \quad \text{where} \quad \nabla\times = \sum_j e_j \times \partial_{e_j}\,.$$

Proof. If we define the matrix $E = (E_{jk})_{jk}$, $E_{jk} := e_j \bullet \mathbf{e}_k$, we see that

$$\begin{aligned}&\sum_j e_j \times \partial_{e_j} v = \sum_{jk} E_{jk}\, e_j \times \partial_k v = \sum_{jkl} E_{jk} E_{jl}\, \mathbf{e}_l \times \partial_k v\\ &= \sum_{kl} \sum_j E_{jk} E_{jl}\, \mathbf{e}_l \times \partial_k v = \sum_{kl} (E^{\mathrm{T}} E)_{kl}\, \mathbf{e}_l \times \partial_k v = \sum_{kl} \delta_{kl} \mathbf{e}_l \times \partial_k v\\ &= \sum_j \mathbf{e}_j \times \partial_j v = \operatorname{rot} v\end{aligned}$$

[2] By $\times$ we denote the 3-dimensional vector product.

since $E^{\mathrm{T}} E = \mathrm{Id}$ (see the proof of I.1.3). □

We now assume that the fluid is under gravity from an outside body (usually it is the Earth) with a potential $\boldsymbol{\phi}$, defined in (I2.10), therefore

$$\mathbf{f} = \varrho_0 \mathfrak{g} \nabla \boldsymbol{\phi} + \mathbf{f}_0 \,, \tag{IV4.3}$$

where $\mathbf{f}_0$ is a classical force (see II.3.8). With this we rewrite the incompressible Euler equation:

4.2 Lemma. If (IV4.3) for the force is used, the incompressible Euler equation (IV4.2) is equivalent to

$$\boxed{\begin{aligned} \mathrm{div} v &= 0 \,, \\ \partial_t v + \boldsymbol{\omega} \times v + \nabla\Big(\frac{1}{2}|v|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\boldsymbol{\phi}\Big) &= \frac{\mathbf{f}_0}{\varrho_0} \,. \end{aligned}} \tag{IV4.4}$$

Proof. With (IV4.3) the momentum equation of the Euler system reads

$$\partial_t v + v\bullet\nabla v + \nabla\Big(\frac{p}{\varrho_0} - \mathfrak{g}\boldsymbol{\phi}\Big) = \frac{\mathbf{f}_0}{\varrho_0} \,.$$

Now

$$v\bullet\nabla v = \Big(\sum_j v_j \partial_j v_i\Big)_i = \underbrace{\Big(\sum_j (\partial_j v_i - \partial_i v_j) v_j\Big)_i}_{= 2\,(\mathrm{D}v)^{\mathrm{A}}\, v} + \underbrace{\Big(\sum_j \partial_i v_j \cdot v_j\Big)_i}_{= \nabla(\frac{1}{2}|v|^2)} \,,$$

therefore

$$\partial_t v + 2\,(\mathrm{D}v)^{\mathrm{A}}\, v + \nabla\Big(\frac{1}{2}|v|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\boldsymbol{\phi}\Big) = \frac{\mathbf{f}_0}{\varrho_0}.$$

What is left is the identity

$$2\,(\mathrm{D}v)^{\mathrm{A}}\, v = \boldsymbol{\omega} \times v \,.$$

This follows from

$$\mathrm{D}v = \begin{bmatrix} v_{1\,\prime 1} & v_{1\,\prime 2} & v_{1\,\prime 3} \\ v_{2\,\prime 1} & v_{2\,\prime 2} & v_{2\,\prime 3} \\ v_{3\,\prime 1} & v_{3\,\prime 2} & v_{3\,\prime 3} \end{bmatrix} ,$$

$$2\,(\mathrm{D}v)^{\mathrm{A}} = \begin{bmatrix} 0 & v_{1\,\prime 2} - v_{2\,\prime 1} & v_{1\,\prime 3} - v_{3\,\prime 1} \\ v_{2\,\prime 1} - v_{1\,\prime 2} & 0 & v_{2\,\prime 3} - v_{3\,\prime 2} \\ v_{3\,\prime 1} - v_{1\,\prime 3} & v_{3\,\prime 2} - v_{2\,\prime 3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\boldsymbol{\omega}_3 & \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 & 0 & -\boldsymbol{\omega}_1 \\ -\boldsymbol{\omega}_2 & \boldsymbol{\omega}_1 & 0 \end{bmatrix} ,$$

$$\begin{bmatrix} 0 & -\boldsymbol{\omega}_3 & \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 & 0 & -\boldsymbol{\omega}_1 \\ -\boldsymbol{\omega}_2 & \boldsymbol{\omega}_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \boldsymbol{\omega} \times v \,.$$

□

From this the vorticity equation can be derived.

4.3 Vorticity equation. It follows from 4.2 that with $\boldsymbol{\omega} = \mathrm{rot}v$

$$(\partial_t + v\bullet\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}\bullet\nabla)\,v + \mathrm{rot}\big(\frac{\mathbf{f}_0}{\varrho_0}\big)\,.$$

Proof. We take the rotation of the momentum equation in (IV4.4) and use the fact that $\mathrm{rot}\nabla h = 0$ for every function h. This gives

$$\partial_t\boldsymbol{\omega} + \mathrm{rot}(\boldsymbol{\omega}\times v) = \mathrm{rot}\big(\frac{\mathbf{f}_0}{\varrho_0}\big)\,.$$

Now we obtain

$$\begin{aligned}
&\mathrm{rot}(\boldsymbol{\omega}\times v) = \nabla\times(\boldsymbol{\omega}\times v) = \sum_j \mathbf{e}_j\times\partial_j(\boldsymbol{\omega}\times v)\\
&= \sum_j\Big(\mathbf{e}_j\times((\partial_j\boldsymbol{\omega})\times v) + \mathbf{e}_j\times(\boldsymbol{\omega}\times\partial_j v)\Big)\\
&\qquad\big(\text{since } \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\bullet\vec{c})\vec{b} - (\vec{a}\bullet\vec{b})\vec{c} \text{ for } \vec{a},\vec{b},\vec{c}\in\mathbb{R}^3\big)\\
&= \sum_j\Big((\mathbf{e}_j\bullet v)\partial_j\boldsymbol{\omega} - (\mathbf{e}_j\bullet\partial_j\boldsymbol{\omega})v + (\mathbf{e}_j\bullet\partial_j v)\boldsymbol{\omega} - (\mathbf{e}_j\bullet\boldsymbol{\omega})\partial_j v\Big)\\
&= (v\bullet\nabla)\boldsymbol{\omega} - (\nabla\bullet\boldsymbol{\omega})v + (\nabla\bullet v)\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\nabla)v\,.
\end{aligned}$$

Since $\nabla\bullet\boldsymbol{\omega} = \nabla\bullet\mathrm{rot}v = 0$ and $\nabla\bullet v = 0$ by the mass conservation in (IV4.4), we infer

$$\mathrm{rot}(\boldsymbol{\omega}\times v) = (v\bullet\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\nabla)v$$

and hence $\partial_t\boldsymbol{\omega} + \mathrm{rot}(\boldsymbol{\omega}\times v) = (\partial_t + v\bullet\nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\nabla)\,v$. □

With this we can formulate

4.4 Bernoulli equation. Let the flow be incompressible (and $n = 3$) so that (IV4.4) is satisfied. Moreover, assume that $\mathbf{f}_0 = 0$ [3] and that the flow domain is simply connected.

(1) If the velocity $v = \nabla\varphi$ in the flow region [4] then

$$\boxed{\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\boldsymbol{\phi} = \mathrm{const}(t)} \qquad \text{(IV4.5)}$$

(2) If the flow is irrotational, that is $\mathrm{rot}v = 0$, and stationary (steady), then

$$\boxed{\frac{1}{2}|v|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\boldsymbol{\phi} = \mathrm{const}} \qquad \text{(IV4.6)}$$

[3] which is true for certain observers which represent an inertial system (see II.3.9)
[4] Then the flow is called ***potential flow***

Remark: A physical reason for a twodimensional flow to be irrotational you will find in 4.8.

Proof. In case (1) it is also $\operatorname{rot} v = \operatorname{rot} \nabla \varphi = 0$, hence in all cases the momentum part of (IV4.4) becomes

$$\partial_t v + \nabla\big(\frac{1}{2}|v|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\phi\big) = 0 \tag{IV4.7}$$

In case (2) the flow is stationary, hence the assertion follows since the domain is simply connected. In case (1) we insert $v = \nabla\varphi$ in (IV4.7)

$$\nabla\big(\partial_t \varphi + \frac{1}{2}|v|^2 + \frac{p}{\varrho_0} - \mathfrak{g}\phi\big) = 0$$

and from there the assertion. □

4.5 Boundary to air. Let a stationary incompressible flow have a connected boundary $\boldsymbol{S} \subset \partial\Omega$ to the air. Then $v \bullet \nu_\Omega = 0$ on $\boldsymbol{S}$ and, if the flow is irrotational, the distributional Euler equation says (without surface tension and $\mathbf{f}_0 = 0$) that

$$\frac{1}{2}|v|^2 = \mathfrak{g}\phi + \text{const} \quad \text{ on } \boldsymbol{S}. \tag{IV4.8}$$

(1) If the gravity term vanishes (i.e. if the $\mathfrak{g}$-term is not there) the modulus of the velocity is constant.

(2) With gravity on the surface of the Earth $|v(x)|^2 = -2g_{\text{Earth}}x_3 + \text{const}$, where $-\mathbf{e}_3$ points to the center of Earth (for g_{Earth} see the end of section I.4).

By Bernoulli's law 4.5(2) the speed of water $|v|$ (assumed as being incompressible) will become larger if the water falls down. Therefore the jet of water (*de:* Wasserstrahl) becomes narrow further down (see Fig. 7). In general there is surface tension between water and air. Therefore far down the jet will split into drops.

Fig. 7: Water under gravity (down is to the right-side)

Proof (2). We have to write the conservation of mass and momentum across the boundary $\boldsymbol{S}$. Since the flow is stationary this means that in an open set $U \subset \mathbb{R}^3$ the distributional mass law

$$\operatorname{div}[\varrho_0 v \mathcal{X}_\Omega] = 0 \text{ in } \mathscr{D}'(U)$$

is satisfied where Ω is the domain occupied by water, and the distributional momentum law (without terms on $\boldsymbol{S}$)

$$\operatorname{div}[(\varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id})\mathcal{X}_\Omega + p_0 \mathrm{Id}\mathcal{X}_{\mathbb{R}^3\setminus\Omega}] = 0 \text{ in } \mathscr{D}'(U)$$

where p_0 is the constant outside pressure. This gives the Euler equations in Ω and the following conditions on the boundary $\boldsymbol{S}$

$$v \bullet \nu_\Omega = 0\,,$$
$$(\varrho_0 v\, v^{\mathrm{T}} + (p - p_0)\mathrm{Id})\nu_\Omega = 0\,,$$

which reduces to $p - p_0 = 0$ on $\boldsymbol{S}$. In Ω the Bernoulli equation reads

$$\frac{1}{2}|v|^2 = \mathfrak{g}\boldsymbol{\phi} - \frac{p}{\varrho_0} + \text{const}\,,$$

so that at the boundary $\boldsymbol{S}$

$$\frac{1}{2}|v|^2 = \mathfrak{g}\boldsymbol{\phi} - \frac{p_0}{\varrho_0} + \text{const}.$$

The outside pressure p_0 is constant and the gravity potential on the surface of the Earth is given by $\mathfrak{g}\boldsymbol{\phi}(x) = -g_{\text{Earth}} x_3 + \text{const}$ (this is the linear approximation). □

4.6 Boundary to a rigid body. Let a stationary incompressible flow have a boundary $\Sigma \subset \partial\Omega$ to a rigid body. Then $v \bullet \nu_\Omega = 0$ on Σ and the force on the rigid body is $p\nu_\Omega$ on Σ. Here $\mathbf{f}_0 = 0$ is assumed. *Remark:* This result is consistent with the distributional approach across Σ, considering the rigid body without any contribution on the surface Σ.

Vortices of fluid flows are natural effects in nature and also natural effects for the mass-momentum equations. In the easist case ($a = 0$ in 4.7) they have a singularity which is a line in space, and in the neighbourhood the velocity v times the distance to the line r goes to a limit as approching the line. Therefore these vortices are so called "vr-vortices" (see section 8 for a viscous version of them). Here is an example.

4.7 Rankine vortex (Example). Let $a > 0$ and $\Omega \in \mathbb{R}$. Consider

$$v(x) := \left\{ \begin{array}{ll} \dfrac{\lambda r}{a^2}\,\mathbf{e}_\theta & \text{if } r \le a \\ \dfrac{\lambda}{r}\,\mathbf{e}_\theta & \text{if } r \ge a \end{array} \right\}, \quad \mathbf{e}_\theta := \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix},$$

where $x = (x_1, x_2, x_3)$ and $(x_1, x_2) = re^{\mathrm{i}\vartheta}$. If $a \to 0$ this converges to a vr-vortex. The center of the vortex is $\{(0,0,s)\,;\ s \in \mathbb{R}\}$ and in the set $\{x \in \mathbb{R}^3\,;\ |(x_1,x_2)| \le a\}$ the velocity is the velocity of a rigid body. This example satisfies the stationary mass-momentum equations without gravity and with $\mathbf{f}_0 = 0$. *Reference:* See Giaiotti & Stel [42].

Proof. As in the previous proof we have to show that in $\mathscr{D}'(\mathbb{R}^3)$

$$\mathrm{div}[\varrho v] = 0\,, \quad \mathrm{div}[\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}] = 0\,,$$

where the mass density is given by

$$\varrho(x) = \begin{cases} \varrho_f \text{ mass density of fluid,} \\ \varrho_s \text{ mass density of solid.} \end{cases}$$

We have $v(x) = v_\vartheta(r)\mathbf{e}_\vartheta$ with

$$v_\vartheta(r) := \begin{cases} \dfrac{\lambda}{r} \text{ in the fluid,} \\ \dfrac{\lambda r}{a^2} \text{ in the solid,} \end{cases}$$

and we obtain

$$\partial_{\mathbf{e}_r} v = v_\vartheta'(r)\mathbf{e}_\vartheta\,, \quad \partial_{\mathbf{e}_\vartheta} v = -\frac{v_\vartheta(r)}{r}\mathbf{e}_r\,,$$

therefore

$$\mathrm{div} v = \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} v + \mathbf{e}_\vartheta \bullet \partial_{\mathbf{e}_\vartheta} v = v_\vartheta'(r)\mathbf{e}_r \bullet \mathbf{e}_\vartheta - \frac{v_\vartheta(r)}{r}\mathbf{e}_r \bullet \mathbf{e}_\vartheta = 0$$

in the entire space. Since v is tangential on the boundary between fluid and solid the mass part is satisfied. The momentum part consists of the differential equations $\mathrm{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = 0$ in the fluid and solid, where

$$p = \begin{cases} p_f \text{ pressure of fluid,} \\ p_s \text{ pressure of solid,} \end{cases}$$

and, since v is tangential on the boundary, the condition $p_f = p_s$ on the interface. The differential equation reduces to $\varrho v \bullet \nabla v + \nabla p = 0$ in both regions. Hence

$$\nabla p = -\varrho v \bullet \nabla v = -\varrho v_\vartheta (\mathbf{e}_\vartheta \bullet \nabla)(v_\vartheta \mathbf{e}_\vartheta) = -\varrho v_\vartheta(r)^2 \partial_{\mathbf{e}_\vartheta} \mathbf{e}_\vartheta = \frac{\varrho}{r} v_\vartheta(r)^2 \mathbf{e}_r$$

since $\partial_{\mathbf{e}_\vartheta}\mathbf{e}_\vartheta = -\frac{1}{r}\mathbf{e}_r$. Therefore p is a function of r only and

$$\partial_r p = \frac{\varrho}{r} v_\vartheta(r)^2 = \begin{cases} \dfrac{\lambda^2 \varrho}{r^3} \text{ in the fluid,} \\ \lambda^2 \varrho r a^4 \text{ in the solid,} \end{cases}$$

This gives

$$p_f(r) = c_f - \frac{\lambda^2 \varrho_f}{2r^2}\,,$$
$$p_s(r) = c_s + \frac{\lambda^2 \varrho_s r^2}{2a^4}\,.$$

If we normalize thw pressure such that $p(r) \to 0$ if $r \to \infty$, we coose $c_f = 0$ an c_s so that the pressure is continuous at the interface, that is,

$$0 = p_s(a) - p_f(a) = c_s + \frac{\lambda^2}{2a^2}\big(\varrho_s + \varrho_f\big)\,.$$

Then the distributional differential equations are satisfied. We compute in the outer region

$$\mathrm{rot} v = \big(\mathbf{e}_r \bullet \partial_{\mathbf{e}_\vartheta} v - \mathbf{e}_\vartheta \bullet \partial_{\mathbf{e}_r} v\big)\mathbf{e}_{x_3} = \Big(-\frac{v_\vartheta(r)}{r} - v_\vartheta'(r)\Big)\mathbf{e}_{x_3} = 0\,,$$

hence the Bernoulli equation 4.4(2) is satisfied in the outer region which is not simply connected. In the inner region (an infinite bar)

$$v(x) = \frac{\lambda}{a^2} r \mathbf{e}_\vartheta = \frac{\lambda}{a^2}(-x_2, x_1, 0)\,,$$

which is a movement of a rigid body. □

Classical aerofoil theory

References: See Acheson [1, 4. Classical aerofoil theory] and Landau & Lifschitz [10, §48 Der Auftrieb eines dünnen Tragflügels]. As modern book see Adams [3, 2.3 Zweidimensionale, inkompressible Potentialströmungen].

We describe now parts of the classical theory. The observer is situated in the wing (see Fig. 9). Since we consider a constant velocity at infinity, this condition will stay unchanged, since the wing is not turning. We consider a stationary flow in 2D, that is,

$$v(x_1, x_2, x_3) = (v_1(x_1, x_2), v_2(x_1, x_2), 0)\,.$$

It follows that

$$\boldsymbol{\omega} = (0, 0, \omega) \text{ with } \omega = \partial_1 v_2 - \partial_2 v_1\,.$$

Then it holds:

4.8 Lemma. Let $\mathbf{f}_0 = 0$. If the incompressible flow is 2D and stationary as in Fig. 8 and if it has constant velocity v_∞ when $x_1 \to -\infty$ then the flow is irrotational, that is $\boldsymbol{\omega} = 0$ in the flow region.

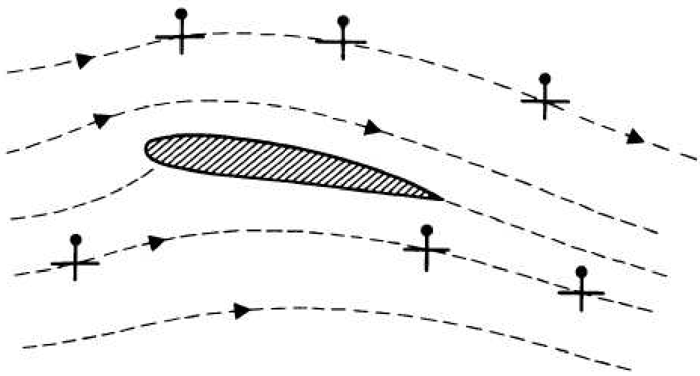


Fig. 8: "The behaviour of a small 'vorticity meter' placed in the steady flow past a fixed wing at small angle of attack. The flow is clearly irrotational." From the book of Acheson [1, Fig. 1.8].

Proof. Since $\boldsymbol{\omega} = (0, 0, \omega)$ it follows $(\boldsymbol{\omega}\bullet\nabla)v = \omega\partial_{x_3}v = 0$. Therefore we conclude from 4.3, if the force $\mathbf{f}_0 = 0$, that $(\partial_t + v\bullet\nabla)\boldsymbol{\omega} = 0$. But the flow is stationary, so $\partial_t v = 0$, and therefore $v\bullet\nabla\boldsymbol{\omega} = 0$. Hence $\boldsymbol{\omega}$ is constant on streamlines (a streamline is a curve $\{\xi(s)\,;\ s \in \mathbb{R}\}$ with $\xi'(s) = v(\xi(s))$). Since by Fig. 8 each streamline goes to the region where the flow gradient approches zero, it follows that $\boldsymbol{\omega} = 0$ on each streamline, hence in the entire flow region. □

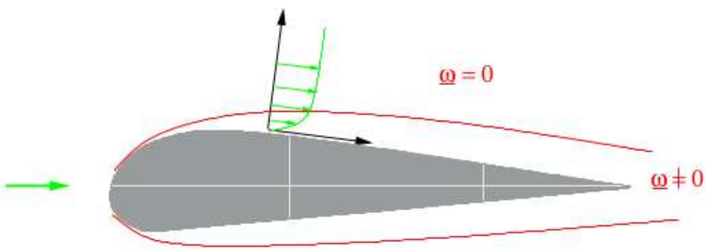


Fig. 9: The vorticity around a wing, aus Adams [3]

As seen in Fig. 9 the proof shows that $\boldsymbol{\omega}$ is 0 only in a subregion of the flow. We consider now a stationary incompressible flow in 2D, which is irrotational, that is, Bernoulli's equation 4.4 holds, and we assume $\mathbf{f}_0 = 0$.

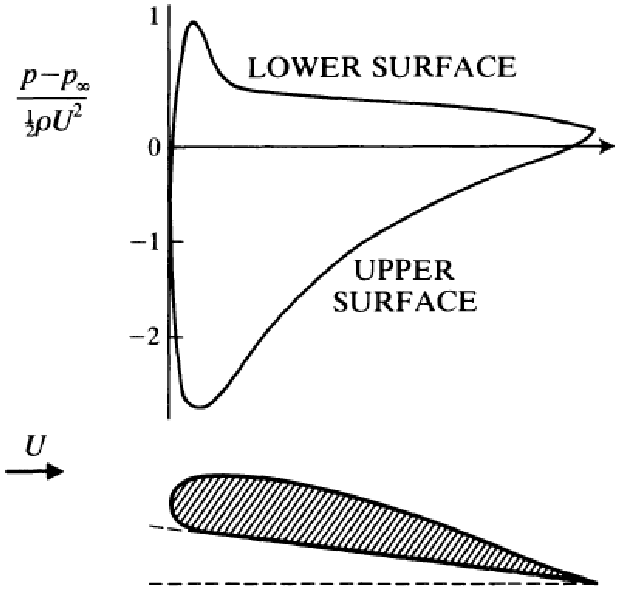


Fig. 10: "Typical pressure distribution on a wing in steady flow" from the book of Acheson [1, Fig. 1.9]. "The pressures on the upper surface are substantially lower than the free-stream value p_∞, while those on the lower surface are a little higher than p_∞. In fact, then, the wing gets most of its lift from a suction effect on its upper surface."

Now we use the special multivalued complex function [5]

$$\boxed{\begin{gathered} W(Z) := U\big(Z\mathrm{e}^{-\mathrm{i}\alpha} + \frac{R^2}{Z}\mathrm{e}^{\mathrm{i}\alpha}\big) - \frac{\mathrm{i}\Gamma}{2\pi}\log Z \\ \text{for } \{Z \in \mathbb{C}\,;\ Z \neq 0\} \text{ with } U, \alpha, R, \Gamma \in \mathbb{R}, \\ \text{where } R > 0,\ U > 0, \text{ and preferably } 0 < \alpha < \frac{\pi}{2}. \end{gathered}} \tag{IV4.9}$$

This function is multivalued, since the logarithm is multivalued. If we set

$$w(z) = W(z)\,, \quad z = x_1 + ix_2\,, \quad w = \varphi + \mathrm{i}\psi\,, \tag{IV4.10}$$

we compute for the velocity

$$v_1 - \mathrm{i}v_2 = \partial_z w = U\big(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{R^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\big) - \frac{\mathrm{i}\Gamma}{2\pi z}\,, \tag{IV4.11}$$

hence $v = (v_1, v_2, 0)$ with $v_1 + \mathrm{i}v_2 = U\mathrm{e}^{\mathrm{i}\alpha}$ as $|z| \to \infty$. That is, at infinity we have a velocity of magnitude U in direction $\mathrm{e}^{\mathrm{i}\alpha}$. The multivalued term

[5] Wir setzen elementare Kenntnisse über komplexe Funktionen voraus.

is a "vortex flow" of strength $\Gamma \in \mathbb{R}$ (a positive Γ means a flow in positive direction), since

$$-\frac{\mathrm{i}\Gamma}{2\pi z} = -\frac{\mathrm{i}\Gamma}{2\pi |z|^2}\overline{z} = -\frac{\Gamma}{2\pi r}\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta},$$

that is, with $\mathbf{e}_\theta = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$,

$$v_1 + \mathrm{i}v_2 = U\big(\mathrm{e}^{\mathrm{i}\alpha} - \frac{R^2}{\overline{z}^2}\mathrm{e}^{-\mathrm{i}\alpha}\big) + \frac{\Gamma}{2\pi r}\mathbf{e}_\theta$$

around the vortex $\{(0,0,s)\,;\ s \in \mathbb{R}\}$ (see 4.7 with $\Omega = \frac{\Gamma}{2\pi}$).

4.9 Flow around a circular cylinder (Lemma). The velocity given in (IV4.11), which is $v = (v_1, v_2, 0)$, is a solution of Bernoulli's equation 4.4 (with vanishing gravity and $\mathbf{f}_0 = 0$) in the domain

$$\Omega := (\mathbb{R}^2 \setminus \overline{\mathrm{B}_R(0)}) \times \mathbb{R} \subset \mathbb{R}^3 .$$

Moreover, we have $v \bullet \nu_\Omega = 0$ on $\partial\Omega$. The total force F on the boundary is

$$F = \int_{\partial \mathrm{B}_R(0)} p\nu_\Omega \,\mathrm{dH}^1 = -\frac{\varrho_0 U\Gamma}{R}\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha},$$

where $U\mathrm{e}^{\mathrm{i}\alpha}$ is the velocity at infinity. This is a ***lift theorem***. *Hint:* See [Hyperphysics: Kutta-Joukowski Lift Theorem] and Acheson [1, Fig. 4.4]. And compare the pressure distribution in Fig. 10.

Proof. The velocity v is given by (IV4.11) that is $v_1 - \mathrm{i}v_2 = \partial_z w$ with a multivalued holomorphic function w, that is $\partial_{\overline{z}} w = 0$, see (IV4.10). Hence $\mathrm{div} v = 0$. If $z = x_1 + \mathrm{i}x_2 = R\mathrm{e}^{i\theta} \in \partial \mathrm{B}_R(0)$ then $\nu_{\mathrm{B}_R(0)}(z) = \frac{z}{|z|} = \frac{z}{R}$, hence

$$\begin{aligned}
-(v \bullet \nu_\Omega)(z) &= (v_1 + iv_2)(z) \bullet \nu_{\mathrm{B}_R(0)}(z) \\
&= U\big(\mathrm{e}^{\mathrm{i}\alpha} - \frac{R^2}{\overline{z}^2}\mathrm{e}^{-\mathrm{i}\alpha}\big) \bullet \nu_{\mathrm{B}_R(0)}(z) + \frac{\Gamma}{2\pi r}\underbrace{\mathbf{e}_\theta \bullet \nu_{\mathrm{B}_R(0)}(z)}_{=0} \\
&= U\big(\mathrm{e}^{\mathrm{i}\alpha} - \frac{R^2}{\overline{z}^2}\mathrm{e}^{-\mathrm{i}\alpha}\big) \bullet \frac{z}{R} = U\mathrm{Re}\Big(\big(\mathrm{e}^{\mathrm{i}\alpha} - \frac{R^2}{\overline{z}^2}\mathrm{e}^{-\mathrm{i}\alpha}\big)\frac{\overline{z}}{R}\Big) \\
&= U\mathrm{Re}\Big(\frac{\overline{z}}{R}\mathrm{e}^{\mathrm{i}\alpha} - \frac{R}{\overline{z}}\mathrm{e}^{-\mathrm{i}\alpha}\Big) = 0 .
\end{aligned}$$

By Bernoulli's law p in the flow region

$$p = \mathrm{const} - \frac{\varrho_0}{2}|v|^2 .$$

And on the boundary of Ω, that is $\overline{z} = R\mathrm{e}^{-\mathrm{i}\theta}$, by (IV4.11) one computes

$$\begin{aligned}
v_1 + \mathrm{i}v_2 &= U\big(\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{2\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\alpha}\big) + \frac{\Gamma}{2\pi R}\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \\
&= \Big(U(1 - \mathrm{e}^{2\mathrm{i}(\theta-\alpha)}) + \frac{\mathrm{i}\Gamma}{2\pi R}\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\Big)\mathrm{e}^{\mathrm{i}\alpha},
\end{aligned}$$

from which it follows that

$$\frac{1}{2\pi R}\int_{\partial \mathrm{B}_R(0)} (v_1 + \mathrm{i}v_2)\,\mathrm{dH}^1 = U\mathrm{e}^{\mathrm{i}\alpha}\,,$$

that is, the velocity of the fluid on the obstacle $\partial\Omega$ in the mean is its value at infinity. We also compute

$$\begin{aligned}
&|v_1 + \mathrm{i}v_2|^2 = \Big|U - U\mathrm{e}^{2\mathrm{i}(\theta-\alpha)} + \frac{\mathrm{i}\Gamma}{2\pi R}\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\Big|^2 \\
&= U^2 - 2U\mathrm{Re}\,\big(U\mathrm{e}^{2\mathrm{i}(\theta-\alpha)} - \frac{\mathrm{i}\Gamma}{2\pi R}\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\big) + \underbrace{\Big|U\mathrm{e}^{2\mathrm{i}(\theta-\alpha)} - \frac{\mathrm{i}\Gamma}{2\pi R}\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\Big|^2}_{= \left|U - \frac{\mathrm{i}\Gamma}{2\pi R}\mathrm{e}^{-\mathrm{i}(\theta-\alpha)}\right|^2} \\
&= 2U^2 - 2U\mathrm{Re}\,\big(U\mathrm{e}^{2\mathrm{i}(\theta-\alpha)}\big) + \frac{U\Gamma}{\pi R}\mathrm{Re}\,\big(\mathrm{i}\mathrm{e}^{\mathrm{i}(\theta-\alpha)} - \mathrm{i}\mathrm{e}^{-\mathrm{i}(\theta-\alpha)}\big) + \Big(\frac{\Gamma}{2\pi R}\Big)^2 \\
&= 2U^2 - 2U^2\cos\,(2(\theta-\alpha)) - \frac{2U\Gamma}{\pi R}\sin\,(\theta-\alpha) + \Big(\frac{\Gamma}{2\pi R}\Big)^2\,.
\end{aligned}$$

Since

$$\int_{\partial \mathrm{B}_R(0)} \nu_{\mathrm{B}_R(0)}\,\mathrm{dH}^1 = 0$$

this gives

$$\begin{aligned}
&\int_{\partial \mathrm{B}_R(0)} p\nu_{\mathbb{R}^2\setminus \mathrm{B}_R(0)}\,\mathrm{dH}^1 = \int_{\partial \mathrm{B}_R(0)} \frac{\varrho_0}{2}|v|^2\nu_{\mathrm{B}_R(0)}\,\mathrm{dH}^1 \\
&= \int_{\partial \mathrm{B}_R(0)} \frac{\varrho_0}{2}|v_1 + \mathrm{i}v_2|^2\nu_{\mathrm{B}_R(0)}\,\mathrm{dH}^1 \\
&= -\varrho_0\int_0^{2\pi}\Big(U^2\cos\,(2(\theta-\alpha)) + \frac{U\Gamma}{\pi R}\sin\,(\theta-\alpha)\Big)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= -\varrho_0\int_0^{2\pi}\Big(U^2\cos\,(2(\theta-\alpha)) + \frac{U\Gamma}{\pi R}\sin\,(\theta-\alpha)\Big)\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\,\mathrm{d}\theta\cdot\mathrm{e}^{\mathrm{i}\alpha} \\
&= -\varrho_0\frac{U\Gamma}{\pi R}\int_0^{2\pi}\sin\,(\theta-\alpha)\mathrm{e}^{\mathrm{i}(\theta-\alpha)}\,\mathrm{d}\theta\cdot\mathrm{e}^{\mathrm{i}\alpha} = -\frac{\varrho_0 U\Gamma}{R}\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}\,.
\end{aligned}$$

□

We want to describe the flow around a wing and for this we have to change the situation. We do this using a simple transformation, which keeps the situation at large z.

4.10 Joukowski transformation. Let $C > 0$.

(1) We consider a transformation from $\zeta \in \mathbb{C}\setminus\{0\}$ to $z \in \mathbb{C}$ by

$$z = J(\zeta) := \zeta + \frac{C^2}{\zeta}\,.$$

The multivalued inverse transformation is

$$\zeta = J^{-1}(z) = \frac{z}{2} + \left(\frac{z^2}{4} - C^2\right)^{\frac{1}{2}} = \frac{z}{2} + \frac{z}{2}\left(1 - \left(\frac{2C}{z}\right)^2\right)^{\frac{1}{2}},$$

where one has to use the "correct value" of the square root, exactly one chooses the branch around 1 for z large, and continues this branch as far as possible, which will be enough for our purpose. *Note:* In the book of Acheson [1, 4.8 Irrotational flow past a finite flat plate] the variable z is called Z and ζ is called z.

(2) Choose $\zeta_1 \in \mathbb{C}$ and define further

$$Z = \zeta - \zeta_1 \,, \quad z = J(\zeta) \,.$$

Here we use in dependence of R only certain values of C and certain ζ_1, that is, they satisfy $\pm C \in \overline{\mathrm{B}_R(\zeta_1)}$, a special choice is

$$C - \zeta_1 \in \partial \mathrm{B}_R(0) \,, \quad -C - \zeta_1 \in \mathrm{B}_R(0) \,, \tag{IV4.12}$$

hence $0 < C < R$, in order to model a wing as in Fig. 12. *Remark:* In Fig. 12 is $a := R$. See also [16_Class_JoukowskiMapping.pdf] from [34].

Proof (1). It is $z = J(\zeta)$ if and only if $\zeta \neq 0$ and

$$0 = \zeta^2 + C^2 - z\zeta = (\zeta - \frac{z}{2})^2 - \frac{z^2}{4} + C^2 \,,$$

that is

$$(\zeta - \frac{z}{2})^2 = \frac{z^2}{4} - C^2 = \frac{z^2}{4}\Big(1 - \Big(\frac{2C}{z}\Big)^2\Big)$$

for $z \neq 0$, which is the formula. □

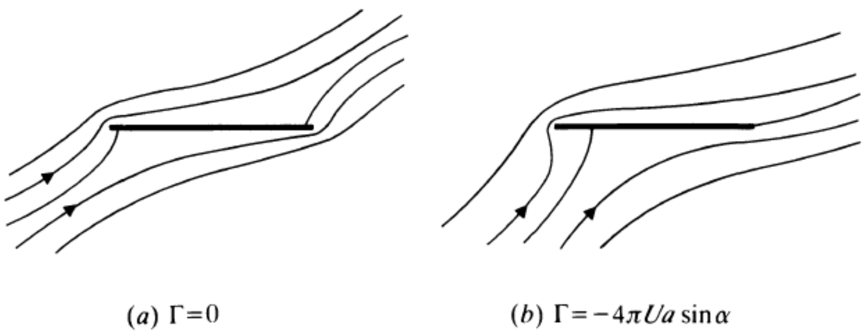


Fig. 11: "Irrotational flow past a finit flat plate" from Acheson [1, Fig. 4.6.].

We now set, with the function W given in (IV4.9),

$$w(z) = W(J^{-1}(z) - \zeta_1)\,, \quad z = x_1 + ix_2\,, \tag{IV4.13}$$

that is, $w(z) = W(Z)$ if z and Z are connected as in 4.10(2).

4.11 Flow around an aerofoil (Lemma). The velocity $v = (v_1, v_2, 0)$, given by $v_1 - iv_2 = \partial_z w(z)$, where w is defined in (IV4.13), is a solution of Bernoulli's equation 4.4 (with vanishing gravity and $\mathbf{f}_0 = 0$) in the domain

$$\Omega := (\mathbb{R}^2 \setminus \overline{D}) \times \mathbb{R} \subset \mathbb{R}^3 \text{ with } \mathbb{R}^2 \setminus D = J(\mathbb{R}^2 \setminus \mathrm{B}_R(\zeta_1))\,.$$

Moreover, we have $v_1 + iv_2 \to U\mathrm{e}^{\mathrm{i}\alpha}$ for $|z| \to \infty$, and v satisfies the boundary condition $v \bullet \nu_\Omega = 0$ on $\partial\Omega$. The total force F on the boundary is

$$F = \int_{\partial D} p\nu_\Omega \,\mathrm{dH}^1 = -\frac{\varrho_0}{2}\int_{\partial \mathrm{B}_R(\zeta_1)} |\partial_Z W(\zeta - \zeta_1)|^2 \frac{\zeta - \zeta_1}{R\,\overline{\partial_\zeta J(\zeta)}}\,\mathrm{dH}^1(\zeta)\,.$$

Remark: In the case described in (IV4.12) one has $C = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta_1}$ for some $\theta_1 \in \mathbb{R}$. The above integral is then Cauchy's principal value at $\zeta = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta_1}$, see 4.12.

In the special case that $\zeta_1 = 0$ and $C = R$ one has the situation in Fig. 11.

Proof. It is $w = W \circ h$ in Ω, where $h(z) := J^{-1}(z) - \zeta_1$ defines a holomorphic map with nonzero derivative (therefore its a conformal transformation), and the holomorphic function W satisfies 4.9. The domain Ω is defined in such a way, that $\partial\Omega$ by the map h will be mapped into $\partial\mathrm{B}_R(0)$, that is, a tangential vector τ_Ω at z of $\partial\Omega$ will be mapped by the derivative $\mathrm{D}h$ at z into a multiple of the tangential vector $\tau_{\mathbb{R}^2\setminus\mathrm{B}_R(0)}$ at $Z := h(z)$ of $\partial\mathrm{B}_R(0)$. Therefore we have with $\lambda = |\partial_z h|$ the formulas

$$\partial_z h\,\tau_\Omega = \mathrm{D}h\,\tau_\Omega = \lambda\tau_{\mathbb{R}^2\setminus\mathrm{B}_R(0)}\,, \quad \partial_z h\,\nu_\Omega = \mathrm{D}h\,\nu_\Omega = \lambda\nu_{\mathbb{R}^2\setminus\mathrm{B}_R(0)}\,.$$

Now, for $Z = h(z)$,

$$\begin{aligned} v_1(z) - \mathrm{i}v_2(z) &= \partial_z w(z) = \partial_z (W \circ h)(z) \\ &= \partial_z h(z)\partial_Z W(Z) = \partial_z h(z)(V_1(Z) - \mathrm{i}V_2(Z)) \end{aligned} \tag{IV4.14}$$

if V is defined by $V_1 - \mathrm{i}V_2 := \partial_Z W$, which is the velocity used in 4.9. Since by 4.9 we know that $V \bullet \nu_{\mathbb{R}^2\setminus\mathrm{B}_R(0)} = 0$, it follows using (IV4.14)

$$\begin{aligned} 0 &= \lambda\, V \bullet \nu_{\mathbb{R}^2\setminus\mathrm{B}_R(0)} = (V_1 + iV_2)\bullet(\lambda\nu_{\mathbb{R}^2\setminus\mathrm{B}_R(0)}) = (V_1 + iV_2)\bullet(\partial_z h\,\nu_\Omega) \\ &= \mathrm{Re}\,((V_1 - iV_2)\partial_z h\,\nu_\Omega) = \mathrm{Re}\,((v_1 - iv_2)\nu_\Omega) = (v_1 + iv_2)\bullet\nu_\Omega = v\bullet\nu_\Omega\,. \end{aligned}$$

This implies that v is tangential at the boundary of the wing $\partial\Omega$, that is, $v \bullet \nu_\Omega = 0$ on $\partial\Omega$, as indicated in Fig. 12.

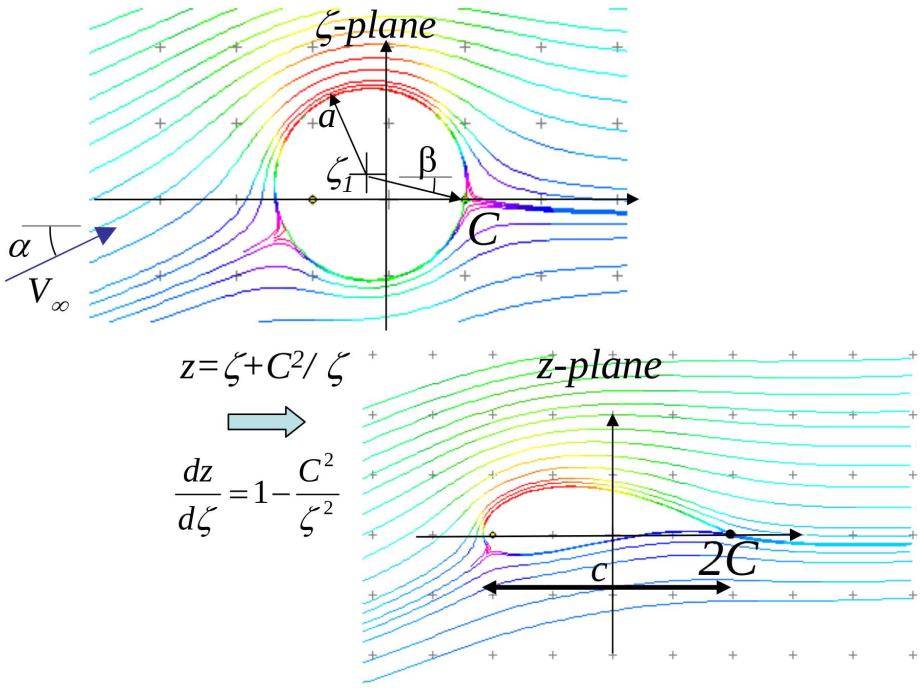


Fig. 12: "The cambered airfoil" from [16_Class_JoukowskiMapping.pdf], "$\operatorname{Re}\zeta_1$ controls thickness, $\operatorname{Im}\zeta_1$ controls camber."

We now compute the total force F on the boundary on a cross section, that is, $F = (F_1, F_2, 0)$. By Bernoulli's law p in the flow region is $p = \text{const} - \frac{\varrho_0}{2}|v|^2 = \text{const} - \frac{\varrho_0}{2}|\partial_z w|^2$. Therefore

$$
\begin{aligned}
F &= \int_{\partial D} p\nu_\Omega \,\mathrm{dH}^1 = -\frac{\varrho_0}{2}\int_{\partial D} |\partial_z w|^2 \nu_{\mathbb{R}^2\setminus D}\,\mathrm{dH}^1(z) \\
&= \frac{\varrho_0}{2}\int_{\partial D} |\partial_z w(z)|^2 \nu_D(z)\,\mathrm{dH}^1(z) \\
&= \frac{\varrho_0}{2}\int_{\partial \mathrm{B}_R(\zeta_1)} |\partial_z w(J(\zeta))|^2 \nu_D(J(\zeta))|\partial_\zeta J(\zeta)|\,\mathrm{dH}^1(\zeta)
\end{aligned}
$$

since $z = J(\zeta)$ and therefore because J is a holomorphic function

$$\mathrm{dH}^1 \llcorner \partial D = |\partial_\zeta J|\,\mathrm{dH}^1 \llcorner \partial \mathrm{B}_R(\zeta_1)\,.$$

The normal satisfies

$$\nu_D = \frac{\partial_\zeta J}{|\partial_\zeta J|}\nu_{\mathrm{B}_R(\zeta_1)}\,,$$

and $w(z) = W(h(z)) = W(J^{-1}(z) - \zeta_1)$ gives

$$\partial_z w(z) = \partial_Z W(h(z))\partial_z h(z) = \frac{\partial_Z W(\zeta - \zeta_1)}{\partial_\zeta J(\zeta)} \text{ for } z = J(\zeta)\,.$$

Thus it is

$$\begin{aligned} F &= \frac{\varrho_0}{2}\int_{\partial \mathrm{B}_R(\zeta_1)} |\partial_z w(J(\zeta))|^2 \nu_D(J(\zeta))|\partial_\zeta J(\zeta)|\,\mathrm{dH}^1(\zeta) \\ &= \frac{\varrho_0}{2}\int_{\partial \mathrm{B}_R(\zeta_1)} \frac{|\partial_Z W(\zeta-\zeta_1)|^2}{|\partial_\zeta J(\zeta)|^2}\partial_\zeta J(\zeta)\nu_{\mathrm{B}_R(\zeta_1)}(\zeta)\,\mathrm{dH}^1(\zeta) \\ &= \frac{\varrho_0}{2}\int_{\partial \mathrm{B}_R(\zeta_1)} |\partial_Z W(\zeta-\zeta_1)|^2 \frac{\nu_{\mathrm{B}_R(\zeta_1)}(\zeta)}{\overline{\partial_\zeta J(\zeta)}}\,\mathrm{dH}^1(\zeta)\,, \end{aligned}$$

quod erat demonstrandum. □

4.12 The case of a sharp trailing edge. This is the case in (IV4.12), hence $C = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta_1}$ for some $\theta_1 \in \mathbb{R}$.

(1) The total force on the wing is finite, since the integral in 4.11 has a Cauchy principle value at the angle θ_1 for every Γ.

(2) The Kutta-Joukowski condition is, that the force at the trailing edge stays finite. This means

$$\Gamma = 4\pi RU \sin{(\alpha - \theta_1)}\,.$$

Then the integrand in 4.11 is integrable.

Notice: See the pictures in Rill [61, Die Kutta Bedingung], which shows that in certain situations this case is stable. Rill writes in [61] according the pictures in Fig. 13: “Das Bild 1 zeigt die Strömung unmittelbar nach dem Start der Strömung. Wir sehen, daß die Strömung an der Hinterkante 'versucht' diese zu umströmen und damit den Ansatz eines Wirbels formt.” “Aus der Theorie der reibungsfreien Strömung resultiert, dass bei der Umströmung einer scharfen Ecke, wie der Hinterkante, unendlich große Geschwindigkeiten auftreten. Dies wird aber von der realen, reibungsbehafteten Strömung im Experiment nicht toleriert. Als Folge davon wandert der Staupunkt auf der Oberseite während des Anfahrvorganges in Richtung Hinterkante. Das Bild 2 zeigt diesen Übergangszustand. Schließlich stellt sich dann die endgültige, stationäre Strömung ein, die wir im Bild 3 beobachten. Wir sehen, daß die Strömung in diesem Fall die Ober- und Unterseite an der Hinterkante glatt verlässt.”

Proof (1). We start with $J(\zeta) = \zeta + \frac{C^2}{\zeta}$ and

$$\partial_\zeta J(\zeta) = 1 - \Big(\frac{C}{\zeta}\Big)^2 = \Big(1+\frac{C}{\zeta}\Big)\Big(1-\frac{C}{\zeta}\Big) = \frac{\zeta + C}{\zeta^2}(\zeta - C)\,.$$

Now write $\zeta \in \partial\mathrm{B}_R(\zeta_1)$ as $\zeta = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta}$ so that for $\theta = \theta_1$ the singularity C will be reached in the integral, that is $C = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta_1}$, and this implies

$$\zeta = \zeta_1 + R\mathrm{e}^{\mathrm{i}\theta} = C + R\mathrm{e}^{\mathrm{i}\theta_1}(\mathrm{e}^{\mathrm{i}(\theta-\theta_1)} - 1)$$

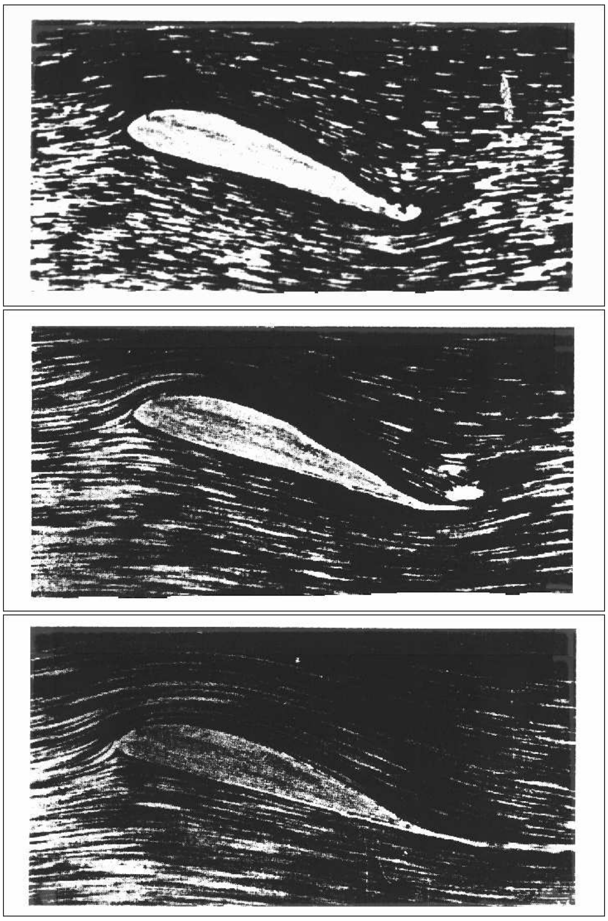


Fig. 13: "Entwicklung von auftriebsbehafteter Profilumströmung" from Rill [61, Die Kutta Bedingung].

and therefore

$$\partial_\zeta J(\zeta) = \frac{\zeta + C}{\zeta^2} R\mathrm{e}^{\mathrm{i}\theta_1}(\mathrm{e}^{\mathrm{i}(\theta-\theta_1)} - 1)\,,$$

which implies the statement. □

Proof (2). We have to show that $\partial_Z W(Z) = 0$ at the trailing edge, that is $\zeta = C$ or $Z = \zeta - \zeta_1 = C - \zeta_1 = -R\mathrm{e}^{\mathrm{i}\theta_1}$. Since

$$\partial_Z W(Z) = U\big(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{R^2}{Z^2}\mathrm{e}^{\mathrm{i}\alpha}\big) - \frac{\mathrm{i}\Gamma}{2\pi Z}$$

it follows that $\partial_Z W(Z) = 0$ is equivalent to

$$\begin{aligned} 0 &= \frac{Z}{RU}\partial_Z W(Z) = \frac{Z}{R}\mathrm{e}^{-\mathrm{i}\alpha} - \frac{R}{Z}\mathrm{e}^{\mathrm{i}\alpha} - \frac{\mathrm{i}\Gamma}{2\pi RU} \\ &= -\mathrm{e}^{\mathrm{i}(\theta_1-\alpha)} + \mathrm{e}^{\mathrm{i}(\alpha-\theta_1)} - \frac{\mathrm{i}\Gamma}{2\pi RU} \\ &= \mathrm{i}\big(2\sin(\alpha-\theta_1) - \frac{\Gamma}{2\pi RU}\big)\,, \end{aligned}$$

which had to be shown. □

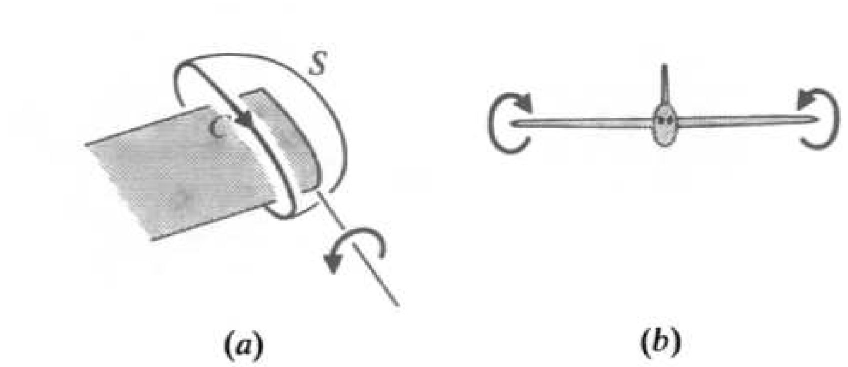


Fig. 14: "Trailing vortices: (a) definition sketch for application of Stoke's theorem; (b) view from some distance ahead of the aircraft;" from Acheson [1, Fig. 1.12.].

The 2D-method described here cannot treat the flow at the end of a wing, see Fig. 14, Fig. 15, and [3, 2.4 Umströmung von Tragflügeln endlicher Spannweite] which is a 3D-phenomenon and important for the flight of aircrafts. But for a flight of a full size glider (*de:* Segelflugzeug) with very long wings the 2D considerations are usefull.
Die hiergemachten Überlegungen gelten in der Realität natürlich nur in einem begrenzten Bereich, da darüberhinaus die Bedingungen der Eulergleichung die Wirklichkeit verlassen. In ihrem Gültigkeitsbereich sind die gemachten Aussagen jedoch überzeugend.

Compressible case

Usually one is concerned with the compressible case and this case contains the conservation laws for mass, momentum and energy:

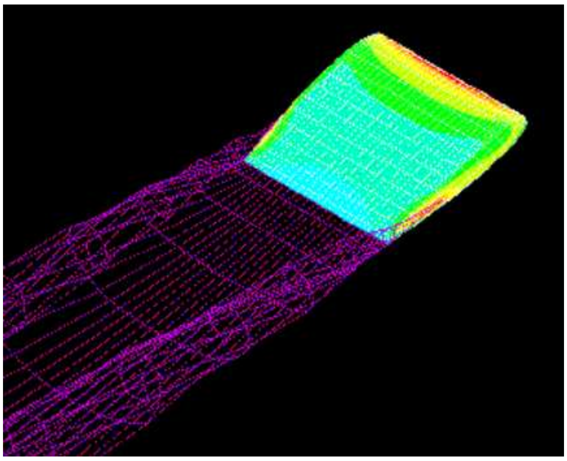


Fig. 15: "Flow over a low aspect ratio wing" from [34, Lifting Line Theory].

(Compressible) Euler equations:

$$\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}((e+p)v) = v \bullet \mathbf{f} \tag{IV4.15}$$

ϱ mass, v velocity, p pressure,

ε internal energy, $e = \varepsilon + \frac{\varrho}{2}|v|^2$ total energy,

$\mathbf{f}$ (classical) force.

Performing the entropy principle 2.2 (with stress tensor S) we obtain (in the limit $S \to 0$) Gibbs relation

$$p = \varrho f'_{\varrho}(\varrho, \theta) - f(\varrho, \theta)\,, \tag{IV4.16}$$

where f is the internal free energy, and the entropy production (IV2.9) reduces to the fact that solutions in general are only L^∞-functions. Therefore we would have to replace the equations by its distributional versions. However, in literature these equations appear in the usual form.

What is left from the entropy principle is that the solutions develop ***shocks*** (de: ***Stoßwellen***), and the entropy production $\sigma \geq 0$ will lead (also in the limit $S \to 0$) to Rankine-Hugoniot conditions on the shock, which I will present in a separate publication [22]. Here I will give at least a short presentation of these shock conditions.

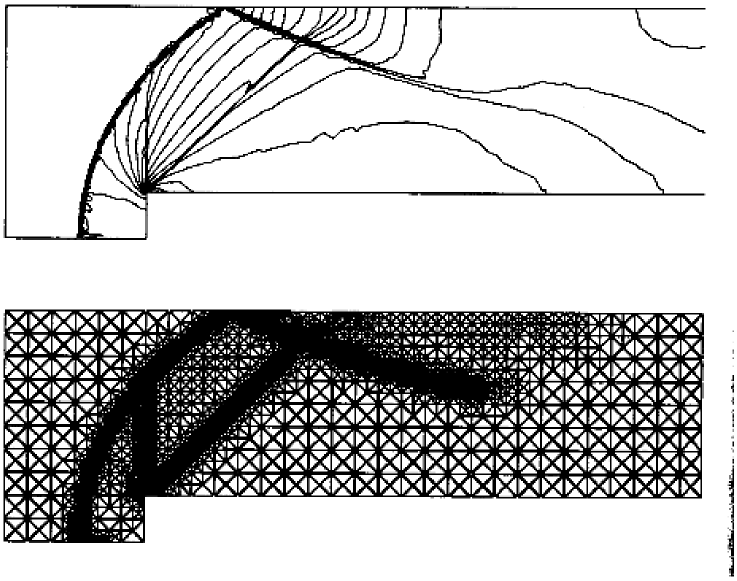


Fig. 16: "Transient phase for the forward-facing step in 2D" from Kröner [51, Example 5.3.9]: "In particular, the grid had to be refined and to be corsened after the shock has passed through."

4.13 Shocks. If we denote the solution of (IV4.1) with $(\varrho^1, v^1, \theta^1)$ and $(\varrho^2, v^2, \theta^2)$ on both sides $\Omega^1, \Omega^2 \subset \mathbb{R} \times \mathbb{R}^n$ of the shock Γ, the following conditions are satisfied on Γ_t:

$$\begin{aligned} v^1_{\tan} &= v^2_{\tan}\,, \\ \mathbf{m} := \varrho^1\lambda^1 &= \varrho^2\lambda^2 \neq 0\,, \\ p^1 + \varrho^1|\lambda^1|^2 &= p^2 + \varrho^2|\lambda^2|^2\,, \\ \frac{\varepsilon^1 + p^1}{\varrho^1} + \frac{|\lambda^1|^2}{2} &= \frac{\varepsilon^2 + p^2}{\varrho^2} + \frac{|\lambda^2|^2}{2}\,, \\ \mathbf{m} \cdot \frac{\eta^1}{\varrho^1} &\geq \mathbf{m} \cdot \frac{\eta^2}{\varrho^2}\,, \end{aligned}$$

where $\lambda^k := (v^k - v_\Gamma)\bullet\nu$, ν a unit normal on Γ_t, and v_Γ defined in I.4.1. Here p is the pressure from (IV4.16) and η the entropy derived from the free energy in (IV4.16) (see III.1.6).

If there are shocks, they have to be reproduced by the numerical schemes (see Fig. 16 and Fig. 17). For numerics one writes this system in a contracted form

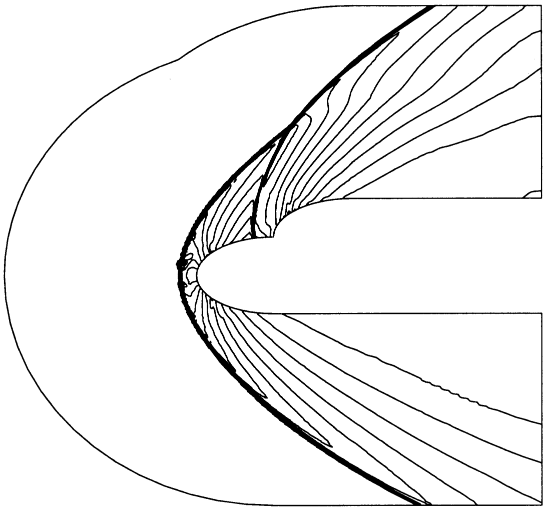


Fig. 17: "Flow around a double ellipsoid" from Kröner [51, Example 5.3.7].

$$\partial_t y + \operatorname{div} F(y) = h,$$
$$y = \begin{bmatrix} \varrho \\ \varrho v \\ e \end{bmatrix}, \quad F(y) = \begin{bmatrix} \varrho v \\ \varrho v\, v^{\mathrm{T}} + p\mathrm{Id} \\ (e+p)v \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ \mathbf{f} \\ v\bullet\mathbf{f} \end{bmatrix}. \tag{IV4.17}$$

This is a hyperbolic system, which normally has L^∞-solutions, thus solutions in the sense of distributions. The introduction of the variable y requires that $\varrho > 0$ for the solution (or what would be better, that ϱ is bounded from below by a positive constant). In the momentum part $p = \widetilde{p}(y)$. We mention that one can say that the schemes introduce a numerical viscosity, which is positive and therefore is consistent with the entropy principle.

References: Numerical schemes of this type you will find in Kröner [51, Chap. 5?????]. The picture Fig. 16 is from T. Geßner: Zeitabhängige Adaption für Finite Volume Verfahren höherer Ordnung am Beispiel der Euler-Gleichungen der Gasdynamik, Diplomthesis IAM Univ. Bonn 1994. And the picture Fig. 17 is from J. Becker: Finite Volume Verfahren in 2-D für Systeme von hyperbolischen Differentialgleichungen mit Flußfunktion von Osher und Solomon, Diplomthesis IAM Univ. Bonn 1995.

5 Nonlinear elasticity

We come back to bodies with large deformations which have been introduced in section I.6. We have shown in I.6.2 that we can also write the conservation laws in reference coordinates (there we had set $\mathbf{J} = 0$). Now with the additional energy equation we have to consider in the physical coordinates the system

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho v) &= \mathbf{r}\,, \\ \partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) &= \mathbf{f} + \mathbf{r} v\,, \\ \partial_t e + \operatorname{div}(e v + \Pi^{\mathrm{T}} v + q) &= \tfrac{\mathbf{r}}{2}|v|^2 + v \bullet \mathbf{f} + g \end{aligned} \tag{IV5.1}$$

where the physical quantities of the mass and momentum conservation as in (I6.4), especially v is a dependent variable. Moreover

$$\mathbf{J} = 0 \text{ and the pressure tensor } \Pi \text{ is symmetric}$$

and the quantities of the energy conservation are defined as

$$e = \varepsilon + \frac{\varrho}{2}|v|^2\,, \quad \varepsilon \text{ objective scalar}, \quad q \text{ objective vector.}$$

The right-hand sides, by (III2.7), have the representation $\widetilde{\mathbf{f}} = \mathbf{f} + \mathbf{r} v$ and $\widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v \bullet \mathbf{f} + g$, where $\mathbf{f}$ is a classical force and g an objective scalar. We want to show now that this system in reference coordinates is equivalent to

Mass-momentum-energy:

$$\begin{aligned} \partial_t \underline{\varrho} &= \underline{\mathbf{r}}\,, \\ \partial_t(\underline{\varrho} V) - \operatorname{div} P &= \underline{\mathbf{f}} + \underline{\mathbf{r}} V\,, \\ \partial_t \underline{e} + \operatorname{div}(-P^{\mathrm{T}} V + \underline{q}) &= \frac{\underline{\mathbf{r}}}{2}|V|^2 + V \bullet \underline{\mathbf{f}} + \underline{g}\,, \end{aligned} \tag{IV5.2}$$

$\underline{\varrho}$, $\underline{\mathbf{r}}$, V, P, $\underline{\mathbf{f}}$ wie in (I6.4),

$\underline{e} := J \cdot (e \circ \tau)\,, \quad \underline{q} := J F^{-1} (q \circ \tau)\,, \quad \underline{g} := J \cdot (g \circ \tau)\,,$

namely, the following holds:

5.1 Lemma. In analogy to I.6.2 and with the same notation as there we show that (IV5.1) is equivalent to (IV5.2).

Proof (First version). We start with the equations (IV5.1) and write it in the form

$$\partial_t \begin{bmatrix} \varrho \\ \varrho v \\ e \end{bmatrix} + \sum_{i=1}^{n} \partial_{x_i} \begin{bmatrix} \varrho v_i \\ \left(\widetilde{\Pi}_{ki}\right)_k \\ \widetilde{q}_i \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ \widetilde{\mathbf{f}} \\ \widetilde{g} \end{bmatrix}, \tag{IV5.3}$$

here

$$\widetilde{\Pi}_{ki} = \varrho v_k v_i + \Pi_{ki}\,, \quad \widetilde{\mathbf{f}} = \mathbf{f} + \mathbf{r} v$$
$$\widetilde{q} = ev + \Pi^{\mathrm{T}}\, v + q\,, \quad \widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v \bullet \mathbf{f} + g\,. \tag{IV5.4}$$

We want to show that this system is equivalent to

$$\partial_t \begin{bmatrix} \underline{\varrho} \\ \underline{\varrho} V \\ \underline{e} \end{bmatrix} + \sum_{i=1}^{n} \partial_{\underline{x}_i} \begin{bmatrix} 0 \\ \left(\underline{\widetilde{\Pi}}_{ki}\right)_k \\ \underline{\widetilde{q}}_i \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{r}} \\ \underline{\widetilde{\mathbf{f}}} \\ \underline{\widetilde{g}} \end{bmatrix}. \tag{IV5.5}$$

According to theorem I.6.1 systems (IV5.3) and (IV5.5) are equivalent, if the quantities satisfy

$$\begin{aligned} \underline{\varrho} &= J\,\varrho\circ\tau\,, & \underline{\mathbf{r}} &= J\,\mathbf{r}\circ\tau\,, \\ \underline{\varrho}\,V &= J\,\varrho\circ\tau\, v\circ\tau\,, & \underline{\widetilde{\mathbf{f}}} &= J\,\widetilde{\mathbf{f}}\circ\tau\,, \\ \underline{e} &= J\,e\circ\tau\,, & \underline{\widetilde{g}} &= J\,\widetilde{g}\circ\tau\,, \end{aligned}$$
$$0 = J\,F^{-1}\left((\varrho v)\circ\tau - \varrho\circ\tau\, V\right),$$
$$\left(\underline{\widetilde{\Pi}}_{ki}\right)_i = J\,F^{-1}\left(\left(\widetilde{\Pi}_{kj}\right)_j\circ\tau - (\varrho v)_k\circ\tau V\right) \quad \text{for all } k,$$
$$\underline{\widetilde{q}} = J\,F^{-1}\left(\widetilde{q}\circ\tau - e\circ\tau V\right).$$

This means that the following known (see I.6.2) relations are satisfied

$$\underline{\varrho} = J\,\varrho\circ\tau\,, \quad V = v\circ\tau\,, \quad \underline{\mathbf{r}} = J\,\mathbf{r}\circ\tau\,.$$

Then the above form of $\widetilde{\Pi}_{ki}$ in (IV5.4) gives that

$$\begin{aligned} \underline{\widetilde{\Pi}}_{ki} &= J\,\sum_j (F^{-1})_{ij}(\widetilde{\Pi}_{kj}\circ\tau - (\varrho v)_k\circ\tau V_j) \\ &= J\,\sum_j (F^{-\mathrm{T}})_{ji}(\Pi_{kj}\circ\tau) = J\left((\Pi\circ\tau)\,F^{-\mathrm{T}}\right)_{ki} = -P_{ki} \end{aligned}$$

and the form of $\widetilde{\mathbf{f}}$ in (IV5.4) gives that

$$\underline{\widetilde{\mathbf{f}}} = J\,\widetilde{\mathbf{f}}\circ\tau = J\,(\mathbf{f} + \mathbf{r}v)\circ\tau = \underline{\mathbf{f}} + \underline{\mathbf{r}}\,V \quad \text{if} \quad \underline{\mathbf{f}} := J\,\mathbf{f}\circ\tau\,.$$

Therefore the first two equations of (IV5.5) are the same as in I.6.2 and in (IV5.2). Thus we have to look at the last equation

$$\partial_t \underline{e} + \sum_{i=1}^{n} \partial_{\underline{x}_i} \underline{\widetilde{q}}_i = \underline{\widetilde{g}}\,,$$
$$\underline{e} = J\,e\circ\tau\,, \quad \underline{\widetilde{q}} = J\,F^{-1}\left(\widetilde{q}\circ\tau - e\circ\tau V\right), \quad \underline{\widetilde{g}} = J\,\widetilde{g}\circ\tau\,.$$

From the definition of $\widetilde{q}$ in (IV5.4) we obtain

$$\begin{aligned} \underline{\widetilde{q}} &= J\,F^{-1}\left((\Pi\circ\tau)^{\mathrm{T}}\,V + q\circ\tau\right) = J\left((\Pi\circ\tau)\,F^{-\mathrm{T}}\right)^{\mathrm{T}} V + J\,F^{-1}\,q\circ\tau \\ &= -\,P^{\mathrm{T}}\,V + \underline{q} \quad \text{if} \quad \underline{q} := J\,F^{-1}\,q\circ\tau\,, \end{aligned}$$

and the definition of $\widetilde{g}$ in (IV5.4) gives

$$\underline{\widetilde{g}} = J\left(\frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + g\right)\circ\tau = \frac{\underline{\mathbf{r}}}{2}|V|^2 + V\bullet\underline{\mathbf{f}} + \underline{g} \quad \text{if} \quad \underline{g} := J\, g\circ\tau\,.$$

Thus we have shown that (IV5.1) is equivalent to (IV5.2). □

Proof (Second version). This is analog to the second proof in I.6.2. The energy equation in (IV5.1) is for test functions $\varsigma \in C_0^\infty(\Omega;\mathbb{R})$

$$\int_\Omega \left(\partial_t\varsigma\cdot e + \nabla\varsigma\bullet(ev + \Pi^{\mathrm{T}} v + q) + \varsigma\cdot\widetilde{g}\right)\,\mathrm{dL}^{n+1} = 0.$$

Defining $\widetilde{\varsigma}(t,\underline{x}) := \varsigma(t,x)$ for $x = \varphi(t,\underline{x})$, this is

$$\int_{t_1}^{t_2}\int_{\mathcal{B}} \Big(\ \partial_t\widetilde{\varsigma}\cdot J(e\circ\tau) + J\cdot((\mathrm{D}\varphi)^{-\mathrm{T}}\,\nabla\widetilde{\varsigma})\bullet((\Pi\circ\tau)^{\mathrm{T}}\, V + q\circ\tau) \\ + \widetilde{\varsigma}\cdot(Jg\circ\tau)\Big)\,\mathrm{dL}^n\,\mathrm{dL}^1 = 0\,.$$

Since

$$\begin{aligned} &J\cdot\Big((\mathrm{D}\varphi)^{-\mathrm{T}}\,\nabla\widetilde{\varsigma}\Big)\bullet\Big((\Pi\circ\tau)^{\mathrm{T}}\, V + q\circ\tau\Big) \\ &= J\cdot\nabla\widetilde{\varsigma}\bullet\Big((\mathrm{D}\varphi)^{-1}\,(\Pi\circ\tau)^{\mathrm{T}}\, V + (\mathrm{D}\varphi)^{-1}\,(q\circ\tau)\Big) \\ &= J\cdot\nabla\widetilde{\varsigma}\bullet\Big(\big(\Pi\circ\tau\,(\mathrm{D}\varphi)^{-\mathrm{T}}\big)^{\mathrm{T}}\, V + (\mathrm{D}\varphi)^{-1}\,(q\circ\tau)\Big) \\ &= \nabla\widetilde{\varsigma}\bullet\left(-P^{\mathrm{T}}\, V + \underline{q}\right)\,, \end{aligned}$$

the result follows. □

As in III.2.3 an equation for ε was derived, we now derive one for $\overline{\varepsilon}$.

5.2 Thermal energy equation. The last equation in (IV5.2) is equivalent modulo the remaining two equations to

$$\partial_t\underline{\varepsilon} + \mathrm{div}\underline{q} = \mathrm{D}V\!:\!P + \underline{g}\,,$$

where the internal energy in reference coordinates $\underline{\varepsilon}$ is given by

$$\underline{e} = \underline{\varepsilon} + \frac{1}{2}\underline{\varrho}|V|^2\,, \quad \underline{\varepsilon} := J\cdot(\varepsilon\circ\tau)\,.$$

Proof. Equation (IV5.2) is

$$\partial_t\underline{e} + \mathrm{div}(-P^{\mathrm{T}}\, V + \underline{q}) = \underline{\widetilde{g}} := \frac{\underline{\mathbf{r}}}{2}|V|^2 + V\bullet\underline{\mathbf{f}} + \underline{g}\,.$$

Now it is

$$\begin{aligned} &\partial_t\underline{e} = \partial_t\underline{\varepsilon} + \frac{1}{2}\partial_t(\underline{\varrho}|V|^2) \\ &= \partial_t\underline{\varepsilon} + \frac{1}{2}|V|^2\partial_t\underline{\varrho} + V\bullet(\underline{\varrho}\partial_t V) = \partial_t\underline{\varepsilon} - \frac{1}{2}|V|^2\partial_t\underline{\varrho} + V\bullet\partial_t(\underline{\varrho}V) \\ &= \partial_t\underline{\varepsilon} - \frac{1}{2}|V|^2\underline{\mathbf{r}} + V\bullet\mathrm{div}P + V\bullet(\underline{\mathbf{f}} + \underline{\mathbf{r}}\,V) \\ &= \partial_t\underline{\varepsilon} + V\bullet\mathrm{div}P + \frac{\underline{\mathbf{r}}}{2}|V|^2 + V\bullet\underline{\mathbf{f}}\,. \end{aligned}$$

It follows

$$\begin{aligned}\widetilde{\underline{g}} &= \partial_t \underline{e} + \mathrm{div}\underline{q} + \mathrm{div}(-P^{\mathrm{T}} V)\\ &= \partial_t \underline{\varepsilon} + \mathrm{div}\underline{q} + V\bullet \mathrm{div}P + \mathrm{div}(-P^{\mathrm{T}} V) + \frac{\underline{\mathbf{r}}}{2}|V|^2 + V\bullet\underline{\mathbf{f}}\\ &= \partial_t \underline{\varepsilon} + \mathrm{div}\underline{q} - (\mathrm{D}V){:}P + \frac{\underline{\mathbf{r}}}{2}|V|^2 + V\bullet\underline{\mathbf{f}}\,,\end{aligned}$$

since $\mathrm{div}(P^{\mathrm{T}} V) = V\bullet \mathrm{div}P + (\mathrm{D}V){:}P$. This implies the assertion. □

We now state the entropy principle in the physical space, but the assumption on the entropy we want to formulate in terms of $\underline{\eta}$. The simplest assumption is $\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F)$ which is related to the definition of thermoelastic bodies, see (IV5.16). If besides thermoelastic bodies one considers other materials with large deformation, $\underline{\eta}$ depends on other variables, and therefore the proof of the entropy principle has to be done again. See for example EX: Viscous Thermoelastic Bodies.

5.3 General entropy principle. We require the entropy principle

$$\sigma := \partial_t \eta + \mathrm{div}\, \psi \geq 0$$

for all solutions of the general problem (IV5.1) (or equivalent (IV5.2)) and define

$$\underline{\sigma} = J \cdot \sigma\circ\tau\,, \quad \underline{\eta} := J \cdot \eta\circ\tau\,, \quad \underline{\psi} := J \cdot F^{-1}(\psi - \eta v)\circ\tau\,.$$

Hence the entropy inequality in reference coordinates is

$$\underline{\sigma} := \partial_t \underline{\eta} + \mathrm{div}\underline{\psi} \geq 0\,.$$

Here the entropy is an objective scalar, that is $\eta\circ Y = \eta^*$, which means that $\underline{\eta}(t,\underline{x}) = \underline{\eta}^*(t^*,\underline{x})$ for $\tau(t,\underline{x}) = Y\circ\tau^*(t^*,\underline{x})$.[6]

We consider now a particular case.

5.4 Theorem. The entropy principle is fulfilled for solutions of (IV5.2) if

$$\underline{\eta} = \widehat{\underline{\eta}}(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F)\,, \quad \underline{\eta}_{,F} F^{\mathrm{T}} \text{ symmetric,} \quad \underline{\psi} = \frac{1}{\underline{\theta}}\underline{q}\,, \tag{IV5.6}$$

and if the residual inequality

$$\boxed{\underline{\theta}\,\underline{\sigma} = \underline{\theta}\,\underline{\eta}_{,\underline{\varrho}}\underline{\mathbf{r}} + (P + \underline{\theta}\,\underline{\eta}_{,F}){:}\mathrm{D}V + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big)\bullet\underline{q} + \underline{g} \geq 0}$$

holds. Here the temperature $\underline{\theta}$ is defined by

$$\underline{\theta} := \theta\circ\tau\,, \quad \text{was äquivalent ist zu} \quad \frac{1}{\underline{\theta}} = \underline{\eta}_{,\underline{\varepsilon}} > 0\,. \tag{IV5.7}$$

[6]Here different observers share the same reference configuration.

Remark: The function $\widehat{\underline{\eta}}$ is objective provided that $\underline{\eta}_{,F}F^{\mathrm{T}}$ is symmetric, which means $\widehat{\underline{\eta}}_{,F}(\underline{x},\underline{\varrho},\underline{\varepsilon},F)\,F^{\mathrm{T}}$ is a symmetric matrix for all $(\underline{x},\underline{\varrho},\underline{\varepsilon},F)$.

The fact that $\widehat{\underline{\eta}}$ depends on $\underline{x}$, is an objective property (see the remark in II.5.3) and results in no extra term in σ. If $\mathbf{r}=0$, then the mass conservation reads $\partial_t\underline{\varrho}=0$. Hence, it follows that $\underline{\varrho}=\underline{\varrho}(\underline{x})$ and therefore, since $\widehat{\underline{\eta}}$ depends already on $\underline{x}$, there is no need for the dependence on $\underline{\varrho}$.

Proof of the entropy principle. It is

$$0\le\sigma=\partial_t\eta+\operatorname{div}\psi=(\partial_t\eta+\operatorname{div}(\eta v))+\operatorname{div}(\psi-\eta v)$$

and therefore it follows with $\underline{\sigma}=J\,\sigma\circ\tau$

$$0\le\underline{\sigma}=\partial_t\underline{\eta}+\operatorname{div}\underline{\psi}=\partial_t\underline{\eta}+\operatorname{div}(\underline{\eta}_{,\underline{\varepsilon}}\underline{q})\,.$$

Dieser Schluss ist analog zur Massenerhaltung. Das Entropieprinzip verlangt auch, dass η ein objektiver Skalar ist, d.h. es gilt $\eta(t,x)=\eta^*(t^*,x^*)$ mit einer Beobachtertransformation $(t,x)=Y(t^*,x^*)$. Da

$$\underline{\eta}(t,\underline{x}):=J(t,\underline{x})\,\eta(t,\varphi(t,\underline{x}))$$

und J ein objektiver Skalar ist, muss nach II.5.1 gelten, dass $\underline{\eta}$ ein objektiver Skalar ist, also

$$\widehat{\underline{\eta}}(\underline{x},\underline{\varrho},\underline{\varepsilon},F)(t,\underline{x})=\underline{\eta}(t,\underline{x})=\underline{\eta}^*(t^*,\underline{x})=\widehat{\underline{\eta}}(\underline{x},\underline{\varrho}^*,\underline{\varepsilon}^*,F^*)(t^*,\underline{x})\,.$$

Da $\underline{\varrho}$ und $\underline{\varepsilon}$ objektive Skalare sind, also $\underline{\varrho}(t,\underline{x})=\underline{\varrho}^*(t^*,\underline{x})$ und entsprechendes für ε, und da $F(t,\underline{x})=Q(t^*)F^*(t^*,\underline{x})$ nach II.5.2, reduziert sich die Gleichung für $\widehat{\underline{\eta}}$ zu

$$\widehat{\underline{\eta}}(\underline{x},\underline{\varrho}^*,\underline{\varepsilon}^*,QF^*)=\widehat{\underline{\eta}}(\underline{x},\underline{\varrho}^*,\underline{\varepsilon}^*,F^*)\,. \tag{IV5.8}$$

Diese Gleichung muss also erfüllt sein. □

Proof of the theorem. Because of $\underline{\eta}=\widehat{\underline{\eta}}(\underline{x},\underline{\varrho},\underline{\varepsilon},F)$ one computes

$$\partial_t\underline{\eta}=\underline{\eta}_{,\underline{\varrho}}\partial_t\underline{\varrho}+\underline{\eta}_{,\underline{\varepsilon}}\partial_t\underline{\varepsilon}+\underline{\eta}_{,F}\boldsymbol{:}\partial_t F\,.$$

Since using 5.2

$$\begin{aligned}
&\partial_t\underline{\varrho}=\underline{\mathbf{r}}\,,\\
&\partial_t\underline{\varepsilon}+\operatorname{div}\underline{q}=\mathrm{D}V\boldsymbol{:}P+\underline{g}\,,\\
&\underline{g}:=\underline{\widetilde{g}}-V\bullet\underline{\widetilde{\mathbf{f}}}+\frac{1}{2}|V|^2\underline{\mathbf{r}}\,,\\
&\partial_t F=\Big(\partial_t\partial_{\underline{x}_j}\varphi_i\Big)_{ij}=\Big(\partial_{\underline{x}_j}(\partial_t\varphi_i)\Big)_{ij}=\Big(\partial_{\underline{x}_j}V_i\Big)_{ij}=\mathrm{D}V\,,
\end{aligned}$$

it follows

$$\begin{aligned}\underline{\sigma} &= \underline{\eta}_{,\underline{\varrho}}\partial_t\underline{\varrho} + \underline{\eta}_{,\underline{\varepsilon}}\partial_t\underline{\varepsilon} + \underline{\eta}_{,F}\mathbin{:}\partial_t F + \operatorname{div}(\underline{\eta}_{,\underline{\varepsilon}}\underline{q}) \\ &= \underline{\eta}_{,\underline{\varrho}}\underline{\mathbf{r}} + \underline{\eta}_{,\underline{\varepsilon}}\big(P\mathbin{:}\mathrm{D}V - \operatorname{div}\underline{q} + \underline{g}\big) + \underline{\eta}_{,F}\mathbin{:}\mathrm{D}V + \operatorname{div}(\underline{\eta}_{,\underline{\varepsilon}}\underline{q}) \\ &= \underline{\eta}_{,\underline{\varrho}}\underline{\mathbf{r}} + (\underline{\eta}_{,\underline{\varepsilon}}P + \underline{\eta}_{,F})\mathbin{:}\mathrm{D}V + \nabla\underline{\eta}_{,\underline{\varepsilon}}\bullet\underline{q} + \underline{\eta}_{,\underline{\varepsilon}}\underline{g}\,.\end{aligned}$$

Now, the definition of θ yields the assertion. □

Proof of the remark. The objectivity of $\underline{\widehat{\eta}}$ requires, see (IV5.8) in the above proof, that

$$\underline{\widehat{\eta}}(\underline{x},\underline{\varrho},\underline{\varepsilon},QF) = \underline{\widehat{\eta}}(\underline{x},\underline{\varrho},\underline{\varepsilon},F)\,.$$

In the following we do not write out the arguments $(\underline{x},\underline{\varrho},\underline{\varepsilon})$), so for simplicity

$$\underline{\widehat{\eta}}(QF) = \underline{\widehat{\eta}}(F) \tag{IV5.9}$$

for all orthonormal Q with positive determinant and all F. We want to show that this is equivalent to

$$\underline{\eta}_{,F}(F)\,F^{\mathrm{T}} \quad \text{is symmetric matrix} \tag{IV5.10}$$

for all F. Let (IV5.9) be true. If we set $Q = \exp(sA)$ with an antisymmetric matrix A, it then follows

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}s}(\underline{\widehat{\eta}}(F)) = \frac{\mathrm{d}}{\mathrm{d}s}(\underline{\widehat{\eta}}(\exp(sA)F)) = \underline{\widehat{\eta}}_{,F}(\exp(sA)F)\mathbin{:}\frac{\mathrm{d}}{\mathrm{d}s}(\exp(sA)F) \\ &= \underline{\widehat{\eta}}_{,F}(\exp(sA)F)\mathbin{:}(A\exp(sA)F) = A\mathbin{:}\big(\underline{\widehat{\eta}}_{,F}(\exp(sA)F)\,(\exp(sA)F)^{\mathrm{T}}\,\big)\,,\end{aligned}$$

and therefore for $s = 0$

$$0 = A\mathbin{:}\big(\underline{\widehat{\eta}}_{,F}(F)\,F^{\mathrm{T}}\,\big)$$

for all antisymmetric A, which means (IV5.10).

$$Q_s = \exp(sA)Q_0\,,\quad s\in[0,1]\,,$$

matrix A (Q_s)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\underline{\widehat{\eta}}(Q_sF) = \frac{\mathrm{d}}{\mathrm{d}s}\underline{\widehat{\eta}}(\exp(sA)F) = \underline{\widehat{\eta}}_{,F}(\exp(sA)F)\mathbin{:}\frac{\mathrm{d}}{\mathrm{d}s}(\exp(sA)F) \\ &= \underline{\widehat{\eta}}_{,F}(\exp(sA)F)\mathbin{:}(A\exp(sA)F) = A\mathbin{:}\big(\underline{\widehat{\eta}}_{,F}(\exp(sA)F)\,(\exp(sA)F)^{\mathrm{T}}\,\big) \\ &= 0\,,\end{aligned}$$

and thus $\underline{\eta}(Q_1F) = \underline{\eta}(Q_0F)$. Then this follows tht for any orthonormal Q with positive determinant $\underline{\eta}(QF) = \underline{\eta}(F)$, that is (IV5.9). □

Now we reformulate the residual inequality for the free energy $\underline{f}$, which is a function on $\underline{\theta}$. The temperature $\underline{\theta}$ is defined in (IV5.7).

5.5 Free energy. We define the ***internal free energy*** in reference coordinates by

$$\underline{f} = \underline{f}(\underline{x}, \underline{\varrho}, \underline{\theta}, F)\,, \quad \underline{f}_{,F} F^{\mathrm{T}} \text{ symmetric,}$$
$$\underline{f} = \underline{\varepsilon} - \underline{\theta}\underline{\eta} \quad \text{für} \quad \underline{\theta} = \frac{1}{\underline{\eta}_{,\underline{\varepsilon}}} > 0\,. \tag{IV5.11}$$

Then the residual inequality in 5.4 reads

$$\boxed{\underline{\theta}\,\underline{\sigma} = -\underline{f}_{,\underline{\varrho}}\underline{\mathbf{r}} + (P - \underline{f}_{,F}):\mathrm{D}V + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big)\bullet\underline{q} + \underline{g} \geq 0.}$$

Proof. By Theorem 5.4 the residual equation is

$$\underline{\theta}\,\underline{\sigma} = \underline{\theta}\,\underline{\eta}_{,\underline{\varrho}}\underline{\mathbf{r}} + (P + \underline{\theta}\,\underline{\eta}_{,F}):\mathrm{D}V + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big)\bullet\underline{q} + \underline{g} \geq 0\,.$$

Now, replacing the derivatives of $\underline{\eta}$ by the following formulas we obtain the assertion. The formulas are:

$$\underline{f}_{,\underline{\theta}} = -\underline{\eta}\,, \quad \underline{f}_{,\underline{\varrho}} = -\underline{\theta}\underline{\eta}_{,\underline{\varrho}}\,, \quad \underline{f}_{,F} = -\underline{\theta}\underline{\eta}_{,F}\,.$$

To prove this we write the formula $\underline{f} = \underline{\varepsilon} - \underline{\theta}\underline{\eta}$ in detail

$$\underline{f}(\underline{x}, \underline{\varrho}, \underline{\theta}(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F), F) = \underline{\varepsilon} - \underline{\theta}(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F)\underline{\eta}(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F)\,,$$

hence differentiating we obtain

$$\partial_{\underline{\varepsilon}}: \quad \underline{f}_{,\underline{\theta}}\underline{\theta}_{,\underline{\varepsilon}} = \underbrace{1 - \underline{\theta}\underline{\eta}_{,\underline{\varepsilon}}}_{=0} - \underline{\theta}_{,\underline{\varepsilon}}\underline{\eta} \quad \Longrightarrow \quad \underline{f}_{,\underline{\theta}} = -\underline{\eta}\,,$$
$$\partial_{\underline{\varrho}}: \quad \underline{f}_{,\underline{\varrho}} + \underline{f}_{,\underline{\theta}}\underline{\theta}_{,\underline{\varrho}} = 0 - \underline{\theta}_{,\underline{\varrho}}\underline{\eta} - \underline{\theta}\underline{\eta}_{,\underline{\varrho}} \quad \Longrightarrow \quad \underline{f}_{,\underline{\varrho}} = -\underline{\theta}\underline{\eta}_{,\underline{\varrho}}\,,$$
$$\partial_F: \quad \underline{f}_{,F} + \underline{f}_{,\underline{\theta}}\underline{\theta}_{,F} = 0 - \underline{\theta}_{,F}\underline{\eta} - \underline{\theta}\underline{\eta}_{,F} \quad \Longrightarrow \quad \underline{f}_{,F} = -\underline{\theta}\underline{\eta}_{,F}\,.$$

(See also exercise III.7.3.) □

Reminder: The quantities are defined by

$$\boxed{\begin{aligned} \underline{\varepsilon} &= J\,\varepsilon\circ\tau\,, & \underline{\theta} &= \theta\circ\tau\,, \\ \underline{\eta} &= J\,\eta\circ\tau\,, & \underline{f} &= J\,f\circ\tau\,. \end{aligned}}$$

References: For the entropy principle see the general description in Marsden & Hughes [55, CH.2 Theorem 5.5] and I-Shih Liu [86, 5.3 Thermodynamics of Elastic Materials]. We recommend Hutter & Jöhnk [47, 5 Material

Equations]. In connection with isotropic bodies see [55, 3.5 Material Symmetries and Isotropic Elasticity] and [47, 5.4 Material Equations for Isotropic Bodies].

We give two realizations of the entropy inequality, one in section 6 about the growth in biology, and the classical version now here where we have no reference mass change, that is $\underline{\mathbf{r}} = 0$.

Thermoelasticity

We had shown the entropy inequality in 5.4 under the assumption that $\underline{\eta}$ depends only on the variables $(\underline{x}, \underline{\varrho}, \underline{\varepsilon}, F)$. Here we assume that

$$\underline{\mathbf{r}} = 0\,, \quad \underline{g} = 0\,,$$

that is, $\underline{\varrho}$ is time independent and the energy is conserved. Then it follows:

5.6 Theorem. Assume that with an internal free energy as in (IV5.11) the following is true for terms in the differential equations

$$P = \underline{f}_{,F}\,, \quad \underline{\varepsilon} = \underline{f} - \underline{\theta}\,\underline{f}_{,\theta}\,. \tag{IV5.12}$$

Then the system (IV5.2) is equivalent to

$$\begin{aligned} &\partial_t \underline{\varrho} = 0\,, \\ &\partial_t(\underline{\varrho} V) = \operatorname{div} P + \underline{\mathbf{f}}\,, \\ &\partial_t \underline{\varepsilon} + \operatorname{div} \underline{q} = \mathrm{D}V \mathbin{:} P\,, \end{aligned} \tag{IV5.13}$$

and the entropy principle for this system is satisfied if

$$\underline{\sigma} = \nabla\Big(\frac{1}{\underline{\theta}}\Big) \bullet \underline{q} \geq 0\,. \tag{IV5.14}$$

Proof. Since $\underline{\mathbf{r}} = 0$ and $P = \underline{f}_{,F}$ the first two terms in the residual entropy inequality are zero. Hence we obtain

$$\underline{\theta}\underline{\sigma} = \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big) \bullet \underline{q} \geq 0$$

as remaining part of the residual inequality. □

Because of (IV5.14) the heat flux $\underline{q}$ must be chosen so that it satisfes this inequality, for example

$$\begin{aligned} \underline{q}(t, \underline{x}) = -\widehat{D}(\underline{x}, \underline{\varrho}, \underline{\theta}, \nabla\underline{\theta}, F)\nabla\underline{\theta}(t, \underline{x}) \quad \text{with} \\ \widehat{D}(\underline{x}, \underline{\varrho}^*, \underline{\theta}^*, \nabla\underline{\theta}^*, F^*) = \widehat{D}(\underline{x}, \underline{\varrho}^*, \underline{\theta}^*, Q\nabla\underline{\theta}^*, QF^*)\,, \end{aligned} \tag{IV5.15}$$

where $D = \widehat{D}(\underline{x}, \underline{\varrho}, \underline{\theta}, \nabla\underline{\theta}, F)$ is a positive semidefinite matrix.

The fact that $\underline{f}_{\prime F}F^{\mathrm{T}}$ is chosen symmetric is devoted to the principle that η has to be an objective scalar. This implies that $\underline{f}_{\prime F}F^{\mathrm{T}} = -\underline{\theta}\,\underline{\eta}_{\prime F}F^{\mathrm{T}}$ is symmetric. The fact that the first Piola-Kirchhoff stress tensor $P = \underline{f}_{\prime F}$ has been chosen in the entropy inequality has the consequence that (recall that $P = FS$) the second Piola-Kirchhoff stress tensor

$$S = F^{-1}P = F^{-1}\underline{f}_{\prime F} = F^{-1}(\underline{f}_{\prime F}F^{\mathrm{T}})F^{-\mathrm{T}}$$

is symmetric ("Boltzmann-Axiom"), since $\underline{f}_{\prime F}F^{\mathrm{T}}$ is symmetric. Therefore also Π is symmetric, since $-\Pi\circ\tau = \frac{1}{J}PF^{\mathrm{T}} = \frac{1}{J}FSF^{\mathrm{T}}$. With this theorem the equations for thermoelastic bodies are:

$$\begin{array}{|l|}\hline
\textbf{\textit{Thermoelastic body}}\text{: In }]t_1,t_2[\times\mathcal{B} \\
\underline{\varrho}\partial_t V - \mathrm{div}_{\underline{x}}P = \underline{\mathbf{f}}\,, \\
\partial_t\underline{\varepsilon} + \mathrm{div}_{\underline{x}}\,\underline{q} = \mathrm{D}V\!:\!P \\ \hline
\underline{\varrho} = \underline{\varrho}(\underline{x}) \text{ reference density (since } \underline{\mathbf{r}} = 0), \\
V = \partial_t\varphi(t,\underline{x}) \text{ velocity, where } \varphi \text{ as in (I6.2)}, \\
\underline{\varepsilon} = \underline{f} - \underline{\theta}\,\underline{f}_{\prime\theta} \text{ where } \underline{f}_{\prime F}F^{\mathrm{T}} \text{ is symmetric}, \\
P = \underline{f}_{\prime F}(\underline{x},\underline{\varrho},\underline{\theta},F) \\ \hline
\end{array} \tag{IV5.16}$$

Nach 5.6 erfüllt also ein thermoelastischer Körper das Entropieprinzip wegen $P = \underline{f}_{\prime F}$, und falls für den Wärmefluss $\underline{q}$ die Residualungleichung (IV5.14) gilt, also z.B. $\underline{q}$ wie in (IV5.15) gewählt wird.

5.7 Lemma. Die Symmetrie von $\underline{f}_{\prime F}F^{\mathrm{T}}$ ist gewährleistet, wenn

$$\underline{f} = \widehat{\underline{f}}(\underline{x},\underline{\varrho},\underline{\theta},F) = \widetilde{\underline{f}}(\underline{x},\underline{\varrho},\underline{\theta},C)\,,$$

wobei $C := F^{\mathrm{T}}F$ der rechte Cauchy-Green Deformationstensor ist. Dabei sei vorausgesetzt, dass $\widetilde{\underline{f}}_{\prime C_{ij}} = \widetilde{\underline{f}}_{\prime C_{ji}}$ ist, d.h. $\widetilde{\underline{f}}_{\prime C}$ symmetrisch ist. Es gilt dann $\widehat{\underline{f}}_{\prime F} = 2F\widetilde{\underline{f}}_{\prime C}$, also nach (IV5.16)

$$P = \widehat{\underline{f}}_{\prime F}F^{\mathrm{T}} = 2F\widetilde{\underline{f}}_{\prime C}F^{\mathrm{T}}\,, \quad S = 2\widetilde{\underline{f}}_{\prime C}\,, \quad -\Pi\circ\tau = \frac{2}{J}F\widetilde{\underline{f}}_{\prime C}F^{\mathrm{T}}\,.$$

Proof. Es gilt, da $C_{ij} = \sum_k F_{ki}F_{kj}$,

$$\begin{aligned}
\widehat{\underline{f}}_{\prime F_{lm}} &= \sum_{i,j}\widetilde{\underline{f}}_{\prime C_{ij}}\frac{\partial C_{ij}}{\partial F_{lm}} = \sum_{i,j,k}\widetilde{\underline{f}}_{\prime C_{ij}}\frac{\partial(F_{ki}F_{kj})}{\partial F_{lm}} \\
&= \sum_{i,j,k}\widetilde{\underline{f}}_{\prime C_{ij}}\big(\delta_{kl}\delta_{im}F_{kj} + F_{ki}\delta_{kl}\delta_{jm}\big) = \sum_{i,j}\widetilde{\underline{f}}_{\prime C_{ij}}\big(\delta_{im}F_{lj} + F_{li}\delta_{jm}\big) \\
&= \sum_{j}\widetilde{\underline{f}}_{\prime C_{mj}}F_{lj} + \sum_{i}\widetilde{\underline{f}}_{\prime C_{im}}F_{li} = \big(F\widetilde{\underline{f}}_{\prime C}{}^{\mathrm{T}} + F\widetilde{\underline{f}}_{\prime C}\big)_{l,m}\,,
\end{aligned}$$

also

$$\widehat{\underline{f}}_{,F} = F\big(\widetilde{\underline{f}}_{,C}{}^{\mathrm{T}} + \widetilde{\underline{f}}_{,C}\big)\,.$$

Now, C is a symmetric matrix, therefore we have assumed that $\widetilde{\underline{f}}_{,C}$ is symmetric, thus $P = \widehat{\underline{f}}_{,F} = 2F\widetilde{\underline{f}}_{,C}$, and $S = F^{-1}P = 2\widetilde{\underline{f}}_{,C}$. And it is obvious that $\widehat{\underline{f}}_{,F}F^{\mathrm{T}} = 2F\widetilde{\underline{f}}_{,C}F^{\mathrm{T}}$ is symmetric. □

We now are going to treat isotropic materials where we follow the definitions in literature, e.g. in Marsden & Hughes [55, 3.5] "By definition, an isotropic free energy constitutive function $\underline{f}$ is to be a 'rotationally invariant' function of the argument C. Since C is symmetric, it can be brought to diagonal form by an orthogonal transformation, so $\underline{f}$ is a function only of the eigenvalues of C; that is, $\underline{f}$ depends only on the principal stretches. Since the eigenvalues are reasonably complicated functions of C, it is sometimes convenient to use the invariants of C." Hence we consider a free energy $\underline{f}$ depending on the invariants of the matrix C. In [55, 3.5] it is written: "To describe non-isotropic materials then requires some 'non-tensorial' constructions or the introduction of additional variables."

5.8 Invarianten einer Matrix A. Wir definieren das charakteristische Polynom zur $n \times n$-Matrix A als das Polynom n-ten Grades (physikalisch ist $n = 3$)

$$p_A(\lambda) := \det(\lambda \mathrm{Id} - A) = \sum_{i=0}^{n} (-1)^i I_i(A)\lambda^{n-i}\,.$$

Die Koeffizienten $I_i(A)$ heißen Invarianten der Matrix A, und sie sind eindeutig durch die Definition bestimmt. Es ist $I_0(A) = 1$. *Erläuterung:* Ist e eine Eigenvektor von A mit zugehörigem Eigenwert λ_e, so gilt $p_A(\lambda_e) = 0$.

Proof der Erläuterung. Ist $Ae = \lambda_e e$, so hat $\lambda_e \mathrm{Id} - A$ einen Vektor $e \neq 0$ im Kern. Daher ist $p_A(\lambda_e) = \det(\lambda_e \mathrm{Id} - A) = 0$. □

5.9 Cayley-Hamilton Theorem. Das charakteristische Polynom zu A kann auch für jede $n \times n$-Matrix M definiert werden als

$$P_A(M) := \sum_{i=0}^{n} (-1)^i I_i(A) M^{n-i}\,.$$

Mit dieser Definition gilt $P_A(A) = 0$. *Hinweis:* Es ist $P_A(\lambda \mathrm{Id}) = p_A(\lambda)\mathrm{Id}$.

Proof für symmetrische Matrizen. Ist die Matrix A symmetrisch, so gibt es eine orthonormale Basis $\{e_1, \ldots, e_n\}$ von Eigenvektoren mit Eigenwerten λ_k für $k = 1, \ldots, n$, das heißt $p_A(\lambda_k) = \det(\lambda_k \mathrm{Id} - A) = 0$. Somit ist also

$$\prod_{k=1}^{n} (\lambda - \lambda_k) = p_A(\lambda) = \sum_{i=0}^{n} (-1)^i I_i(A)\lambda^{n-i}\,.$$

Aus dieser Identität folgt durch Koeffizientenvergleich, was $I_i(A)$ ist, siehe den Beweis von . Nun zum Beweis dieses Satzes. Wegen

$$A = \sum_k \lambda_k e_k {e_k}^{\mathrm{T}} \,, \quad \mathrm{Id} = \sum_k e_k {e_k}^{\mathrm{T}} \,,$$

gilt

$$\begin{aligned} 0 &= \sum_k p_A(\lambda_k) e_k {e_k}^{\mathrm{T}} = \sum_{i=0}^{n} (-1)^i I_i(A) \sum_k \lambda_k^{n-i} e_k {e_k}^{\mathrm{T}} \\ &= \sum_{i=0}^{n} (-1)^i I_i(A) \Big(\sum_k \lambda_k e_k {e_k}^{\mathrm{T}} \Big)^{n-i} = P_A(A) \,, \end{aligned}$$

da $\{e_1, \ldots, e_n\}$ ein Orthonormalsystem ist. □

Proof für allgemeine Matrizen. Sei A eine $n \times n$-Matrix. Die Adjunkte von A (siehe [Wikipedia: Adjunkte]) ist definiert durch $\mathrm{adj}\,(A) = (\widetilde{a}_{j,i})_{ij}$, also ist $(\mathrm{adj}\,(A))^{\mathrm{T}} = (\widetilde{a}_{i,j})_{ij}$, wobei für $i, j = 1, \ldots, n$

$$\widetilde{a}_{i,j} = \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1.n} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{n.n} \end{bmatrix} .$$

Dann ist

$$\begin{aligned} \sum_i \widetilde{a}_{i,j} a_{i,k} &= \sum_i \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1.n} \\ 0 & \cdots & 0 & a_{i,k} & 0 & \cdots & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{n.n} \end{bmatrix} \\ &= \sum_i \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1.n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,k} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1.n} \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{n.n} \end{bmatrix} \\ &= \det \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,k} & a_{1,j+1} & \cdots & a_{1.n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,k} & a_{n,j+1} & \cdots & a_{n.n} \end{bmatrix} = \det\,(A)\,\delta_{j,k} \,. \end{aligned}$$

Entsprechend erhält man

$$\sum_j a_{k,j}\widetilde{a}_{i,j} = \det(A)\,\delta_{k,i}\,.$$

Also ist

$$\det(A)\,\mathrm{Id} = (\mathrm{adj}\,(A))\,A = A\,\mathrm{adj}\,(A)\,. \tag{IV5.17}$$

Now we apply this to the $n\times n$-Matrix $\lambda\mathrm{Id}-A$, that is

$$\det(\lambda\mathrm{Id}-A)\,\mathrm{Id} = (\mathrm{adj}\,(\lambda\mathrm{Id}-A))\,(\lambda\mathrm{Id}-A)\,,$$

wobei λ eine (reelle oder komplexe) Zahl ist. So wie

$$p_A(\lambda) = \det(\lambda\mathrm{Id}-A) = \sum_{k=0}^{n} c_k\lambda^k\,,\quad c_n = 1\,,$$

ein Polynom n-ten Grades ist, so sind die Komponenten $(\mathrm{adj}\,(\lambda\mathrm{Id}-A))_{ij}$ Polynome $(n-1)$-ten Grades in λ (das folgt aus der Definition der Determinante), also gibt es Matrizen B_k, $k=0,\dots,n-1$, mit

$$\mathrm{adj}\,(\lambda\mathrm{Id}-A) = \sum_{k=0}^{n-1}\lambda^k B_k\,.$$

Nun ist (siehe [Wikipedia: Cayley-Hamilton Theorem])

$$\sum_{k=0}^{n}\lambda^k c_k\mathrm{Id} = p_A(\lambda)\mathrm{Id} = \det(\lambda\mathrm{Id}-A)\,\mathrm{Id} = \mathrm{adj}\,(\lambda\mathrm{Id}-A)\,(\lambda\mathrm{Id}-A)$$
$$= \sum_{k=0}^{n-1}\lambda^k B_k(\lambda\mathrm{Id}-A) = \sum_{k=1}^{n}\lambda^k B_{k-1} - \sum_{k=0}^{n-1}\lambda^k B_k A\,.$$

Comparison of the coefficients gives

$$\begin{aligned} B_{n-1} &= c_n\mathrm{Id} = \mathrm{Id}\,,\\ B_{k-1}-B_kA &= c_k\mathrm{Id}\quad\text{for } 1\le k\le n-1,\\ -B_0A &= c_0\mathrm{Id}\,. \end{aligned}$$

These Identities imply

$$\begin{aligned} &\sum_{k=0}^{n} c_kA^k = \sum_{k=0}^{n}(c_k\mathrm{Id})A^k\\ &= B_{n-1}A^n + \sum_{k\,:\,1\le k\le n-1}(B_{k-1}-B_kA)A^k \quad - B_0A\\ &= \sum_{k=1}^{n} B_{k-1}A^k - \sum_{k=0}^{n-1} B_kA^{k+1} = 0\,, \end{aligned}$$

hence

$$\sum_{k=0}^{n} c_kA^k = 0\,.$$

The c_k are the coefficients of the charakteristic polynomial p_A, das heißt $c_{n-i} = (-1)^i I_i(A)$. □

Wie benötigen dies für die folgende Darstellung von S.

5.10 Isotropes S. Nimm an, dass

$$\underline{f} = \underline{\widetilde{f}}(\underline{x}, \underline{\varrho}, \underline{\theta}, C) = \underline{\breve{f}}(\underline{x}, \underline{\varrho}, \underline{\theta}, I_1(C), \ldots, I_n(C)) \,,$$

also hängt $\underline{\breve{f}}$ nur von den Invarianten $I_i(C)$ der Matrix C ab. Dann gilt

$$S = F^{-1} P = F^{-1} \underline{\widehat{f}}_{,F} = 2\underline{\widetilde{f}}_{,C} = 2 \sum_i \underline{\breve{f}}_{,I_i} \frac{\partial I_i}{\partial C} \,.$$

Im physikalischen Fall $n = 3$ gilt also

$$S = 2\underline{\breve{f}}_{,I_1} \frac{\partial I_1}{\partial C} + 2\underline{\breve{f}}_{,I_2} \frac{\partial I_2}{\partial C} + 2\underline{\breve{f}}_{,I_3} \frac{\partial I_3}{\partial C} \,,$$

wobei

$$\begin{aligned} I_1(C) &:= \mathrm{trace}\, C \,, & I_2(C) &:= \mathrm{trace}\ \mathrm{adj}\, C \,, & I_3(C) &:= \det C \,, \\ \frac{\partial I_1}{\partial C} &= \mathrm{Id} \,, & \frac{\partial I_2}{\partial C} &= I_1 \mathrm{Id} - C \,, & \frac{\partial I_3}{\partial C} &= I_3 C^{-1} \,. \end{aligned}$$

Proof. Siehe [55, CH.3: 5.8-5.15]. Es ist $\underline{\widetilde{f}}_{,C} = \sum_i \underline{\breve{f}}_{,I_i} \frac{\partial I_i}{\partial C}$. Also haben wir die Matrizen $\frac{\partial I_i}{\partial C}$ zu berechnen. Sei A eine $n \times n$-Matrix. Dann ist (das folgt aus der Definition der Determinante)

$$\frac{\partial \det A}{\partial A^{\mathrm{T}}} = \mathrm{adj}\, A$$

also für jedes A nach (IV5.17)

$$\frac{\partial \det A}{\partial A^{\mathrm{T}}} A = (\mathrm{adj}\, A)\, A = (\det A)\, \mathrm{Id} \,.$$

Now we apply this to the $n \times n$ matrix $A - \lambda \mathrm{Id}$, that ish

$$\frac{\partial \det (A - \lambda \mathrm{Id})}{\partial A^{\mathrm{T}}} (A - \lambda \mathrm{Id}) = \big(\det (A - \lambda \mathrm{Id})\big) \mathrm{Id}$$

hence since $p_A(\lambda) = (-1)^n \det (A - \lambda \mathrm{Id})$

$$\frac{\partial\, p_A(\lambda)}{\partial A^{\mathrm{T}}} (A - \lambda \mathrm{Id}) = p_A(\lambda)\, \mathrm{Id} \,.$$

With the coefficients $c_k = \widehat{c}_k(A)$, which we have introduced in the proof of 5.9, this is

$$\begin{aligned} \sum_{k=0}^{n} c_k \lambda^k\, \mathrm{Id} &= p_A(\lambda)\, \mathrm{Id} = \frac{\partial}{\partial A^{\mathrm{T}}} \Big(\sum_{k=0}^{n} c_k \lambda^k \Big) (A - \lambda \mathrm{Id}) \\ &= \Big(\sum_{k=0}^{n-1} \frac{\partial c_k}{\partial A^{\mathrm{T}}} \lambda^k \Big) (A - \lambda \mathrm{Id}) \qquad (\text{da } c_n = 1) \\ &= \sum_{k=0}^{n-1} \lambda^k \frac{\partial c_k}{\partial A^{\mathrm{T}}} A - \sum_{k=1}^{n} \lambda^k \frac{\partial c_{k-1}}{\partial A^{\mathrm{T}}} \,. \end{aligned}$$

Identifying the coefficients gives

$$\begin{aligned}
-\frac{\partial c_{n-1}}{\partial A^{\mathrm{T}}} &= c_n \mathrm{Id} = \mathrm{Id}\,, \\
\frac{\partial c_k}{\partial A^{\mathrm{T}}} A - \frac{\partial c_{k-1}}{\partial A^{\mathrm{T}}} &= c_k \mathrm{Id} \quad \text{for } 1 \le k \le n-1, \\
\frac{\partial c_0}{\partial A^{\mathrm{T}}} A &= c_0 \mathrm{Id}\,.
\end{aligned} \tag{IV5.18}$$

These Identities implies the formula for S. We do this in the following proof for $n = 3$. □

Proof im physikalischen Fall. Sei A wieder eine beliebige $n \times n$-Matrix. Mit $M := \lambda \mathrm{Id} - A$, also $m_{ii} = \lambda - a_{ii}$ und $m_{ij} = -a_{ij}$ füt $j \neq i$, ist

$$\begin{aligned}
&\det M = m_{11}m_{22}m_{33} + m_{12}m_{23}m_{31} + m_{13}m_{21}m_{32} \\
&\qquad\qquad -m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32} \\
&= \lambda^3 - \lambda^2(a_{11} + a_{22} + a_{33}) \\
&+\lambda(a_{22}a_{33} + a_{11}a_{33} + a_{11}a_{22} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32}) \\
&-\det(A) \\
&= \lambda^3 - \lambda^2 \operatorname{trace}(A) + \lambda \operatorname{trace} \operatorname{adj}(A) - \det(A)\,,
\end{aligned}$$

also

$$I_1(A) = \operatorname{trace}(A)\,, \quad I_2(A) = \operatorname{trace}(\operatorname{adj}(A))\,, \quad I_3(A) = \det(A)\,.$$

Für die Ableitungen der Invarianten gilt für die symmetrische Matrix C wegen $I_i(C) = (-1)^i c_{n-i}(C)$ und $n = 3$ nach (IV5.18)

$$\begin{aligned}
\frac{\partial I_1}{\partial C} = -\frac{\partial c_2}{\partial C} &= \mathrm{Id}\,, \\
-\frac{\partial I_1}{\partial C} C - \frac{\partial I_2}{\partial C} = \frac{\partial c_2}{\partial C} C - \frac{\partial c_1}{\partial C} &= c_2 \mathrm{Id} = -I_1 \mathrm{Id}\,, \\
\frac{\partial I_2}{\partial C} C + \frac{\partial I_3}{\partial C} = \frac{\partial c_1}{\partial C} C - \frac{\partial c_0}{\partial C} &= c_1 \mathrm{Id} = I_2 \mathrm{Id}\,, \\
-\frac{\partial I_3}{\partial C} C = \frac{\partial c_0}{\partial C} C &= c_0 \mathrm{Id} = -I_3 \mathrm{Id}\,,
\end{aligned}$$

und daraus

$$\begin{aligned}
\frac{\partial I_3}{\partial C} &= I_3 C^{-1}\,, \\
\frac{\partial I_2}{\partial C} &= \Big(-\frac{\partial I_3}{\partial C} + I_2 \mathrm{Id}\Big) C^{-1} = -I_3 C^{-2} + I_2 C^{-1}\,, \\
\frac{\partial I_1}{\partial C} &= \Big(-\frac{\partial I_2}{\partial C} + I_1 \mathrm{Id}\Big) C^{-1} = I_3 C^{-3} - I_2 C^{-2} + I_1 C^{-1}\,, \\
\frac{\partial I_1}{\partial C} &= \mathrm{Id}\,.
\end{aligned}$$

Das Cayley-Hamilton Theorem ist darin enthalten, also für die Matrix C

$$0 = \mathrm{Id} - I_3(C)\,C^{-3} + I_2(C)\,C^{-2} - I_1(C)\,C^{-1} \,.$$

Im Übrigen handelt es sich um 3 Gleichungen, wenn man in der zweiten Gleichung $-I_3\,C^{-2} + I_2\,C^{-1} = -C + I_1\mathrm{Id}$ auf Grund des Cayley-Hamilton Prinzips ausnutzt, nämlich

$$\frac{\partial I_1}{\partial C} = \mathrm{Id}\,, \quad \frac{\partial I_2}{\partial C} = I_1\,\mathrm{Id} - C\,, \quad \frac{\partial I_3}{\partial C} = I_3\; C^{-1} \,.$$

□

Also erhalten wir in $n = 3$ für den zweiten Piola-Kirchhoff Spannungstensor

$$\boxed{S = 2\underline{f}_{,I_1}\mathrm{Id} + 2\underline{f}_{,I_2}\Big(I_1(C)\,\mathrm{Id} - C\Big) + 2\underline{f}_{,I_3} I_3(C)\; C^{-1}} \tag{IV5.19}$$

und für den negativen Drucktensor, der $-J\Pi{\circ}\tau = F S\, F^{\mathrm{T}}$ erfüllt, gilt

$$\boxed{-\Pi{\circ}\tau = \tfrac{2}{J}\underline{f}_{,I_1} B + \tfrac{2}{J}\underline{f}_{,I_2}\Big(I_1(B)\,B - B^2\Big) + \tfrac{2}{J}\underline{f}_{,I_3} I_3(B)\,\mathrm{Id}\,,} \tag{IV5.20}$$

was äquivalent zur ersten Gleichung ist.

5.11 Lemma. Die Gleichungen (IV5.19) und (IV5.20) sind äquivalent.

Proof. Es folgt wegen $B = F\,F^{\mathrm{T}}$, da $FC^k\,F^{\mathrm{T}} = B^{k+1}$ für $k \in \mathbb{Z}$,

$$\begin{aligned}
&F S\, F^{\mathrm{T}} = 2F\Big(\underline{f}_{,I_1}\mathrm{Id} + \underline{f}_{,I_2}(I_1\,\mathrm{Id} - C) + \underline{f}_{,I_3} I_3\; C^{-1}\Big)\,F^{\mathrm{T}} \\
&= 2\underline{f}_{,I_1} F\,F^{\mathrm{T}} + 2\underline{f}_{,I_2}(I_1\,F\,F^{\mathrm{T}} - FC\,F^{\mathrm{T}}) + 2\underline{f}_{,I_3} I_3\,F\,C^{-1}\,F^{\mathrm{T}} \\
&= 2\underline{f}_{,I_1} B + 2\underline{f}_{,I_2}(I_1\,B - B^2) + 2\underline{f}_{,I_3} I_3\,\mathrm{Id}\,,
\end{aligned}$$

also die Dastellung von $-\Pi{\circ}\tau$ bis auf die Invarianten I_i, $i = 1, 2, 3$. Since for an arbitrary matrix A

$$\begin{aligned}
I_1(A) &= \mathrm{trace}\,A\,, \\
I_2(A) &= \frac{1}{2}\big((\mathrm{trace}\,A)^2 - \mathrm{trace}\,(A^2)\big)\,, \\
I_3(A) &= \frac{1}{6}\big((\mathrm{trace}\,A)^3 - 3\,\mathrm{trace}\,(A)\cdot\mathrm{trace}\,(A^2) + 2\,\mathrm{trace}\,(A^3)\big)\,,
\end{aligned}$$

see [Wikipedia: Invariants of tensors] and ex??? we have to show

$$\mathrm{trace}\,(B^m) = \mathrm{trace}\,(C^m) \text{ for } m = 1, 2, 3.$$

Since for arbitrary matrices A and M (see [Wikipedia: Trace])

$$\mathrm{trace}\,(AM) = \sum_{i,k} A_{ik}M_{ki} = \sum_{i,k} M_{ik}A_{ki} = \mathrm{trace}\,(MA)$$

it follows for $m \geq 1$

$$\operatorname{trace}(B^m) = \operatorname{trace}(FC^{m-1}F^{\mathrm{T}}) = \operatorname{trace}(C^{m-1}F^{\mathrm{T}}F) = \operatorname{trace}(C^m)\,.$$

Und auch $J = \det F = \sqrt{\det B} = \sqrt{\det C}$. □

Diese Formeln stimmen überein mit denen in der Literatur.

References: [69, IV.2 Static Universal Solutions] and [55, 3.5 Material Symmetries and Isotropic Elasticity] and [47, 5.4 Material Equations for Isotropic Bodies].

6 Tissue growth

We refer to the system in 5.5 where we have derived the conseqences of the entropy principle for a single species. We will now proceed with several species, which in connection with the growth of a body will contain the reacting substances which are present, see Fig. 18 for example. Herewith we model a heat-driven growth. See the references for more detail. Therefore we have to renew the entropy principle. The special feature here is that we combine in the residual inequality, that is in the inequality $\underline{\sigma} \geq 0$, the $\mathbf{r}$-term with parts of the $\mathrm{D}V$-term to obtain a matrix which has to be positive semidefinit. This method is a quite general procedure and of general interest because it makes clear how important the entropy principle in physics is.

References: For use in the biology of bone growth see Maršík & Klika & Chlup [56]. A somewhat different approach one finds in Ambrosi & Guillou [24].

That is, we have in physical coordinates, i.e. Euler coordinates, a system of mass conservations with index α, and for the total mass a momentum conservation, and in addition, the conservation of energy. With the notation in section III.3 we deal with a mixture of class I. The system in Euler coordinates is

$$\begin{aligned}
&\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \dots, m \;, \\
&\varrho = \sum_\alpha \varrho_\alpha \text{ the total mass,} \\
&\mathbf{r} = \sum_\alpha \mathbf{r}_\alpha \text{ the total reaction rate,} \\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{r} v + \mathbf{f}\,, \\
&\partial_t e + \operatorname{div}(e v + \Pi^{\mathrm{T}}\, v + q) = \frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + g \\
&e = \varepsilon + \frac{\varrho}{2}|v|^2 \text{ the total energy.}
\end{aligned} \tag{IV6.1}$$

Here we have used

$$\widetilde{\mathbf{f}} = \mathbf{r} v + \mathbf{f} \quad \text{ and } \quad \widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet\mathbf{f} + g\,.$$

The equation for the total mass is the sum of the individual mass equations

$$\partial_t \varrho + \operatorname{div}(\varrho v) = \mathbf{r}\,,$$

hence no independent equation. We show

6.1 Lemma. System (IV6.1) is equivalent to (IV6.2).

Growth of a body:

$$\partial_t \underline{\varrho}_\alpha = \underline{\mathbf{r}}_\alpha \text{ for } \alpha = 1,\dots,m \ ,$$
$$\partial_t(\underline{\varrho} V) = \operatorname{div} P + \underline{\mathbf{f}}\,,$$
$$\partial_t \underline{e} + \operatorname{div}(-P^{\mathrm{T}} V + \underline{q}) = \underline{g}$$

$\underline{\varrho}_\alpha = J \cdot \varrho_\alpha \circ \tau\,,\quad \underline{\varrho} = \sum_\alpha \underline{\varrho}_\alpha$ the total reference mass,

$\underline{\mathbf{r}}_\alpha$ the mass change of constituent α,

$\underline{e} = \underline{\varepsilon} + \frac{\underline{\varrho}}{2}|V|^2$ the total energy,

V the velocity,

$\underline{\varepsilon}$ will eventually be "replaced" by $\underline{\theta}$,

$\underline{\mathbf{r}}_\alpha$, P, $\underline{q}$ are material quantities to be determined,

$\underline{\mathbf{f}}$ an external force density, $\underline{g}$ in most cases $= 0$.

(IV6.2)

Proof. For each α we derive the mass equation for this constituent as in I.6.2. The equation for the entire mass and entire momentum then follows as in I.6.2. The energy equation follows as in 5.1. □

We now see that the entropy principle is satisfied for a free energy depending on the density of all involved species.

6.2 Theorem. The entropy principle for system (IV6.1) (and thus for (IV6.2)) is satisfied with

$$\underline{f} = \widehat{\underline{f}}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\theta}, F)\,,\quad \underline{f}_{,F} F^{\mathrm{T}} \text{ symmetric,}$$
$$\underline{\varepsilon} = \underline{f} - \underline{\theta}\underline{f}_{,\underline{\theta}}\,,\quad \underline{f}_{,\underline{\theta}\underline{\theta}} < 0\,,$$
$$V = \partial_t\varphi\,,\quad F = \mathrm{D}\varphi\,,$$

and if the residual entropy inequality

$$\underline{\theta}\,\underline{\sigma} = -\sum_\alpha \underline{f}_{,\underline{\varrho}_\alpha} \underline{\mathbf{r}}_\alpha + (P - \underline{f}_{,F}) \mathbin{:} \mathrm{D}V + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big) \bullet \underline{q} + \underline{g} \geq 0$$

is satisfied.

Proof. The difference to the proof of 5.4 is that the entropy depends on the densities ϱ_1 to ϱ_m instead of the total density. That is, here we have to replace $\underline{\eta}_{,\underline{\varrho}}\underline{\mathbf{r}}$ by

$$\sum_\alpha \underline{\eta}_{,\underline{\varrho}_\alpha} \underline{\mathbf{r}}_\alpha\,.$$

We define the entropy in reference coordinates by

$$\underline{\eta}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\varepsilon}, F) := -\underline{f}_{,\theta}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\theta}, F)$$
$$\text{for } \underline{\varepsilon} = \underline{f}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\theta}, F) - \underline{\theta}\underline{f}_{,\theta}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\theta}, F)\,.$$

It follows

$$\partial_t \underline{\eta} = \sum_\alpha \underline{\eta}_{,\underline{\varrho}_\alpha} \underbrace{\partial_t \underline{\varrho}_\alpha}_{= \underline{\mathbf{r}}_\alpha} + \underline{\eta}_{,\underline{\varepsilon}} \partial_t \underline{\varepsilon} + \underline{\eta}_{,F} \bullet \partial_t F.$$

We obtain with the entropy flux $\underline{\psi} = \underline{\eta}_{,\underline{\varepsilon}} \underline{q}$ for the entropy production in analogy to 5.4

$$\underline{\theta}\,\underline{\sigma} = \underline{\theta} \sum_\alpha \underline{\eta}_{,\underline{\varrho}_\alpha} \underline{\mathbf{r}}_\alpha + (P + \underline{\theta}\,\underline{\eta}_{,F}) \mathbin{\bullet\!\!\bullet} \mathrm{D}V + \underline{\theta} \nabla\Big(\frac{1}{\underline{\theta}}\Big) \bullet \underline{q} + \underline{g} \geq 0\,.$$

Due to $\underline{f}_{,\underline{\varrho}_\alpha} = -\underline{\theta}\underline{\eta}_{,\underline{\varrho}_\alpha}$ and $\underline{f}_{,F} = -\underline{\theta}\underline{\eta}_{,F}$ (see exercise III.7.3) the assertion follows. □

Now we will choose $P = P_{el} + P_{dis}$ satisfying the entropy inequality.

6.3 Theorem. Let $\underline{g} = 0$ and assume in 6.2 that

$$V = \partial_t \varphi\,, \quad F = \mathrm{D}\varphi$$
$$P = P_{el} + P_{dis}\,, \quad P_{el} = \underline{f}_{,F}\,, \quad P_{dis} = p\,F^{-\mathrm{T}}\,,$$
$$\mathcal{V} := (\,\mathrm{div} v) \circ \tau\,, \quad \mathcal{A}_\alpha := -\underline{f}_{,\underline{\varrho}_\alpha}\,.$$

Moreover we assume that for p and $\underline{\mathbf{r}}_\alpha$, $\alpha = 1, \dots, m$,

$$p = l_{00}\mathcal{V} + \sum_\beta l_{0\beta}\mathcal{A}_\beta\,,$$
$$\underline{\mathbf{r}}_\alpha = l_{\alpha 0}\mathcal{V} + \sum_\beta l_{\alpha\beta}\mathcal{A}_\beta\,.$$

Then the entropy principle for the system (IV6.2) is satisfied if

$$\underline{\sigma} = \frac{1}{\underline{\theta}} \mathcal{A} \bullet L \mathcal{A} + \nabla\Big(\frac{1}{\underline{\theta}}\Big) \bullet \underline{q} \geq 0\,,$$

where

$$\mathcal{A} := \begin{bmatrix} \mathcal{V} \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_m \end{bmatrix}, \quad L := \begin{bmatrix} l_{00} & l_{01} & \cdots & l_{0m} \\ l_{10} & l_{11} & \cdots & l_{1m} \\ \vdots & & \vdots & \\ l_{m0} & l_{m1} & \cdots & l_{mm} \end{bmatrix}.$$

Comment: An anti-symmetric part of the matrix L does not produce additional entropy. This makes such a growth process more likely. Thus $(l_{\alpha\beta})_{\alpha\geq 1, \beta \geq 1}$ can be symmetric, but the case $l_{0\alpha} = -l_{\alpha 0}$ for $\alpha \geq 1$ seems to be the most realistic one (compare the reference in 6.4).

The system (IV6.2) becomes

$$
\begin{array}{rcl}
\partial_t \underline{\varrho}_\alpha &=& \underline{\mathbf{r}}_\alpha \quad \text{for } \alpha = 1,\dots,m, \\
\partial_t(\underline{\varrho} V) &=& \operatorname{div} P + \underline{\mathbf{f}} \\
\partial_t(\underline{f} - \theta \underline{f}_{,\theta}) &=& \operatorname{div}(c\nabla\theta) + \mathrm{D}V{:}P
\end{array}
$$

$$
\begin{array}{l}
V = \partial_t \varphi \text{ the velocity,} \\
F = \mathrm{D}\varphi \text{ the deformation gradient,} \\
\underline{f} = \widehat{\underline{f}}(\underline{x}, \underline{\varrho}_1, \dots, \underline{\varrho}_m, \underline{\theta}, F) \text{ the free energy,} \\
\underline{\varrho} = \sum_\alpha \underline{\varrho}_\alpha \text{ the total mass,} \\
P = p\, F^{-\mathrm{T}} + \underline{f}_{,F}\,, \\
p = l_{00}(\operatorname{div} v)\circ\tau - \sum_\beta l_{0\beta} \underline{f}_{,\underline{\varrho}\beta}\,, \\
\underline{\mathbf{r}}_\alpha = l_{\alpha 0}(\operatorname{div} v)\circ\tau - \sum_\beta l_{\alpha\beta} \underline{f}_{,\underline{\varrho}\beta}\,, \\
(l_{kl})_{k,l=0,\dots,m} \geq 0\,, \quad c \geq 0\,.
\end{array}
\tag{IV6.3}
$$

This system one has to solve.

Proof. Since $\mathcal{A}_\alpha := -\underline{f}_{,\underline{\varrho}_\alpha}$ the entropy inequality 6.2 equals

$$\underline{\theta}\,\underline{\sigma} = \sum_\alpha \underline{\mathbf{r}}_\alpha \mathcal{A}_\alpha + (P - \underline{f}_{,F}){:}\mathrm{D}V + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big){\bullet}\underline{q} \geq 0\,.$$

We decompose P into an elastic part P_{el} and a dissipative part P_{dis}, i.e.

$$P = P_{el} + P_{dis}\,, \quad P_{el} = \underline{f}_{,F}\,, \quad P_{dis} = p\, F^{-\mathrm{T}}\,.$$

This implies, since $DV = \big((Dv)\circ\tau\big)F$,

$$(P - \underline{f}_{,F}){:}\mathrm{D}V = P_{dis}{:}\mathrm{D}V = p \operatorname{trace}(\mathrm{D}V\, F^{-1}) = p\,(\operatorname{div} v)\circ\tau\,.$$

Thus, if the (unknown) functions p and $\underline{\mathbf{r}}_\alpha$ fulfill the identities

$$
\begin{aligned}
p &= l_{00}(\operatorname{div} v)\circ\tau + \sum_\beta l_{0\beta}\mathcal{A}_\beta\,, \\
\underline{\mathbf{r}}_\alpha &= l_{\alpha 0}(\operatorname{div} v)\circ\tau + \sum_\beta l_{\alpha\beta}\mathcal{A}_\beta\,,
\end{aligned}
$$

it follows with $\mathcal{V} := (\operatorname{div} v)\circ\tau$

$$p\,\mathcal{V} + \sum_\alpha \underline{\mathbf{r}}_\alpha \mathcal{A}_\alpha = \begin{bmatrix} \mathcal{V} \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_m \end{bmatrix} \bullet \begin{bmatrix} l_{00} & l_{01} & \cdots & l_{0m} \\ l_{10} & l_{11} & \cdots & l_{1m} \\ \vdots & & \vdots & \\ l_{m0} & l_{m1} & \cdots & l_{mm} \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_m \end{bmatrix}.$$

Therefore, the formula for the entropy production is

$$\underline{\theta}\,\underline{\sigma} = \begin{bmatrix} \mathcal{V} \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_m \end{bmatrix} \bullet \begin{bmatrix} l_{00} & l_{01} & \cdots & l_{0m} \\ l_{10} & l_{11} & \cdots & l_{1m} \\ \vdots & & \vdots & \\ l_{m0} & l_{m1} & \cdots & l_{mm} \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_m \end{bmatrix} + \underline{\theta}\nabla\Big(\frac{1}{\underline{\theta}}\Big)\bullet \underline{q} \geq 0\,,$$

so for example, if L is positive semi-definite and $\underline{q} = -c\nabla\underline{\theta}$ with $c \geq 0$. □

The essentials here is, that two terms in the entropy production have been combined, that is, only the sum of these terms is nonnegative.

$$\frac{\mathrm{d}N_{MCELL}}{\mathrm{d}\tau} = -\delta_1(\beta_1 + N_{MCELL})N_{MCELL} + \mathcal{J}_3 + \mathcal{J}_{New\_B} - \mathcal{D}_1$$

$$\begin{aligned}\frac{\mathrm{d}N_{Old\_B}}{\mathrm{d}\tau} = &-(\beta_3 - N_{MCELL} + N_{Old\_B} + N_{Activ\_OB} + N_{Osteoid} + N_{New\_B})N_{Old\_B} - \delta_3(\beta_7 - N_{Old\_B} \\ &- 2(N_{Activ\_OB} + N_{Osteoid} + N_{14}))N_{Old\_B} + 2\mathcal{J}_{New\_B} - \mathcal{D}_2 - \mathcal{D}_3\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}N_{Activ\_OB}}{\mathrm{d}\tau} = &\,\delta_3(\beta_7 - N_{Old\_B} - 2(N_{Activ\_OB} + N_{Osteoid} + N_{New\_B}))N_{Old\_B} \\ &-\delta_4(\beta_{10} - N_{Osteoid} - N_{New\_B})N_{Activ\_OB} + \mathcal{D}_3 - \mathcal{D}_4\end{aligned}$$

$$\frac{\mathrm{d}N_{Osteoid}}{\mathrm{d}\tau} = \delta_4(\beta_{10} - N_{Osteoid} - N_{New\_B})N_{Activ\_OB} - \delta_5(\beta_{13} - N_{New\_B})N_{Osteoid} + \mathcal{D}_4 - \mathcal{D}_3$$

$$\frac{\mathrm{d}N_{New\_B}}{\mathrm{d}\tau} = \delta_5(\beta_{13} - N_{New\_B})N_{Osteoid} - \mathcal{J}_{New\_B} + \mathcal{D}_5.$$

Fig. 18: Aus Marsik & Klika & Chlup [56]

6.4 Referenz. In Maršík & Klika & Chlup [56] one finds the realization of the mass equations in Fig. 18. Here N_α is the molar density of phase α, see the definition in III.3.1(2). The statement 6.3 is presented with

$$p = l_{00}(\,\mathrm{div}v)\circ\tau + \sum_\beta l_{0\beta}\mathcal{A}_\beta\,,$$

$$\underline{\mathbf{r}}_\alpha = l_{\alpha 0}(\,\mathrm{div}v)\circ\tau + l_{\alpha\alpha}\mathcal{A}_\alpha\,,$$

i.e. with $l_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Reference is also made to Onsager's principle, but this is not necessary, only the positivity of the entropy production counts.

7 Sound waves

Der Schall ist für uns allgegenwärtig, wir hören ihn mit unseren Ohren, er wird von allen uns umgebenden Maschinen erzeugt, insbesondere durch Reibung und Stöße, sowie von Explosionen verursacht. Der Schall wird aber auch durch die Natur erzeugt wie das Vogelzwitschern und das Waldesrauschen, sowie der Donner. Er wird übertragen durch das uns umgebende Gas, d.h. die Luft. Er ist eine Störung der Dichte dieses Gases.

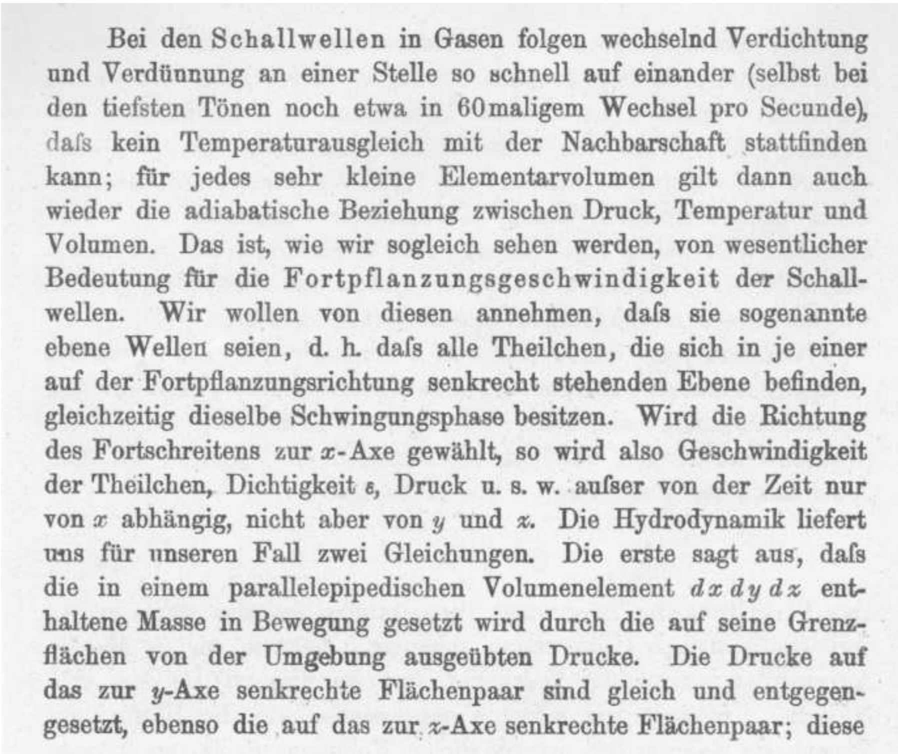

Bei den Schallwellen in Gasen folgen wechselnd Verdichtung und Verdünnung an einer Stelle so schnell auf einander (selbst bei den tiefsten Tönen noch etwa in 60maligem Wechsel pro Secunde), dafs kein Temperaturausgleich mit der Nachbarschaft stattfinden kann; für jedes sehr kleine Elementarvolumen gilt dann auch wieder die adiabatische Beziehung zwischen Druck, Temperatur und Volumen. Das ist, wie wir sogleich sehen werden, von wesentlicher Bedeutung für die Fortpflanzungsgeschwindigkeit der Schallwellen. Wir wollen von diesen annehmen, dafs sie sogenannte ebene Wellen seien, d. h. dafs alle Theilchen, die sich in je einer auf der Fortpflanzungsrichtung senkrecht stehenden Ebene befinden, gleichzeitig dieselbe Schwingungsphase besitzen. Wird die Richtung des Fortschreitens zur x-Axe gewählt, so wird also Geschwindigkeit der Theilchen, Dichtigkeit ε, Druck u. s. w. aufser von der Zeit nur von x abhängig, nicht aber von y und z. Die Hydrodynamik liefert uns für unseren Fall zwei Gleichungen. Die erste sagt aus, dafs die in einem parallelepipedischen Volumenelement $dx\,dy\,dz$ enthaltene Masse in Bewegung gesetzt wird durch die auf seine Grenzflächen von der Umgebung ausgeübten Drucke. Die Drucke auf das zur y-Axe senkrechte Flächenpaar sind gleich und entgegengesetzt, ebenso die auf das zur z-Axe senkrechte Flächenpaar; diese

Fig. 19: From the book of Helmholtz [110, §48]

Bei dieser Störung wechseln sich Verdichtung (*en:* compression, condensation) und Verdünnung (*en:* rarefaction, dilation) des Gases wechselseitig ab. Dies kann sich dann wie eine Welle ausbreiten.

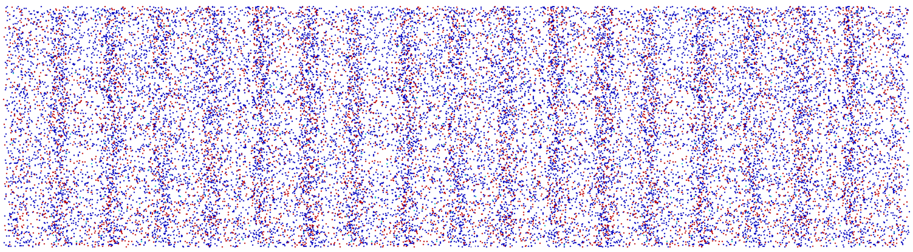


Fig. 20: Compression and rarefaction of a planar wave, from [126]

Wir modellieren den Schall durch die Massen-Impuls-Energie-Erhaltung eines Gases oder einer Flüssigkeit, wobei wir im Allgemeinen die Zähigkeit und Wärmeleitfähigkeit berücksichtigen. Das Differentialgleichungssystem ist also (IV2.2)

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,, \\ &\partial_t(\varepsilon + \tfrac{\varrho}{2}|v|^2) + \operatorname{div}((\varepsilon + \tfrac{\varrho}{2}|v|^2)v + \Pi^{\mathrm{T}}\, v + q) = \mathbf{f}\bullet v\,, \end{aligned} \tag{IV7.1}$$

und es gilt das Entropieprinzip 2.2

$$\partial_t \eta + \operatorname{div}\psi =: \sigma \geq 0\,,$$

wobei wohlbekannt ist (siehe u.a. 2.2 und (IV2.15)), dass

$$\begin{aligned} &\eta = \widehat{\eta}(\varrho, \varepsilon)\,, \quad f = \widehat{f}(\varrho, \theta)\,, \quad \theta = \frac{1}{\eta'_\varepsilon}\,, \quad p = \widetilde{p}(\varrho, \theta)\,, \\ &p = \varrho f'_\varrho - f\,, \quad \varepsilon = f - \theta f'_\theta\,, \quad \eta = -f'_\theta\,, \\ &\Pi = p\mathrm{Id} - S\,, \quad S \text{ symmetrischer Spannungstensor}, \\ &\psi = \eta v + \frac{1}{\theta} q\,, \quad q \text{ Wärmeleitfähigkeit, ,} \\ &\sigma := \frac{1}{\theta}\mathrm{D}v\bullet S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0\,, \quad \sigma \text{ Entropieproduktion.} \end{aligned} \tag{IV7.2}$$

Nun hatten wir in 2.3 (siehe (IV2.16)) gezeigt, dass (IV7.1) äquivalent ist zu

$$\begin{aligned} &\partial_{(1,v)} \varrho + \varrho \operatorname{div} v = 0\,, \\ &\varrho\, \partial_{(1,v)} v + \operatorname{div} \Pi = \mathbf{f}\,, \\ &\partial_{(1,v)} \varepsilon + \varepsilon \operatorname{div} v + \operatorname{div} q + \mathrm{D}v{:}\Pi = 0\,, \end{aligned} \tag{IV7.3}$$

wobei wir jetzt $\partial_{(1,v)} h = (\partial_t + v\bullet\nabla)h = \overset{\circ}{h}$ für jede Funktion h schreiben. Diese Gleichungen mit Energieerhaltung (IV7.3) sind wiederum äquivalent zu Gleichungen mit der Entropieidentität (dies werden wir in Lemma 7.1 zeigen), wobei s die spezifische Entropie ist:

$$\boxed{\begin{aligned} &\partial_{(1,v)} \varrho + \varrho \operatorname{div} v = 0\,, \\ &\varrho\, \partial_{(1,v)} v + \nabla p - \operatorname{div} S = \mathbf{f}\,, \\ &\varrho\, \partial_{(1,v)} s + \operatorname{div}\big(\frac{1}{\theta} q\big) = \sigma\,. \end{aligned}} \tag{IV7.4}$$

7.1 Lemma. Sei s die ***spezifische Entropie***, d.h.

$$\eta = \varrho s \quad \text{or} \quad s := \eta^{\mathrm{sp}} = \frac{\eta}{\varrho} = -\frac{1}{\varrho} f'_\theta = f^{\mathrm{sp}}{}'_\theta\,.$$

Dann sind die Gleichungssysteme (IV7.3) und (IV7.4) zueinander äquivalent.

Proof. Dies folgt daraus, dass für alle Funktionen die Entropieidentität (III4.5) eine Linearkombination der Masse-, Impuls- und Energieerhaltung ist, wobei der Koeffizient vor der Energiegleichung gleich $\Lambda_e = \eta'_\varepsilon = \frac{1}{\theta}$ (siehe (III4.4)) ungleich Null ist, also kann man ebenso die Energieerhaltung als eine Linearkombination der Masse-, Impuls- und Entropiegleichung schreiben. Dies ergibt dann dass (IV7.1) (was äquivalent zu (IV7.3) ist) und die Gleichung (IV7.4) äquivalent sind. Denn es gilt für den Term in der Entropieidentität

$$\partial_t \eta + \operatorname{div}(\eta v) = s\big(\partial_t \varrho + \operatorname{div}(\varrho v)\big) + \varrho(\partial_t s + v \bullet \nabla s)\,,$$

und $\partial_t \varrho + \operatorname{div}(\varrho v) = 0$ nach der Massenerhaltung. Also ist die Entropiegleichung die letzte Gleichung von (IV7.4). □

Referenzen: Landau & Lifschitz [10, Kapitel VIII Der Schall]. Ausbreitung ohne Dämpfung: Landau & Lifschitz [10, VIII§63 Schallwellen], DeGroot & Mazur [6, Chap XII §3–§5], Hutter [8, 7.2 Ausbreitung kleiner Störungen in einem Gas], Müller [11, 10.4 Schall], Acheson [1, 3.6 Sound waves]. Und als mathematische Literatur Feireisl & Novotny [40, 4.4 Acoustic waves].

Wir betrachten in diesem Abschnitt Lösungen (ϱ, v, θ) der Form

$$\begin{bmatrix} \varrho \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} \underline{\varrho} + \varrho'_\delta \\ \underline{v} + u'_\delta \\ \underline{\theta} + \theta'_\delta \end{bmatrix},$$

wobei $(\underline{\varrho}, \underline{v}, \underline{\theta})$ eine Lösung ist und $(\varrho'_\delta, u'_\delta, \theta'_\delta)$ eine kleine Störung davon. Es ist klar, dass als Bestandteil von Lösungen die Größen $v = \underline{v} + u'_\delta$ sowie $\underline{v}$ Geschwindigkeiten sind, und darum ist u'_δ ein objektiver Vektor. Wir nehmen nun an, dass

$$\begin{bmatrix} \varrho'_\delta \\ u'_\delta \\ \theta'_\delta \end{bmatrix} = \delta \begin{bmatrix} \varrho' \\ u' \\ \theta' \end{bmatrix} + \mathcal{O}(\delta)\,,$$

und berechnen die erste Variation

$$\lim_{\delta \to 0} \frac{1}{\delta}\Big(L(\varrho, v, \theta) - L(\underline{\varrho}, \underline{v}, \underline{\theta})\Big)\,,$$

wobei von (IV7.4) gilt

$$L(\varrho, v, \theta) := \begin{bmatrix} \partial_{(1,v)}\varrho + \varrho \operatorname{div} v \\ \varrho\, \partial_{(1,v)} v + \nabla p - \operatorname{div} S - \mathbf{f} \\ \varrho\, \partial_{(1,v)} s + \operatorname{div}\big(\frac{1}{\theta} q\big) - \sigma \end{bmatrix}. \tag{IV7.5}$$

Es ist also $L(\varrho, v, \theta) = 0$ und $L(\underline{\varrho}, \underline{v}, \underline{\theta}) = 0$.

Schall ohne Dämpfung

Wir nehmen zunächst an, dass der Schall, den wir betrachten, über große Entfernungen hörbar ist, und dann ist es gerechtfertigt in (IV7.5)

$$S = 0\,, \quad q = 0$$

anzunehmen, also ist die Situation adiabatisch (siehe den Text in Fig. 19) und die Entropieproduktion $\sigma = 0$. Es wird dann (IV7.4) zu

$$\begin{aligned} &\partial_{(1,v)}\varrho + \varrho\,\mathrm{div}\,v = 0\,,\\ &\varrho\,\partial_{(1,v)}v + \nabla p = \mathbf{f}\,,\\ &\partial_{(1,v)}s = 0\,, \end{aligned} \tag{IV7.6}$$

und natürlich entsprechend auch in (IV7.5). Man erhält dann für die Störung das Resultat (siehe das System [40, (4.25)]):

7.2 Theorem. Für eine Störung

$$(\varrho, v, \theta) = (\underline{\varrho}, \underline{v}, \underline{\theta}) + \delta(\varrho', u', \theta') + o(\delta)\,, \quad \mathbf{f} = \underline{\mathbf{f}} + \delta\mathbf{f}' + o(\delta)\,,$$

folgt aus (IV7.6)

$$\begin{aligned} &\partial_{(1,\underline{v})}\varrho' + \mathrm{div}(\underline{\varrho}\,u') = -\varrho'\,\mathrm{div}\underline{v}\,,\\ &\underline{\varrho}\big(\partial_{(1,\underline{v})}u' + u'\bullet\nabla\underline{v}\big) + \nabla p' = \mathbf{f}' - \varrho'\partial_{(1,\underline{v})}\underline{v}\,,\\ &\partial_{(1,\underline{v})}s' + u'\bullet\nabla\underline{s} = 0\,, \end{aligned} \tag{IV7.7}$$

wobei

$$p = \underline{p} + \delta p' + o(\delta)\,, \quad s = \underline{s} + \delta s' + o(\delta)\,.$$

Für p und s gelten nach (IV7.2) noch konstitutive Gleichungen, die in 7.3 und 7.4 zu Aussagen über p' und s' führen.

Proof. Es ist

$$\begin{aligned} \partial_{(1,v)}\varrho &= \partial_{(1,\underline{v}+\delta u')}(\underline{\varrho} + \delta\varrho') + o(\delta)\\ &= \partial_{(1,\underline{v})}\underline{\varrho} + \delta(\partial_{(1,\underline{v})}\varrho' + u'\bullet\nabla\underline{\varrho}) + o(\delta)\,, \end{aligned}$$

entsprechendes gilt für $\partial_{(1,v)}v$ und $\partial_{(1,v)}s$. Und es gilt

$$\begin{aligned} \varrho\,\mathrm{div}\,v &= (\underline{\varrho} + \delta\varrho')\,\mathrm{div}(\underline{v} + \delta u') + o(\delta)\\ &= \underline{\varrho}\,\mathrm{div}\,\underline{v} + \delta(\underline{\varrho}\,\mathrm{div}\,u' + \varrho'\,\mathrm{div}\,\underline{v}) + o(\delta)\,, \end{aligned}$$

sowie unter Benutzung der Formel für $\partial_{(1,v)}v$

$$\begin{aligned} \varrho\,\partial_{(1,v)}v &= (\underline{\varrho} + \delta\varrho')(\partial_{(1,\underline{v})}\underline{v} + \delta(\partial_{(1,\underline{v})}u' + u'\bullet\nabla\underline{v})) + o(\delta)\\ &= \underline{\varrho}\partial_{(1,\underline{v})}\underline{v} + +\delta(\underline{\varrho}(\partial_{(1,\underline{v})}u' + u'\bullet\nabla\underline{v}) + \varrho'\partial_{(1,\underline{v})}\underline{v}) + o(\delta)\,. \end{aligned}$$

Aus diesen Rechnungen folgt die Behauptung. □

The differential equation $\partial_{(1,v)}s = 0$ suggests that $s = \text{const}$ und damit auch $\underline{s} = \text{const}$. Using this we can determine the sound speed.

7.3 Annahme an Entropie. Zur Konstanz von s.

(1) Wir nehmen an, dass $s_0 \in \mathbb{R}$ mit $s = s_0$, also auch $\underline{s} = s_0$ und $s' = 0$. Damit ist die dritte Gleichung in (IV7.7) erfüllt. Außerdem ist damit der Wert der Entropie derselbe für alle betrachteten Lösungen.

(2) Es sei $s = \widehat{s}(\varrho, \varepsilon) = \widetilde{s}(\varrho, \theta)$. Zum Wert s_0 gibt es Funktionen $\varrho \mapsto \widetilde{\theta}(\varrho)$ und $\varrho \mapsto \widetilde{\varepsilon}(\varrho)$, so dass $\widehat{s}(\varrho, \varepsilon) = \widetilde{s}(\varrho, \theta) = \underline{s} = s_0$ genau dann, wenn $\theta = \widetilde{\theta}(\varrho)$ und $\varepsilon = \widetilde{\varepsilon}(\varrho)$.

Proof (1). Aus der Differentialgleichung $\partial_{(1,v)}s = 0$ folgt, dass s konstant ist auf Stromlinien. Wenn nun $s = s_0$ lokal an dem Ursprung dieser Stromlinien so ist $s = s_0$ überall. Nun gilt $s = \underline{s} + \delta s' + \mathcal{O}(\delta)$, so dass also $\underline{s} = s_0$ und $s' = 0$. □

Proof (2). Die spezifische Entropie $s = s_0$ kann als Funktion von (ϱ, ε) und als Funktion von (ϱ, θ) angesehen werden, wir schreiben $s = \widehat{s}(\varrho, \varepsilon)$ und $s = \widetilde{s}(\varrho, \theta)$. Wenn, wie hier $s = s_0$, ist das als Nebenbedingung für die Parameter zu sehen, weshalb es dann Funktionen $\widetilde{\varepsilon}$ und $\widetilde{\theta}$ gibt (was von der Gestalt von s abhängt), so dass $\theta = \widetilde{\theta}(\varrho)$ und $\varepsilon = \widetilde{\varepsilon}(\varrho)$. □

7.4 Schallgeschwindigkeit. Es gelte 7.3. Dann ist für eine Störung

$$p' = c_S^2 \varrho' \,, \quad c_S > 0 \text{ die } \textbf{\textit{Schallgeschwindigkeit}}.$$

(1) Sie ist gegeben durch

$$c_S(\underline{\varrho}, \underline{\theta})^2 = \widetilde{p}'_{\varrho}(\underline{\varrho}, \underline{\theta}) - \frac{\widetilde{s}'_{\varrho}(\underline{\varrho}, \underline{\theta})}{\widetilde{s}'_{\theta}(\underline{\varrho}, \underline{\theta})} \widetilde{p}'_{\theta}(\underline{\varrho}, \underline{\theta}) = \frac{1}{\widetilde{s}'_{\theta}(\underline{\varrho}, \underline{\theta})} \det \begin{bmatrix} \widetilde{p}'_{\varrho} & \widetilde{p}'_{\theta} \\ \widetilde{s}'_{\varrho} & \widetilde{s}'_{\theta} \end{bmatrix} (\underline{\varrho}, \underline{\theta})$$
$$= \frac{\mathrm{d}}{\mathrm{d}\varrho} \left(\widetilde{p}(\varrho, \widehat{\theta}(\varrho)) \right)\Big|_{\varrho = \underline{\varrho}} = -\frac{\underline{\theta}}{\underline{\varrho}} \begin{bmatrix} \underline{\varrho} \\ \underline{\varepsilon} + \underline{p} \end{bmatrix} \bullet \left(\mathrm{D}^2 \eta(\underline{\varrho}, \underline{\varepsilon}) \begin{bmatrix} \underline{\varrho} \\ \underline{\varepsilon} + \underline{p} \end{bmatrix} \right) > 0 \,,$$

wenn die Entropie $(\varrho, \varepsilon) \mapsto \eta(\varrho, \varepsilon)$ eine konkave Funktion ist.

(2) Im Falle eines idealen Gases 2.5 (beachte auch 2.4) gilt

$$c_S = \sqrt{\gamma R \underline{\theta}} \,, \quad \gamma = \frac{c_P}{c_V} \,, \quad R = c_P - c_V \,, \quad \underline{\theta} = \text{const} \cdot \underline{\varrho}^{\gamma - 1} \,,$$

siehe die Formel [10, (63,15)].

2.01b Schallgeschwindigkeit in Flüssigkeiten

Flüssigkeit	t °C	ϱ g/cm³	c_L km/s	Flüssigkeit	t °C	ϱ g/cm³	c_L km/s
Aceton	20	0,79	1,19	Sauerstoff	−182,9	1,14	0,91
Ammoniak (konz.)	16	0,88	1,66	Stickstoff	−197	0,82	0,87
Benzin	17	0,68	1,17	Toluol	20	0,86	1,32
Benzol	20	0,88	1,32	Wasser	0	1,000	1,403
Ethylalkohol	20	0,79	1,16	(destilliert)	20	0,988	1,483
Glyzerin	20	1,26	1,90		40	0,992	1,529
Kochsalzlösung					60	0,983	1,551
1%	25	1,01	1,52		80	0,971	1,555
25%	25	1,19	1,77		100	0,958	1,543
Paraffinöl	33,5	0,8	1,42	Wasser			
Petroleum	15	0,8	1,33	(Seewasser)	20	1,03	1,522
Quecksilber	20	13,5	1,45	(schweres)	19,8	1,10	1,383
Salzsäure (konzentriert)	15,5	1,19	1,52	Wasserstoff	−252	0,07	1,150

2.01c Schallgeschwindigkeit in Gasen und Dämpfen unter Normdruck 101,3 kPa

Gas/Dampf	t °C	ϱ kg/m³	c_L m/s	Gas/Dampf	t °C	ϱ kg/m³	c_L m/s
Acetylen	0	1,17	327	Luft	20	1,21	344
Ammoniak	0	0,77	415		40		355
Argon	0	1,78	319		100		387
Brom	58		149	Methan	0	0,72	430
Chlor	0	3,22	206	Neon	0	0,90	435
Helium	0	0,18	971	Propan	0	2,02	238
Kohlenoxid	0	1,25	338	Sauerstoff	0	1,43	316
Kohlendioxid	0	1,98	259	Stickstoff	0	1,25	334
	18		266	Wasserdampf	134		494
Leuchtgas	0		453	Wasserstoff	0	0,09	1284
Luft	−40		307		18		1301
	−20		319	Wasserstoff	0		890
	0	1,29	331	(schwerer)			

Fig. 21: Schallgeschwindigkeit, aus Kohlrausch [83]

Proof (1). Die dritte Darstellung von c_S^2 folgt aus der Tatsache, dass

$$\widetilde{s}(\varrho,\widetilde{\theta}(\varrho)) = \widehat{s}(\varrho,\widetilde{\varepsilon}(\varrho)) = s = s_0$$

Indem wir mit $p = \underline{p} + \delta p' + \mathcal{O}(\delta)$ die Ableitung nach δ bilden, erhalten wir die folgenden Identitäten

$$p' = \frac{\mathrm{d}}{\mathrm{d}\varrho}\Big(\widetilde{p}(\varrho,\widetilde{\theta}(\varrho))\Big)\Big|_{\varrho=\underline{\varrho}}\,\varrho' = \frac{\mathrm{d}}{\mathrm{d}\varrho}\Big(\widehat{p}(\varrho,\widetilde{\varepsilon}(\varrho))\Big)\Big|_{\varrho=\underline{\varrho}}\,\varrho'$$

Die erste Darstellung von c_S^2 erhalten wir, wenn wir von den beiden Identitäten $p = \widetilde{p}(\varrho,\theta)$ und $s_0 = s = \widetilde{s}(\varrho,\theta)$ ausgehen und mit

$$p = \underline{p} + \delta p' + \mathcal{O}(\delta)\,, \quad s = \underline{s} + \delta s' + \mathcal{O}(\delta)$$

die Ableitung nach δ ausrechnen:

$$\begin{aligned} p' &= \widetilde{p}'_{\varrho}(\underline{\varrho},\underline{\theta})\varrho' + \widetilde{p}'_{\theta}(\underline{\varrho},\underline{\theta})\theta'\,, \\ 0 = s' &= \widetilde{s}'_{\varrho}(\underline{\varrho},\underline{\theta})\varrho' + \widetilde{s}'_{\theta}(\underline{\varrho},\underline{\theta})\theta'\,, \end{aligned}$$

und damit

$$p' = \Big(\widetilde{p}'_{\varrho}(\underline{\varrho},\underline{\theta}) - \frac{\widetilde{s}'_{\varrho}(\underline{\varrho},\underline{\theta})}{\widetilde{s}'_{\theta}(\underline{\varrho},\underline{\theta})}\widetilde{p}'_{\theta}(\underline{\varrho},\underline{\theta})\Big)\varrho'\,.$$

Die letzte Darstellung von c_S^2 erhalten wir, indem wir die Gibbs Relation schreiben als

$$\varepsilon + p = \frac{\eta - \varrho\eta'_\varrho}{\eta'_\varepsilon}$$

so dass also aus

$$\frac{\widehat{\eta}(\varrho, \widetilde{\varepsilon}(\varrho))}{\varrho} = \widehat{s}(\varrho, \widetilde{\varepsilon}(\varrho)) = s_0$$

folgt

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varrho}\Big(\widehat{s}(\varrho, \widetilde{\varepsilon}(\varrho))\Big) = -\frac{\widehat{\eta}}{\varrho^2} + \frac{\widehat{\eta}'_\varrho + \widehat{\eta}'_\varepsilon \widetilde{\varepsilon}'_\varrho}{\varrho}\,,$$

das heißt

$$\widetilde{\varepsilon}'_\varrho = \frac{\widehat{\eta} - \varrho\widehat{\eta}'_\varrho}{\varrho\widehat{\eta}'_\varepsilon} = \frac{\eta - \varrho\eta'_\varrho}{\varrho\eta'_\varepsilon} = \frac{\varepsilon + p}{\varrho}\,.$$

Daraus schließen wir nun

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varrho}\Big(\widehat{p}(\varrho, \widetilde{\varepsilon}(\varrho))\Big) = \widehat{p}'_\varrho + \widetilde{\varepsilon}'_\varrho \widehat{p}'_\varepsilon = \widehat{p}'_\varrho + \frac{\varepsilon + p}{\varrho}\widehat{p}'_\varepsilon \\
&= \Big(\frac{\eta - \varrho\eta'_\varrho}{\eta'_\varepsilon} - \varepsilon\Big)'_\varrho + \frac{\varepsilon + p}{\varrho}\Big(\frac{\eta - \varrho\eta'_\varrho}{\eta'_\varepsilon} - \varepsilon\Big)'_\varepsilon \quad (\text{es ist } p = \widehat{p}(\varrho, \varepsilon)) \\
&= -\frac{\varrho\eta'_{\varrho\varrho}}{\eta'_\varepsilon} - \frac{\eta - \varrho\eta'_\varrho}{\eta'^2_\varepsilon}\eta'_{\varepsilon\varrho} + \frac{\varepsilon + p}{\varrho}\Big(\frac{\eta'_\varepsilon - \varrho\eta'_{\varrho\varepsilon}}{\eta'_\varepsilon} - 1 - \frac{\eta - \varrho\eta'_\varrho}{\eta'^2_\varepsilon}\eta'_{\varepsilon\varepsilon}\Big) \\
&= -\frac{\varrho}{\eta'_\varepsilon}\eta'_{\varrho\varrho} - 2\frac{\varepsilon + p}{\eta'_\varepsilon}\eta'_{\varrho\varepsilon} - \frac{(\varepsilon + p)^2}{\varrho\eta'_\varepsilon}\eta'_{\varepsilon\varepsilon} \\
&= -\frac{1}{\varrho\eta'_\varepsilon}\Big(\varrho^2\eta'_{\varrho\varrho} + 2\varrho(\varepsilon + p)\eta'_{\varrho\varepsilon} + (\varepsilon + p)^2\eta'_{\varepsilon\varepsilon}\Big) \\
&= -\frac{\theta}{\varrho}\begin{bmatrix}\varrho \\ \varepsilon + p\end{bmatrix} \bullet \Big(\begin{bmatrix}\eta'_{\varrho\varrho} & \eta'_{\varrho\varepsilon} \\ \eta'_{\varrho\varepsilon} & \eta'_{\varepsilon\varepsilon}\end{bmatrix}\begin{bmatrix}\varrho \\ \varepsilon + p\end{bmatrix}\Big) \geq 0\,,
\end{aligned}$$

d.h. die Schallgeschwindigkeit ist positiv. □

Proof (2). Nach 2.5 gilt für alle Lösungen

$$\varepsilon = c_V\theta\varrho\,, \quad \text{wobei} \quad \theta = \frac{1}{\eta'_\varepsilon}\,,$$

und nach der Gibbs Relation ist mit $\eta = \varrho s$ (ebenfalls in 2.5)

$$\begin{aligned}
&s = \widehat{s}(\varrho, \varepsilon) = c_V \log\varepsilon - c_P \log\varrho + c \quad (c \in \mathbb{R}) \\
&\widetilde{s}(\varrho, \theta) = c_V \log\theta + c_V \log(c_V\varrho) - c_P \log\varrho + c \\
&= c_V \log\theta - R \log\varrho + c_V \log c_V + c
\end{aligned}$$

sowie $p = R\theta\varrho$. Dies gilt auch für die Lösug $(\underline{\varrho}, \underline{v}, \underline{\theta})$, weshalb

$$c_S^2 = p'_\varrho - \frac{s'_\varrho}{s'_\theta}p'_\theta = R\underline{\theta} + \frac{R\underline{\theta}}{c_V\underline{\varrho}}R\underline{\varrho} = R\underline{\theta}\frac{c_P}{c_V}\,.$$

Nun haben wir wegen der Bedingung $\underline{s} = s_0$

$$c_V \log \underline{\varepsilon} - c_P \log \underline{\varrho} = s_0 - c, \quad \text{was} \quad \log \underline{\theta} = \frac{R}{c_V} \log \underline{\varrho} + \text{const}$$

mit const $= \frac{1}{c_V}(s_0 + c_V \log c_V - c)$ bedeutet. Dies war zu zeigen. □

7.5 Wellengleichung. If 7.3 holds and in 7.2 in addition $(\underline{\varrho}, \underline{v}, \underline{\theta}) = \text{const}$ (we mention that then also $c_S = \text{const}$) then the equations (IV7.7) are equivalent to

$$\begin{aligned} &\partial_{(1,\underline{v})} \varrho' + \underline{\varrho}\, \text{div} u' = 0\,, \\ &\underline{\varrho} \partial_{(1,\underline{v})} u' + \nabla p' = \mathbf{f}'\,. \end{aligned} \tag{IV7.8}$$

If the observer is so that $\text{div}\,\mathbf{f}' = 0$ it follows

$$\partial^2_{(1,\underline{v})} \varrho' - \Delta p' = 0 \quad \text{or} \quad \frac{1}{c_S^2} \partial^2_{(1,\underline{v})} p' - \Delta p' = 0 \tag{IV7.9}$$

Remark: We have $p' = c_S^2 \varrho'$.

Proof. Under the assumptions (IV7.7) reduces to (IV7.8). Then

$$\begin{aligned} 0 &= \partial_{(1,\underline{v})}(\partial_{(1,\underline{v})} \varrho' + \underline{\varrho}\, \text{div} u') = \partial^2_{(1,\underline{v})} \varrho' + \text{div}(\underline{\varrho}\, \partial_{(1,\underline{v})} u') \\ &= \partial^2_{(1,\underline{v})} \varrho' - \Delta p' + \text{div}\mathbf{f}'\,, \end{aligned}$$

which is the first equation in (IV7.9). From this the second equation follows since $\varrho' = \frac{1}{c_S^2} p'$. □

7.6 Potential flow. If in 7.5 the perturbation $u' = \nabla \varphi'$, then

$$\frac{1}{c_S^2} \partial^2_{(1,\underline{v})} \varphi' - \Delta \varphi' = 0 \quad \text{and} \quad \frac{1}{c_S^2} \partial^2_{(1,\underline{v})} u' - \Delta u' = 0$$

Erinnerung: u' is an objective vector, therefore φ' an objective scalar.

Die Übertragung von Schall besteht nun in einer Überlagerung von Wellen verschiedener Wellenlänge und verschiedener Richtung.

Monochromatische Welle

Wir machen die Voraussetzungen wie in 7.6 (also auch $\text{div}\,\mathbf{f}' = 0$) und zusätzlich $\underline{v} = 0$. Eine Lösung der Wellengleichung (IV7.8) ist dann gegeben durch

$$\varphi'(t, x) = g(x \bullet \mathbf{k} - \omega t)\,,$$

wobei $\tau \mapsto g(\tau)$ eine beliebige Funktion ist. Es ist

$$\partial_t \varphi' = -\omega g'_\tau\,, \quad u'(t, x) = \nabla \varphi' = g'_\tau(x \bullet \mathbf{k} - \omega t)\, \mathbf{k}$$

also ist die erste Gleichung von (IV7.8)

$$\partial_t \varrho' = -\operatorname{div}(\underline{\varrho} u') = -\operatorname{div}(\underline{\varrho} g'{}_\tau \mathbf{k}) = -\mathbf{k}\bullet\nabla(\underline{\varrho} g'{}_\tau) = -\underline{\varrho} g'{}_{\tau\tau} \cdot |\mathbf{k}|^2$$

erfüllt wenn

$$\varrho'(t,x) = \frac{\underline{\varrho}|\mathbf{k}|^2}{\omega} g'{}_\tau(x\bullet\mathbf{k} - \omega t)$$

und es gilt die zweite Gleichung von (IV7.8)

$$-\underline{\varrho}\omega g'{}_{\tau\tau}\mathbf{k} = \underline{\varrho}\partial_t u' = -\nabla p'$$

wenn

$$p'(t,x) = \underline{\varrho}\omega g'{}_\tau(x\bullet\mathbf{k} - \omega t) = \frac{\omega^2}{|\mathbf{k}|^2}\varrho' .$$

Also ist

$$c_S = \frac{\omega}{|\mathbf{k}|} \quad \text{somit} \quad |\mathbf{k}| = \frac{\omega}{c_S}$$

d.h. mit einem Einheitsvektor $\mathbf{n}$ gilt für den ***Wellenzahlvektor*** $\mathbf{k}$

$$\mathbf{k} = \frac{\omega}{c_S}\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n}\,, \quad \lambda := \frac{2\pi c_S}{\omega}\,,$$

wobei λ die ***Wellenlänge*** ist und ω die ***Frequenz*** (siehe [10, (63,20)]). Wir haben also

$$\varphi'(t,x) = g\big(\omega(\frac{x\bullet\mathbf{n}}{c_S} - t)\big)$$

Wir definieren jetzt $g(\tau) := \operatorname{Re}\big(a\exp(\mathrm{i}\tau)\big)$ und daher ist speziell

$$\varphi'(t,x) = \operatorname{Re}\Big(a\exp\big(\mathrm{i}\omega(x\bullet\frac{\mathbf{n}}{c_S} - t)\big)\Big)$$

wobei $a \in \mathbb{C}$ die ***(komplexe) Amplitude*** von φ' ist.

Reflexion einer Welle

Wie verhält sich der Schall an einer Wand oder am Übergang von verschiedenen Medien? Wir betrachten hier das Verhalten einer monochromatischen Welle, wie das linke Bild in Fig. 23 zeigt, wie es an einer Materialgrenze teilweise reflektiert wird und teilweise transmittiert.
Dazu betrachten wir die beiden Gebiete $\Omega^1 = \{(t,x)\,;\; x_1 < \gamma(t,x_2,x_3)\}$ und $\Omega^2 = \{(t,x)\,;\; x_1 > \gamma(t,x_2,x_3)\}$, wobei deren gemeinsamer Rand die Hyperfläche $\Gamma = \{(t,x)\,;\; x_1 = \gamma(t,x_2,x_3)\}$ sei mit einer Graphenfunktion

$$\begin{aligned} &\gamma = \underline{\gamma} + \delta\gamma' + o(\delta)\,, \quad \underline{\gamma} = 0 \\ &\tau(t,x) := (t, x_1 + \gamma(t,x_2,x_3), x_2, x_3)\,. \end{aligned} \tag{IV7.10}$$

Trifft eine Schallwelle auf die Grenze zwischen zwei verschiedenen Medien (Flüssigkeiten oder Gasen), dann wird ein Teil von ihr reflektiert und ein Teil gebrochen. Außer der einfallenden Welle entstehen also noch zwei Wellen; die eine (die reflektierte) breitet sich von der Grenzfläche rückwärts in das erste Medium aus, die zweite (die gebrochene) breitet sich von der Grenzfläche in das zweite Medium aus. Im ersten Medium entsteht folglich eine Überlagerung zweier Wellen (einfallende und reflektierte); im zweiten Medium ist eine gebrochene Welle vorhanden.

Fig. 22: Aus Landau & Lifschitz [10, aus §65]

Dabei beteht der Limes aus den stationären Gebieten $\underline{\Omega}^1 = \{(t,x)\,;\ x_1 < 0\}$ und $\underline{\Omega}^2 = \{(t,x)\,;\ x_1 > 0\}$ mit Rand $\underline{\Gamma} = \{(t,x)\,;\ x_1 = 0\}$, und es ist $\Omega^k = \tau(\underline{\Omega}^k)$ und $\Gamma = \tau(\underline{\Gamma})$. In Ω^k haben wir Lösungen $(\varrho^k, v^k, \theta^k)$ von System (IV7.4) oder dazu äquivalent des ursprünglichen Systems (IV7.1), natürlich mit $S = 0$ und $q = 0$. In dem Gebiet $\Omega = \Omega^1 \cup \Gamma \cup \Omega^2$ gelten nun die ursprüngliche Masse-Impuls-Energieerhaltungen (IV7.1) im distributionellen Sinne. Dabei ergeben sich auf Γ Randbedingungen, die im Limes $\delta \to 0$ die Form (IV7.11) annehmen.

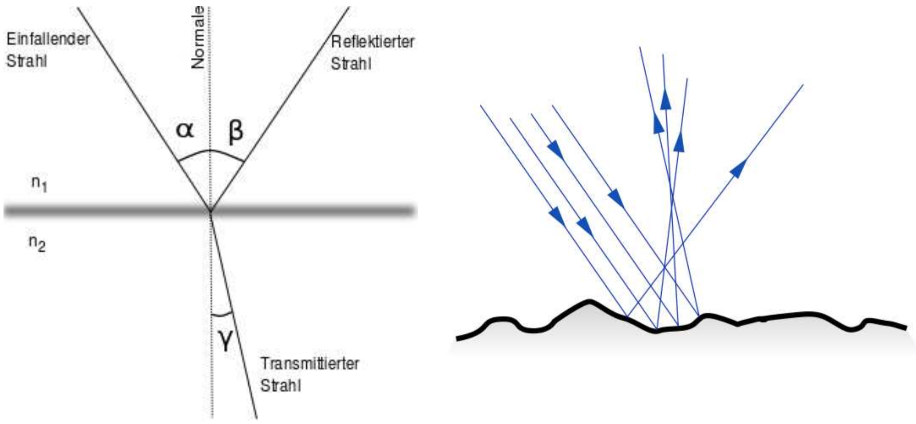


Fig. 23: Reflexion einer Welle, aus Wikipedia [132]

7.7 Randbedingungen. Für die Funktionen $(\varrho^k, v^k, \theta^k)$ in Ω^k seien die adiabatischen distributionellen Gleichungen (IV7.1) in Ω (mit $S = 0$, $q = 0$) erfüllt. Es gebe keinen Massenfluss über Γ hinaus, d.h. die Massenerhaltung von (IV7.1) ist separat für $k = 1, 2$ erfüllt. Ist dann

$$(\varrho^k, v^k, \theta^k)\circ\tau = (\underline{\varrho}^k, \underline{v}^k, \underline{\theta}^k) + \delta(\varrho'_k, u'_k, \theta'_k) + \mathcal{O}(\delta)\,, \quad \underline{v}^k = 0\,,$$

und sind die Voraussetzungen in 7.5 erfüllt, so gelten die folgenden Randbedingungen für Masse und Impuls

$$\left.\begin{aligned} u'_1\bullet\nu &= u'_2\bullet\nu \\ p'_1 &= p'_2 \end{aligned}\right\}\text{auf } \underline{\Gamma}, \tag{IV7.11}$$

wobei ν eine Normale auf $\underline{\Gamma}$ ist. Es ist auch $\underline{p}^1 = \underline{p}^2$ auf $\underline{\Gamma}$ erfüllt.

Remark: Diese Randbedingungen sind in [10, §65 Reflexion und Brechung von Schallwellen] zu finden.

Proof. Für die Massenerhaltung der Phase k (da zwischen den Massen keine Masse ausgetauscht wird) gilt, wenn $\eta \in C_0^\infty(\Omega;\mathbb{R})$,

$$\begin{aligned} 0 &= \int_{\Omega^k} \big(\partial_t\eta\cdot\varrho^k + \nabla\eta\bullet(\varrho^k v^k)\big)\,\mathrm{dL}^4 \\ &= \int_{\Omega^k} (\partial_t,\nabla)\eta\bullet(\varrho^k,\varrho^k v^k)\,\mathrm{dL}^4 \\ &= \textit{(Volumenterm)} + \int_\Gamma \eta\, n_{\Omega^k}\bullet(\varrho^k,\varrho^k v^k)\,\mathrm{dH}^3\,, \end{aligned}$$

wobei n_{Ω^k} die äußere Normale von Ω^k in Raumzeit an Γ ist. Da η beliebig ist, ist der Volumenterm gleich Null (das ist die Differentialgleichung in Ω^k) und es folgt für den Γ-Term

$$0 = n_{\Omega^k}\bullet(\varrho^k,\varrho^k v^k) \quad \text{auf } \Gamma.$$

Da (siehe Abschnitt I.4 ??????????)

$$n_{\Omega^k} = \frac{(-v_\Gamma\bullet\nu^k,\nu^k)}{\sqrt{1+|v_\Gamma|^2}}\,,$$

wobei ν^k die äußere Normale von $\Omega^k(t)$ an $\Gamma(t)$ ist und v_Γ die Geschwindigkeit von $t\mapsto\Gamma(t)$, folgt

$$0 = n_{\Omega^k}\bullet(\varrho^k,\varrho^k v^k) = \varrho^k\frac{(v^k - v_\Gamma)\bullet\nu^k}{\sqrt{1+|v_\Gamma|^2}}\,.$$

Da nun

$$\begin{aligned} v^k\circ\tau &= \underline{v}^k + \delta u_k' + \mathcal{O}(\delta)\,, \quad \underline{v}^k = 0\,, \\ \nu^k\circ\tau &= \underline{\nu}^k + \mathcal{O}(1)\,, \quad \underline{\nu}^k = (-1)^{k-1}\mathbf{e}_1\,, \\ v_\Gamma\circ\tau &= \underline{v}_\Gamma + \delta v_\Gamma' + \mathcal{O}(\delta)\,, \quad \underline{v}_\Gamma = 0\,, \end{aligned}$$

folgt also mit $\nu = \nu^1 = -\nu^2$, da $\varrho^k > 0$,

$$(u_k' - v_\Gamma')\bullet\nu = 0 \text{ auf } \underline{\Gamma}\,.$$

Da dies für $k=1$ und $k=2$ gilt, erhalten wir $(u_1' - u_2')\bullet\nu = 0$.

Für die Impulserhaltung gilt für $\zeta \in C_0^\infty(\Omega;\mathbb{R}^3)$ analog

$$\begin{aligned} 0 &= \sum_{k,i}\int_{\Omega^k}\big(\partial_t\zeta_i\cdot\varrho^k v_i^k + \sum_j \partial_j\zeta_i\cdot(\varrho^k v_i^k v_j^k + p^k\delta_{i,j})\big)\,\mathrm{dL}^4 \\ &= \sum_{k,i}\int_{\Omega^k}(\partial_t,\nabla)\zeta_i\bullet(\varrho^k v_i^k,\varrho^k v_i^k v^k + p^k\mathbf{e}_j)\,\mathrm{dL}^4 \\ &= \textit{(Volumenterme)} + \sum_i\int_\Gamma \zeta_i\sum_k \nu_{\Omega^k}\bullet(\varrho^k v_i^k,\varrho^k v_i^k v^k + p^k\mathbf{e}_i)\,\mathrm{dH}^3\,. \end{aligned}$$

Da ζ_i beliebig ist, sind die Volumenterme gleich Null (das sind die Differentialgleichungen in Ω^k) und es folgt für den Γ-Term für i

$$0 = \sum_k n_{\Omega^k} \bullet (\varrho^k v_i^k, \varrho^k v_i^k v^k + p^k \mathbf{e}_i) \quad \text{auf } \Gamma.$$

Die Formel für n_{Ω^k} von oben ergibt mit der Tatsache, dass $(v^k - v_\Gamma) \bullet \nu^k = 0$,

$$0 = \sum_k p^k \nu^k = (p^1 - p^2)\nu\,.$$

Da $p^k = \underline{p}^k + \delta p'_k + \mathcal{O}(\delta)$ folgt $\underline{p}^1 = \underline{p}^2$ und dann $p'_1 = p'_2$. □

Wie geben nun folgende monochromatische Welle vor, wobei im linken Bild der Fig. 23 die Koordinate x_1 nach unten und x_2 nach rechts zeigt, und wobei $c_k := c_S(\underline{\varrho}^k, \underline{\theta}^k)$ als Abkürzung für die Schallgeschwindigkeit im Gebiet Ω^k benutzt wird:

$$\begin{aligned}
\varphi'_1 &= \mathrm{Re}\Big(a_1 \exp\big(\mathrm{i}\omega(x \bullet \frac{\mathbf{n_1}}{c_1} - t)\big)\Big) \quad \text{(Einfallender Strahl)} \\
&\quad + \mathrm{Re}\Big(\widetilde{a}_1 \exp\big(\mathrm{i}\omega(x \bullet \frac{\widetilde{\mathbf{n}}_\mathbf{1}}{c_1} - t)\big)\Big) \quad \text{(Reflektierter Strahl)}, \\
\mathbf{n_1} &= (\cos\alpha_1, \sin\alpha_1, 0)\,, \quad \widetilde{\mathbf{n}}_\mathbf{1} = (-\cos\alpha_1, \sin\alpha_1, 0)\,, \\
\varphi'_2 &= \mathrm{Re}\Big(a_2 \exp\big(\mathrm{i}\omega(x \bullet \frac{\mathbf{n_2}}{c_2} - t)\big)\Big) \quad \text{(Transmittierter Strahl)}, \\
\mathbf{n_2} &= (\cos\alpha_2, \sin\alpha_2, 0)\,.
\end{aligned} \tag{IV7.12}$$

Hier ist die Lösung bis auf einige Randbedingungen schon gegeben. Man könnte auch den einfallenden Strahl vorgeben und dann danach fragen, wie der Respons an Γ ist, was natürlich von dessen Struktur abhängt. Hier soll aber gezeigt werden, dass die Oberfläche so geartet ist, dass eine reflektierter und ein transmittierender Strahl auftritt, also der Strahl die hergeleiteten Randbedingungen erfüllt.

Mit $g_1(\tau) := \mathrm{Re}\big(a_1 \exp(\mathrm{i}\tau)\big)$ und entsprechend $\widetilde{g}_1$ und g_2 berechnen wir

$$\begin{aligned}
\varphi'_1 &= g_1(\tau_1) + \widetilde{g}_1(\widetilde{\tau}_1)\,, \quad \tau_1 = \omega\big(x \bullet \frac{\mathbf{n_1}}{c_1} - t\big)\,, \quad \widetilde{\tau}_1 = \omega\big(x \bullet \frac{\widetilde{\mathbf{n}}_\mathbf{1}}{c_1} - t\big) \\
\varphi'_2 &= g_2(\tau_2)\,, \quad \tau_2 = \omega\big(x \bullet \frac{\mathbf{n_2}}{c_2} - t\big) \\
u'_1 &= \nabla\varphi'_1 = \frac{\omega {g_1}'_\tau(\tau_1)}{c_1}\mathbf{n_1} + \frac{\omega {\widetilde{g}_1}{}'_\tau(\widetilde{\tau}_1)}{c_1}\widetilde{\mathbf{n}}_\mathbf{1} \\
u'_2 &= \nabla\varphi'_2 = \frac{\omega {g_2}'_\tau(\tau_2)}{c_2}\mathbf{n_2} \\
p'_1 &= \underline{\varrho}^1\omega({g_1}'_\tau(\tau_1) + {\widetilde{g}_1}{}'_\tau(\widetilde{\tau}_1))\,, \quad p'_2 = \underline{\varrho}^2\omega {g_2}'_\tau(\tau_2)\,.
\end{aligned}$$

Also lauten die Randbedingungen auf Γ: Für die Masse ist $u_1'\bullet\nu = u_2'\bullet\nu$ äquivalent zu

$$\frac{g_1{'}_\tau(\tau_1)}{c_1}\mathbf{n_1}\bullet\nu + \frac{\widetilde{g}_1{'}_\tau(\widetilde{\tau}_1)}{c_1}\widetilde{\mathbf{n}}_\mathbf{1}\bullet\nu = \frac{g_2{'}_\tau(\tau_2)}{c_2}\mathbf{n_2}\bullet\nu\,,$$

wobei $\nu = \nu^1 = -\nu^2$ die Normale an den Rand sei. Da

$$\mathbf{n_1}\bullet\nu = \cos\alpha_1\,,\quad \widetilde{\mathbf{n}}_\mathbf{1}\bullet\nu = -\cos\alpha_1\,,\quad \mathbf{n_2}\bullet\nu = \cos\alpha_2\,,$$

heißt dies

$$\frac{\cos\alpha_1}{c_1}(g_1{'}_\tau(\tau_1) - \widetilde{g}_1{'}_\tau(\widetilde{\tau}_1)) = \frac{\cos\alpha_2}{c_2}g_2{'}_\tau(\tau_2)\,. \tag{IV7.13}$$

Für den Impuls ist $p_1' = p_2'$ die Bedingung, also

$$\underline{\varrho}^1(g_1{'}_\tau(\tau_1) + \widetilde{g}_1{'}_\tau(\widetilde{\tau}_1)) = \underline{\varrho}^2 g_2{'}_\tau(\tau_2)\,. \tag{IV7.14}$$

Aus diesen beiden Gleichungen ergibt sich

7.8 Lemma. Die Randbedingungen liefern für den Strahl in (IV7.12)

$$\begin{aligned} \frac{\sin\alpha_1}{c_1} &= \frac{\sin\alpha_2}{c_2} && \text{in } \mathbb{R},\\ \underline{\varrho}^1(a_1+\widetilde{a}_1) &= \underline{\varrho}^2 a_2 && \text{in } \mathbb{C},\\ \frac{\cos\alpha_1}{c_1}(a_1-\widetilde{a}_1) &= \frac{\cos\alpha_2}{c_2}a_2 && \text{in } \mathbb{C}. \end{aligned}$$

Proof. Since

$$g_1{'}_\tau(\tau) = \mathrm{Re}(\mathrm{i}a_1\mathrm{e}^{\mathrm{i}\tau})\,,\quad \widetilde{g}_1{'}_\tau(\tau) = \mathrm{Re}(\mathrm{i}\widetilde{a}_1\mathrm{e}^{\mathrm{i}\tau})\,,\quad g_2{'}_\tau(\tau) = \mathrm{Re}(\mathrm{i}a_2\mathrm{e}^{\mathrm{i}\tau})$$

the property (IV7.14) reads

$$\underline{\varrho}^1\mathrm{Re}\big(\mathrm{i}a_1\mathrm{e}^{\mathrm{i}\tau_1} + \mathrm{i}\widetilde{a}_1\mathrm{e}^{\mathrm{i}\widetilde{\tau}_1}\big) = \underline{\varrho}^2\mathrm{Re}\big(\mathrm{i}a_2\mathrm{e}^{\mathrm{i}\tau_2}\big)$$

on Γ. Now $x_1 = 0$ on the boundary Γ and therefore

$$\tau_1 = \omega x_2\frac{\sin\alpha_1}{c_1} - \omega t = \widetilde{\tau}_1\,,\quad \tau_2 = \omega x_2\frac{\sin\alpha_2}{c_2} - \omega t\,. \tag{IV7.15}$$

Hence the equation becomes

$$\underline{\varrho}^1\mathrm{Re}\Big(\mathrm{i}(a_1+\widetilde{a}_1)\mathrm{e}^{-\mathrm{i}\omega t}\exp\big(\mathrm{i}\omega x_2\frac{\sin\alpha_1}{c_1}\big)\Big) = \underline{\varrho}^2\mathrm{Re}\Big(\mathrm{i}a_2\mathrm{e}^{-\mathrm{i}\omega t}\exp\big(\mathrm{i}\omega x_2\frac{\sin\alpha_2}{c_2}\big)\Big)$$

and since this holds for all t, it holds for $t = 0$ and $t = \frac{\pi}{2\omega}$, we get

$$\underline{\varrho}^1(a_1+\widetilde{a}_1)\exp\big(\mathrm{i}\omega x_2\frac{\sin\alpha_1}{c_1}\big) = \underline{\varrho}^2 a_2\exp\big(\mathrm{i}\omega x_2\frac{\sin\alpha_2}{c_2}\big)\,.$$

First setting $x_2 = 0$ we get

$$\underline{\varrho}^1(a_1 + \widetilde{a}_1) = \underline{\varrho}^2 a_2 \,.$$

Inserting this result in the equation and assuming $a_2 \neq 0$ we get

$$\exp\big(\mathrm{i}\omega x_2 \frac{\sin\alpha_1}{c_1}\big) = \exp\big(\mathrm{i}\omega x_2 \frac{\sin\alpha_2}{c_2}\big)\,.$$

This holds if for every $x_2 \in \mathbb{R}$ there is a $k \in \mathbb{Z}$ such that

$$\omega x_2 \Big(\frac{\sin\alpha_1}{c_1} - \frac{\sin\alpha_2}{c_2}\Big) = 2\pi k\,.$$

Since $\omega \neq 0$ this is possible only if

$$\frac{\sin\alpha_1}{c_1} - \frac{\sin\alpha_2}{c_2} = 0 \tag{IV7.16}$$

Now to the equation (IV7.13) which reads

$$\frac{\cos\alpha_1}{c_1}\mathrm{Re}\big(\mathrm{i}a_1\mathrm{e}^{\mathrm{i}\tau_1} - \mathrm{i}\widetilde{a}_1\mathrm{e}^{\mathrm{i}\widetilde{\tau}_1}\big) = \frac{\cos\alpha_2}{c_2}\mathrm{Re}\big(\mathrm{i}a_2\mathrm{e}^{\mathrm{i}\tau_2}\big)$$

Using (IV7.16) we get $\tau_1 = \widetilde{\tau}_1 = \tau_2$ from (IV7.15), and the argument on $e^{-\mathrm{i}\omega t}$ gives that this equation becomes

$$\frac{\cos\alpha_1}{c_1}(a_1 - \widetilde{a}_1) = \frac{\cos\alpha_2}{c_2}a_2\,.$$

This finishes the proof. □

Das Verhalten des Schalls wird bei einer rauhen Oberfläche von den Oberflächeneigenschaften der Wand bestimmt, siehe zum Beispiel das rechte Bild in Fig. 23. Dies werden wir in [22] darstellen. Es sei auch auf [40, Chap 7 Interaction of Acoustic Waves with Boundary] verwiesen.

Kugelwellen

Die bisherigen Schallwellen bezogen sich auf einen vom Ursprungsort weit entfernten Beobachter. Wir sind nun in der Nähe eines Ursprungortes. Dort betrachten wir eine Kugelwelle

$$\varphi'(t,x) = \frac{g(r - c_S t)}{r}\,, \quad r := |x|\,, \tag{IV7.17}$$

wobei wir die Voraussetzungen in 7.5 machen mit $\underline{v} = 0$ und $\mathbf{f}' = 0$, und natürlich machen wir die Voraussetzung $u' = \nabla\varphi'$.

7.9 Theorem. Es sei $\tau \mapsto g(\tau)$ eine beliebige Funktion und definiere φ' gemaß (IV7.18). Dann können die anderen Störungsfunktionen auf analoge Weise definiert werden, so dass diese Funktionen sdie Wellengleichung (IV7.8) in $\{(t,x)\,;\ r > 0\}$ erfüllen. *Hinweis:* Siehe 7.11 für $r = 0$.

Proof. Es ist

$$\partial_t\varphi' = -\frac{c_S}{r}g'_T\,, \quad r^2\partial_r\varphi' = r^2\Big(\frac{g'_T}{r} - \frac{g}{r^2}\Big) = rg'_T - g$$

also ist die erste Gleichung von (IV7.8)

$$\partial_t\varrho' = -\underline{\varrho}\operatorname{div}u' = -\underline{\varrho}\Delta\varphi' = -\frac{\underline{\varrho}}{r^2}\partial_r(r^2\partial_r\varphi') = -\frac{\underline{\varrho}}{r}g'_{TT}$$

erfüllt, wenn

$$\varrho' = \frac{\underline{\varrho}}{c_s r}g'_T(r - c_S t)\,,$$

und es gilt die zweite Gleichung von (IV7.8)

$$0 = \underline{\varrho}\partial_t u' + \nabla p' = \nabla(\underline{\varrho}\partial_t\varphi' + p')\,,$$

wenn

$$p' = -\underline{\varrho}\partial_t\varphi' = \frac{c_S\underline{\varrho}}{r}g'_T = c_S^2\varrho'\,.$$

Dies alles gilt nur wenn $r > 0$. □

Wenn wir für Kugelwellen die Wellengleichung im ganzen Raum erklären wollen, sind wir auf Distributionen angewiesen, d.h. auf die Formulierung der Erhaltungssätze (IV7.1) für Distributionen

$$\begin{aligned} &\partial_t[\varrho] + \operatorname{div}[\varrho v] = \mathbf{r}\,, \\ &\partial_t[\varrho v] + \operatorname{div}[\varrho v\,v^{\mathrm{T}} + p\mathrm{Id}] = \mathbf{f} + \mathbf{r}v\,, \end{aligned} \tag{IV7.18}$$

wobei $S = 0$, $q = 0$ und $s = s_0$ für die Entropie gilt und wobei $\mathbf{r}$ eine Distribution mit Träger im Ursprung sein wird, da, wie gerade in 7.9 gezeigt, die Kugelwelle eine Lösung von (IV7.18) außerhalb $\{0\}$ darstellt.

7.10 Lemma. Es sei $\mathbf{f} = 0$. Für eine Störung wie in 7.2 und $\mathbf{r} = \delta\mathbf{r}' + \mathcal{O}(\delta)$ (also $\underline{\mathbf{r}} = 0$) gilt im Falle $\underline{v} = 0$ im ganzen Raumzeitgebiet

$$\begin{aligned} &\partial_t[\varrho'] + \operatorname{div}[\underline{\varrho}u'] = \mathbf{r}\,, \\ &\partial_t[\underline{\varrho}u'] + \operatorname{div}[p'\mathrm{Id}] = 0\,. \end{aligned} \tag{IV7.19}$$

Proof. (Zur Beweismethode erinnern wir an den Beweis von 7.2.) Für die Massenerhaltung gilt $\partial_t[\varrho'] + \operatorname{div}[\underline{\varrho}u'] = \mathbf{r}$ und für die Impulserhaltung folgt wegen $\mathbf{r}v = \delta\mathbf{r}'\underline{v} + \mathcal{O}(\delta) = \mathcal{O}(\delta)$ und wegen $\varrho v\,v^{\mathrm{T}} = \mathcal{O}(\delta^2)$ die Gleichung $\partial_t[\underline{\varrho}u'] + \operatorname{div}[p'\mathrm{Id}] = 0$. □

Es gilt der folgende Satz, wobei in $\{(t,x)\,;\ r = 0\}$ nur die Funktion $\operatorname{div}u'$ eine nennenswerte Singularität hat, das andere kann man mit L^1-Funktionen erklären.

7.11 Theorem. Für Kugelwellen gelten die Gleichungen (IV7.19) mit

$$\mathbf{r} := \underline{\varrho} h \boldsymbol{\mu}_0 , \quad h(t) := 4\pi \cdot g(-c_S t) .$$

Hinweis: $\boldsymbol{\mu}_0$ is defined in I.2.8.

Proof. Es ist $\varphi' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^3)$, ebenso $\varrho' = \frac{\underline{\varrho}}{c_S r} g{'}_\tau(r - c_S t)$ und $p' = c_S^2 \varrho'$. Somit ist

$$\partial_r \varphi' = \Big(\frac{g}{r}\Big)_{'r} = \frac{g{'}_\tau}{r} - \frac{g}{r^2} \in L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^3)$$

und außerdem

$$u' = \nabla\varphi' = \Big(\frac{g{'}_\tau}{r} - \frac{g}{r^2}\Big)\mathbf{e}_r , \quad \partial_t u' = -c_S \Big(\frac{g{'}_{\tau\tau}}{r} - \frac{g{'}_\tau}{r^2}\Big)\mathbf{e}_r$$

in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^3)$. Also können wir auch schreiben

$$\left.\begin{aligned} [\partial_t \varrho'] + \underline{\varrho}\,\mathrm{div}[u'] &= [\mathbf{r}] , \\ [\underline{\varrho}\partial_t u' + \nabla p'] &= 0 \end{aligned}\right\} \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3) .$$

Es bleibt der Term $\mathrm{div}[u']$ zu bestimmen, der eine distributionelle Singularität besitzt. Für Testfunktionen $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$ ist

$$\big\langle \zeta , [\partial_t \varrho'] + \underline{\varrho}\,\mathrm{div}[u'] \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^3)} = \int_{\mathbb{R}\times(\mathbb{R}^3\setminus\{0\})} (\zeta\partial_t\varrho' + \underline{\varrho}\nabla\zeta\bullet u')\,\mathrm{dL}^4$$

$$= \lim_{\delta\to 0}\Big(\int_{\mathbb{R}\times(\mathbb{R}^3\setminus \mathrm{B}_\delta(0))} \zeta\,\underbrace{(\partial_t\varrho' - \underline{\varrho}\,\mathrm{div}u')}_{=0}\,\mathrm{dL}^4 - \int_{\mathbb{R}\times\partial\mathrm{B}_\delta(0)} \zeta\underline{\varrho}\nu_{\partial\mathrm{B}_\delta(0)}\bullet u')\,\mathrm{dH}^3\Big)$$

$$= -\lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\partial\mathrm{B}_\delta(0)} \zeta\underline{\varrho}\nu_{\partial\mathrm{B}_\delta(0)}\bullet u'\,\mathrm{dH}^2\,\mathrm{dL}^1 = \int_{\mathbb{R}} \zeta(t,0)\underline{\varrho}h(t)\,\mathrm{dL}^1(t) ,$$

falls

$$\begin{aligned} h(t) &:= \lim_{\delta\to 0}\int_{\partial\mathrm{B}_\delta(0)} (-\nu_{\partial\mathrm{B}_\delta(0)}\bullet u')\,\mathrm{dH}^2 \\ &= \lim_{\delta\to 0}\int_{\partial\mathrm{B}_\delta(0)} \Big(\frac{g(r-c_S t)}{r^2} - \frac{g{'}_\tau(r-c_S t)}{r}\Big)\,\mathrm{dH}^2 \\ &= \lim_{\delta\to 0}\int_{\partial\mathrm{B}_1(0)} \big(g(\delta - c_S t) - \delta g{'}_\tau(\delta - c_S t)\big)\,\mathrm{dH}^2 = 4\pi\cdot g(-c_S t) . \end{aligned}$$

Also gilt

$$\langle \zeta , \mathbf{r} \rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^3)} := \big\langle \zeta , \underline{\varrho} h \boldsymbol{\mu}_0 \big\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^3)} = \int_{\mathbb{R}} \zeta(t,0)\underline{\varrho}h(t)\,\mathrm{dL}^1(t) ,$$

was zu beweisen war. □

Wir sehen also, dass eine Kugelwelle in der Massenerhaltung in einer im Nullpunkt konzentrierten Massenzu- bzw. -abnahme besteht. Bei einem Quellterm mit

$$g(\tau) := \mathrm{Re}\big(a \exp\big(\tfrac{\mathrm{i}\omega}{c_S}\tau\big)\big) \tag{IV7.20}$$

kann man sich $\mathbf{r}$ zum Beispiel vorstellen als eine mikroskopisch an- und abschwellende Kugel. Wir sehen auch den Doppler Effekt, wenn ein Beobachter sich mit $t \mapsto \xi(t)$ in einer Kugelwelle bewegt.

7.12 Doppler Effekt. Vom Ursprung gehe eine Welle mit Frequenz ω aus (siehe (IV7.17) und (IV7.20)). Ein Beobachter, der sich mit $t \mapsto \xi(t) \in \mathbb{R}^3$ bewegt, nimmt die Welle mit einer Frequenz $\omega_{\xi(t)}$ wahr, für die gilt

$$\omega_{\xi(t)} = \omega - \dot{\xi}(t)\bullet\mathbf{k}(\xi(t))\,. \tag{IV7.21}$$

Dabei ist der Wellenzahlvektor $\mathbf{k}$ definiert durch

$$\mathbf{k}(x) := \frac{\omega}{c_S}\mathbf{n}(x)\,, \quad \mathbf{n}(x) = \frac{x}{|x|} = \mathbf{e}_r(x)\,.$$

Genauer ist für t nahe t_0

$$p'(t,x) = \underline{\varrho}\,\mathrm{Re}\big(\mathrm{i}a\omega\frac{\mathrm{e}^{-\mathrm{i}\omega\alpha(t,x)}}{|x|}\big)\,, \quad \alpha(t,x) := 1 - \frac{1}{c_S}|x|\,,$$
$$\alpha(t,\xi(t)) = \alpha_0 + \frac{\omega_{\xi(t_0)}}{\omega}(t-t_0) + \mathcal{O}(t-t_0)\,,$$

mit $\alpha_0 = t_0 - \frac{1}{c_S}|\xi(t_0)|$.
Achtung: Die Schallwelle ist anders (siehe [21, ??]), wenn sich der Ursprung der Welle bewegt und der Beobachter etwa still steht.

Der Wellenzahlvektor ist also derjenige, der zu der in dem Punkte linearisierten Welle gehört.

Proof. Es ist mit g von (IV7.20)

$$p'(t,x) = -\underline{\varrho}\partial_t\varphi'(t,x) = \frac{c_S\underline{\varrho}}{r}g{\prime}_{\tau}(r - c_S t)\,,$$
$$= \underline{\varrho}\,\mathrm{Re}\big(\mathrm{i}a\omega\frac{\mathrm{e}^{-\mathrm{i}\omega\alpha(t,x)}}{|x|}\big)\,, \quad \alpha(t,x) := t - \frac{1}{c_S}|x|\,,$$

also wenn $x = \xi(t)$

$$\alpha(t,\xi(t)) = t - \frac{1}{c_S}|\xi(t)| = t - \frac{1}{c_S}(|\xi(t_0) + \xi{\prime}_{\tau}(t_0)(t-t_0)| + \mathcal{O}(t-t_0))$$
$$= t - \frac{1}{c_S}(|\xi(t_0)| + \xi{\prime}_{\tau}(t_0)\bullet\mathbf{n}(\xi(t_0))(t-t_0) + \mathcal{O}(t-t_0))$$
$$= t_0 - \frac{1}{c_S}|\xi(t_0)| + (1 - \frac{1}{c_S}\dot{\xi}(t_0)\bullet\mathbf{n}(\xi(t_0))(t-t_0) + \mathcal{O}(t-t_0)\,.$$

□

Schall mit Dämpfung

Wir gehen aus von dem System (IV7.4) (see [40, 4.1 Scaling and scaled equations]) jetzt aber mit "dissipativen" Termen

$$\begin{aligned}&\partial_{(1,v)}\varrho + \varrho \operatorname{div} v = 0\,,\\ &\varrho\, \partial_{(1,v)} v + \nabla p - \operatorname{div} S = \mathbf{f}\,,\\ &\varrho\, \partial_{(1,v)} s + \operatorname{div}\big(\frac{1}{\theta}q\big) = \sigma := \frac{1}{\theta}\mathrm{D}v \bullet S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q\,,\end{aligned} \tag{IV7.22}$$

mit den Gleichungen in (IV7.2). Für Störungen

$$\begin{aligned}&(\varrho, v, \theta) = (\underline{\varrho}, \underline{v}, \underline{\theta}) + \delta(\varrho', v', \theta') + o(\delta)\,,\\ &p = \underline{p} + \delta p' + o(\delta)\,, \quad S = \underline{S} + \delta S' + o(\delta)\,,\\ &s = \underline{s} + \delta s' + o(\delta)\,, \quad q = \underline{q} + \delta S' + o(\delta)\,,\end{aligned} \tag{IV7.23}$$

wie ergeben sich in erster Näherung die Gleichungen

$$\begin{aligned}&\partial_{(1,\underline{v})}\varrho' + \operatorname{div}(\underline{\varrho}\, u') = -\varrho' \operatorname{div}\underline{v}\,,\\ &\underline{\varrho}\big(\partial_{(1,\underline{v})}u' + u'\bullet\nabla\underline{v}\big) + \nabla p' - \operatorname{div} S' = \mathbf{f}' - \varrho'\partial_{(1,\underline{v})}\underline{v}\,,\\ &\underline{\varrho}\big(\partial_{(1,\underline{v})}s' + u'\bullet\nabla\underline{s}\big) + \operatorname{div}\big(\tfrac{1}{\underline{\theta}}q' - \tfrac{\theta'}{\underline{\theta}^2}q\big) = \sigma'\,,\\ &\sigma = \underline{\sigma} + \delta\sigma' + o(\delta)\,.\end{aligned} \tag{IV7.24}$$

In dem Falle, dass das Referenzsystem im "Equilibrium" ist, d.h.

$$(\underline{\varrho}, \underline{v}, \underline{\theta}) = \text{const}\,, \quad \underline{S} = 0\,, \quad \underline{q} = 0 \tag{IV7.25}$$

ist $\underline{\sigma} = 0$ und wir erhalten für die Störung

$$\begin{aligned}&\partial_{(1,\underline{v})}\varrho' + \operatorname{div}(\underline{\varrho}\, u') = 0\,,\\ &\underline{\varrho}\partial_{(1,\underline{v})}u' + \nabla p' - \operatorname{div} S' = \mathbf{f}'\,,\\ &\underline{\varrho}\partial_{(1,\underline{v})}s' + \operatorname{div}\big(\tfrac{1}{\underline{\theta}}q'\big) = \sigma'\,.\end{aligned} \tag{IV7.26}$$

Aus (IV7.24) folgt das aus den in (IV7.25) gemachten Annahmen bis auf die Aussagen $\underline{s} = \text{const}$ und $\sigma' = 0$, die jetzt bewiesen werden unter der Annahme, dass für S und q die Gleichungen (IV2.14)

$$S = 2a\,(\mathrm{D}v)^{\mathrm{S}} + b\operatorname{div}(v)\,\mathrm{Id}\,, \quad q = -c\nabla\theta\,, \tag{IV7.27}$$

erfüllt sind mit Konstanten a, b, c (oder mit Funktionen, die sich auch wie in (IV7.23) entwickeln lassen).

7.13 Lemma. Ist (IV7.27) erfüllt, so gelten die Aussagen:

(1) $\underline{s} = \text{const.}$

(2) $\sigma' = 0$, in more detail

$$\sigma = \frac{\delta^2}{\underline{\theta}}\Big(\frac{\underline{a}}{2}|\mathrm{D}u' + (\mathrm{D}u')^{\mathrm{T}}|^2 + \underline{b}|\operatorname{div}(u')|^2 + \frac{\underline{c}}{\underline{\theta}}|\nabla\theta'|^2\Big)\,.$$

Remark: Siehe [10, §77].

Proof (2). Die Voraussetzungen an S und q zusammen mit $\underline{S} = 0$ und $\underline{q} = 0$ liefern

$$S' = \Big(\underline{a}(\mathrm{D}u' + (\mathrm{D}u')^{\mathrm{T}}) + \underline{b}\operatorname{div}(u')\,\mathrm{Id}\Big)\,, \quad q' = -\underline{c}\nabla\theta'\,.$$

Dies ergibt, da $\mathrm{D}v = \delta\mathrm{D}u' + \mathcal{O}(\delta)$,

$$\begin{aligned}\sigma &= \frac{1}{\theta}\mathrm{D}v\bullet S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q\\ &= \delta^2\Big(\frac{1}{\underline{\theta}}\mathrm{D}u'\mathbin{:}S' + \nabla\Big(\frac{-\theta'}{\underline{\theta}^2}\Big)\bullet q'\Big)\\ &= \delta^2\Big(\frac{\underline{a}}{2\underline{\theta}}|\mathrm{D}u' + (\mathrm{D}u')^{\mathrm{T}}|^2 + \frac{\underline{b}}{\underline{\theta}}|\operatorname{div}(u')|^2 + \frac{\underline{c}}{\underline{\theta}^2}|\nabla\theta'|^2\Big)\,.\end{aligned}$$

Also ist $\sigma' = 0$. □

Proof (1). Wegen $\underline{S} = 0$ und $\underline{q} = 0$ ist $\underline{\sigma} = 0$ und die Entropiegleichung sagt $\underline{\varrho}\partial_{(1,\underline{v})}\underline{s} = 0$. Daraus folgt $\underline{s} = \text{const}$ wie in 7.3(1). □

The equations (IV7.26) for the perturbation are like the Stokes-Fourier equation and therefore the wave will dissipate.

8 vr-Vortices

We consider in this section simple vortices, that are those solutions of the incompressible Navier-Stokes equations that have a singularity in an one-dimensional curve. Or even simpler, we consider situations in which the core of the singularity is located on a straight line, i.e. we consider flows around an axis, which we take as the x_3-axis. In this case the velocity close to the axis is very large. Therefore the flow should be described by a compressible fluid (see e.g. Bershader [25] where also the connection to the Rankine vortex 4.7 is discussed). But we will not do this here, we rather stay with an incompressible approximation having a singularity.

Fig. 24: From [Wikipedia: Tornado]: A tornado recorded from an aircraft.

Here we will first consider the flow between two cylinders (see 8.1) and let the radius of the inner cylinder go to zero. Then we will present an explicit solution (see 8.4), which describes a hurricane or a typhoon (*de:* Taifun). This hurricane is realistic near the point of impact.

In the first example we consider two concentric cylinders Z_R and $Z_{R'}$ where

$$Z_r = \{x \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 < r^2\} \text{ for } r > 0\,,$$

and in $Z_R \setminus \overline{Z_{R'}}$ consider a stationary liquid which is modeled by the incompressible Navier-Stokes equations with constant viscosity coefficients and with force $\mathbf{f} = 0$ (see (IV3.7)). Hence we describe the flow for an observer which himself considers as an inertial frame. Hence gravity is neglected.

8.1 Couette flow. In the space between the cylinders we have a stationary velocity field of the form

$$v(x) = v_\vartheta(r)\mathbf{e}_\vartheta\,, \quad r = \sqrt{x_1^2 + x_2^2}\,,$$
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r\cos\vartheta \\ r\sin\vartheta \\ x_3 \end{bmatrix}, \quad \mathbf{e}_\vartheta := \begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{bmatrix}$$

and we show the following:

(1) It is (v,p), with v as above, a solution of the stationary incompressible Navier-Stokes equation (as just defined) if

$$v_\vartheta(r) = Ar + \frac{B}{r}\,, \quad A, B \in \mathbb{R},$$
$$\partial_r p(r) = \frac{\varrho_0}{r}|v_\vartheta(r)|^2\,, \quad p = p(r)\,.$$

(2) If the fluid has Dirichlet boundary conditions, that is, it rotates at ∂Z_R with angular velocity ω, and at $\partial Z_{R'}$ with angular velocity ω', then

$$A = \frac{\omega R^2 - \omega' R'^2}{R^2 - R'^2}\,, \quad B = \frac{\omega' - \omega}{\frac{1}{R'^2} - \frac{1}{R^2}} = \frac{R^2 R'^2(\omega' - \omega)}{R^2 - R'^2}\,.$$

Referenzen: (See Hutter [8, 4.3.1 Couette Viskosimeter] and Hutter & Wang [9, 7.3.1 Couette Viscometer]). See also [Wikipedia: Taylor-Couette flow].

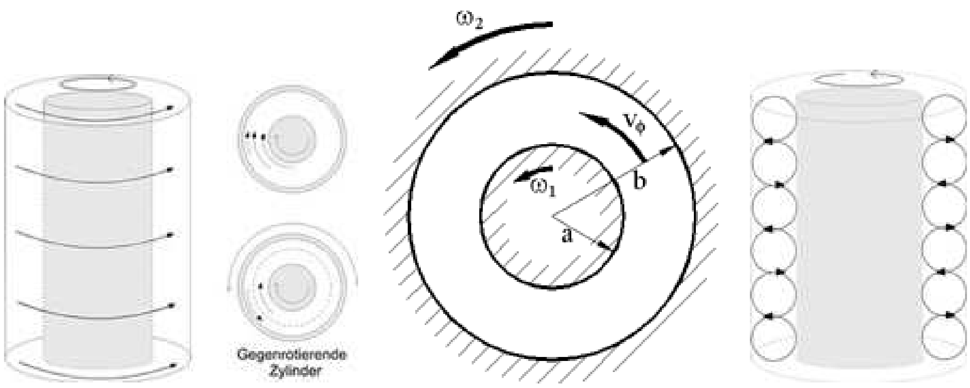


Fig. 25: *Center:* $\omega_1 = \omega'$, $\omega_2 = \omega$, $a = R'$, $b = R$. *Left:* Couette flow (see [Wikipedia: Couette flow]). *Right:* Not considered here the Taylor-Couette flow (see [Wikipedia: Taylor-Couette flow]).

In Fig. 25 the left image shows a flow as it is treated here, see also the center of the image. For large values of $|\omega - \omega'|$ the flow is indicated on the right image, a case which is not treated here.

Proof (1). The differential equations which have to be solved are (IV3.7), in our case these are $\operatorname{div} v = 0$ and

$$\varrho_0 (v \bullet \nabla) v + \nabla p - a \Delta v = 0 \tag{IV8.1}$$

with $a = \text{const} > 0$. For the unit vectors

$$\mathbf{e}_r := \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{bmatrix}, \quad \mathbf{e}_\vartheta := \begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{bmatrix},$$

if they are considered as functions of x, it is (see I.1.5)

$$\begin{aligned} \partial_{\mathbf{e}_r} \mathbf{e}_r &= 0\,, \quad \partial_{\mathbf{e}_\vartheta} \mathbf{e}_r = \frac{1}{r} \mathbf{e}_\vartheta\,, \\ \partial_{\mathbf{e}_r} \mathbf{e}_\vartheta &= 0\,, \quad \partial_{\mathbf{e}_\vartheta} \mathbf{e}_\vartheta = -\frac{1}{r} \mathbf{e}_r\,, \end{aligned}$$

or, if they are considered as functions of (r, ϑ, x_3),

$$\begin{aligned} \partial_r \mathbf{e}_r &= 0\,, \quad \partial_\vartheta \mathbf{e}_r = \mathbf{e}_\vartheta\,, \\ \partial_r \mathbf{e}_\vartheta &= 0\,, \quad \partial_\vartheta \mathbf{e}_\vartheta = -\mathbf{e}_r\,. \end{aligned}$$

Thus our velocity has the divergence

$$\begin{aligned} \operatorname{div} v &= \mathbf{e}_r \bullet \partial_{\mathbf{e}_r} v + \mathbf{e}_\vartheta \bullet \partial_{\mathbf{e}_\vartheta} v \\ &= \mathbf{e}_r \bullet \partial_r v + \frac{1}{r} \mathbf{e}_\vartheta \bullet \partial_\vartheta v = v_\vartheta' \mathbf{e}_r \bullet \mathbf{e}_\vartheta - \frac{v_\vartheta}{r} \mathbf{e}_\vartheta \bullet \mathbf{e}_r = 0\,, \end{aligned}$$

since $\mathbf{e}_r \bullet \mathbf{e}_\vartheta = 0$. The differential equation (IV8.1) we solve cosecutively for v and then for p, that is specially, we write

$$\begin{aligned} \Delta v = 0\,, \quad v &= v_\vartheta \mathbf{e}_\vartheta\,, \\ \nabla p &= -\varrho_0 (v \bullet \nabla) v\,. \end{aligned}$$

As we will see, the second equation can be solved for p because of the special solution v of the first equation, therefore this decomposition is consistent with the one in 3.6. The Laplace operator of v is

$$\begin{aligned} \Delta v &= \partial_r^2 v + \frac{1}{r} \partial_r v + \frac{1}{r^2} \partial_\vartheta^2 v \\ &= v_\vartheta'' \mathbf{e}_\vartheta + \frac{1}{r} v_\vartheta' \mathbf{e}_\vartheta + \frac{v_\vartheta}{r^2} \underbrace{\partial_\vartheta \underbrace{\partial_\vartheta \mathbf{e}_\vartheta}_{= -\mathbf{e}_r}}_{= -\mathbf{e}_\vartheta} \\ &= \big(v_\vartheta'' + \frac{1}{r} v_\vartheta' - \frac{1}{r^2} v_\vartheta \big) \mathbf{e}_\vartheta\,, \end{aligned}$$

and the general solution of

$$v_\vartheta'' + \frac{1}{r}v_\vartheta' - \frac{1}{r^2}v_\vartheta = 0$$

is

$$v_\vartheta(r) = Ar + \frac{B}{r}\,.$$

The second equation for the pressure gives

$$\begin{aligned}\nabla p &= -\varrho_0(v\bullet\nabla)v = -\varrho_0 v_\vartheta(\mathbf{e}_\vartheta\bullet\nabla)(v_\vartheta\mathbf{e}_\vartheta)\\ &= -\varrho_0 v_\vartheta^2\partial_{\mathbf{e}_\vartheta}\mathbf{e}_\vartheta = \frac{\varrho_0}{r}v_\vartheta^2\mathbf{e}_r\,,\end{aligned}$$

and therefore

$$p = p(r)\,,\quad \partial_r p(r) = \frac{\varrho_0}{r}|v_\vartheta(r)|^2\,.$$

Thus, the statements are proved. □

Proof (2). The assumption is that we have the following Dirichlet conditions

$$v_\vartheta(R) = \omega R\,,\quad v_\vartheta(R') = \omega' R'\,.$$

Plugging this into (1) we obtain

$$AR + \frac{B}{R} = \omega R\,,\quad AR' + \frac{B}{R'} = \omega' R'\,,$$

what is equivalent to the above formulas for A and B. □

When the inner cylinder $Z_{R'}$ degenerates, that is, if $R' \to 0$ and ω' stays bounded, then the coefficient A converges to ω while B converges to 0. In this case a linear velocity v_ϑ is left, that is,

$$v_\vartheta = \omega r\,.$$

But if the angular speed goes to infinity, $\omega' \to \infty$ as $R' \to 0$, lets say we have

$$R'^2\omega' \to C \text{ with } C \in \mathbb{R}\,,$$

then it follows

$$\begin{aligned} B &= \frac{1}{1-\frac{R'^2}{R^2}}\big(R'^2\omega' - R'^2\omega\big) \to C \\ A &= \frac{1}{1-\frac{R'^2}{R^2}}\big(\omega - \frac{\omega' R'^2}{R^2}\big) \to \omega - \frac{C}{R^2} \end{aligned} \qquad \text{for } R' \to 0,$$

and the solution (v,p) converges to the ***free vr-vortex***

$$\boxed{\begin{aligned} v_\vartheta(r) &= (\omega - \frac{C}{R^2})r + \frac{C}{r}\,, \\ \partial_r p(r) &= \frac{\varrho_0 |v_\vartheta(r)|^2}{r}\,. \end{aligned}} \tag{IV8.2}$$

Flows with this property are referred to as "vr-vortex", that is, the velocity v goes to infinity towards the singularity, while obtaining a bounded vector field, if one multiplies the velocity v with the distance r from the singularity. Thus $v_\vartheta(r)\cdot r$ remains bounded. The vortex in 8.1 in this limit is then called "free vr-vortex". This limit can also be obtained by writing the equations for a system, that consists of the fluid and a rigid body $Z_{R'}$ (see the Rankine vortex 4.7 and the paper of Giaiotti & Stel [42]). If this system rotates rapidly with angular velocity

$$\omega' = \frac{C + \mathcal{O}(1)}{R'^2},$$

then one proceeds to the limit $R' \to 0$.

One can show the following:

8.2 Theorem. For the solution of the free vr-vortex in (IV8.2) is in $\mathscr{D}'(\mathbb{R}^3)$

$$\operatorname{div}\big(v\mu_{\mathbb{R}^3\setminus W}\big) = 0\,,$$
$$\operatorname{div}\big(\lim_{\varepsilon\searrow 0} M(v)\mu_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)} - N\mu_W\big) = 0\,,$$

where the limit in $\mathscr{D}'(\mathbb{R}^3)$ exists. Here

$$M(v) := \varrho_0 v v^{\mathrm{T}} + p\mathrm{Id} - 2a\,(\mathrm{D}v)^{\mathrm{S}} \ \text{ in } \mathbb{R}^3 \setminus W\,,$$
$$N := \frac{C\pi}{2}\begin{bmatrix} C\varrho_0 & 4a & 0 \\ -4a & C\varrho_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\langle\, \eta\,,\, \mu_W\,\rangle_{\mathscr{D}(\mathbb{R}^3)} := \int_{\mathbb{R}} \eta(0,0,x_3)\,\mathrm{dL}^1(x_3)$$

with the singularity set $W := \{(0,0,x_3)\,;\ x_3 \in \mathbb{R}\}$. We denote by $\mu_{\mathbb{R}^3\setminus W}$, respectively $\mu_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)}$, the Lebesque measure L^3 on $\mathbb{R}^3 \setminus W$, respectively $\mathbb{R}^3 \setminus \mathrm{B}_\varepsilon(W)$, where $\mathrm{L}^3(W) = 0$. The measure μ_W denotes the Hausdorff measure H^1 on W. Pay heed to the singularity of v.

Observe: The limit "$\lim_{\varepsilon\searrow 0}$" in the assertion stands after the divergence operator, i.e. "$\operatorname{div}\lim_{\varepsilon\searrow 0}$", although in the proof first it is considered before the divergence operator, i.e. "$\lim_{\varepsilon\searrow 0}\operatorname{div}$". Therefore the part "Existence of the limit" of the proof is necessary. *Definition:* The specified functions in $\mathbb{R}^3 \setminus W$ are not defined in the singularity W. The derivative $\mathrm{D}v$ is only defined in the open set $\mathbb{R}^3 \setminus W$.

The nontrivial part of the matrix N acts in the horizontal directions. This part is

$$\begin{bmatrix} C\varrho_0 & 4a \\ -4a & C\varrho_0 \end{bmatrix} = C\varrho_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It consists of an expansion of order $C\varrho_0$ and an antisymmetric part of order $4a$. The theorem states that the mass and momentum conservation are satisfied in the whole space, and that in the distributional sense, where the described Cauchy limit is a distribution (to this Cauchy principal value see [71, 6 Cauchy's principal value] and from classical literature [79, (for $n = 1$)]). It should be noted that here we rely on the original form of momentum conservation, that is, we use the term $\mathrm{div}S$ with the stress tensor $S = a\,(\mathrm{D}v)^{\mathrm{S}}$ and not $a\Delta v$. Outside the singularity W, the solution fulfills the stationary incompressible Navier-Stokes equations

$$\mathrm{div}v = 0\,,$$
$$\mathrm{div}M(v) = 0\,.$$

Incidentally, it is an open problem to approximate the vortex by solutions of the compressible Navier-Stokes equations.

Proof (Conservation of mass). Let $\widetilde{A} := \omega - \frac{C}{R^2}$. Then

$$v = v_\vartheta \mathbf{e}_\vartheta = \frac{v_\vartheta}{r} \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} = \Big(\widetilde{A} + \frac{C}{r^2}\Big) \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}, \tag{IV8.3}$$

which is of order $\frac{1}{r}$ near W, and thus v is integrable over $\mathbb{R}^3$. It holds $\mathrm{div}v = 0$ in $\mathbb{R}^3 \setminus W$ and for $\zeta \in \mathscr{D}(\mathbb{R}^3;\mathbb{R})$

$$\begin{aligned}
&\big\langle\, \zeta\,,\ \mathrm{div}(v\mu_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)})\,\big\rangle_{\mathscr{D}(\mathbb{R}^3)} = -\big\langle\, \nabla\zeta\,,\ v\mu_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)}\,\big\rangle_{\mathscr{D}(\mathbb{R}^3)} \\
&= -\int_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)} \nabla\zeta \bullet v \,\mathrm{dL}^3 = \int_{\mathbb{R}} \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta\, \nu_{\mathrm{B}_\varepsilon(0)} \bullet v \,\mathrm{dH}^1\,\mathrm{dL}^1 \\
&= \int_{\mathbb{R}} \Big(\widetilde{A}\varepsilon + \frac{C}{\varepsilon}\Big) \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta\ \underbrace{\underbrace{\nu_{\mathrm{B}_\varepsilon(0)}}_{=\mathbf{e}_r} \bullet \mathbf{e}_\vartheta}_{=0}\ \mathrm{dH}^1\,\mathrm{dL}^1 = 0\,,
\end{aligned}$$

thus also $\big\langle\, \zeta\,,\ \mathrm{div}(v\mu_{\mathbb{R}^3\setminus W})\,\big\rangle_{\mathscr{D}(\mathbb{R}^3)} = 0$. □

Proof (Conservation of momentum). The solution of (IV8.3) fulfills in $\mathbb{R}^3 \setminus W$

$$\begin{aligned} \mathrm{D}v &= \mathrm{D}\Big((\widetilde{A} + \frac{C}{r^2}) \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \Big) \\ &= (\widetilde{A} + \frac{C}{r^2}) \mathrm{D}\Big(\begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \Big) + \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \mathrm{D}\Big(\frac{C}{r^2} \Big) \\ &= (\widetilde{A} + \frac{C}{r^2}) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{2C}{r^4} \underbrace{\begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} [\, x_1 \quad x_2 \quad 0 \,]}_{= r^2 \mathbf{e}_\vartheta \mathbf{e}_r{}^{\mathrm{T}}} \,, \end{aligned}$$

and hence ($\mathbf{e}_r$, $\mathbf{e}_\vartheta$ are defined in the proof of 8.1)

$$(\mathrm{D}v)^{\mathrm{S}} = \frac{1}{2} (\mathrm{D}v + (\mathrm{D}v)^{\mathrm{T}}) = -\frac{C}{r^2} (\mathbf{e}_\vartheta \mathbf{e}_r{}^{\mathrm{T}} + \mathbf{e}_r \mathbf{e}_\vartheta{}^{\mathrm{T}}) \,.$$

Then in $\mathbb{R}^3 \setminus W$

$$\begin{aligned} M(v) &:= \varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - 2a\, (\mathrm{D}v)^{\mathrm{S}} \\ &= \varrho_0 v_\vartheta^2 \mathbf{e}_\vartheta \mathbf{e}_\vartheta{}^{\mathrm{T}} + p\mathrm{Id} + \tfrac{2aC}{r^2} (\mathbf{e}_\vartheta \mathbf{e}_r{}^{\mathrm{T}} + \mathbf{e}_r \mathbf{e}_\vartheta{}^{\mathrm{T}}) \,, \end{aligned} \tag{IV8.4}$$

and therewith

$$\begin{aligned} &\mathrm{div} M(v) = (\mathbf{e}_r \bullet \partial_{\mathbf{e}_r} + \mathbf{e}_\vartheta \bullet \partial_{\mathbf{e}_\vartheta}) M(v) \\ &= (\partial_{\mathbf{e}_r} M(v))\, \mathbf{e}_r + (\partial_{\mathbf{e}_\vartheta} M(v))\, \mathbf{e}_\vartheta \\ &= \varrho_0 \partial_r (v_\vartheta^2) \underbrace{(\mathbf{e}_\vartheta \mathbf{e}_\vartheta{}^{\mathrm{T}})\, \mathbf{e}_r}_{= 0} + \varrho_0 v_\vartheta^2 \partial_{\mathbf{e}_\vartheta} (\mathbf{e}_\vartheta \mathbf{e}_\vartheta{}^{\mathrm{T}})\, \mathbf{e}_\vartheta + \partial_r p(r)\, \mathbf{e}_r \\ &\quad \underbrace{+ \partial_r \Big(\frac{2aC}{r^2} \Big) (\mathbf{e}_\vartheta \mathbf{e}_r{}^{\mathrm{T}} + \mathbf{e}_r \mathbf{e}_\vartheta{}^{\mathrm{T}})\, \mathbf{e}_r + \frac{2aC}{r^2} \partial_{\mathbf{e}_\vartheta} (\mathbf{e}_\vartheta \mathbf{e}_r{}^{\mathrm{T}} + \mathbf{e}_r \mathbf{e}_\vartheta{}^{\mathrm{T}})\, \mathbf{e}_\vartheta}_{= \frac{2aC}{r^2} \big(-\frac{2}{r} \mathbf{e}_\vartheta + \mathbf{e}_\vartheta \bullet (\partial_{\mathbf{e}_\vartheta} \mathbf{e}_r) \mathbf{e}_\vartheta + \partial_{\mathbf{e}_\vartheta} \mathbf{e}_r \big) = 0} \\ &= \Big(-\frac{\varrho_0 v_\vartheta^2}{r} + \partial_r p(r) \Big)\, \mathbf{e}_r = 0 \,, \end{aligned}$$

since p is a solution of (IV8.2). Now for $\zeta \in \mathscr{D}(\mathbb{R}^3; \mathbb{R}^3)$ it holds

$$\begin{aligned} &\big\langle \zeta \,,\ \mathrm{div}(M(v) \mu_{\mathbb{R}^3 \setminus \mathrm{B}_\varepsilon(W)}) \big\rangle_{\mathscr{D}(\mathbb{R}^3)} = -\big\langle \mathrm{D}\zeta \,,\ M(v) \mu_{\mathbb{R}^3 \setminus \mathrm{B}_\varepsilon(W)}) \big\rangle_{\mathscr{D}(\mathbb{R}^3)} \\ &= -\int_{\mathbb{R}^3 \setminus \mathrm{B}_\varepsilon(W)} \sum_{ij} \partial_j \zeta_i\, M_{ij}(v)\, \mathrm{dL}^3 \\ &= \sum_{ij} \int_{\mathbb{R}} \int_{\partial \mathrm{B}_\varepsilon(0)} (\nu_{\mathrm{B}_\varepsilon(0)} \bullet \mathbf{e}_j) \zeta_i\, M_{ij}(v)\, \mathrm{dH}^1\, \mathrm{dL}^1 \\ &= \int_{\mathbb{R}} \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta \bullet (M(v) \mathbf{e}_r)\, \mathrm{dH}^1\, \mathrm{dL}^1 \,. \end{aligned}$$

Since from (IV8.4)

$$M(v)\mathbf{e}_r = p\mathbf{e}_r + \frac{2aC}{r^2}\mathbf{e}_\vartheta$$

this is

$$= \int_{\mathbb{R}} \int_{\partial B_\varepsilon(0)} \zeta \bullet \Big(p(\varepsilon)\mathbf{e}_r + \frac{2aC}{\varepsilon^2}\mathbf{e}_\vartheta \Big)\, d\mathrm{H}^1\, d\mathrm{L}^1 = (*)\,.$$

Now obviously ($\mathbf{e}_r$ and $\mathbf{e}_\vartheta$ only depend on $\mathrm{e}^{\mathrm{i}\vartheta}$)

$$\int_{\partial B_\varepsilon(0)} \Big(p(\varepsilon)\mathbf{e}_r + \frac{2aC}{\varepsilon^2}\mathbf{e}_\vartheta \Big)\, d\mathrm{H}^1 = 0\,,$$

and since

$$\begin{aligned} \zeta(x_1,x_2,x_3) &= \zeta(0,x_3) + \sum_{j=1,2} x_j \partial_j \zeta(0,x_3) + \mathcal{O}(r^2) \\ &= \zeta(0,x_3) + r \sum_{j=1,2,3} (\mathbf{e}_r)_j \partial_j \zeta(0,x_3) + \mathcal{O}(r^2)\,, \end{aligned}$$

and since from (IV8.2)

$$\partial_r p(r) = \varrho_0 \widetilde{A}^2 r + \frac{2\varrho_0 \widetilde{A} C}{r} + \frac{\varrho_0 C^2}{r^3}\,,$$

which implies

$$p(r) = -\frac{\varrho_0 C^2}{2r^2} + \mathcal{O}(|\log r|) \text{ für } r \to 0\,, \tag{IV8.5}$$

this gives for $\varepsilon \to 0$

$$\begin{aligned} (*) &= \int_{\mathbb{R}} \frac{1}{\varepsilon} \int_{\partial B_\varepsilon(0)} \frac{\zeta(x_1,x_2,x_3) - \zeta(0,x_3)}{\varepsilon} \bullet \Big(\varepsilon^2 p(\varepsilon)\mathbf{e}_r + 2aC\mathbf{e}_\vartheta \Big) \\ &\qquad\qquad d\mathrm{H}^1(x_1,x_2)\, d\mathrm{L}^1(x_3) \\ &= \sum_{ij} \int_{\mathbb{R}} \partial_j \zeta_i(0,x_3) \frac{1}{\varepsilon} \int_{\partial B_\varepsilon(0)} (\mathbf{e}_r)_j \Big(\varepsilon^2 p(\varepsilon)\mathbf{e}_r + 2aC\mathbf{e}_\vartheta \Big)_i d\mathrm{H}^1\, d\mathrm{L}^1(x_3) + o(1) \\ &= -\int_{\mathbb{R}} D\zeta(0,x_3) \,\mathbf{:}\, \underbrace{\int_{\partial B_1(0)} \big(\frac{\varrho_0 C^2}{2} \mathbf{e}_r {\mathbf{e}_r}^{\mathrm{T}} - 2aC \mathbf{e}_\vartheta {\mathbf{e}_r}^{\mathrm{T}} \big)\, d\mathrm{H}^1}_{=:\,N}\; d\mathrm{L}^1(x_3) + o(1)\,, \end{aligned}$$

since $\varepsilon^2 \log\varepsilon \to 0$. In the limit $\varepsilon \to 0$ it follows

$$\lim_{\varepsilon \searrow 0} \big\langle D\zeta\,,\, M(v)\mu_{\mathbb{R}^3 \setminus B_\varepsilon(W)} \big\rangle_{\mathscr{D}(\mathbb{R}^3)} = \big\langle D\zeta\,,\, N\mu_W \big\rangle_{\mathscr{D}(\mathbb{R}^3)}\,.$$

The claimed representation of N follows if one uses

$$\int_{\partial B_1(0)} \mathbf{e}_r {\mathbf{e}_r}^{\mathrm{T}}\, d\mathrm{H}^1 = \pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \int_{\partial B_1(0)} \mathbf{e}_\vartheta {\mathbf{e}_r}^{\mathrm{T}}\, d\mathrm{H}^1 = \pi \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

Proof (Existence of the limit). We still have to show that

$$\lim_{\varepsilon\searrow 0}\left\langle\, \Xi\,,\ M(v)\mu_{\mathbb{R}^3\setminus \mathrm{B}_\varepsilon(W)}\,\right\rangle_{\mathscr{D}(\mathbb{R}^3)}$$

exists for all $\Xi\in\mathscr{D}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$. (So far, this is only shown for $\Xi=\mathrm{D}\zeta$ with $\zeta\in\mathscr{D}(\mathbb{R}^3;\mathbb{R}^3)$.) We let $\Xi\in\mathscr{D}(\mathbb{R}\times \mathrm{B}_1(0)\,;\mathbb{R}^{3\times 3})$ because we only have to consider parts which are non-integrable in the limit. It is

$$\begin{aligned}
M(v) &= \varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - 2a\,(\mathrm{D}v)^{\mathrm{S}}\\
&= \underbrace{\frac{\varrho_0 C^2}{r^2}\mathbf{e}_\vartheta\,{\mathbf{e}_\vartheta}^{\mathrm{T}} - \frac{\varrho_0 C^2}{2r^2}\mathrm{Id} + \frac{2aC}{r^2}(\mathbf{e}_\vartheta\,{\mathbf{e}_r}^{\mathrm{T}} + \mathbf{e}_r\,{\mathbf{e}_\vartheta}^{\mathrm{T}})}_{=:\,\widetilde{M}(r,\vartheta)} + \mathcal{O}(|\log r|)\,.
\end{aligned}$$

All three terms of $\widetilde{M}$ are not integrable near W, and we decompose

$$\begin{aligned}
&\widetilde{M}(r,\vartheta) = M^0(r,\vartheta) + M^1(r,\vartheta)\,,\\
&M^0(r,\vartheta) := \frac{\varrho_0 C^2}{r^2}\begin{bmatrix}\sin^2\vartheta & -\sin\vartheta\cos\vartheta & 0\\ -\sin\vartheta\cos\vartheta & \cos^2\vartheta & 0\\ 0 & 0 & 0\end{bmatrix} - \frac{\varrho_0 C^2}{2r^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\\
&\qquad\qquad + \frac{2aC}{r^2}\begin{bmatrix}-2\sin\vartheta\cos\vartheta & \cos^2\vartheta-\sin^2\vartheta & 0\\ \cos^2\vartheta-\sin^2\vartheta & -2\sin\vartheta\cos\vartheta & 0\\ 0&0&0\end{bmatrix}\,,\\
&M^1(r,\vartheta) := -\frac{\varrho_0 C^2}{2r^2}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\,.
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\int_{\mathrm{B}_1(0)\setminus \mathrm{B}_\varepsilon(0)} M^0\,\mathrm{dL}^2\\
&= \int_\varepsilon^1\int_0^{2\pi}\Bigg(\frac{\varrho_0 C^2}{r}\begin{bmatrix}\sin^2\vartheta & -\sin\vartheta\cos\vartheta & 0\\ -\sin\vartheta\cos\vartheta & \cos^2\vartheta & 0\\ 0 & 0 & 0\end{bmatrix} - \frac{\varrho_0 C^2}{2r}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\\
&\qquad + \frac{2aC}{r}\begin{bmatrix}-2\sin\vartheta\cos\vartheta & \cos^2\vartheta-\sin^2\vartheta & 0\\ \cos^2\vartheta-\sin^2\vartheta & -2\sin\vartheta\cos\vartheta & 0\\ 0&0&0\end{bmatrix}\Bigg)\,\mathrm{d}\vartheta\,\mathrm{d}r\\
&= 2\pi\int_\varepsilon^1\Bigg(\frac{\varrho_0 C^2}{r}\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&0\end{bmatrix} - \frac{\varrho_0 C^2}{2r}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\\
&\qquad + \frac{2aC}{r}\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Bigg)\,\mathrm{d}r\\
&= 0\,.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\int_{\mathbb{R}\times(\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0))} \Xi\!:\!M^0\,\mathrm{dL}^3\\
&= \int_{\mathbb{R}}\int_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)} (\Xi(x_1,x_2,x_3)-\Xi(0,0,x_3))\!:\!M^0\,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3\\
&= \int_{\mathbb{R}}\int_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)}\int_0^1 \partial_{\mathbf{e}_r}\Xi(sx_1,sx_2,x_3)\,\mathrm{d}s\!:\!(rM^0)\,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3
\end{aligned}$$

where $rM_0 = \mathcal{O}(\frac{1}{r})$ is integrable over $\mathbb{R}\times\mathrm{B}_1(0)$, that is,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}\times(\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0))} \Xi\!:\!M^0\,\mathrm{dL}^3$$

exists. To deal with the term

$$\int_{\mathbb{R}\times(\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0))} \Xi\!:\!M^1\,\mathrm{dL}^3 = -\frac{\varrho_0 C^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^2} \Xi_{33}\mathcal{X}_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)}\frac{1}{r^2}\,\mathrm{dL}^2\,\mathrm{dL}^1$$

we choose $\varphi_\varepsilon:\mathbb{R}^2\to\mathbb{R}$ as the solution of

$$-\Delta\varphi_\varepsilon = \frac{1}{r^2}\mathcal{X}_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)} \text{ in } \mathrm{B}_1(0)\ , \quad \varphi_\varepsilon = 0 \text{ on } \partial\mathrm{B}_1(0)\ ,$$

that is, $\varphi_\varepsilon \to \varphi$ (it is $\varphi_\varepsilon = \varphi$ on $\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)$) where

$$\varphi = \widehat{\varphi}(r) = \int_r^1 \frac{|\log r'|}{r'}\,\mathrm{d}r'$$

is an integrable function in $\mathrm{B}_1(0)$. From this we conclude

$$\begin{aligned}
\int_{\mathbb{R}^2} \Xi_{33}\mathcal{X}_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)}\frac{1}{r^2}\,\mathrm{dL}^2 &= -\int_{\mathrm{B}_1(0)} \Xi_{33}\Delta\varphi_\varepsilon\,\mathrm{dL}^2\\
= -\int_{\mathrm{B}_1(0)} \Delta\Xi_{33}\cdot\varphi_\varepsilon\,\mathrm{dL}^2 &\longrightarrow -\int_{\mathrm{B}_1(0)} \Delta\Xi_{33}\cdot\varphi\,\mathrm{dL}^2\ ,
\end{aligned}$$

so that also

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}\times(\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0))} \Xi\!:\!M^1\,\mathrm{dL}^3$$

exists. □

The Swirling Vortex

We now give an example of a vortex touching the ground. At the ground the boundary condition $v = 0$ is required. The gravitational term is (justifiably) neglected, as well as the centrifugal forces with respect to an observer in the center. Note that we now use polar coordinates in $\mathbb{R}^3$.[7] which are defined

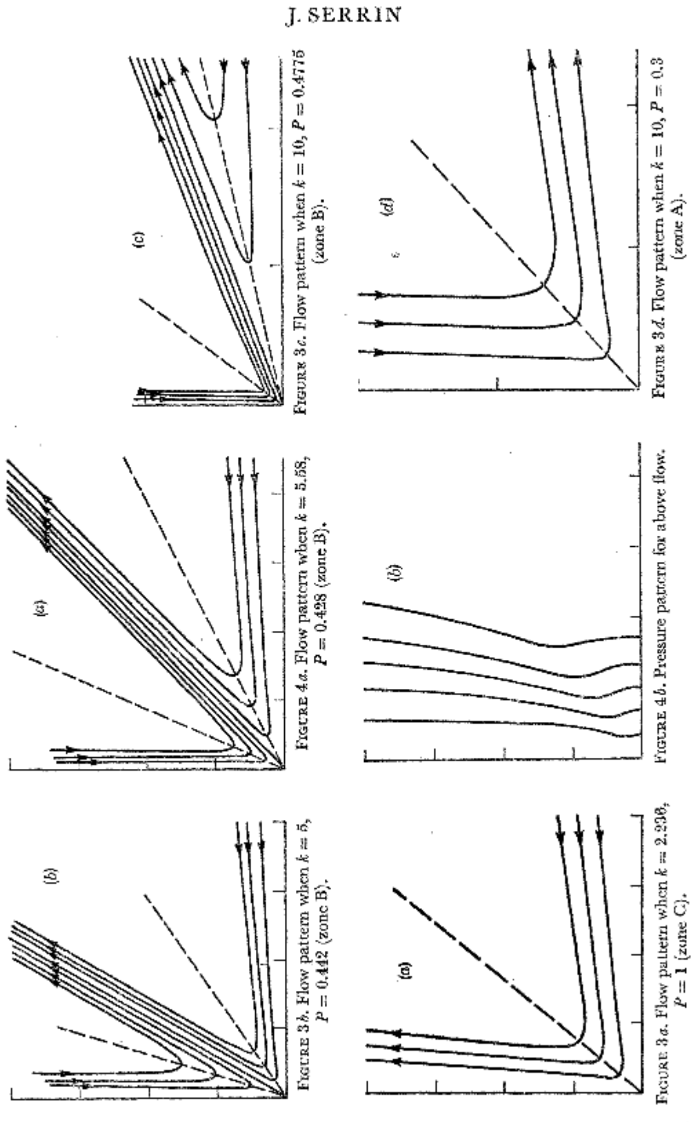


Fig. 26: Page from Serrin [63]

by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \tau \begin{bmatrix} \widetilde{r} \\ \alpha \\ \vartheta \end{bmatrix} = \begin{bmatrix} \widetilde{r}\sin\alpha\cos\vartheta \\ \widetilde{r}\sin\alpha\sin\vartheta \\ \widetilde{r}\cos\alpha \end{bmatrix},$$

$$\widetilde{r} > 0, \quad 0 < \alpha < \tfrac{\pi}{2}, \quad \vartheta \in \mathbb{R}, \quad \text{and} \tag{IV8.6}$$

$$\mathbf{e}_{\widetilde{r}} = \begin{bmatrix} \sin\alpha\cos\vartheta \\ \sin\alpha\sin\vartheta \\ \cos\alpha \end{bmatrix}, \quad \mathbf{e}_\alpha = \begin{bmatrix} \cos\alpha\cos\vartheta \\ \cos\alpha\sin\vartheta \\ -\sin\alpha \end{bmatrix}, \quad \mathbf{e}_\vartheta = \begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \\ 0 \end{bmatrix},$$

thus $\{\mathbf{e}_{\widetilde{r}}, \mathbf{e}_\alpha, \mathbf{e}_\vartheta\}$ is the corresponding orthonormal system with respect to these polar coordinates. The flow region is $\{x \in \mathbb{R}^3\,;\ x_3 > 0\}$. It is

$$\begin{aligned}
&\partial_{\widetilde{r}}\mathbf{e}_{\widetilde{r}} = 0\,, \quad \partial_\alpha\mathbf{e}_{\widetilde{r}} = \mathbf{e}_\alpha\,, \qquad \partial_\vartheta\mathbf{e}_{\widetilde{r}} = \sin\alpha\,\mathbf{e}_\vartheta\,,\\
&\partial_{\widetilde{r}}\mathbf{e}_\alpha = 0\,, \quad \partial_\alpha\mathbf{e}_\alpha = -\mathbf{e}_{\widetilde{r}}\,, \quad \partial_\vartheta\mathbf{e}_\alpha = \cos\alpha\,\mathbf{e}_\vartheta\,,\\
&\partial_{\widetilde{r}}\mathbf{e}_\vartheta = 0\,, \quad \partial_\alpha\mathbf{e}_\vartheta = 0\,, \qquad \partial_\vartheta\mathbf{e}_\vartheta = -\sin\alpha\,\mathbf{e}_{\widetilde{r}} - \cos\alpha\,\mathbf{e}_\alpha\,.
\end{aligned}$$

Define $\widetilde{v} := v\circ\tau$ and $\widetilde{p} := p\circ\tau$. Then (we use the basis vectors in both coordinate systems)

$$\partial_{\widetilde{r}}\widetilde{v} = (\partial_{\mathbf{e}_{\widetilde{r}}}v)\circ\tau\,, \quad \partial_\alpha\widetilde{v} = \widetilde{r}(\partial_{\mathbf{e}_\alpha}v)\circ\tau\,, \quad \partial_\vartheta\widetilde{v} = r(\partial_{\mathbf{e}_\vartheta}v)\circ\tau\,,$$

and similarly for p.

8.3 Laplace operator in polar coordinates in $\mathbb{R}^3$. It holds

$$\begin{aligned}
(\Delta v)\circ\tau \;&=\; \frac{1}{\widetilde{r}^2\sin\alpha}\Big(\partial_{\widetilde{r}}\big(\widetilde{r}^2\sin\alpha\,\partial_{\widetilde{r}}\widetilde{v}\big) + \partial_\alpha\big(\sin\alpha\,\partial_\alpha\widetilde{v}\big) + \partial_\vartheta\big(\frac{1}{\sin\alpha}\partial_\vartheta\widetilde{v}\big)\Big)\\
&=\; \partial_{\widetilde{r}}^2\widetilde{v} + \frac{2}{\widetilde{r}}\partial_{\widetilde{r}}\widetilde{v} + \frac{1}{\widetilde{r}^2}\partial_\alpha^2\widetilde{v} + \frac{\cot\alpha}{\widetilde{r}^2}\partial_\alpha\widetilde{v} + \frac{1}{\widetilde{r}^2\sin\alpha}\partial_\vartheta^2\widetilde{v}\,.
\end{aligned}$$

Proof. Here for $l = 1, 2, 3$ the equation in (I5.11) with $Z = 1$ and $N = 1$ is

$$\operatorname{div} q = \mathbf{r} \qquad \text{with } u = 0,\ q = -\nabla v_l,\ \mathbf{r} = -\Delta v_l,$$

and the transformation $x = \tau(x^*)$ where $x^* = (\widetilde{r}, \alpha, \vartheta)$. The invariance of the equation gives

$$\operatorname{div} q^* = \mathbf{r}^* \qquad \text{with } u^* = 0,$$

$$\text{if} \quad q\circ\tau = \frac{1}{J}\mathrm{D}\tau\, q^*\,, \quad \mathbf{r}\circ\tau = \frac{1}{J}\mathbf{r}^*\,, \quad J = \det \mathrm{D}\tau\,.$$

Now $v_l\circ\tau = \widetilde{v}_l$ and this implies

$$\begin{aligned}
&\partial_j\widetilde{v}_l = \sum_i \tau_{i'j}(\partial_i v_l)\circ\tau = -\sum_i \tau_{i'j}q_i\circ\tau = -\frac{1}{J}\sum_{ik}\tau_{i'j}\tau_{i'k}q_k^*\\
&= -\frac{1}{J}\sum_k c_{jk}q_k^* \text{ with } c_{jk} := \sum_i \tau_{i'j}\tau_{i'k} = \tau_{'j}\bullet\tau_{'k}\,.
\end{aligned}$$

[7] also named “spherical coordinates”

Then, if

$$\tau'_j \bullet \tau'_k = \lambda_j \delta_{jk} \,, \qquad \text{(IV8.7)}$$

this implies $-q^*_j = \frac{J}{\lambda_j}\partial_j \widetilde{v}_l$, hence the differential equation

$$J(\Delta v_l)\circ\tau = -J\,\mathbf{r}\circ\tau = -\mathbf{r}^* = -\operatorname{div} q^* = \sum_j \partial_j\big(\frac{J}{\lambda_j}\partial_j\widetilde{v}_l\big)$$

$$= \partial_{\widetilde{r}}\big(\frac{J}{\lambda_1}\partial_{\widetilde{r}}\widetilde{v}_l\big) + \partial_\alpha\big(\frac{J}{\lambda_2}\partial_\alpha\widetilde{v}_l\big) + \partial_\vartheta\big(\frac{J}{\lambda_3}\partial_\vartheta\widetilde{v}_l\big)\,.$$

Here $J = \det \mathrm{D}\tau = \widetilde{r}^2 \sin\alpha > 0$ and (IV8.7) is satisfied with

$$\lambda_1 = |\tau'_1|^2 = 1\,, \quad \lambda_2 = |\tau'_2|^2 = \widetilde{r}^2\,, \quad \lambda_3 = |\tau'_3|^2 = \widetilde{r}^2\sin^2\alpha\,.$$

(*Remark:* The classical alternative is a proof by direct computation.) □

8.4 J. Serrin: "The Swirling Vortex". We consider stationary solutions of the incompressible Navier-Stokes equation in

$$\{x \in \mathbb{R}^3\,;\; x_3 > 0 \text{ und } r = \sqrt{x_1^2 + x_2^2} > 0\}$$

of the form

$$v(x) = \frac{G(s)}{r}\mathbf{e}_{\widetilde{r}} + \frac{F(s)}{r}\mathbf{e}_\alpha + \frac{\Omega(s)}{r}\mathbf{e}_\vartheta$$

with polar coordinates $(\widetilde{r}, \alpha, \vartheta)$ (see (IV8.6)) and

$$r = \sqrt{x_1^2 + x_2^2} = \widetilde{r}\sin\alpha\,, \quad s = \cos\alpha\,.$$

Here F, G, and Ω are bounded functions. The following applies:

(1) For the gradient of the velocity

$$\partial_{\widetilde{r}} v = -\frac{G}{\widetilde{r}r}\mathbf{e}_{\widetilde{r}} - \frac{F}{\widetilde{r}r}\mathbf{e}_\alpha - \frac{\Omega}{\widetilde{r}r}\mathbf{e}_\vartheta\,,$$

$$\partial_\alpha v = \Big(\partial_\alpha\Big(\frac{G}{r}\Big) - \frac{F}{r}\Big)\mathbf{e}_{\widetilde{r}} + \Big(\partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r}\Big)\mathbf{e}_\alpha + \partial_\alpha\Big(\frac{\Omega}{r}\Big)\mathbf{e}_\vartheta\,,$$

$$\partial_\vartheta v = -\frac{\Omega\sin\alpha}{r}\mathbf{e}_{\widetilde{r}} - \frac{\Omega\cos\alpha}{r}\mathbf{e}_\alpha + \Big(\frac{G\sin\alpha}{r} + \frac{F\cos\alpha}{r}\Big)\mathbf{e}_\vartheta\,.$$

(2) The boundary condition $v = 0$ on $\{x \in \mathbb{R}^3\,;\; x_3 = 0,\, \widetilde{r} > 0\}$ is satisfied if

$$G(s) \to 0,\; F(s) \to 0,\; \Omega(s) \to 0 \text{ as } s \to 0.$$

(3) The mass conservation is satisfied in $\{x \in \mathbb{R}^3\,;\; r > 0,\, x_3 > 0\}$ if

$$G(\cos\alpha) = \sin\alpha \cdot F'(\cos\alpha).$$

(4) The incompressible Navier-Stokes equation in $\{x \in \mathbb{R}^3\,;\ r > 0,\, x_3 > 0\}$ is equivalent to

$$\begin{aligned}
FF'' + F'^2 + (F^2 + \Omega^2)\operatorname{cosec}^2\alpha &= \frac{\widetilde{r}^3}{\varrho_0}\partial_{\widetilde{r}}p - \frac{a}{\varrho_0}(F'''\sin^2\alpha - 2F''\cos\alpha)\,,\\
FF' + (F^2 + \Omega^2)\cot\alpha\cdot\operatorname{cosec}\alpha &= \frac{\widetilde{r}^2\sin\alpha}{\varrho_0}\partial_\alpha p + \frac{a}{\varrho_0}F''\sin^2\alpha\,,\\
F\Omega' &= \frac{\widetilde{r}^2}{\varrho_0}\partial_\vartheta p - \frac{a}{\varrho_0}\Omega''\sin^2\alpha\,.
\end{aligned}$$

Reference: See Serrin [63, Chap I. Formulation of the problem, 1. Basic equations]. See also the rest of the paper of [63] and the exercise in [21, "The swirling vortex"].

Proof (1). The definition of $\widetilde{v}$ is

$$\widetilde{v}(\widetilde{r},\alpha,\vartheta) = \frac{G(s)}{r}\mathbf{e}_{\widetilde{r}}(\alpha,\vartheta) + \frac{F(s)}{r}\mathbf{e}_\alpha(\alpha,\vartheta) + \frac{\Omega(s)}{r}\mathbf{e}_\vartheta(\vartheta)\,,$$
$$r = \widetilde{r}\sin\alpha\,,\quad s = \cos\alpha\,.$$

This implies

$$\begin{aligned}
\partial_{\widetilde{r}}\widetilde{v} &= -\frac{1}{\widetilde{r}}\widetilde{v}\,,\\
\partial_\alpha\widetilde{v} &= \partial_\alpha\Big(\frac{G}{r}\Big)\mathbf{e}_{\widetilde{r}} + \partial_\alpha\Big(\frac{F}{r}\Big)\mathbf{e}_\alpha + \partial_\alpha\Big(\frac{\Omega}{r}\Big)\mathbf{e}_\vartheta + \frac{G}{r}\partial_\alpha\mathbf{e}_{\widetilde{r}} + \frac{F}{r}\partial_\alpha\mathbf{e}_\alpha\,,\\
&= \Big(\partial_\alpha\Big(\frac{G}{r}\Big) - \frac{F}{r}\Big)\mathbf{e}_{\widetilde{r}} + \Big(\partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r}\Big)\mathbf{e}_\alpha + \partial_\alpha\Big(\frac{\Omega}{r}\Big)\mathbf{e}_\vartheta\,,\\
\partial_\vartheta\widetilde{v} &= \frac{G}{r}\partial_\vartheta\mathbf{e}_{\widetilde{r}} + \frac{F}{r}\partial_\vartheta\mathbf{e}_\alpha + \frac{\Omega}{r}\partial_\vartheta\mathbf{e}_\vartheta\,,\\
&= -\frac{\Omega\sin\alpha}{r}\mathbf{e}_{\widetilde{r}} - \frac{\Omega\cos\alpha}{r}\mathbf{e}_\alpha + \Big(\frac{G\sin\alpha}{r} + \frac{F\cos\alpha}{r}\Big)\mathbf{e}_\vartheta\,.
\end{aligned}$$

In the following proofs we do not write $\widetilde{v}$ and $\widetilde{p}$. □

Proof (2). Is obvious. □

Proof (3). We compute using (1)

$$\begin{aligned}
0 &= (\operatorname{div}v)\circ\tau = \big(\mathbf{e}_{\widetilde{r}}\bullet\partial_{\mathbf{e}_{\widetilde{r}}}v + \mathbf{e}_\alpha\bullet\partial_{\mathbf{e}_\alpha}v + \mathbf{e}_\vartheta\bullet\partial_{\mathbf{e}_\vartheta}v\big)\\
&= \mathbf{e}_{\widetilde{r}}\bullet\partial_{\widetilde{r}}v + \frac{1}{\widetilde{r}}\mathbf{e}_\alpha\bullet\partial_\alpha v + \frac{1}{r}\mathbf{e}_\vartheta\bullet\partial_\vartheta v\\
&= -\frac{G}{\widetilde{r}r} + \frac{1}{\widetilde{r}}\Big(\partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r}\Big) + \frac{1}{r}\Big(\frac{G\sin\alpha}{r} + \frac{F\cos\alpha}{r}\Big)\\
&= \frac{1}{\widetilde{r}^2}\Big(\partial_\alpha\Big(\frac{F}{\sin\alpha}\Big) + \frac{G}{\sin\alpha} + \frac{F\cos\alpha}{\sin^2\alpha}\Big) = \frac{1}{\widetilde{r}^2\sin\alpha}\Big(\partial_\alpha F + G\Big)\,,
\end{aligned}$$

which implies the assertion $\partial_\alpha F + G = 0$. □

Proof (4). With

$$M(v) := \varrho_0 v\, v^{\mathrm{T}} + p\mathrm{Id} - 2a\,(\mathrm{D}v)^{\mathrm{S}}$$

the stationary Navier-Stokes equation reads $\mathrm{div}v = 0$ and

$$0 = \mathrm{div}M = \varrho_0 (v\bullet\nabla)v + \nabla p - a\,\mathrm{div}\mathrm{D}v\,.$$

We compute the individual terms. With $p = \widetilde{p}(\widetilde{r},\alpha,\vartheta)$ it is

$$\begin{aligned}\nabla p &= \partial_{\mathbf{e}_{\widetilde{r}}} p\,\mathbf{e}_{\widetilde{r}} + \partial_{\mathbf{e}_\alpha} p\,\mathbf{e}_\alpha + \partial_{\mathbf{e}_\vartheta} p\,\mathbf{e}_\vartheta \\ &= \partial_{\widetilde{r}} p\,\mathbf{e}_{\widetilde{r}} + \tfrac{1}{\widetilde{r}}\partial_\alpha p\,\mathbf{e}_\alpha + \tfrac{1}{r}\partial_\vartheta p\,\mathbf{e}_\vartheta\,.\end{aligned} \tag{IV8.8}$$

Next we deal with the nonlinear term. Using (1) we get

$$\begin{aligned}&(v\bullet\nabla)v = v\bullet\mathbf{e}_{\widetilde{r}}\partial_{\mathbf{e}_{\widetilde{r}}} v + v\bullet\mathbf{e}_\alpha\partial_{\mathbf{e}_\alpha} v + v\bullet\mathbf{e}_\vartheta\partial\mathbf{e}_\vartheta v \\ &= v\bullet\mathbf{e}_{\widetilde{r}}\partial_{\widetilde{r}} v + \frac{v\bullet\mathbf{e}_\alpha}{\widetilde{r}}\partial_\alpha v + \frac{v\bullet\mathbf{e}_\vartheta}{r}\partial_\vartheta v \\ &= \frac{G}{r}\Big(-\frac{G}{\widetilde{r}r}\mathbf{e}_{\widetilde{r}} - \frac{F}{\widetilde{r}r}\mathbf{e}_\alpha - \frac{\Omega}{\widetilde{r}r}\mathbf{e}_\vartheta\Big) \\ &+\frac{F}{\widetilde{r}r}\Big(\Big(\partial_\alpha\Big(\frac{G}{r}\Big) - \frac{F}{r}\Big)\mathbf{e}_{\widetilde{r}} + \Big(\partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r}\Big)\mathbf{e}_\alpha + \partial_\alpha\Big(\frac{\Omega}{r}\Big)\mathbf{e}_\vartheta\Big) \\ &+\frac{\Omega}{r^2}\Big(-\frac{\Omega\sin\alpha}{r}\mathbf{e}_{\widetilde{r}} - \frac{\Omega\cos\alpha}{r}\mathbf{e}_\alpha + \Big(\frac{G\sin\alpha}{r} + \frac{F\cos\alpha}{r}\Big)\mathbf{e}_\vartheta\Big) \\ &= \Big(-\frac{G^2}{\widetilde{r}r^2} + \frac{F}{\widetilde{r}r}\Big(\partial_\alpha\Big(\frac{G}{r}\Big) - \frac{F}{r}\Big) - \frac{\Omega^2}{r^3}\sin\alpha\Big)\mathbf{e}_{\widetilde{r}} \\ &+\Big(-\frac{GF}{\widetilde{r}r^2} + \frac{F}{\widetilde{r}r}\Big(\partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r}\Big) - \frac{\Omega^2}{r^3}\cos\alpha\Big)\mathbf{e}_\alpha \\ &+\Big(-\frac{G\Omega}{\widetilde{r}r^2} + \frac{F}{\widetilde{r}r}\partial_\alpha\Big(\frac{\Omega}{r}\Big) + \frac{\Omega}{r^3}\big(G\sin\alpha + F\cos\alpha\big)\Big)\mathbf{e}_\vartheta\,.\end{aligned} \tag{IV8.9}$$

Now we treat the viscous term. Using 8.3 we have the following representation for the Laplace operator in polar coordinates

$$\begin{aligned}
&\widetilde{r}r(\,\mathrm{divD}v)\circ\tau = \partial_{\widetilde{r}}\big(\widetilde{r}^2\sin\alpha\,\partial_{\widetilde{r}}v\big) + \partial_\alpha\big(\sin\alpha\,\partial_\alpha v\big) + \partial_\vartheta\big(\tfrac{1}{\sin\alpha}\partial_\vartheta v\big)\\
&= \partial_{\widetilde{r}}\big(-G\mathbf{e}_{\widetilde{r}} - F\mathbf{e}_\alpha - \Omega\mathbf{e}_\vartheta\big)\\
&+\partial_\alpha\Big(\sin\alpha\Big(\partial_\alpha\Big(\tfrac{G}{r}\Big) - \tfrac{F}{r}\Big)\mathbf{e}_{\widetilde{r}} + \sin\alpha\Big(\partial_\alpha\Big(\tfrac{F}{r}\Big) + \tfrac{G}{r}\Big)\mathbf{e}_\alpha + \sin\alpha\partial_\alpha\Big(\tfrac{\Omega}{r}\Big)\mathbf{e}_\vartheta\Big)\\
&+\partial_\vartheta\Big(-\tfrac{\Omega}{r}\mathbf{e}_{\widetilde{r}} - \tfrac{\Omega\cot\alpha}{r}\mathbf{e}_\alpha + \Big(\tfrac{G}{r} + \tfrac{F\cot\alpha}{r}\Big)\mathbf{e}_\vartheta\Big)\\
&= \partial_\alpha\Big(\sin\alpha\Big(\partial_\alpha\Big(\tfrac{G}{r}\Big) - \tfrac{F}{r}\Big)\Big)\mathbf{e}_{\widetilde{r}} + \partial_\alpha\Big(\sin\alpha\Big(\partial_\alpha\Big(\tfrac{F}{r}\Big) + \tfrac{G}{r}\Big)\Big)\mathbf{e}_\alpha\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\partial_\alpha\Big(\sin\alpha\cdot\partial_\alpha\Big(\tfrac{\Omega}{r}\Big)\Big)\mathbf{e}_\vartheta\\
&+\sin\alpha\Big(\partial_\alpha\Big(\tfrac{G}{r}\Big) - \tfrac{F}{r}\Big)\partial_\alpha\mathbf{e}_{\widetilde{r}} + \sin\alpha\Big(\partial_\alpha\Big(\tfrac{F}{r}\Big) + \tfrac{G}{r}\Big)\partial_\alpha\mathbf{e}_\alpha\\
&\qquad\qquad\qquad\qquad\qquad\qquad\big(\,\partial_\alpha\mathbf{e}_{\widetilde{r}} = \mathbf{e}_\alpha\,,\ \partial_\alpha\mathbf{e}_\alpha = -\mathbf{e}_{\widetilde{r}}\,,\ \partial_\alpha\mathbf{e}_\vartheta = 0\,\big)\\
&-\tfrac{\Omega}{r}\partial_\vartheta\mathbf{e}_{\widetilde{r}} - \tfrac{\Omega\cot\alpha}{r}\partial_\vartheta\mathbf{e}_\alpha + \Big(\tfrac{G}{r} + \tfrac{F\cot\alpha}{r}\Big)\partial_\vartheta\mathbf{e}_\vartheta\\
&\qquad\qquad\big(\,\partial_\vartheta\mathbf{e}_{\widetilde{r}} = \sin\alpha\mathbf{e}_\vartheta\,,\ \partial_\vartheta\mathbf{e}_\alpha = \cos\alpha\mathbf{e}_\vartheta\,,\ \partial_\vartheta\mathbf{e}_\vartheta = -\sin\alpha\mathbf{e}_{\widetilde{r}} - \cos\alpha\mathbf{e}_\alpha\,\big)\\
&= \Big(\partial_\alpha\Big(\sin\alpha\Big(\partial_\alpha\Big(\tfrac{G}{r}\Big) - \tfrac{F}{r}\Big)\Big) - \sin\alpha\Big(\partial_\alpha\Big(\tfrac{F}{r}\Big) + \tfrac{G}{r}\Big) - \sin\alpha\Big(\tfrac{G}{r} + \tfrac{F\cot\alpha}{r}\Big)\Big)\mathbf{e}_{\widetilde{r}}\\
&+\Big(\partial_\alpha\Big(\sin\alpha\Big(\partial_\alpha\Big(\tfrac{F}{r}\Big) + \tfrac{G}{r}\Big)\Big) + \sin\alpha\Big(\partial_\alpha\Big(\tfrac{G}{r}\Big) - \tfrac{F}{r}\Big) - \cos\alpha\Big(\tfrac{G}{r} + \tfrac{F\cot\alpha}{r}\Big)\Big)\mathbf{e}_\alpha\\
&+\Big(\partial_\alpha\Big(\sin\alpha\,\partial_\alpha\Big(\tfrac{\Omega}{r}\Big)\Big) - \tfrac{\Omega}{r\sin\alpha}\Big)\mathbf{e}_\vartheta\,.
\end{aligned}$$

Now one computes

$$\partial_\alpha\Big(\sin\alpha\,\partial_\alpha\Big(\tfrac{\Omega}{r}\Big)\Big) - \tfrac{\Omega}{r\sin\alpha} = \tfrac{\Omega''\sin^2\alpha}{\widetilde{r}}$$

and (after some computation) using the mass conservation $G = \sin\alpha\cdot F'$

$$\begin{aligned}
&A := \partial_\alpha\Big(\frac{F}{r}\Big) + \frac{G}{r} = -\frac{1}{\widetilde{r}}\frac{F\cos\alpha}{\sin^2\alpha}\\
&B := \partial_\alpha\Big(\frac{G}{r}\Big) - \frac{F}{r} = -\frac{1}{\widetilde{r}}\Big(F''\sin\alpha + \frac{F}{\sin\alpha}\Big)\\
&E := \frac{G}{r} + \frac{F\cot\alpha}{r} = \frac{1}{\widetilde{r}}\Big(F' + \frac{F\cos\alpha}{\sin^2\alpha}\Big)\\
&\partial_\alpha(\sin\alpha\cdot B) - \sin\alpha\cdot A - \sin\alpha\cdot E = \frac{1}{\widetilde{r}}\Big(F'''\sin^2\alpha - F''\partial_\alpha(\sin^2\alpha)\Big)\\
&\partial_\alpha(\sin\alpha\cdot A) + \sin\alpha\cdot B - \cos\alpha\cdot E = -\frac{1}{\widetilde{r}}F''\sin^2\alpha\,.
\end{aligned}$$

Altogether this gives

$$\begin{aligned}
&(\,\mathrm{divD}v)\circ\tau = \frac{1}{\widetilde{r}r}\Big(\partial_{\widetilde{r}}\big(\widetilde{r}^2\sin\alpha\,\partial_{\widetilde{r}}v\big) + \partial_\alpha\big(\sin\alpha\,\partial_\alpha v\big) + \partial_\vartheta\big(\frac{1}{\sin\alpha}\partial_\vartheta v\big)\Big)\\
&= \frac{1}{\widetilde{r}^3}\Big(\big(F'''\sin^2\alpha - 2F''\cos\alpha\big)\,\mathbf{e}_{\widetilde{r}} - F''\sin\alpha\,\mathbf{e}_\alpha + \Omega''\sin\alpha\,\mathbf{e}_\vartheta\Big)\,.
\end{aligned}$$

We do the same for the $(v\bullet\nabla)v$-term to get

$$\begin{aligned}
-\frac{G^2}{\widetilde{r}r^2}+\frac{F}{\widetilde{r}r}B-\frac{\Omega^2}{r^3}\sin\alpha &= -\frac{1}{\widetilde{r}^3}\Big(F'^2+F''F+\frac{F^2+\Omega^2}{\sin^2\alpha}\Big)\,,\\
-\frac{GF}{\widetilde{r}r^2}+\frac{F}{\widetilde{r}r}A-\frac{\Omega^2}{r^3}\cos\alpha &= -\frac{1}{\widetilde{r}^3}\Big(\frac{F'F}{\sin\alpha}+\frac{(F^2+\Omega^2)\cos\alpha}{\sin^3\alpha}\Big)\,,\\
-\frac{G\Omega}{\widetilde{r}r^2}+\frac{F}{\widetilde{r}r}\partial_\alpha\Big(\frac{\Omega}{r}\Big)+\frac{\Omega}{r^3}(G\sin\alpha+F\cos\alpha) &= -\frac{\Omega'F}{\widetilde{r}^3\sin\alpha}\,.
\end{aligned}$$

With this we finally add all terms of M and obtain

$$\begin{aligned}
0 &= \widetilde{r}^3(\operatorname{div}M)\circ\tau\\
&= -\varrho_0\Big(\Big(F'^2+F''F+\frac{F^2+\Omega^2}{\sin^2\alpha}\Big)\mathbf{e}_{\widetilde{r}}\\
&\qquad\qquad +\Big(\frac{F'F}{\sin\alpha}+\frac{(F^2+\Omega^2)\cos\alpha}{\sin^3\alpha}\Big)\mathbf{e}_\alpha+\frac{\Omega'F}{\sin\alpha}\mathbf{e}_\vartheta\Big)\\
&\quad +\widetilde{r}^2\Big(\widetilde{r}\partial_{\widetilde{r}}p\,\mathbf{e}_{\widetilde{r}}+\partial_\alpha p\,\mathbf{e}_\alpha+\frac{\partial_\vartheta p}{\sin\alpha}\,\mathbf{e}_\vartheta\Big)\\
&\quad -a\Big(\big(F'''\sin^2\alpha-2F''\cos\alpha\big)\,\mathbf{e}_{\widetilde{r}}-F''\sin\alpha\,\mathbf{e}_\alpha+\Omega''\sin\alpha\,\mathbf{e}_\vartheta\Big)\,.
\end{aligned} \tag{IV8.10}$$

We only have to collect now the terms in the $\mathbf{e}_{\widetilde{r}}$, in the $\mathbf{e}_\alpha$, and the $\mathbf{e}_\vartheta$ directions to achieve the three equations in the assertion ($\operatorname{cosec}\alpha := (\sin\alpha)^{-1}$ and the second and third equation is multiplied by $\sin\alpha$.) . ☐

The function v is a vr-vortex with the positive x_3-axis as singularity. The pressure satisfies the following statement which one can compare with the result in (IV8.5) for the free vr-vortex.

8.5 The pressure. From the equations in 8.4(4) we conclude that

$$p=\widetilde{p}(\widetilde{r},\alpha,\vartheta)=\frac{\boldsymbol{\pi}(\cos\alpha)}{\widetilde{r}^2\sin^2\alpha}=\frac{\boldsymbol{\pi}}{r^2}$$

where $s\mapsto\boldsymbol{\pi}(s)$ is the reduced pressure with finite value $\boldsymbol{\pi}(0)$. Here the pressure p is normalized to be 0 at infinity.

The pressure on the ground is

$$p=\frac{\boldsymbol{\pi}(0)}{\widetilde{r}}\ \text{ on } \{x\,;\ \widetilde{r}>0,\ x_3=0\}\,,$$

and near the singularity $\{x\,;\ \widetilde{r}>0,\ r=0\}$ the pressure satisfies

$$\partial_\alpha p=-\frac{a}{\widetilde{r}^2}F''\sin\alpha+\frac{\varrho_0}{\widetilde{r}^2}\Big(\frac{F'F}{\sin\alpha}+\frac{(F^2+\Omega^2)\cos\alpha}{\sin^3\alpha}\Big)\,.$$

Here, since the boundary condition $v=0$ was selected, the solution shown is realistic especially near the ground.

Fig. 27: "Tornado at Elbow Lake, Minnesota, 5 September 1969 (Photograph by Olaf Dybdal)" from [63]

Proof. Wir betrachten die Funktionen in $\{x;\, \widetilde{r} > 0,\, 0 < \alpha \leq \frac{\pi}{2}\}$ und nehmen an, dass $p = \widetilde{p}(\widetilde{r}, \alpha, \vartheta)$. Nach der zweiten Gleichung in 8.4(4) ist

$$\partial_\alpha(\widetilde{r}^2 p) = \widetilde{r}^2 \partial_\alpha p = \mathrm{fcn}_2(\alpha)$$

daher

$$\widetilde{r}^2 p = \mathrm{fcn}\,(\alpha) + \mathrm{fcn}_0(\widetilde{r}, \vartheta)\,.$$

Now the third equality in 8.4(4) says

$$\mathrm{fcn}_3(\alpha) = \widetilde{r}^2 \partial_\vartheta p = \partial_\vartheta(\widetilde{r}^2 p) = \partial_\vartheta \mathrm{fcn}_0\,.$$

Now $\partial_\vartheta \mathrm{fcn}_0$ is independent of α, so that $\partial_\vartheta(\widetilde{r}^2 p) = \mathrm{con}_3$, a constant. We conclude that $\widetilde{r}^2 p$ is linear in ϑ, but is 2π-periodic in ϑ, hence it is independent of ϑ. Therefore $\mathrm{fcn}_0 = \mathrm{fcn}_0(\widetilde{r})$, so that

$$p = \frac{\mathrm{fcn}\,(\alpha) + \mathrm{fcn}_0(\widetilde{r})}{\widetilde{r}^2}\,.$$

Let us use the first equation in 8.4(4) which means

$$\mathrm{fcn}_1(\alpha) = \widetilde{r}^3 \partial_{\widetilde{r}} p = -2\mathrm{fcn}\,(\alpha) - 2\mathrm{fcn}_0(\widetilde{r}) + \widetilde{r}\partial_{\widetilde{r}} \mathrm{fcn}_0(\widetilde{r})$$

which says that $\widetilde{r}\partial_{\widetilde{r}}\mathrm{fcn}_0(\widetilde{r}) - 2\mathrm{fcn}_0(\widetilde{r})$ is independent of $\widetilde{r}$ and this is equivalent to

$$\mathrm{fcn}_0(\widetilde{r}) = \mathrm{con}_0 \widetilde{r}^2 + c_0$$

with constants con_0 and c_0. Thus we obtain for p

$$p = \frac{\mathrm{fcn}\,(\alpha) + c_0}{\widetilde{r}^2} + \mathrm{con}_0\,.$$

If we normalize p by 0 at infinity and define $\boldsymbol{\pi} := (\mathrm{fcn}\,(\alpha) + c_0)\sin^2\alpha$, we get the assertion. □

The flow developes a singularity near the positive x_3-axis. In contrary to 8.2 this singularity is 8.4 not considered in the differential equations, we do this belatedly in [21, "The swirling vortex"].

A solution using the compressible Navier-Stokes equation will be an approximation of the singular behavior described here.

9 Fractionation

Wir betrachten eine Mischung von verschiedenen Flüssigkeiten mit der Eigenschaft, dass die Anziehungskräfte der Moleküle derselben Sorte der Mischung dominant ist. Dann erfordert diese Mischung mehrere Impulsbilanzen, und zwar eine für jede einzelne Komponente. Wenn ϱ_α die Dichte der α-ten Komponente der Mischung und v_α deren Geschwindigkeit ist, ist der allgemeine Ansatz der Massen- und Impulsbilanzen (see II.3.13)

$$\boxed{\begin{gathered} \partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v_\alpha + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \,, \\ \partial_t(\varrho_\alpha v_\alpha) + \operatorname{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \widetilde{\mathbf{f}}_\alpha \,, \\ \widetilde{\mathbf{f}}_\alpha := \mathbf{r}_\alpha v_\alpha + \mathrm{D} v_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha \end{gathered}} \tag{IV9.1}$$

für $\alpha = 1, \ldots, m$, where the representation for $\widetilde{\mathbf{f}}_\alpha$ is from (II3.17). Hierbei ist m die Anzahl der Komponenten. Neben den Scheinkräften wird hierbei $\mathbf{f}_\alpha$ hauptsächlich aus der Schwerkraft bestehen, und Π_α sind zunächst beliebige, auch nichtsymmetrische, Matrizen (siehe 9.2). Darüberhiaus gibt es noch eine Gleichung für die (totale) Energie e, die weiter unten hergeleitet wird.

Das System (IV9.1) wurde schon in (III3.9) als Mischung der Klasse II dargestellt, also ist in der Massenerhaltung

$$\varrho_\alpha v_\alpha + \mathbf{J}_\alpha = \varrho_\alpha v + (\varrho_\alpha u_\alpha + \mathbf{J}_\alpha)$$

where the two objective vectors $\varrho_\alpha u_\alpha$ and $\mathbf{J}_\alpha$ describe the relative movement and the diffusion of the species α. Wir wollen in diesem Abschnitt systematisch die Energiegleichung und die Residualungleichung, welche aus dem Entropieprinzip folgt, herleiten. Wir kommen somit zu Bedingungen, welche die Terme $\mathbf{J}^\alpha$ und Π^α sowie $\mathbf{r}^\alpha$ und $\mathbf{f}^\alpha$ für verschiedene Komponenten α miteinander koppeln. Wir können also entgültig sagen, dass die Gleichungen des Systems wie vermutet voneinander abhängen.

References: The mixture theory has been originally described in two papers by Green & Naghdi [107] and I.Müller [117], where “each used a different entropy inequality” [27, 1.7. Comments on the Formulation of Mixture Theories]. For the non-uniqueness of the entropy principle, which is meant here, we make the statement 9.8, and we use in the case that there is no diffusion an entropy flux which was already proposed by Clausius-Duhem, see below 9.6. Later publications, all following the I.Müller paper, you find in I.Müller [87, 6 Thermodynamics of Mixtures of Non-viscous Fluids] where also shortly the history is addressed, and in Hutter & Jöhnk [47, 7 Theory of Mixtures] where the different classes of a mixture are introduced. Bothe & Dreyer give in [26] a detailed theory of the mixture problem. All these publications are based on the fact that $\mathbf{J}_\alpha = \varrho_\alpha u_\alpha$ is meant.

Define the total mass density and the mean velocity (this is III.3.1(1)) by

$$\varrho := \sum_\alpha \varrho_\alpha \,, \quad v := \frac{1}{\varrho}\sum_\alpha \varrho_\alpha v_\alpha \,, \quad \varrho > 0 \,, \tag{IV9.2}$$

and the "relative velocities" (these are objective vectors) by

$$u_\alpha := v_\alpha - v \quad \text{so that} \quad \sum_\alpha \varrho_\alpha u_\alpha = 0 \,, \tag{IV9.3}$$

which follows from the definitions of ϱ and v in (IV9.2)

$$\sum_\alpha \varrho_\alpha u_\alpha = \sum_\alpha \varrho_\alpha (v_\alpha - v) = \sum_\alpha \varrho_\alpha v_\alpha - \big(\sum_\alpha \varrho_\alpha\big) v = 0 \,.$$

With these definitions we obtain

9.1 Total mass-momentum system. Define

$$\mathbf{J} := \sum_\alpha \mathbf{J}_\alpha \,, \quad \mathbf{r} := \sum_\alpha \mathbf{r}_\alpha \,, \quad \widetilde{\mathbf{f}} := \sum_\alpha \widetilde{\mathbf{f}}_\alpha \,.$$

Then as sum of the individual balance laws one gets

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r} \,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\, \mathbf{J}^{\mathrm{T}} + \Pi_{mix}) = \widetilde{\mathbf{f}} \,, \end{aligned} \tag{IV9.4}$$

where

$$\begin{aligned} &\Pi_{mix} := \sum_\alpha (\varrho_\alpha u_\alpha\, {u_\alpha}^{\mathrm{T}} + u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) \,, \\ &\widetilde{\mathbf{f}} = \mathbf{r} v + \mathrm{D} v \mathbf{J} + \mathbf{f} \,, \quad \mathbf{f} := \sum_\alpha (\mathbf{r}_\alpha u_\alpha + \mathrm{D} u_\alpha \mathbf{J}_\alpha) + \sum_\alpha \mathbf{f}_\alpha \,. \end{aligned} \tag{IV9.5}$$

Proof. Since u_α satisfy (IV9.3) we obtain

$$\begin{aligned} &\sum_\alpha \varrho_\alpha v_\alpha\, {v_\alpha}^{\mathrm{T}} = \sum_\alpha \varrho_\alpha (v + u_\alpha)\,(v + u_\alpha)^{\mathrm{T}} \\ &= \sum_\alpha \varrho_\alpha v\, v^{\mathrm{T}} + \sum_\alpha \varrho_\alpha u_\alpha\, {u_\alpha}^{\mathrm{T}} = \varrho v\, v^{\mathrm{T}} + \sum_\alpha \varrho_\alpha u_\alpha\, {u_\alpha}^{\mathrm{T}} \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_\alpha (\varrho_\alpha v_\alpha\, {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) \\ &= \varrho v\, v^{\mathrm{T}} + \sum_\alpha (\varrho_\alpha u_\alpha\, {u_\alpha}^{\mathrm{T}} + (v + u_\alpha)\, {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) \\ &= \varrho v\, v^{\mathrm{T}} + v\, \mathbf{J}^{\mathrm{T}} + \Pi_{mix} \,, \end{aligned}$$

if the matrix Π_{mix} is defined as in the statement. Moreover

$$\begin{aligned} &\widetilde{\mathbf{f}} = \sum_\alpha \widetilde{\mathbf{f}}_\alpha = \sum_\alpha (\mathbf{r}_\alpha v_\alpha + \mathrm{D} v_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha) \\ &= \sum_\alpha (\mathbf{r}_\alpha v + \mathrm{D} v \mathbf{J}_\alpha) + \sum_\alpha (\mathbf{r}_\alpha u_\alpha + \mathrm{D} u_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha) \\ &= \mathbf{r} v + \mathrm{D} v \mathbf{J} + \mathbf{f} \end{aligned}$$

if the force $\mathbf{f}$ is defined as in the statement. Thus the formula for the momentum balance follows. □

9.2 Symmetry of Π. Assume $\Pi_\alpha = \Pi_\alpha^{sym} + \Pi_\alpha^{rest}$ and

$$\Pi_{mix} = \underbrace{\sum_\alpha (\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}} + \Pi_\alpha^{sym})}_{=:\,\Pi_{mix}^{sym}} + \underbrace{\sum_\alpha (u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha^{rest})}_{=:\,\Pi_{mix}^{rest}}$$

with a symmetric part Π_α^{sym} and a rest Π_α^{rest}, both objective tensors. Such a splitting is performed in II.3.13. *Hinweis:* Falls $\mathbf{J}_\alpha = 0$ kann Π_α und dann auch Π_{mix} als symmetrisch angenommen werden.

In addition to the mass-momentum equations in (IV9.1) we now assume for the mixture a total energy equation

$$\begin{aligned} &\partial_t e + \mathrm{div}\widetilde{q} = \widetilde{g}\,, \quad e := \varepsilon_{mix} + \frac{\varrho}{2}|v|^2\,, \\ &\widetilde{q} = ev + \frac{1}{2}|v|^2\mathbf{J} + {\Pi_{mix}}^{\mathrm{T}} v + q_{mix}\,, \\ &\widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet \mathrm{D}v\mathbf{J} + v\bullet\mathbf{f} + \mathrm{D}v{:}\Pi_{mix}^{rest} + g\,, \end{aligned} \tag{IV9.6}$$

subject to the condition that this energy equation together with the total mass-momentum equation (IV9.4) is a mass-momentum-energy system as in II.3.13. In particular it follows that ε_{mix} and g are objective scalars and q_{mix} an objective vector. Eventually, we want the objective scalar g set to 0 in order to satisfy the total energy conservation.

Hypothetical energy equation

But now we ask the question, what should be the constitutive relation of these quantities ε_{mix} and q_{mix} in the energy equation? As answer we write down the (hypothetical) energy balance of the phase α and treat the sum of these equations as total energy equation. This is an idea going back to Truesdell [122], see I.Müller [87, 3.2.2.7 Metaphysical principles] and the text in Hutter & Jöhnk [47, 7.1 General Introduction], also see Case III in section III.3. We follow this procedure. That is, we state the equation

$$\begin{aligned} &\partial_t e_\alpha + \mathrm{div}\widetilde{q}_\alpha = \widetilde{g}_\alpha\,, \\ &e_\alpha = \varepsilon_\alpha + \frac{\varrho_\alpha}{2}|v_\alpha|^2\,, \quad \varepsilon_\alpha \text{ objective scalar,} \\ &\widetilde{q}_\alpha = e_\alpha v_\alpha + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v_\alpha + q_\alpha\,, \\ &\widetilde{g}_\alpha = \frac{\mathbf{r}_\alpha}{2}|v_\alpha|^2 + v_\alpha\bullet \mathrm{D}v_\alpha\mathbf{J}_\alpha + v_\alpha\bullet\mathbf{f}_\alpha + \mathrm{D}v_\alpha{:}\Pi_\alpha^{rest} + g_\alpha\,, \end{aligned} \tag{IV9.7}$$

where q_α is an objective vector and g_α an objective scalar. This energy equation for phase α together with the mass-momentum equation of phase α is a mass-momentum-energy system as in II.3.13. Therefore also the sum over α of these systems stayes to be a mass-momentum-energy system as in II.3.13. This means that we are able to define the quantities of the total energy equation by

$$e := \sum_\alpha e_\alpha\,, \quad \widetilde{q} := \sum_\alpha \widetilde{q}_\alpha\,, \quad \widetilde{g} := \sum_\alpha \widetilde{g}_\alpha\,. \tag{IV9.8}$$

Thus by construction these terms have the transformation properties described above and satisfy (IV9.6), where the mixing terms are as follows.

9.3 Theorem. With definitions (IV9.8) the energy equation (IV9.6) is satisfied with

$$\varepsilon_{mix} = \sum_\alpha \frac{\varrho_\alpha}{2}|u_\alpha|^2 + \sum_\alpha \varepsilon_\alpha\,,$$

$$q_{mix} = \sum_\alpha \big(\frac{\varrho_\alpha}{2}|u_\alpha|^2 u_\alpha + \frac{1}{2}|u_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}}\, u_\alpha\big) + \sum_\alpha(\varepsilon_\alpha u_\alpha + q_\alpha)\,,$$

$$g = \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}u_\alpha : \Pi_\alpha^{rest} + g_\alpha\big)\,.$$

Proof. For e we have by definition, since $v_\alpha = v + u_\alpha$,

$$\begin{aligned} e &= \sum_\alpha e_\alpha = \sum_\alpha \big(\varepsilon_\alpha + \frac{\varrho_\alpha}{2}|v+u_\alpha|^2\big) \\ &= \sum_\alpha \big(\varepsilon_\alpha + \frac{\varrho_\alpha}{2}|v|^2 + \varrho_\alpha u_\alpha \bullet v + \frac{\varrho_\alpha}{2}|u_\alpha|^2\big) \\ &= \sum_\alpha \varepsilon_\alpha + \frac{\varrho}{2}|v|^2 + \sum_\alpha \frac{\varrho_\alpha}{2}|u_\alpha|^2\,, \end{aligned}$$

hence

$$e_{mix} = \sum_\alpha \varepsilon_\alpha + \sum_\alpha \frac{\varrho_\alpha}{2}|u_\alpha|^2\,.$$

Next we have for $\widetilde{q}$

$$\begin{aligned} \widetilde{q} &= \sum_\alpha \widetilde{q}_\alpha = \sum_\alpha \big(e_\alpha v_\alpha + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}}\, v_\alpha + q_\alpha\big) \\ &= \sum_\alpha \big(e_\alpha(v+u_\alpha) + \frac{1}{2}|v+u_\alpha|^2\mathbf{J}_\alpha\big) + \sum_\alpha \big({\Pi_\alpha}^{\mathrm{T}}\,(v+u_\alpha) + q_\alpha\big) \\ &= \sum_\alpha \big(e_\alpha v + \frac{1}{2}|v|^2\mathbf{J}_\alpha\big) + \sum_\alpha \big(e_\alpha u_\alpha + (v \bullet u_\alpha)\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}}\, v\big) \\ &\qquad + \sum_\alpha \big(\frac{1}{2}|u_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}}\, u_\alpha + q_\alpha\big)\,, \end{aligned}$$

Now using the identity of e_α in (IV9.7)

$$\sum_\alpha e_\alpha u_\alpha = \sum_\alpha \varepsilon_\alpha u_\alpha + \sum_\alpha \frac{\varrho_\alpha}{2}|v+u_\alpha|^2 u_\alpha$$

$$= \underbrace{\sum_\alpha \frac{1}{2}|v|^2 \varrho_\alpha u_\alpha}_{=0} + \sum_\alpha \varrho_\alpha (v\bullet u_\alpha) u_\alpha + \sum_\alpha \big(\varepsilon_\alpha + \frac{\varrho_\alpha}{2}|u_\alpha|^2\big) u_\alpha \,,$$

it follows that $\widetilde{q}$ equals

$$= \sum_\alpha \big(e_\alpha v + \frac{1}{2}|v|^2 \mathbf{J}_\alpha\big) + \sum_\alpha \big(\varrho_\alpha (v\bullet u_\alpha) u_\alpha + (v\bullet u_\alpha)\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v\big)$$

$$+ \sum_\alpha \big(\big(\varepsilon_\alpha + \frac{\varrho_\alpha}{2}|u_\alpha|^2\big) u_\alpha + \frac{1}{2}|u_\alpha|^2 \mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} u_\alpha + q_\alpha\big)$$

$$= ev + \frac{1}{2}|v|^2 \mathbf{J} + {\Pi_{mix}}^{\mathrm{T}} v$$

$$+ \sum_\alpha \big(\frac{\varrho_\alpha}{2}|u_\alpha|^2 u_\alpha + \frac{1}{2}|u_\alpha|^2 \mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} u_\alpha\big) + \sum_\alpha (\varepsilon_\alpha u_\alpha + q_\alpha) \,,$$

hence

$$q_{mix} = \sum_\alpha \big(\frac{\varrho_\alpha}{2}|u_\alpha|^2 u_\alpha + \frac{1}{2}|u_\alpha|^2 \mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} u_\alpha\big) + \sum_\alpha (\varepsilon_\alpha u_\alpha + q_\alpha) \,.$$

For the right side we get since $\mathbf{f} = \sum_\alpha (\mathbf{r}_\alpha u_\alpha + \mathrm{D}u_\alpha \mathbf{J}_\alpha) + \sum_\alpha \mathbf{f}_\alpha$ by (IV9.5) and writing $v_\alpha = v + u_\alpha$

$$\widetilde{g} = \sum_\alpha \widetilde{g}_\alpha = \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|v_\alpha|^2 + v_\alpha \bullet \mathrm{D}v_\alpha \mathbf{J}_\alpha + v_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}v_\alpha : \Pi_\alpha^{rest} + g_\alpha\big)$$

$$= \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|v|^2 + \mathbf{r}_\alpha v\bullet u_\alpha + + v\bullet \mathrm{D}v \mathbf{J}_\alpha + v\bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + v\bullet \mathbf{f}_\alpha\big)$$

$$+ \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}v \mathbf{J}_\alpha + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}v_\alpha : \Pi_\alpha^{rest} + g_\alpha\big)$$

$$= \frac{\mathbf{r}}{2}|v|^2 + v\bullet \mathrm{D}v \mathbf{J} + v\bullet \mathbf{f} + \sum_\alpha u_\alpha \bullet \mathrm{D}v \mathbf{J}_\alpha$$

$$+ \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}v_\alpha : \Pi_\alpha^{rest} + g_\alpha\big) \,,$$

Since by 9.2

$$\mathrm{D}v : \Pi_{mix}^{rest} = \sum_\alpha \mathrm{D}v : \big(u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha^{rest}\big) = \sum_\alpha u_\alpha \bullet \mathrm{D}v \mathbf{J}_\alpha + \sum_\alpha \mathrm{D}v : \Pi_\alpha^{rest}$$

it follows again since $v_\alpha = v + u_\alpha$

$$\widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet \mathrm{D}v \mathbf{J} + v\bullet \mathbf{f} + \mathrm{D}v : \Pi_{mix}^{rest} + g \,,$$

$$g := \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha - \mathrm{D}v : \Pi_\alpha^{rest} + \mathrm{D}v_\alpha : \Pi_\alpha^{rest} + g_\alpha\big)$$

$$= \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}u_\alpha : \Pi_\alpha^{rest} + g_\alpha\big) \,.$$

The assertion follows. □

Thus we have derived, as wanted, a total energy equation, but with one essential exception: The last terms of ε_{mix} and q_{mix} contain quantities which are known only in this derivation and not in the original mixture system. Therefore we define

$$\begin{gathered}\varepsilon := \sum_\alpha \varepsilon_\alpha\,, \quad q := \sum_\alpha (\varepsilon_\alpha u_\alpha + q_\alpha)\,,\\ g = \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}u_\alpha : \Pi_\alpha^{rest} + g_\alpha\big)\,,\end{gathered} \tag{IV9.9}$$

and one can understand this definition as the fact that mixtures of Case III are also mixtures of Case II. For us it is important that replacing the terms ε_α, q_α and g_α in 9.3 by ε, q and g defined in (IV9.9), we obtain an equation which is relevant for the Case II mixture. Therefore in the following we take ε and q as independent variables, and forget about (IV9.9).

References: The equation in 9.3, resp. (IV9.10), of different internal energies one also finds in DeGroot & Mazur [6, Chap.III §4], in Hutter & Jöhnk [47, Theory of mixtures (7.6.15)], and in Bothe & Dreyer [26, Mixture balances, after (19)]. In [26, eq.(38)] the entropy is assumed to be a function of this "thermal energy" ε and the densities ϱ_α, as we will do in (IV9.15) and in (IV10.7).

Mixture system of Class II

We come in a sense back to the mixture problem at the beginning. In addition to the mass-momentum equations in (IV9.1) we assume for the mixture the single energy equation (IV9.6), in which we define

$$\begin{aligned}\varepsilon_{mix} &:= \varepsilon + \sum_\alpha \frac{\varrho_\alpha}{2}|u_\alpha|^2\,,\\ q_{mix} &:= q + \sum_\alpha \big(\frac{\varrho_\alpha}{2}|u_\alpha|^2 u_\alpha + \frac{1}{2}|u_\alpha|^2 \mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} u_\alpha\big)\,.\end{aligned} \tag{IV9.10}$$

Hence (IV9.1) together with (IV9.6), and with the free variables ε and q in (IV9.10), describes the full mass-momentum and energy system of Case II mixture. We write this down again

$$\begin{array}{|l|}\hline \textbf{\textit{Mixture system of Class II:}} \\ \partial_t\varrho_\alpha + \mathrm{div}(\varrho_\alpha v_\alpha + \mathbf{J}_\alpha) = \mathbf{r}_\alpha\,, \\ \partial_t(\varrho_\alpha v_\alpha) + \mathrm{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \widetilde{\mathbf{f}}_\alpha\,, \\ \partial_t e + \mathrm{div}\widetilde{q} = \widetilde{g}\,, \quad e := \varepsilon_{mix} + \frac{\varrho}{2}|v|^2 \\ \hline \widetilde{\mathbf{f}}_\alpha := \mathbf{r}_\alpha v_\alpha + \mathrm{D}v_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha\,, \text{ zu } \mathbf{f} \text{ and } \widetilde{\mathbf{f}} \text{ siehe (IV9.5)}, \\ \widetilde{q} = ev + \frac{1}{2}|v|^2\mathbf{J} + {\Pi_{mix}}^{\mathrm{T}} v + q_{mix}\,, \\ \widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet \mathrm{D}v\mathbf{J} + v\bullet\mathbf{f} + \mathrm{D}v : \Pi^{rest}_{mix} + g\,, \\ \hline \end{array} \tag{IV9.11}$$

where Π_{mix} is defined in (IV9.5) and ε_{mix} and q_{mix} in (IV9.10). We want to consider the energy conservation for the total energy equation, which is represented by the objective scalar g. But before we discuss this further we first study the entropy principle

$$\sigma := \partial_t\eta + \mathrm{div}\psi \geq 0$$

for this system (IV9.11). The entropy will depend on the variable ε in (IV9.10). Therefore we need the following

9.4 Thermal energy equation. The energy equation of (IV9.11) can be written, modulo the other equations of (IV9.11), as an equation for the objective scalar ε

$$\begin{aligned} \partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) = g - \sum_\alpha \mathrm{D}v_\alpha : \Pi^{sym}_\alpha \\ - \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha\bullet \mathrm{D}u_\alpha\mathbf{J}_\alpha + u_\alpha\bullet\mathbf{f}_\alpha + \mathrm{D}u_\alpha : \Pi^{rest}_\alpha\big)\,. \end{aligned}$$

Here Π_α is an in general antisymmetric matrix. The right-hand side of this equation is, as it should be, an objective scalar.
Remark: This identity is identical with [26, Eq.(21)], and essentially identical with [60, Eq.(2.46)] (e.g. replace there velocities by relative velocities).

Proof. It is by (IV9.10)

$$e = \varepsilon_{mix} + \frac{\varrho}{2}|v|^2 = \varepsilon + \sum_\alpha \frac{\varrho_\alpha}{2}(|u_\alpha|^2 + |v|^2) = \varepsilon + \sum_\alpha \frac{\varrho_\alpha}{2}|v_\alpha|^2\,.$$

The second summand satifies the differential identity III.2.2 (follows from the mass and momentum equation)

$$\begin{aligned} \partial_t\Big(\frac{\varrho_\alpha}{2}|v_\alpha|^2\Big) + \mathrm{div}\Big(\frac{\varrho_\alpha}{2}|v_\alpha|^2 + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v_\alpha\Big) \\ = v_\alpha\bullet\mathbf{f}_\alpha + \frac{\mathbf{r}_\alpha}{2}|v_\alpha|^2 + v_\alpha\bullet(\mathrm{D}v_\alpha\mathbf{J}_\alpha) + \mathrm{D}v_\alpha : \Pi_\alpha\,. \end{aligned}$$

Subtracting this from the differential equation (IV9.6) we obtain

$$\begin{aligned} &\partial_t\varepsilon + \operatorname{div}\big(\widetilde{q} - \sum_\alpha \big(\frac{\varrho_\alpha}{2}|v_\alpha|^2 v_\alpha + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v_\alpha\big)\big) \\ &= \widetilde{g} - \sum_\alpha \big(\frac{1}{2}|v_\alpha|^2\mathbf{r}_\alpha + v_\alpha \bullet (\mathrm{D}v_\alpha \mathbf{J}_\alpha) + v_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}v_\alpha : \Pi_\alpha\big)\,. \end{aligned} \tag{IV9.12}$$

Since by (IV9.5)

$$\Pi_{mix} := \sum_\alpha (\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}} + u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}}) + \sum_\alpha \Pi_\alpha\,,$$

the flux is by (IV9.11)

$$\begin{aligned} &\widetilde{q} - \sum_\alpha \big(\frac{\varrho_\alpha}{2}|v_\alpha|^2 v_\alpha + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v_\alpha\big) \\ &= \big(\varepsilon + \sum_\alpha \frac{\varrho_\alpha}{2}|v_\alpha|^2\big)v + \frac{1}{2}|v|^2 \sum_\alpha \mathbf{J}_\alpha + {\Pi_{mix}}^{\mathrm{T}} v + q_{mix} \\ &\qquad\qquad - \sum_\alpha \big(\frac{\varrho_\alpha}{2}|v_\alpha|^2 v_\alpha + \frac{1}{2}|v_\alpha|^2\mathbf{J}_\alpha + {\Pi_\alpha}^{\mathrm{T}} v_\alpha\big) \\ &= \varepsilon v - \sum_\alpha \frac{\varrho_\alpha}{2}|v_\alpha|^2 u_\alpha - \sum_\alpha \big(\frac{1}{2}|u_\alpha|^2 + v\bullet u_\alpha\big)\mathbf{J}_\alpha - \sum_\alpha {\Pi_\alpha}^{\mathrm{T}} v_\alpha \\ &\qquad\qquad + \sum_\alpha (\varrho_\alpha v\bullet u_\alpha)u_\alpha + \sum_\alpha (v\bullet u_\alpha)\mathbf{J}_\alpha + \sum_\alpha {\Pi_\alpha}^{\mathrm{T}} v + q_{mix} \\ &= \varepsilon v - \underbrace{\sum_\alpha \frac{\varrho_\alpha}{2}|v|^2 u_\alpha}_{=0} - \sum_\alpha \frac{\varrho_\alpha}{2}|u_\alpha|^2 u_\alpha - \sum_\alpha \frac{1}{2}|u_\alpha|^2\mathbf{J}_\alpha - \sum_\alpha {\Pi_\alpha}^{\mathrm{T}} u_\alpha + q_{mix} \\ &= \varepsilon v + q \end{aligned}$$

where we have used (IV9.10) for q_{mix}. Since

$$\mathrm{D}v : \Pi_{mix}^{rest} = \sum_\alpha \mathrm{D}v : (u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha^{rest}) = \sum_\alpha u_\alpha \bullet (\mathrm{D}v \mathbf{J}_\alpha) + \sum_\alpha \mathrm{D}v : \Pi_\alpha^{rest}$$

we obtain for the right-hand side of (IV9.12) by the formula for $\widetilde{g}$ in (IV9.11) and since $\mathbf{f} = \sum_\alpha (\mathbf{r}_\alpha u_\alpha + \mathrm{D}u_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha)$

$$\begin{aligned}
&\widetilde{g} - \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2} |v_\alpha|^2 + v_\alpha \bullet (\mathrm{D}v_\alpha \mathbf{J}_\alpha) + v_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}v_\alpha : \Pi_\alpha \big) \\
&= \frac{\mathbf{r}}{2}|v|^2 + v \bullet (\mathrm{D}v\mathbf{J}) + v \bullet \mathbf{f} + \mathrm{D}v : \Pi^{rest}_{mix} + g \\
&\qquad - \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|v_\alpha|^2 + v_\alpha \bullet (\mathrm{D}v_\alpha \mathbf{J}_\alpha) + v_\alpha \bullet \mathbf{f}_\alpha \big) - \sum_\alpha \mathrm{D}v_\alpha : \Pi_\alpha \\
&= \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|v|^2 - \frac{\mathbf{r}_\alpha}{2}|v_\alpha|^2 \big) + \sum_\alpha \big(v \bullet (\mathbf{r}_\alpha u_\alpha + \mathrm{D}u_\alpha \mathbf{J}_\alpha + \mathbf{f}_\alpha) - v_\alpha \bullet \mathbf{f}_\alpha \big) \\
&\qquad + \sum_\alpha \big(v \bullet (\mathrm{D}v \mathbf{J}_\alpha) + u_\alpha \bullet (\mathrm{D}v \mathbf{J}_\alpha) - v_\alpha \bullet (\mathrm{D}v_\alpha \mathbf{J}_\alpha) \big) \\
&\qquad + \sum_\alpha \mathrm{D}v : \Pi^{rest}_\alpha + g - \sum_\alpha \mathrm{D}v_\alpha : \Pi_\alpha \\
&= - \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathbf{f}_\alpha \big) - \sum_\alpha u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha \\
&\qquad + \sum_\alpha \mathrm{D}v : \Pi^{rest}_\alpha + g - \sum_\alpha \mathrm{D}v_\alpha : \Pi_\alpha \\
&= - \sum_\alpha \big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + u_\alpha \bullet \mathrm{D}u_\alpha \mathbf{J}_\alpha + u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D}u_\alpha : \Pi^{rest}_\alpha \big) \\
&\qquad + g - \sum_\alpha \mathrm{D}v_\alpha : (\Pi_\alpha - \Pi^{rest}_\alpha) \,,
\end{aligned}$$

the assertion. □

Proof of objectivity. It is $\sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha$ an objective scalar since

$$\begin{aligned}
&\big(\sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha \big) \circ Y = \sum_\alpha (u_\alpha \circ Y) \bullet (\mathbf{f}_\alpha \circ Y) \\
&= \sum_\alpha (Q u^*_\alpha) \bullet (\varrho^*_\alpha \ddot{X} + 2\varrho^*_\alpha \dot{Q}(v^* + u^*_\alpha)) + \sum_\alpha (Q u^*_\alpha) \bullet (Q \mathbf{f}^*_\alpha) \\
&= \big(Q \underbrace{\sum_\alpha \varrho^*_\alpha u^*_\alpha}_{=0} \big) \bullet (\ddot{X} + 2\dot{Q} v^*) + 2 \sum_\alpha \varrho^*_\alpha \underbrace{(Q u^*_\alpha) \bullet (\dot{Q} u^*_\alpha)}_{= Q^{\mathrm{T}} \dot{Q} : u^*_\alpha {u^*_\alpha}^{\mathrm{T}} = 0} + \sum_\alpha u^*_\alpha \bullet \mathbf{f}^*_\alpha \,,
\end{aligned}$$

since $Q^{\mathrm{T}} \dot{Q}$ is antisymmetric. And the functions $\mathrm{D}u_\alpha : \Pi^{rest}_\alpha$ are objective scalars, since Π^{rest}_α is an objective tensor by 9.2, and the fact that u_α is an objective scalar implies that $\mathrm{D}u_\alpha$ is an objective tensor. □

Wir behandeln nun zunächst nichtreagierende Substanzen, zu den reagierenden siehe den nächsten Abschnitt 10.

Entropy principle for non reacting systems

We now consider the case that there is no reaction or diffusion

$$\mathbf{r}_\alpha = 0 \,, \quad \mathbf{J}_\alpha = 0 \,, \quad \text{hence} \quad \mathbf{f} = \sum_\alpha \mathbf{f}_\alpha \,, \tag{IV9.13}$$

and we assume (for simplicity) that Π_α and therefore also Π_{mix} are symmetric. In this situation the mixture system (IV9.11), with the energy equation replaced by the thermal energy equation from 9.4, reads

$$\begin{aligned}
&\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v_\alpha) = 0\,,\\
&\partial_t(\varrho_\alpha v_\alpha) + \operatorname{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \mathbf{f}_\alpha\,,\\
&\partial_t \varepsilon + \operatorname{div}(\varepsilon v + q) = g - \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha - \sum_\alpha \mathrm{D}v_\alpha : \Pi_\alpha\,,
\end{aligned} \tag{IV9.14}$$

where ϱ_α, v_α, and ε are the unknown variables. We now assume that the entropy itself is a function of these variables. Therefore we have to use the following theorem.

9.5 Theorem. System (IV9.14) is equivalent to the system

$$\begin{aligned}
&\overset{\circ}{\varrho}_\alpha + u_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha = 0\,,\\
&\varrho_\alpha(\overset{\circ}{v}_\alpha + u_\alpha \bullet \nabla v_\alpha) + \operatorname{div}_x \Pi_\alpha = \mathbf{f}_\alpha\,,\\
&\overset{\circ}{\varepsilon} + \varepsilon \operatorname{div}_x v + \operatorname{div}_x q = g - \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha - \sum_\alpha \mathrm{D}v_\alpha : \Pi_\alpha\,.
\end{aligned}$$

Here $\overset{\circ}{h} := (\partial_t + v \bullet \nabla_x) h$ for every function h.

Proof. The mass equation of species α is with $v_\alpha = v + u_\alpha$

$$\begin{aligned}
0 &= \partial_t \varrho_\alpha + v_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha\\
&= (\partial_t + v \bullet \nabla)\varrho_\alpha + u_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha\,,
\end{aligned}$$

the momentum equation

$$\begin{aligned}
\mathbf{f}_\alpha &= \big(\underbrace{\partial_t \varrho_\alpha + \operatorname{div}_x(\varrho_\alpha v_\alpha)}_{=0}\big) v_\alpha + \varrho_\alpha(\partial_t v_\alpha + v_\alpha \bullet \nabla v_\alpha) + \operatorname{div}_x \Pi_\alpha\\
&= \varrho_\alpha(\partial_t + v \bullet \nabla) v_\alpha + \varrho_\alpha u_\alpha \bullet \nabla v_\alpha + \operatorname{div}_x \Pi_\alpha\,,
\end{aligned}$$

and the energy equation in the form of (IV9.14) with

$$\partial_t \varepsilon + \operatorname{div}_x(\varepsilon v) = (\partial_t + v \bullet \nabla)\varepsilon + \varepsilon \operatorname{div}_x v\,.$$

□

We have to make sure, that the entropy is an objective scalar. The simplest way to achieve this is to assune that

$$\eta = \widehat{\eta}((\varrho_\beta)_\beta\,, \varepsilon)\,. \tag{IV9.15}$$

Then we obtain

$$
\begin{aligned}
\sigma &= \partial_t \eta + \operatorname{div}_x \psi = \overset{\circ}{\eta} + \eta \operatorname{div}_x v + \operatorname{div}_x(\psi - \eta v) \\
&= \sum_\alpha \eta'_{\varrho_\alpha} \overset{\circ}{\varrho}_\alpha + \eta'_\varepsilon \overset{\circ}{\varepsilon} + \eta \operatorname{div}_x v + \operatorname{div}_x(\psi - \eta v) \\
&= -\sum_\alpha \eta'_{\varrho_\alpha}(u_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha) - \eta'_\varepsilon \big(\varepsilon \operatorname{div}_x v + \sum_\alpha (u_\alpha \bullet \mathbf{f}_\alpha + \mathrm{D} v_\alpha \mathbin{:} \Pi_\alpha)\big) \\
&\qquad + \eta'_\varepsilon(-\operatorname{div}_x q + g) + \eta \operatorname{div}_x v + \operatorname{div}_x(\psi - \eta v) \\
&= \big(\eta - \varepsilon \eta'_\varepsilon\big) \operatorname{div}_x v - \eta'_\varepsilon \operatorname{div}_x q + \operatorname{div}_x(\psi - \eta v) + \eta'_\varepsilon g \\
&\qquad - \sum_\alpha \varrho_\alpha \eta'_{\varrho_\alpha} \operatorname{div}_x v_\alpha - \eta'_\varepsilon \sum_\alpha \mathrm{D} v_\alpha \mathbin{:} \Pi_\alpha - \sum_\alpha u_\alpha \bullet \big(\eta'_{\varrho_\alpha} \nabla \varrho_\alpha + \eta'_\varepsilon \mathbf{f}_\alpha\big) \\
&= \big(\eta - \varepsilon \eta'_\varepsilon\big) \operatorname{div}_x v - \eta'_\varepsilon \operatorname{div}_x q + \operatorname{div}_x(\psi - \eta v) + \eta'_\varepsilon g \\
&\qquad + \sum_\alpha \mathrm{D} v_\alpha \mathbin{:} \big(-\varrho_\alpha \eta'_{\varrho_\alpha} \mathrm{Id} - \eta'_\varepsilon \Pi_\alpha\big) - \sum_\alpha u_\alpha \bullet \big(\eta'_{\varrho_\alpha} \nabla \varrho_\alpha + \eta'_\varepsilon \mathbf{f}_\alpha\big) .
\end{aligned}
$$

Because

$$
\boxed{v = \sum_\alpha c_\alpha v_\alpha , \quad c_\alpha := \frac{\varrho_\alpha}{\varrho} , \quad \sum_\alpha c_\alpha = 1 ,} \tag{IV9.16}
$$

we obtain

$$
\mathrm{D} v = \sum_\alpha c_\alpha \mathrm{D} v_\alpha + \sum_\alpha v_\alpha (\nabla c_\alpha)^{\mathrm{T}} ,
$$

and, since $v_\alpha = v + u_\alpha$ and $\nabla(\sum_\alpha c_\alpha) = 0$,

$$
\sum_\alpha v_\alpha \nabla c_\alpha{}^{\mathrm{T}} = \underbrace{\sum_\alpha v (\nabla c_\alpha)^{\mathrm{T}}}_{=0} + \sum_\alpha u_\alpha (\nabla c_\alpha)^{\mathrm{T}} .
$$

So we arrive at the formula

$$
\boxed{\mathrm{D} v = \sum_\alpha c_\alpha \mathrm{D} v_\alpha + \sum_\alpha u_\alpha (\nabla c_\alpha)^{\mathrm{T}}} \tag{IV9.17}
$$

and consequently $\operatorname{div}_x v = \sum_\alpha (c_\alpha \operatorname{div}_x v_\alpha + u_\alpha \bullet \nabla c_\alpha)$, a formula one finds in [26, (62)]. Plugging this in the expression for the entropy production σ we obtain

$$
\begin{aligned}
\sigma &= \big(\eta - \varepsilon \eta'_\varepsilon\big) \operatorname{div}_x v - \eta'_\varepsilon \operatorname{div}_x q + \operatorname{div}_x(\psi - \eta v) + \eta'_\varepsilon g \\
&\quad + \sum_\alpha \mathrm{D} v_\alpha \mathbin{:} \big(-\varrho_\alpha \eta'_{\varrho_\alpha} \mathrm{Id} - \eta'_\varepsilon \Pi_\alpha\big) - \sum_\alpha u_\alpha \bullet \big(\eta'_{\varrho_\alpha} \nabla \varrho_\alpha + \eta'_\varepsilon \mathbf{f}_\alpha\big) \\
&= -\eta'_\varepsilon \operatorname{div}_x q + \operatorname{div}_x(\psi - \eta v) + \eta'_\varepsilon g \\
&\qquad + \sum_\alpha \mathrm{D} v_\alpha \mathbin{:} \big(\big(c_\alpha(\eta - \varepsilon \eta'_\varepsilon) - \varrho_\alpha \eta'_{\varrho_\alpha}\big) \mathrm{Id} - \eta'_\varepsilon \Pi_\alpha\big) \\
&\qquad + \sum_\alpha u_\alpha \bullet \big((\eta - \varepsilon \eta'_\varepsilon) \nabla c_\alpha - \eta'_{\varrho_\alpha} \nabla \varrho_\alpha - \eta'_\varepsilon \mathbf{f}_\alpha\big) \\
&= \operatorname{div}_x(\psi - \eta v - \eta'_\varepsilon q) + \eta'_\varepsilon g + \nabla \eta'_\varepsilon \bullet q \\
&\qquad + \sum_\alpha \eta'_\varepsilon \mathrm{D} v_\alpha \mathbin{:} S_\alpha + \sum_\alpha \eta'_\varepsilon u_\alpha \bullet \widetilde{s}_\alpha
\end{aligned} \tag{IV9.18}
$$

where we have performed the well known (classical) identity

$$-\eta'_\varepsilon \operatorname{div}_x q = \operatorname{div}_x(-\eta'_\varepsilon q) + \nabla\eta'_\varepsilon \bullet q \,,$$

and we have defined S_α and $\widetilde{s}_\alpha$ by

$$\begin{aligned}\eta'_\varepsilon \Pi_\alpha &= \big(c_\alpha(\eta - \varepsilon\eta'_\varepsilon) - \varrho_\alpha \eta'_{\varrho_\alpha}\big)\mathrm{Id} - \eta'_\varepsilon S_\alpha \\ \eta'_\varepsilon \mathbf{f}_\alpha &= (\eta - \varepsilon\eta'_\varepsilon)\nabla c_\alpha - \eta'_{\varrho_\alpha}\nabla\varrho_\alpha - \eta'_\varepsilon \widetilde{s}_\alpha \,.\end{aligned}$$

This we can define because the absolute temperature θ, given by

$$\frac{1}{\theta} := \eta'_\varepsilon((\varrho_\beta)_\beta \,, \varepsilon) > 0 \,,$$

is positive. Thus multiplying the definitions by θ gives

$$\begin{aligned}\Pi_\alpha &:= \big(c_\alpha(\theta\eta - \varepsilon) - \varrho_\alpha\theta\eta'_{\varrho_\alpha}\big)\mathrm{Id} - S_\alpha \\ \mathbf{f}_\alpha &:= (\theta\eta - \varepsilon)\nabla c_\alpha - \theta\eta'_{\varrho_\alpha}\nabla\varrho_\alpha - \widetilde{s}_\alpha \,.\end{aligned} \tag{IV9.19}$$

Also since

$$\sum_\alpha \varrho_\alpha u_\alpha = 0 \,,$$

we can define with any arbitrary vector field λ

$$\widetilde{s}_\alpha = s_\alpha - c_\alpha \lambda$$

and obtain in the equation of the entropy production

$$\sum_\alpha u_\alpha \bullet \widetilde{s}_\alpha = \sum_\alpha u_\alpha \bullet s_\alpha \,.$$

By (IV9.13) it follows

$$\begin{aligned}\mathbf{f} = \sum_\alpha \mathbf{f}_\alpha &= \sum_\alpha \big((\theta\eta - \varepsilon)\nabla c_\alpha - \theta\eta'_{\varrho_\alpha}\nabla\varrho_\alpha - s_\alpha\big) + \sum_\alpha c_\alpha\lambda \\ &= \lambda - \sum_\alpha \big(\theta\eta'_{\varrho_\alpha}\nabla\varrho_\alpha + s_\alpha\big)\end{aligned}$$

which gives λ in terms of the overall force $\mathbf{f}$. Thus the terms in the momentum equation become

$$\begin{aligned}\Pi_\alpha &:= \big(c_\alpha(\theta\eta - \varepsilon) - \theta\varrho_\alpha\eta'_{\varrho_\alpha}\big)\mathrm{Id} - S_\alpha \\ \mathbf{f}_\alpha &:= (\theta\eta - \varepsilon)\nabla c_\alpha - \theta\eta'_{\varrho_\alpha}\nabla\varrho_\alpha - s_\alpha + c_\alpha\lambda \\ \lambda &:= \mathbf{f} + \sum_\beta \big(\theta\eta'_{\varrho_\beta}\nabla\varrho_\beta + s_\beta\big) \,.\end{aligned} \tag{IV9.20}$$

These are the restrictions in the system (IV9.14) which come from the entropy principle. Besides these equations there is the residual inequality

$$\begin{aligned}0 \le \sigma = \;& \operatorname{div}_x(\psi - \eta v - \eta'_\varepsilon q) + \eta'_\varepsilon g + \nabla\eta'_\varepsilon \bullet q \\ &+ \sum_\alpha \eta'_\varepsilon \mathrm{D}v_\alpha : S_\alpha + \sum_\alpha \eta'_\varepsilon u_\alpha \bullet \widetilde{s}_\alpha \,.\end{aligned} \tag{IV9.21}$$

So far there are no other assumptions on the system (IV9.14) than the constitutive relation (IV9.15) on the entropy. Now making assumptions we obtain from the above calculations:

9.6 Theorem. For system (IV9.14) the entropy principle is satisfied for the entropy and the entropy flux

$$\eta = \widehat{\eta}((\varrho_\beta)_\beta, \varepsilon), \quad \psi := \eta v + \eta'_\varepsilon q,$$

if the following holds:

(1) In system (IV9.14) the identities (IV9.20) are true for Π_α and $\mathbf{f}_\alpha$ with symmetric tensors S_α and vector fields s_α. In the energy equation $g := 0$. The actual system you will find in (IV9.25).

(2) The functions in (IV9.14) fulfill the residual inequality

$$\sigma = \nabla\Big(\frac{1}{\theta}\Big)\bullet q + \frac{1}{\theta}\sum_\alpha \mathrm{D}v_\alpha : S_\alpha + \frac{1}{\theta}\sum_\alpha u_\alpha \bullet s_\alpha \geq 0. \tag{IV9.22}$$

Important: The entropy flux ψ has the form of Clausius-Duhem.

The fact that the entropy flux is of the Clausius-Duhem form reflects the fact that it has this form for a single fluid and this here is a simple generalization to mixtures of fluids.

Proof. See the above computation of σ resulting in the inequality (IV9.21). Then one can set $\psi = \eta v + \eta'_\varepsilon q$ and $g = 0$. □

Having performed the entropy principle we can formulate the reaction-free mixture system. We do this with the free energy (it is $f = \widehat{f}((\varrho_\beta)_\beta, \theta)$)

$$f := \varepsilon - \theta\eta, \quad f'_\theta = -\eta, \quad f'_{\varrho_\alpha} = -\theta\eta'_{\varrho_\alpha}. \tag{IV9.23}$$

Then the assumptions (IV9.20) read

$$\boxed{\begin{aligned} & f = \widehat{f}((\varrho_\beta)_\beta, \theta), \quad \varepsilon = \widehat{\varepsilon}((\varrho_\beta)_\beta, \theta), \\ & p_\alpha := \varrho_\alpha f'_{\varrho_\alpha} - c_\alpha f, \quad \Pi_\alpha := p_\alpha \mathrm{Id} - S_\alpha, \\ & \mathbf{f}_\alpha := f'_{\varrho_\alpha}\nabla\varrho_\alpha - f\nabla c_\alpha - s_\alpha + c_\alpha\lambda, \\ & \lambda := \mathbf{f} - \sum_\beta \big(f'_{\varrho_\beta}\nabla\varrho_\beta - s_\beta\big), \end{aligned}} \tag{IV9.24}$$

and the system without reactions and with the assumptions in theorem 9.6 becomes

Non-reacting mixtures:

$$\partial_t \varrho_\alpha + \mathrm{div}(\varrho_\alpha v_\alpha) = 0\,,$$
$$\partial_t(\varrho_\alpha v_\alpha) + \mathrm{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + p_\alpha \mathrm{Id} - S_\alpha) = \mathbf{f}_\alpha\,,$$
$$\partial_t \varepsilon + \mathrm{div}(\varepsilon v + q) + \sum_\alpha p_\alpha \,\mathrm{div} v_\alpha = \sum_\alpha \mathrm{D} v_\alpha \mathbin{\bullet\!\!\bullet} S_\alpha - \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha \tag{IV9.25}$$

p_α and $\mathbf{f}_\alpha$ and ε as in (IV9.24),

q and s_α and S_α as in (IV9.22).

Here the gradient of the partial pressure ∇p_α and the partial force $\mathbf{f}_\alpha$ have certain common terms. Therefore these terms, because they are equal, will not contribute to the momentum equation. We show

9.7 Lemma. If for a non-reacting mixture (IV9.24) holds and

$$p_\alpha^{\mathrm{sp}} := \frac{p_\alpha}{\varrho_\alpha}\,, \quad p := \sum_\alpha p_\alpha\,, \tag{IV9.26}$$

then for every species α

$$\nabla p_\alpha - \mathbf{f}_\alpha = \varrho_\alpha\big(\nabla p_\alpha^{\mathrm{sp}} - \sum_\beta c_\beta \nabla p_\beta^{\mathrm{sp}}\big) + c_\alpha\big(\nabla p - \mathbf{f}\big) + \big(s_\alpha - c_\alpha \sum_\beta s_\beta\big)\,.$$

Proof. We have that $p_\alpha = \varrho_\alpha p_\alpha^{\mathrm{sp}}$ therefore

$$\nabla p_\alpha = \varrho_\alpha \nabla p_\alpha^{\mathrm{sp}} + p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha\,.$$

By (IV9.24) we get for

$$p_\alpha = \varrho_\alpha f'_{\varrho_\alpha} - c_\alpha f = \varrho_\alpha\big(f'_{\varrho_\alpha} - \frac{f}{\varrho}\big)$$

that

$$p_\alpha^{\mathrm{sp}} = f'_{\varrho_\alpha} - \frac{f}{\varrho}\,.$$

Now, since

$$\nabla c_\alpha = \nabla\Big(\frac{\varrho_\alpha}{\varrho}\Big) = \frac{1}{\varrho}\nabla\varrho_\alpha + \varrho_\alpha\nabla\big(\frac{1}{\varrho}\big) = \frac{1}{\varrho}\nabla\varrho_\alpha - \frac{c_\alpha}{\varrho}\sum_\beta \nabla\varrho_\beta\,,$$

we get also from (IV9.24) that

$$\begin{aligned}
\mathbf{f}_\alpha &= f'_{\varrho_\alpha}\nabla\varrho_\alpha - f\nabla c_\alpha - s_\alpha + c_\alpha\big(\mathbf{f} - \sum_\beta f'_{\varrho_\beta}\nabla\varrho_\beta + \sum_\beta s_\beta\big) \\
&= \big(f'_{\varrho_\alpha} - \frac{f}{\varrho}\big)\nabla\varrho_\alpha + c_\alpha\big(\mathbf{f} - \sum_\beta \big(f'_{\varrho_\beta} - \frac{f}{\varrho}\big)\nabla\varrho_\beta\big) - \big(s_\alpha - c_\alpha\sum_\beta s_\beta\big) \\
&= p_\alpha^{\mathrm{sp}}\nabla\varrho_\alpha + c_\alpha\big(\mathbf{f} - \sum_\beta p_\beta^{\mathrm{sp}}\nabla\varrho_\beta\big) - \big(s_\alpha - c_\alpha\sum_\beta s_\beta\big)\,.
\end{aligned}$$

This implies

$$\nabla p_\alpha - \mathbf{f}_\alpha = \varrho_\alpha \nabla p_\alpha^{\mathrm{sp}} - c_\alpha\big(\mathbf{f} - \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta\big) + \big(s_\alpha - c_\alpha \sum_\beta s_\beta\big)\,.$$

Now

$$\sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta = \nabla\big(\underbrace{\sum_\beta p_\beta^{\mathrm{sp}} \varrho_\beta}_{=\sum_\beta p_\beta = p}\big) - \sum_\beta \varrho_\beta \nabla p_\beta^{\mathrm{sp}}$$

and $c_\alpha \varrho_\beta = \varrho_\alpha c_\beta$. Therefore we obtain

$$\nabla p_\alpha - \mathbf{f}_\alpha = \varrho_\alpha\big(\nabla p_\alpha^{\mathrm{sp}} - \sum_\beta c_\beta \nabla p_\beta^{\mathrm{sp}}\big) + c_\alpha\big(\nabla p - \mathbf{f}\big) + \big(s_\alpha - c_\alpha \sum_\beta s_\beta\big)\,.$$

This is the assertion. □

By the way, the entropy inequality has no unique representation, although the answer given here is physically plausible. One should realize that a different entropy flux also implies a different momentum flux, hence it is important which entropy one takes. (Another problem with nonuniqueness of entropy one finds in section 12.)

9.8 Non uniqueness of entropy principle. We can add a term

$$0 = \sum_\alpha \big(\operatorname{div}_x(-\pi_\alpha u_\alpha) + \mathrm{D}v_\alpha \mathbin{\bullet\!\!\bullet} (\widetilde{\pi}_\alpha \mathrm{Id}) + u_\alpha \bullet \nabla \widetilde{\pi}_\alpha\big)\,, \quad \widetilde{\pi}_\alpha := \pi_\alpha - c_\alpha \sum_\beta \pi_\beta\,, \tag{IV9.27}$$

to the identity (IV9.18). This will lead to a family of solutions with

$$\begin{aligned} \Pi_\alpha &= \big(c_\alpha(\theta\eta - \varepsilon) - \varrho_\alpha \theta \eta'_{\varrho_\alpha} + \theta \widetilde{\pi}_\alpha\big)\mathrm{Id} - S_\alpha \\ \mathbf{f}_\alpha &= (\theta\eta - \varepsilon)\nabla c_\alpha - \theta \eta'_{\varrho_\alpha} \nabla \varrho_\alpha + \theta \nabla \widetilde{\pi}_\alpha - \widetilde{s}_\alpha \end{aligned} \tag{IV9.28}$$

in place of the representation (IV9.19). The entropy flux will be

$$\psi = \eta v + \eta'_\varepsilon q + \sum_\alpha \pi_\alpha u_\alpha\,,$$

that is, the entropy flux will depend on the relative velocities.
Remark: In literature one finds $\pi_\alpha := \varrho_\alpha \eta'_{\varrho_\alpha}$, which results in

$$\begin{aligned} \Pi_\alpha &= c_\alpha\big(\theta\eta - \varepsilon - \theta \textstyle\sum_\beta \varrho_\beta \eta'_{\varrho_\beta}\big)\mathrm{Id} - S_\alpha \\ \mathbf{f}_\alpha &= \big(\theta\eta - \varepsilon - \theta \textstyle\sum_\beta \varrho_\beta \eta'_{\varrho_\beta}\big)\nabla c_\alpha + \theta \varrho_\alpha \nabla \eta'_{\varrho_\alpha} - \widetilde{s}_\alpha\,, \end{aligned}$$

and it would imply $p_\alpha = c_\alpha p$ and $\mathbf{f}_\alpha$ would depend on $\nabla\theta$.

Proof. To prove the identity we compute

$$\sum_\alpha \operatorname{div}_x(\pi_\alpha u_\alpha) = \sum_\alpha \operatorname{div}_x(\widetilde{\pi}_\alpha u_\alpha) = \sum_\alpha \nabla \widetilde{\pi}_\alpha \bullet u_\alpha + \sum_\alpha \widetilde{\pi}_\alpha \operatorname{div}_x u_\alpha$$

and

$$\sum_\alpha \widetilde{\pi}_\alpha \operatorname{div}_x u_\alpha = \sum_\alpha \widetilde{\pi}_\alpha \operatorname{div}_x v_\alpha - \underbrace{\sum_\alpha \widetilde{\pi}_\alpha \operatorname{div}_x v}_{=0}\,,$$

since $\sum_\alpha \widetilde{\pi}_\alpha = 0$, which gives the assertion (IV9.27). This added to (IV9.18) gives

$$\sigma = -\eta'_\varepsilon \,\mathrm{div}_x q + \mathrm{div}_x\big(\psi - \eta v - \sum_\alpha \pi_\alpha u_\alpha\big) + \eta'_\varepsilon g$$
$$+ \sum_\alpha \mathrm{D}v_\alpha \mathbin{:} \big((c_\alpha(\eta - \varepsilon\eta'_\varepsilon) - \varrho_\alpha \eta'_{\varrho_\alpha} + \widetilde{\pi}_\alpha)\mathrm{Id} - \eta'_\varepsilon \Pi_\alpha\big)$$
$$+ \sum_\alpha u_\alpha \bullet \big((\eta - \varepsilon\eta'_\varepsilon)\nabla c_\alpha - \eta'_{\varrho_\alpha}\nabla\varrho_\alpha + \nabla\widetilde{\pi}_\alpha - \eta'_\varepsilon \mathbf{f}_\alpha\big)$$

leading to (IV9.28). □

Equilibria

We call an equilibrium a stationary solution with all $v_\alpha = 0$, hence $v = 0$ and all $u_\alpha = 0$, and in addition $S_\alpha = 0$ and $s_\alpha = 0$. Thus the mass-momentum equations (IV9.25) are equivalent to

$$\boxed{\nabla p_\alpha = \mathbf{f}_\alpha \quad \text{for all } \alpha} \tag{IV9.29}$$

and the energy equation $\mathrm{div} q = 0$. Then the following holds.

9.9 Lemma. Consider a region where all $\varrho_\alpha > 0$. Then equilibria satisfy

$$\nabla p_\alpha^{\mathrm{sp}} = \textstyle\sum_\beta c_\beta \nabla p_\beta^{\mathrm{sp}} \quad \text{for all } \alpha. \tag{IV9.30}$$

That is, the gradient of the specific partial pressure $\nabla p_\alpha^{\mathrm{sp}}$ is independent of the species α, therefore it is a function of the mixture, or (IV9.30) is equivalent to:

$$\nabla p_\alpha^{\mathrm{sp}} \text{ is the same function for all } \alpha. \tag{IV9.31}$$

Proof. Since $\mathbf{f} = \sum_\alpha \mathbf{f}_\alpha$ and $p = \sum_\alpha p_\alpha$, we also have $\nabla p - \mathbf{f} = 0$. Hence the result of 9.7 is

$$\varrho_\alpha\big(\nabla p_\alpha^{\mathrm{sp}} - \sum_\beta c_\beta \nabla p_\beta^{\mathrm{sp}}\big) = 0\,.$$

If $\varrho_\alpha > 0$ the assertion follows. Since $\sum_\beta c_\beta = 1$ the result (IV9.30) is equivalent to (IV9.31). □

Therefore if we consider a region where all $\varrho_\alpha > 0$, we can state that the system (IV9.25), and thus also (IV9.29), is equivalent to

$$\boxed{\begin{gathered} \nabla p = \mathbf{f}\,, \quad p = \sum_\alpha p_\alpha\,, \quad \mathbf{f} = \sum_\alpha \mathbf{f}_\alpha\,, \\ \nabla p_\alpha^{\mathrm{sp}} \quad \text{is the same function for all } \alpha, \end{gathered}} \tag{IV9.32}$$

9.10 Fundamental law. It follows from (IV9.32) that there are constants $\widetilde{d}_\alpha \in \mathbb{R}$ such that

$$\boxed{\widetilde{p}_\alpha^{\mathrm{sp}} \quad \text{is the same function for all } \alpha,} \tag{IV9.33}$$

where $\widetilde{p}_\alpha$ are the "normalized" pressures

$$\widetilde{p}_\alpha := \varrho_\alpha \widetilde{f}{\prime}_{\varrho_\alpha} - c_\alpha \widetilde{f}\,, \quad \widetilde{f} := f - \sum_\beta \widetilde{d}_\beta\, \varrho_\beta\,.$$

Remark: It is assumed that the space domain is connected. Here we talk about equilibria, in this case there is no time variable.
Note: Because $\widetilde{d}_\alpha$ are numbers, the energy equation does not change, if we choose $\widetilde{f} = \widetilde{e} - \theta\eta$, that is, if we replace e by $\widetilde{e}$.

It is

$$\widetilde{p}_\alpha^{\mathrm{sp}} = \widetilde{f}{\prime}_{\varrho_\alpha} - \frac{\widetilde{f}}{\varrho} \quad \text{where} \quad \widetilde{\mu}_\alpha := \widetilde{f}{\prime}_{\varrho_\alpha}\,,$$

therefore the chemical potentials $\widetilde{\mu}_\alpha$ also are independent of α. See the paper of Huggins, specially the equations [109, (1)-(4)].

Proof. The equations (IV9.32) say that for all α and β

$$\nabla(p_\alpha^{\mathrm{sp}} - p_\beta^{\mathrm{sp}}) = 0 \text{ in the domain,}$$

so there are constants $k_{\alpha\beta} \in \mathbb{R}$ with

$$p_\alpha^{\mathrm{sp}} - p_\beta^{\mathrm{sp}} = k_{\alpha\beta}\,. \tag{IV9.34}$$

Then for a given γ_0 define

$$\widetilde{d}_\alpha := k_{\alpha\gamma_0}\,.$$

Since

$$p_\alpha^{\mathrm{sp}} = k_{\alpha\beta} + p_\beta^{\mathrm{sp}} = k_{\alpha\beta} + k_{\beta\gamma} + p_\gamma^{\mathrm{sp}}$$

and also $p_\alpha^{\mathrm{sp}} = k_{\alpha\gamma} + p_\gamma^{\mathrm{sp}}$, we conclude

$$k_{\alpha\gamma} = k_{\alpha\beta} + k_{\beta\gamma}\,,$$

hence

$$k_{\alpha\beta} = k_{\alpha\gamma_0} - k_{\beta\gamma_0} = \widetilde{d}_\alpha - \widetilde{d}_\beta\,.$$

Then (IV9.34) can be written as

$$p_\alpha^{\mathrm{sp}} - \widetilde{d}_\alpha = p_\beta^{\mathrm{sp}} - \widetilde{d}_\beta\,.$$

We compute for $\widetilde{p}_\alpha$ defined in the statement

$$\widetilde{p}_\alpha^{\mathrm{sp}} = \frac{\widetilde{p}_\alpha}{\varrho_\alpha} = \widetilde{f}{\prime}_{\varrho_\alpha} - \frac{\widetilde{f}}{\varrho} = \underbrace{f{\prime}_{\varrho_\alpha} - \frac{f}{\varrho} - \widetilde{d}_\alpha}_{= p_\alpha^{\mathrm{sp}} - \widetilde{d}_\alpha} + \sum_\beta \widetilde{d}_\beta c_\beta\,,$$

and from there the assertion. □

Wir nehmen nun an, dass auf die Mischung die Schwerkraft wirkt, z.B. die Erdanziehungskraft, wir bezeichnen daher den Kraftterm in der allgemeinen Situation mit

$$\mathbf{f} = \varrho\nabla(\mathfrak{g}\phi)\,, \tag{IV9.35}$$

wir sind also ein Beobachter, der keine Scheinkräfte bewirkt, oder für den die Scheinkräfte vernachlässigbar sind. Befinden wir uns in einer Zentrifuge so haben wir die Zentrifugalkraft als eine Scheinkraft, also muss dann $\mathbf{f}$ entsprechend abgeändert werden.

9.11 Simple mixture of ideal gases. Let us consider an ideal mixture of gases with free energy (see 2.5(4))

$$f((\varrho_\beta)_\beta\,,\theta) := \sum_\alpha \varrho_\alpha\big(R^\alpha\theta\log\varrho_\alpha - c_V^\alpha\theta\log\theta + d^\alpha\theta\big)\,, \tag{IV9.36}$$

where $R^\alpha = c_P^\alpha - c_V^\alpha > 0$, and c_P^α, c_V^α and d^α are constants. Then it follows for equilibria that the pressure p and the internal energy ε is given by

$$\begin{aligned} \Big(\sum_\alpha \varrho_\alpha f'_{\varrho_\alpha}\Big) - f = \ p\ &= \sum_\alpha R^\alpha\theta\varrho_\alpha\,,\\ f - \theta f'_\theta = \ \varepsilon\ &= \sum_\alpha c_V^\alpha\theta\varrho_\alpha\,. \end{aligned} \tag{IV9.37}$$

Remark: Here only Gibbs relation, the relation between f and p, is used. The relation $f - \theta f'_\theta = \varepsilon$ was a definition, see (III1.7) or (IV9.23).

If f satisfies (IV9.37) it follows that f is of the form (IV9.36) plus a term $\theta\varrho\, d(\frac{\vec\varrho}{|\vec\varrho|})$, which contains terms $\sum_\alpha\theta\varrho_\alpha d^\alpha(\vec\varrho)$, provided $\big(\partial_{\varrho_\beta}d^\alpha\big)_{\alpha,\beta}$ satisfies $\sum_{\alpha\beta}\varrho_\alpha\varrho_\beta\partial_{\varrho_\beta}d^\alpha(\vec\varrho) = 0$.

Proof (IV9.36)⇒(IV9.37). Wir nehmen an, dass die freie Energie

$$f = \sum_\alpha \varrho_\alpha h_\alpha\,, \quad h_\alpha := a_\alpha(\theta)\log\varrho_\alpha - b_\alpha(\theta)\,,$$

ist, wobei $a_\alpha(\theta) := R^\alpha\theta$ und $b_\alpha(\theta) := c_V^\alpha\theta\log\theta - d^\alpha\theta$. Es folgt

$$\begin{aligned} &\mu_\alpha := f'_{\varrho_\alpha} = h_\alpha + \varrho_\alpha h_\alpha{}'_{\varrho_\alpha} = h_\alpha + a_\alpha\,,\\ &p_\alpha^{\mathrm{sp}} = f'_{\varrho_\alpha} - \frac{f}{\varrho} = h_\alpha - \sum_\beta c_\beta h_\beta + a_\alpha\,,\\ &p = \sum_\alpha \varrho_\alpha p_\alpha^{\mathrm{sp}} = \varrho\sum_\alpha c_\alpha p_\alpha^{\mathrm{sp}} = \varrho\sum_\alpha c_\alpha a_\alpha = \sum_\alpha \varrho_\alpha a_\alpha\,, \end{aligned} \tag{IV9.38}$$

and

$$\varepsilon = f + \theta\eta = f - \theta f'_\theta = -\theta^2\Big(\frac{f}{\theta}\Big)'_\theta = \theta^2\Big(\sum_\alpha \varrho_\alpha c_V^\alpha\log\theta\Big)'_\theta = \sum_\alpha \theta\varrho_\alpha c_V^\alpha\,.$$

Hence (IV9.37) is proved. □

Proof of the other direction. Let $f = \widehat{f}(\vec{\varrho}, \theta)$ be any function satisfying (IV9.37). If we subtract from this the function in (IV9.36) we are left with a function f satisfying the homogeneous differential equations

$$f - \theta f_{\prime\theta} = 0\,, \quad \big(\sum_\alpha \varrho_\alpha f_{\prime\varrho_\alpha}\big) - f = 0\,.$$

The first equation means $\big(\frac{f}{\theta}\big)_{\prime\theta} = 0$ hence $f(\vec{\varrho}, \theta) = \theta d_1(\vec{\varrho})$. The second equation gives

$$0 = \sum_\alpha \varrho_\alpha \Big(\frac{f}{\varrho}\Big)_{\prime\varrho_\alpha} = \vec{\varrho}\bullet\nabla_{\vec{\varrho}}\Big(\frac{f}{\varrho}\Big) = \theta\vec{\varrho}\bullet\nabla_{\vec{\varrho}}\Big(\frac{d_1\vec{\varrho}}{\varrho}\Big)$$

hence

$$\frac{d_1(\vec{\varrho})}{\varrho} = d\big(\frac{\vec{\varrho}}{|\vec{\varrho}|}\big)\,.$$

Therefore $f(\vec{\varrho}, \theta) = \theta\varrho d(\frac{\vec{\varrho}}{|\vec{\varrho}|})$. A function $f = \sum_\alpha \theta\varrho_\alpha d^\alpha(\vec{\varrho})$ means that $d = \sum_\alpha c_\alpha d^\alpha$ and

$$0 = \sum_\beta \varrho_\beta \partial_{\varrho_\beta} d = \sum_\alpha \Big(\underbrace{\sum_\beta \varrho_\beta \partial_{\varrho_\beta} c_\alpha}_{=0} \Big) d^\alpha + \sum_{\alpha\beta} \varrho c_\beta c_\alpha \partial_{\varrho_\beta} d^\alpha\,.$$

□

Warning: It is $p = \sum_\alpha p_\alpha$ where p_α is the pressure of the moment equation of species α in (IV9.25), and also $p = \sum_\alpha R^\alpha\theta\varrho_\alpha$ in 9.11, but p_α is not $R^\alpha\theta\varrho_\alpha$ as one can see in the above proof.

9.12 Equilibrium for ideal gases. Let the free energy be given as in 9.11 and define $a_\alpha(\theta) := R^\alpha\theta$. Then in the isothermal situation $\theta = \text{const}$ the system (IV9.32) under gravity (IV9.35) is equivalent to

$$\begin{aligned}&\sum_\alpha a_\alpha \nabla c_\alpha + \big(\sum_\alpha a_\alpha c_\alpha\big)\nabla\log\varrho = \nabla(\mathfrak{g}\phi)\,,\\ &\frac{a_\alpha}{c_\alpha}\nabla c_\alpha + a_\alpha\nabla\log\varrho \quad \text{is the same function for all } \alpha.\end{aligned} \tag{IV9.39}$$

Proof. Die äußere Kraft $\mathbf{f}$ ist in (IV9.35) gegeben und mit dem Druck p in (IV9.38) sind die Gleichungen in (IV9.32) äquivalent zu

$$\begin{aligned}&\nabla\big(\sum_\alpha a_\alpha\varrho_\alpha\big) = \varrho\nabla(\mathfrak{g}\phi)\,,\\ &\nabla(a_\alpha\log\varrho_\alpha) \text{ is the same function for all } \alpha,\end{aligned} \tag{IV9.40}$$

denn, da wir den isothermen Fall behandeln,

$$\begin{aligned}&\nabla p_\alpha^{\text{sp}} = \nabla f_{\prime\varrho_\alpha} - \nabla\big(\frac{f}{\varrho}\big) \text{ is the same function for all } \alpha,\\ \iff\quad &\nabla f_{\prime\varrho_\alpha} = \nabla h_\alpha = \nabla(a_\alpha\log\varrho_\alpha) \text{ is the same function for all } \alpha,\\ \iff\quad &\nabla(a_\alpha\log\varrho_\alpha) = \nabla(a_\alpha\log(c_\alpha\varrho)) = a_\alpha\nabla\log c_\alpha + a_\alpha\nabla\log\varrho\\ &\qquad\qquad\text{is the same function for all } \alpha.\end{aligned}$$

Da

$$\nabla\big(\sum_\alpha a_\alpha\varrho_\alpha\big) = \nabla\big(\sum_\alpha a_\alpha c_\alpha\varrho\big) = \sum_\alpha a_\alpha(\varrho\nabla c_\alpha + c_\alpha\nabla\varrho) = \varrho\big(\sum_\alpha a_\alpha\nabla c_\alpha + \big(\sum_\alpha a_\alpha c_\alpha\big)\nabla\log\varrho\big)$$

ist (IV9.40) äquivalent zu

$$\sum_\alpha a_\alpha\nabla c_\alpha + \big(\sum_\alpha a_\alpha c_\alpha\big)\nabla\log\varrho = \nabla(\mathfrak{g}\phi)\,,$$
$$\frac{a_\alpha}{c_\alpha}\nabla c_\alpha + a_\alpha\nabla\log\varrho \text{ is the same function for all } \alpha,$$

□

9.13 Binäre Mischung. Für eine binäre Mischung, d.h. $\alpha = 1, 2$, in 9.12 folgt im Fall $R^1 \neq R^2$ und im isothermen Fall aus (IV9.39)

$$\log\frac{c}{1-c} = \Big(\frac{1}{a_1} - \frac{1}{a_2}\Big)\cdot(-\mathfrak{g}\phi) + \frac{k_\phi}{a_1 a_2} \quad \text{where } c := c_2\,,$$
$$\log\varrho = \frac{a_1}{a_2 - a_1}\log c_1 + \frac{a_2}{a_1 - a_2}\log c_2 + \frac{k_\varrho}{a_2 - a_1}\,.$$

Here k_ϕ and k_ϱ are constants.

Proof. Da $c_1 + c_2 = 1$ ist die zweite Gleichung von (IV9.39) äquivalent zu

$$\frac{a_2}{c_2}\nabla c_2 + a_2\nabla\log\varrho = \frac{a_1}{c_1}\nabla c_1 + a_1\nabla\log\varrho\,,$$

also fogt mit $c := c_2$, dass (IV9.39) äquivalent ist zu

$$(a_2 - a_1)\nabla c + (a_1 c_1 + a_2 c_2)\nabla\log\varrho = \nabla(\mathfrak{g}\phi)\,,$$
$$\big(\frac{a_1}{c_1} + \frac{a_2}{c_2}\big)\nabla c + (a_2 - a_1)\nabla\log\varrho = 0\,.$$

Taking $\nabla\log\varrho$ from the second equation and inserting it in the first equation one obtains

$$(a_2 - a_1)\nabla(\mathfrak{g}\phi) = \underbrace{\Big(\,(a_2 - a_1)^2 - (a_1 c_1 + a_2 c_2)\big(\frac{a_1}{c_1} + \frac{a_2}{c_2}\big)\Big)}_{= -a_1 a_2\left(\frac{1}{c_1} + \frac{1}{c_2}\right) \text{ since } c_1 + c_2 = 1}\nabla c\,,$$

hence the system is equivalent to

$$(a_2 - a_1)\nabla(\mathfrak{g}\phi) = -a_1 a_2\big(\frac{1}{c_1} + \frac{1}{c_2}\big)\nabla c = a_1 a_2(\nabla\log c_1 - \nabla\log c_2)\,,$$
$$(a_2 - a_1)\nabla\log\varrho = -\big(\frac{a_1}{c_1} + \frac{a_2}{c_2}\big)\nabla c = a_1\nabla\log c_1 - a_2\nabla\log c_2\,.$$

Now $a_1(\theta) = R^1\theta \neq R^2\theta = a_2(\theta)$ and a_1 and a_2 are constant, since it is assumed that the processes are isothermal. Therefore

$$\nabla\big((a_1 - a_2)\mathfrak{g}\phi + a_1 a_2(\log c_1 - \log c_2)\big) = 0\,,$$
$$\nabla\big((a_1 - a_2)\log\varrho + (a_1\log c_1 - a_2\log c_2)\big) = 0\,.$$

It follows that there are constants $k_\phi, k_\varrho \in \mathbb{R}$ so that

$$(a_1 - a_2)\mathfrak{g}\phi + a_1 a_2(\log c_1 - \log c_2) + k_\phi = 0\,,$$
$$(a_1 - a_2)\log\varrho + (a_1\log c_1 - a_2\log c_2) + k_\varrho = 0\,,$$

or

$$\log\frac{c}{1-c} = \log c_2 - \log c_1 = \frac{(a_1 - a_2)\mathfrak{g}\phi + k_\phi}{a_1 a_2}$$
$$= \Big(\frac{1}{a_1} - \frac{1}{a_2}\Big)\cdot(-\mathfrak{g}\phi) + \frac{k_\phi}{a_1 a_2}\,,$$
$$\log\varrho = \frac{a_1}{a_2 - a_1}\log c_1 + \frac{a_2}{a_1 - a_2}\log c_2 + \frac{k_\varrho}{a_2 - a_1}\,,$$

also die Behauptung. □

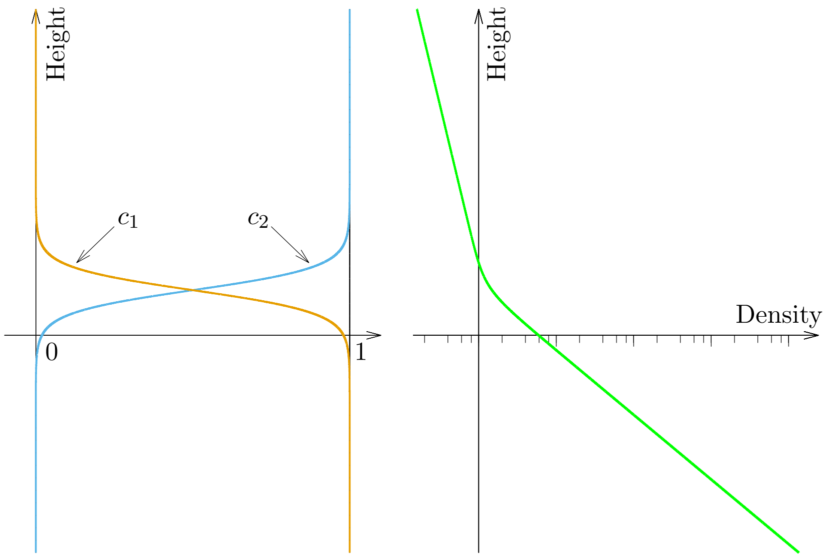


Fig. 28: Es wird die "Binäre Mischung" aus dem Text gezeigt. *Links:* Konzentrationen c_2 and c_1. *Rechts:* Dichte ϱ im logarithmischen Maßstab. In einer vertikalen Säule setzt sich die "schwerere" Substanz nach unten ab und die "leichtere" nach oben.

Daraus folgt, dass ein Equilibrium von idealen Gasen die "Fraktionierung" auf der Erdoberfläche erlärt, sie wird verursacht durch die Gravitation.

9.14 Fractionation. Consider equilibria as in 9.13. On Earth's surface if $\mathfrak{g}\phi(x) = \text{const} - g_{\text{Earth}}\, x_3$, where g_{Earth} is the Earth's gravitation, and if $c := c_2$,

(1) it is

$$\frac{\partial_{x_3} c}{c(1-c)} = \frac{g_{\text{Earth}}}{\mathcal{R}\theta}(M_1 - M_2)\,,$$

which is positive if $M_1 > M_2$.

(2) it is

$$\partial_{x_3} c(x_3) = \frac{g_{\text{Earth}}}{4\mathcal{R}\theta}(M_1 - M_2) \quad \text{if} \quad c(x_3) = \frac{1}{2}\,,$$

which goes to $+\infty$, if $M_1 - M_2 \to \infty$.

(3) is $\varrho_1 = \widehat{\varrho}_1(x_3)$ and satisfies

$$\log \varrho_1 = -\frac{g_{\text{Earth}}}{\mathcal{R}\theta} M_1 x_3 + \text{const.}$$

Wir betrachten hier nur eindimensionale Lösungen, und die Gravitation ist durch eine lineare Funktion approximiert. Siehe dazu Fig. 28.

Proof (1). It is from the equation in 9.13

$$\begin{aligned}\Big(\log \frac{c}{1-c}\Big)_{\prime x_3} &= \Big(\frac{1}{a_1} - \frac{1}{a_2}\Big) \cdot (-\mathfrak{g}\phi(x))_{\prime x_3} \\ &= \Big(\frac{1}{a_1} - \frac{1}{a_2}\Big) g_{\text{Earth}} = \frac{g_{\text{Earth}}}{\mathcal{R}\theta}(M_1 - M_2)\end{aligned}$$

and

$$\Big(\log \frac{c}{1-c}\Big)_{\prime x_3} = \frac{c_{\prime x_3}}{c(1-c)}\,.$$

□

Proof (3). It is

$$\begin{aligned}\log \varrho_1 &= \log \varrho + \log c_1 = \big(\frac{a_1}{a_2 - a_1} + 1\big)\log c_1 + \frac{a_2}{a_1 - a_2}\log c_2 \\ &= \underbrace{\big(\frac{a_1}{a_2 - a_1} + 1 + \frac{a_2}{a_1 - a_2}\big)}_{=0} \log c_1 + \frac{a_2}{a_1 - a_2} \underbrace{\log \frac{c_2}{c_1}}_{= \log \frac{c}{1-c}} + \text{const} \\ &= \frac{a_2}{a_1 - a_2}\Big(\frac{1}{a_1} - \frac{1}{a_2}\Big) \cdot (-\mathfrak{g}\phi(x)) + \text{const} = -\frac{1}{a_1} \cdot (-\mathfrak{g}\phi(x)) + \text{const} \\ &= -\frac{1}{R^1\theta} g_{\text{Earth}} x_3 + \text{const} = -\frac{M_1 g_{\text{Earth}}}{\mathcal{R}\theta} x_3 + \text{const.}\end{aligned}$$

□

10 Unsteady mixtures

Wir betrachten die Mischung von verschiedenen Gasen und benutzen die Theorie, die in Abschnitt 9 entwickelt wurde. Wir hatten dort als Anwendung Equilibria untersucht, d.h. stationäre Lösungen, die sich in Ruhe befinden, also mit $v = 0$ und relativer Bewegung $u_\alpha = 0$. Hier betrachten wir nun Gase, für die die relative Bewegungen $u_\alpha \neq 0$ sind, aber nach wie vor nehmen wir $S_\alpha = 0$, also $\Pi_\alpha = p_\alpha \mathrm{Id}$, an.

References: Bothe & Dreyer [26].

Zwei gegenläufige Substanzen

Here in the case $s_\alpha = 0$ we consider only two substances so that u_α have opposite direction, that is, one substance is moving in the opposite direction of the other. Hence the mass-momentum-energy equations (IV9.25) are equivalent to

$$\begin{aligned} \mathrm{div}(\varrho_\alpha u_\alpha) &= 0\,, \\ \mathrm{div}(\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}} + p_\alpha \mathrm{Id}) &= \mathbf{f}_\alpha\,, \\ \mathrm{div} q + \sum_\alpha p_\alpha \,\mathrm{div} u_\alpha + \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha &= 0 \end{aligned} \tag{IV10.1}$$

for $\alpha = 1, 2$, where $\mathbf{f}_\alpha$ has the representation in (IV9.24), that is,

$$\begin{aligned} \mathbf{f}_\alpha &:= f'_{\varrho_\alpha} \nabla \varrho_\alpha - f \nabla c_\alpha + c_\alpha \lambda\,, \\ \lambda &:= \mathbf{f} - \sum_\beta f'_{\varrho_\beta} \nabla \varrho_\beta\,, \quad \mathbf{f} = \sum_\alpha \mathbf{f}_\alpha\,. \end{aligned} \tag{IV10.2}$$

Da $\varrho_1 u_1 + \varrho_2 u_2 = 0$ definieren wir

$$w := \varrho_1 u_1 = -\varrho_2 u_2\,, \quad u_1 = \frac{w}{\varrho_1}\,, \quad u_2 = -\frac{w}{\varrho_2}\,,$$

also sind die Massenerhaltungen $\mathrm{div}(\varrho_\alpha u_\alpha) = 0$ äquivalent zu $\mathrm{div} w = 0$. Die Impulserhaltung für α wird zu

$$\mathbf{f}_\alpha - \nabla p_\alpha = \mathrm{div}(\varrho_\alpha u_\alpha {u_\alpha}^{\mathrm{T}}) = \mathrm{div}\big(\frac{1}{\varrho_\alpha} w w^{\mathrm{T}}\big)\,,$$

also gelten für die Massen- und Impulserhaltung die Gleichungen

$$\begin{aligned} \mathrm{div} w &= 0\,, \\ \mathrm{div}\big(\frac{1}{\varrho_\alpha} w w^{\mathrm{T}}\big) &= \mathbf{f}_\alpha - \nabla p_\alpha \ \text{ für } \alpha = 1, 2. \end{aligned} \tag{IV10.3}$$

Daraus folgt

10.1 Lemma. Für zwei Substanzen sind die Massen und Impulserhaltung äquivalent zu

$$\mathrm{div} w = 0\,,$$
$$\mathrm{div}\Big(\big(\frac{1}{\varrho_1} + \frac{1}{\varrho_2}\big) w\, w^{\mathrm{T}}\Big) = \mathbf{f} - \nabla p\,,$$
$$\frac{1}{\varrho_1}\,\mathrm{div}\big(\frac{1}{\varrho_1} w\, w^{\mathrm{T}}\big) + \nabla p_1^{\mathrm{sp}} = \frac{1}{\varrho_2}\,\mathrm{div}\big(\frac{1}{\varrho_2} w\, w^{\mathrm{T}}\big) + \nabla p_2^{\mathrm{sp}}\,,$$

where p is defined as in (IV9.26).

This is identical to the equations (IV9.32) in the case of equilibria, that is, to the case $w = 0$.

Proof. Die Summe der zweiten Gleichung in (IV10.3) ergibt die zweite Gleichung. Zur Herleitung der dritten Gleichung schreiben wir 9.7 als

$$\begin{aligned}&\mathbf{f}_\alpha - \nabla p_\alpha - c_\alpha \sum_\beta (\mathbf{f}_\beta - \nabla p_\beta)\\ &= \mathbf{f}_\alpha - \nabla p_\alpha - c_\alpha\big(\mathbf{f} - \nabla p\big) = -\varrho_\alpha\big(\nabla p_\alpha^{\mathrm{sp}} - \sum_\beta c_\beta \nabla p_\beta^{\mathrm{sp}}\big)\,,\end{aligned}$$

und daher für $\alpha = 1$

$$c_2(\mathbf{f}_1 - \nabla p_1) - c_1(\mathbf{f}_2 - \nabla p_2) = -\varrho_1 c_2 \nabla p_1^{\mathrm{sp}} + \varrho_1 c_2 \nabla p_2^{\mathrm{sp}}\,,$$

was äquivalent ist zu

$$c_2\big(\mathbf{f}_1 - \nabla p_1 + \varrho_1 \nabla p_1^{\mathrm{sp}}\big) = c_1\big(\mathbf{f}_2 - \nabla p_2 + \varrho_2 \nabla p_2^{\mathrm{sp}}\big)\,.$$

Division durch $\varrho c_1 c_2$ ergibt

$$\frac{1}{\varrho_1}(\mathbf{f}_1 - \nabla p_1) + \nabla p_1^{\mathrm{sp}} = \frac{1}{\varrho_2}(\mathbf{f}_2 - \nabla p_2) + \nabla p_2^{\mathrm{sp}}\,,$$

was mit (IV10.3) äquivalent zur dritten Gleichung ist. □

Die Gleichungen (IV10.1) sind auch äquivalen zu den Folgenden.

10.2 Lemma. Für zwei Substanzen ist das Masse-Impuls-Energie System (IV10.1) äquivalent zu

$$\mathrm{div}\, w = 0\,, \qquad \mathrm{div}\, q = 0\,,$$
$$\nabla p_\alpha + \mathrm{div}\big(\frac{1}{\varrho_\alpha} w\, w^{\mathrm{T}}\big) = c_\alpha \mathbf{f} + p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha - c_\alpha \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta$$

für $\alpha = 1, 2$.

Proof. The identity (IV10.2) gives for the partial forces

$$\mathbf{f}_\alpha - c_\alpha \mathbf{f} = f'_{\varrho_\alpha} \nabla \varrho_\alpha - f \nabla c_\alpha - c_\alpha \sum_\beta f'_{\varrho_\beta} \nabla \varrho_\beta \,,$$

and since the specific pressures satisfy

$$p_\alpha^{\mathrm{sp}} = f'_{\varrho_\alpha} - \frac{f}{\varrho} \,,$$

we obtain

$$\begin{aligned}
& p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha - c_\alpha \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta \\
&= f'_{\varrho_\alpha} \nabla \varrho_\alpha - \frac{f}{\varrho} \nabla \varrho_\alpha - c_\alpha \sum_\beta f'_{\varrho_\beta} \nabla \varrho_\beta + c_\alpha \frac{f}{\varrho} \sum_\beta \nabla \varrho_\beta \\
&= f'_{\varrho_\alpha} \nabla \varrho_\alpha - f \underbrace{\Big(\frac{1}{\varrho} \nabla \varrho_\alpha + \frac{1}{\varrho} c_\alpha \nabla \varrho \Big)}_{= \nabla c_\alpha} - c_\alpha \sum_\beta f'_{\varrho_\beta} \nabla \varrho_\beta \,,
\end{aligned}$$

hence

$$\mathbf{f}_\alpha - c_\alpha \mathbf{f} = p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha - c_\alpha {\textstyle\sum_\beta} p_\beta^{\mathrm{sp}} \nabla \varrho_\beta \,. \tag{IV10.4}$$

Now the second equation of (IV10.1) gives by (IV10.4)

$$\begin{aligned}
& \nabla p_\alpha + \operatorname{div}\Big(\frac{1}{\varrho_\alpha} w\, w^{\mathrm{T}} \Big) - c_\alpha \mathbf{f} \\
&= \mathbf{f}_\alpha - c_\alpha \mathbf{f} = p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha - c_\alpha \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta \,,
\end{aligned}$$

which is part of the assertion. Since, using (IV10.4),

$$\begin{aligned}
& \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha = \sum_\alpha u_\alpha \bullet \Big(c_\alpha \mathbf{f} + p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha - c_\alpha \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta \Big) \\
&= \underbrace{\sum_\alpha c_\alpha u_\alpha}_{=0} \bullet \Big(\mathbf{f} - \sum_\beta p_\beta^{\mathrm{sp}} \nabla \varrho_\beta \Big) + \sum_\alpha u_\alpha \bullet (p_\alpha^{\mathrm{sp}} \nabla \varrho_\alpha) = \sum_\alpha p_\alpha^{\mathrm{sp}} u_\alpha \bullet \nabla \varrho_\alpha \,,
\end{aligned}$$

the third equation of (IV10.1) gives, using $\operatorname{div} w = 0$,

$$\begin{aligned}
& -\operatorname{div} q = \sum_\alpha p_\alpha \operatorname{div} u_\alpha + \sum_\alpha u_\alpha \bullet \mathbf{f}_\alpha \\
&= \sum_\alpha p_\alpha \operatorname{div} u_\alpha + \sum_\alpha p_\alpha^{\mathrm{sp}} u_\alpha \bullet \nabla \varrho_\alpha \\
&= p_1 w \bullet \nabla \Big(\frac{1}{\varrho_1} \Big) - p_2 w \bullet \nabla \Big(\frac{1}{\varrho_2} \Big) + \frac{p_1^{\mathrm{sp}}}{\varrho_1} w \bullet \nabla \varrho_1 - \frac{p_2^{\mathrm{sp}}}{\varrho_2} w \bullet \nabla \varrho_2 = 0 \,,
\end{aligned}$$

which is the energy equation. □

Hence the momentum balance of the α-th constituent has as force the α-part of the total force plus and an additional term whose sum over α is 0. Now we consider ideal Gases as in 9.11 and 9.12.

10.3 Ideale Mischung aus zwei Substanzen. Betrachte eine Mischung aus idealen Gasen, d.h. mit der freien Energie in (IV9.36). Außerdem sei $q = 0$ in isothermen Fall, d.h. $\theta = \text{const}$. Dann sind die Differentialgleichungen in 10.2 äquivalent zu der Massenerhaltung $\text{div}w = 0$ und den Impulserhaltungen

$$a_\alpha \nabla \varrho_\alpha + \text{div}\big(\frac{1}{\varrho_\alpha} w\, w^{\text{T}}\big) = c_\alpha \mathbf{f} \quad \text{für } \alpha = 1, 2.$$

Proof. Betrachte die Impulserhaltung von 10.2

$$\nabla p_\alpha - \big(p_\alpha^{\text{sp}} \nabla \varrho_\alpha - c_\alpha \sum_\beta p_\beta^{\text{sp}} \nabla \varrho_\beta\big) = -\,\text{div}\big(\frac{1}{\varrho_\alpha} w\, w^{\text{T}}\big) + c_\alpha \mathbf{f}\,.$$

Wegen $p_\alpha = \varrho_\alpha p_\alpha^{\text{sp}}$ ist

$$\nabla p_\alpha = \nabla(\varrho_\alpha p_\alpha^{\text{sp}}) = p_\alpha^{\text{sp}} \nabla \varrho_\alpha + \varrho_\alpha \nabla p_\alpha^{\text{sp}}\,,$$

also ist die linke Seite der Impulserhaltung

$$= \varrho_\alpha \nabla p_\alpha^{\text{sp}} + c_\alpha \sum_\beta p_\beta^{\text{sp}} \nabla \varrho_\beta = \sum_\beta \big(\varrho_\alpha p^{\text{sp}}_{\alpha}{}'_{\varrho_\beta} + c_\alpha p_\beta^{\text{sp}}\big) \nabla \varrho_\beta + p^{\text{sp}}_{\alpha}{}'_{\theta} \nabla \theta\,,$$

da $p_\alpha^{\text{sp}} = \widehat{p}_\alpha^{\text{sp}}((\varrho_\beta)_\beta, \theta)$. Dies ist deshalb richtig, da die freie Energie (IV9.36) impliziert, dass

$$\begin{aligned} p_\alpha^{\text{sp}} &= a_\alpha + h_\alpha - \sum_\gamma c_\gamma h_\gamma\,, \quad a_\alpha(\theta) := R^\alpha \theta \\ h_\alpha &:= a_\alpha(\theta) \log \varrho_\alpha - b_\alpha(\theta)\,, \quad b_\alpha(\theta) := c_V^\alpha \theta \log \theta - d^\alpha \theta \end{aligned}$$

(siehe den Beweis von 9.11, eine Aussage, die in der Situation hier gilt). Nun ist $\nabla\theta = 0$, da $\theta = \text{const}$ angenommen wurde. also ist die linke Seite obiger Impulserhaltung

$$= \sum_\beta \big(\varrho_\alpha p^{\text{sp}}_{\alpha}{}'_{\varrho_\beta} + c_\alpha p_\beta^{\text{sp}}\big) \nabla \varrho_\beta = \varrho_\alpha \sum_\beta \big(p^{\text{sp}}_{\alpha}{}'_{\varrho_\beta} + \frac{1}{\varrho} p_\beta^{\text{sp}}\big) \nabla \varrho_\beta\,.$$

Wegen

$$\begin{aligned} p^{\text{sp}}_{\alpha}{}'_{\varrho_\beta} &= h_\alpha{}'_{\varrho_\beta} - \sum_\gamma c_\gamma h_\gamma{}'_{\varrho_\beta} - \sum_\gamma c_\gamma{}'_{\varrho_\beta} h_\gamma \\ &= \frac{a_\alpha}{\varrho_\alpha}\delta_{\alpha,\beta} - c_\beta \frac{a_\beta}{\varrho_\beta} - \sum_\gamma \big(\frac{\delta_{\gamma,\beta}}{\varrho} - \frac{\varrho_\gamma}{\varrho^2}\big) h_\gamma \\ &= \frac{a_\alpha}{\varrho_\alpha}\delta_{\alpha,\beta} - \frac{a_\beta}{\varrho} - \frac{h_\beta}{\varrho} + \sum_\gamma \frac{\varrho_\gamma}{\varrho^2} h_\gamma = \frac{a_\alpha}{\varrho_\alpha}\delta_{\alpha,\beta} - \frac{p_\beta^{\text{sp}}}{\varrho} \end{aligned}$$

ist

$$p^{\mathrm{sp}}_{\alpha}{}'_{\varrho_\beta} + \frac{p^{\mathrm{sp}}_{\beta}}{\varrho} = \frac{a_\alpha}{\varrho_\alpha}\delta_{\alpha,\beta}\,,$$

also die linke Seite obiger Impulserhaltung

$$= \varrho_\alpha \sum_\beta \frac{a_\alpha}{\varrho_\alpha}\delta_{\alpha,\beta}\nabla\varrho_\beta = a_\alpha\nabla\varrho_\alpha\,,$$

also die Behauptung. □

10.4 Bemerkung. Im Falle $w = 0$ lautet 10.3

$$a_\alpha\nabla\varrho_\alpha = c_\alpha\mathbf{f} \quad \text{für } \alpha = 1,2.$$

Dies ist unter den Voraussetzungen in 9.13 äquivalent zu den beiden Gleichungen in 9.13.

Proof. .... □

In Bearbeitung

....

Entropy principle for a systems of reacting substances

We now consider the general case, where the substances will react with each other or/and will diffuse, that is,

$$\mathbf{r}_\alpha \text{ arbitrary}, \quad \mathbf{J}_\alpha \text{ arbitrary}, \quad \text{hence} \quad \widetilde{\mathbf{f}} = \sum_\alpha \widetilde{\mathbf{f}}_\alpha\,. \tag{IV10.5}$$

In this situation the pressure tensors Π_α and consequently also Π_{mix} are assumed to be nonsymmetric. Replacing the energy equation in (IV9.11) by the thermal energy equation in 9.4, the system to solve is

$$\boxed{\begin{aligned} &\partial_t\varrho_\alpha + \operatorname{div}(\varrho_\alpha v_\alpha + \mathbf{J}_\alpha) = \mathbf{r}_\alpha\,,\\ &\partial_t(\varrho_\alpha v_\alpha) + \operatorname{div}(\varrho_\alpha v_\alpha {v_\alpha}^{\mathrm{T}} + v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) = \widetilde{\mathbf{f}}_\alpha\,,\\ &\qquad\qquad \widetilde{\mathbf{f}}_\alpha := \mathbf{r}_\alpha v_\alpha + \mathrm{D}v_\alpha\mathbf{J}_\alpha + \mathbf{f}_\alpha\,,\\ &\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) = g - \sum_\alpha \mathrm{D}v_\alpha \mathbin{:} \Pi_\alpha^{sym}\\ &-\sum_\alpha\Big(\frac{\mathbf{r}_\alpha}{2}|u_\alpha|^2 + \mathrm{D}u_\alpha \mathbin{:} (\Pi_\alpha^{rest} + u_\alpha{\mathbf{J}_\alpha}^{\mathrm{T}}) + u_\alpha\bullet\mathbf{f}_\alpha\Big)\,,\end{aligned}} \tag{IV10.6}$$

where g is the quantity which we will set to 0 in order to satisfy the energy principle for the entire energy

$$e = \varepsilon + \sum_\alpha \frac{\varrho_\alpha}{2}|v_\alpha|^2 = \Big(\varepsilon + \sum_\alpha\frac{\varrho_\alpha}{2}|u_\alpha|^2\Big) + \frac{\varrho}{2}|v|^2$$

and ε will be the internal energy independent of the $u_\alpha = v_\alpha - v$, that is, we will have the constitutive equation $\varepsilon = \widehat{\varepsilon}((\varrho_\beta)_\beta, \theta)$. But the main part is to exploit the entropy principle, where we will also assume that $\eta = \widehat{\eta}((\varrho_\beta)_\beta, \theta)$. Therefore we need in analogy to 9.5

10.5 Theorem. System (IV10.6) is equivalent to the system

$$\begin{aligned}
&\overset{\circ}{\varrho}_\alpha + u_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha + \operatorname{div} \mathbf{J}_\alpha = \mathbf{r}_\alpha \,, \\
&\varrho_\alpha(\overset{\circ}{v}_\alpha + u_\alpha \bullet \nabla v_\alpha) + \operatorname{div}_x \Pi_\alpha = \mathbf{f}_\alpha \,, \\
&\overset{\circ}{\varepsilon} + \varepsilon \operatorname{div}_x v + \operatorname{div}_x q = g - \sum_\alpha \mathrm{D} v_\alpha : \Pi_\alpha^{sym} \\
&- \sum_\alpha \Big(\frac{\mathbf{r}_\alpha}{2} |u_\alpha|^2 + \mathrm{D} u_\alpha : (\Pi_\alpha^{rest} + u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}}) + u_\alpha \bullet \mathbf{f}_\alpha \Big) \,.
\end{aligned}$$

Here $\overset{\circ}{h} := (\partial_t + v \bullet \nabla_x) h$ for every function h.

Remark: The right-hand side of the thermal energy equation is an objective scalar, see the proof of 9.4.

Proof. The mass equation of species α is with $v_\alpha = v + u_\alpha$

$$\begin{aligned}
\mathbf{r}_\alpha &= \partial_t \varrho_\alpha + v_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha + \operatorname{div} \mathbf{J}_\alpha \\
&= (\partial_t + v \bullet \nabla) \varrho_\alpha + u_\alpha \bullet \nabla \varrho_\alpha + \varrho_\alpha \operatorname{div}_x v_\alpha + \operatorname{div} \mathbf{J}_\alpha \,,
\end{aligned}$$

the momentum equation

$$\begin{aligned}
\widetilde{\mathbf{f}}_\alpha &= \underbrace{\big(\partial_t \varrho_\alpha + \operatorname{div}_x(\varrho_\alpha v_\alpha) \big)}_{= -\operatorname{div}_x \mathbf{J}_\alpha + \mathbf{r}_\alpha} v_\alpha + \varrho_\alpha(\partial_t v_\alpha + v_\alpha \bullet \nabla v_\alpha) + \operatorname{div}_x(v_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} + \Pi_\alpha) \\
&= \mathbf{r}_\alpha v_\alpha + \varrho_\alpha(\partial_t v_\alpha + v \bullet \nabla v_\alpha + u_\alpha \bullet \nabla v_\alpha) + (\mathrm{D}_x v_\alpha) \mathbf{J}_\alpha + \operatorname{div}_x \Pi_\alpha \\
&= \mathbf{r}_\alpha v_\alpha + (\mathrm{D}_x v_\alpha) \mathbf{J}_\alpha + \varrho_\alpha \overset{\circ}{v}_\alpha + \varrho_\alpha u_\alpha \bullet \nabla v_\alpha + \operatorname{div}_x \Pi_\alpha \,,
\end{aligned}$$

hence

$$\mathbf{f}_\alpha = \varrho_\alpha(\overset{\circ}{v}_\alpha + u_\alpha \bullet \nabla v_\alpha) + \operatorname{div}_x \Pi_\alpha \,.$$

And the energy equation in the form of (IV10.6) with

$$\partial_t \varepsilon + \operatorname{div}_x(\varepsilon v) = (\partial_t + v \bullet \nabla) \varepsilon + \varepsilon \operatorname{div}_x v = \overset{\circ}{\varepsilon} + \varepsilon \operatorname{div}_x v \,.$$

□

We assume that the entropy depends on the same variables as before, that is (IV9.15)

$$\eta = \widehat{\eta}((\varrho_\beta)_\beta, \varepsilon) \tag{IV10.7}$$

is satisfied. Then defining

$$M_\alpha := \Pi_\alpha^{rest} + u_\alpha {\mathbf{J}_\alpha}^{\mathrm{T}} \,, \quad M := \textstyle\sum_\beta M_\beta \,, \tag{IV10.8}$$

(finally M_α will be chosen to be 0) we obtain

In Bearbeitung

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11 Reaction-diffusion systems

Wir betrachten ein System von Massen mit Massendichten ϱ_α (mit $\alpha = 1, \ldots, m$), für die wir die Massenbilanz

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \ldots, m \tag{IV11.1}$$

fordern. Dies heißt, dass die einzelnen Massen miteinander reagieren (mit

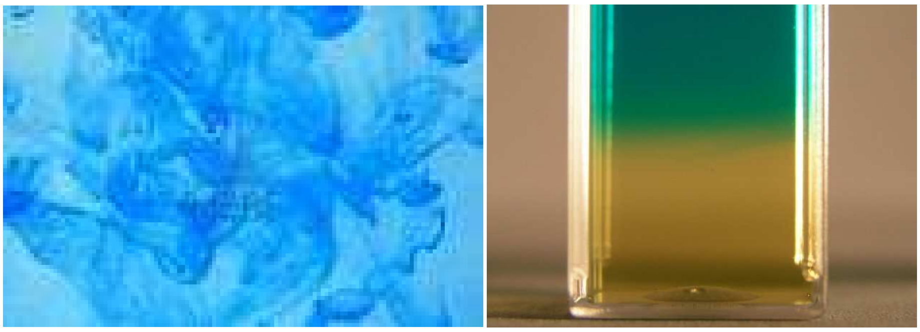

Fig. 29: *Left:* Movement of mixture. *Right:* Diffusion

Rate $\mathbf{r}_\alpha$), und dass sie räumlich expandieren, also diffundieren (mit Fluss $\mathbf{J}_\alpha$). Wir werden hier von einer Mischung der Klasse I im Sinne von Abschnitt III.3 ausgehen, d.h. die Massen werden (nur) mit einer Geschwindigkeit v transportiert, und es ist die Frage, was die Geschwindigkeit v ist. Wir werden in diesem Abschnitt die folgenden Geschwindigkeitsmittel von III.3.1 behandeln:

- Geschwindigkeit einer dominierenden Komponente (siehe III.3.1(3)),
- Baryzentrische Geschwindigkeit (siehe III.3.1(1)).

Referenzen: Landau & Lifschitz [10, Kapitel VI Diffusion], DeGroot & Mazur [6, Chap. XI Heat Conduction, Diffusion and Cross-Effects], I. Müller [87, 6.6.1.1 Phenomenological equations]. Und aus der mathematischen Literatur mit vielen Beispielen B. Perthame [90] und G.R. Gavalas [41, Chap. 2 Distributed Chemical Reaction Systems].

Insgesamt werden zu den Gleichungen in (IV11.1) für die Gesamtmasse

$$\varrho := \sum_\alpha \varrho_\alpha$$

eine einzelne Impulserhaltung und eine gemeinsame Energieerhaltung gefordert, so dass wir also ein System aus $m+2$ Gleichungen erhalten:

Allgemeines Diffusionssystem:

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1,\dots,m\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + v\,\mathbf{J}^{\mathrm{T}} + \Pi) = \widetilde{\mathbf{f}}\,,$$
$$\partial_t e + \operatorname{div}(ev + \frac{1}{2}|v|^2\mathbf{J} + \Pi^{\mathrm{T}}\, v + q) = \widetilde{g}\,, \tag{IV11.2}$$

Π symmetrischer Drucktensor (siehe weiter unten),

$e = \varepsilon + \frac{\varrho}{2}|v|^2$ Energie,

zu $\mathbf{J}$, $\mathbf{r}$, $\widetilde{\mathbf{f}}$, $\widetilde{g}$ siehe (IV11.3) und (IV11.4).

Hier ist die Energiebilanz aufgeführt, was einen Grund hat, nämlich dass die Temperatur bei Reaktionen teilweise eine wesentliche Rolle spielt, so etwa bei der Flammenausbreitung. Aber auch das Zusammenschütten zweier Substanzen, die dann reagieren, geschieht in der Regel mit einer Temperaturänderung. Die rechten Seiten $\widetilde{f}$ und $\widetilde{g}$ erfüllen (II3.32), da Π symmetrisch ist,

$$\widetilde{\mathbf{f}} = (\mathbf{r} + \mathbf{J}\bullet\nabla)v + \mathbf{f}\,, \quad \mathbf{f} \text{ a } \textbf{\textit{force}} \text{ (see II.3.8)},$$
$$\widetilde{g} = \frac{\mathbf{r}}{2}|v|^2 + v\bullet \mathrm{D}v\mathbf{J} + v\bullet\mathbf{f} + g\,, \quad g \text{ an objective scalar,} \tag{IV11.3}$$

und $\mathbf{J}$ und $\mathbf{r}$ sind die aufsummierten Größen

$$\mathbf{J} := \textstyle\sum_\alpha \mathbf{J}_\alpha\,, \quad \mathbf{r} := \sum_\alpha \mathbf{r}_\alpha\,. \tag{IV11.4}$$

Es sei bemerkt, dass die Massenerhaltung für die Gesamtmasse lautet

$$\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \tag{IV11.5}$$

was durch Aufsummieren der Gleichungen für die Partialmassen folgt. Diese Massengleichung (IV11.5) und die Impuls-Energie Gleichungen transformieren sich wie das Masse-Impuls-Energie-System in III.2.1 bzw. in II.3.13. Das bestimmt die vorhandenen Größen Π als objektiven Tensor und q als objektiven Vektor. Wenn wir den abhängigen Erhaltungssatz (IV11.5) fordern, bleiben noch $m-1$ unabhängige Gleichungen für die einzelnen Komponenten und die folgende Versionen der Impulserhaltung und der Energieerhaltung für die innere Energie ε (siehe III.2.3):

$$\partial_t \varrho + \operatorname{div}(\varrho v + \mathbf{J}) = \mathbf{r}\,, \quad \varrho \text{ Gesamtmasse,}$$
$$\varrho(\partial_t v + v\bullet\nabla v) + \operatorname{div}\Pi = \mathbf{f}\,, \quad \Pi \text{ symmetrisch,} \tag{IV11.6}$$
$$\partial_t \varepsilon + \operatorname{div}(\varepsilon v + q) = -\mathrm{D}v\bullet\Pi + g\,, \quad \varepsilon \text{ innere Energie.}$$

Schließlich postulieren wir das Entropieprinzip

$$\sigma := \partial_t \eta + \mathrm{div}\psi \geq 0 \tag{IV11.7}$$

für Lösungen des Systems (IV11.2), wobei η die Entropie und ψ der Entropiefluss ist, beides sind Größen, die wir noch zu bestimmen haben. Bei dieser Bestimmung sind folgende Größen von eminenter Bedeutung:

11.1 Chemical potential. Falls für die Entropie $\eta = \widehat{\eta}(\vec{\varrho}, \varepsilon)$ gilt, wobei $\vec{\varrho} = (\varrho_\alpha)_\alpha$ der Vektor der Dichten, ist das ***chemische Potential*** μ_α der Spezies $\alpha \in \{1, \ldots, m\}$ gegeben durch

$$\boxed{\mu_\alpha := -\theta \eta'_{\varrho_\alpha} = f'_{\varrho_\alpha} \,,}$$

wobei entsprechend $f = \widehat{f}(\vec{\varrho}, \theta)$ mit $f = \varepsilon - \theta\eta$ gilt und wie immer die ***Temperatur*** θ gegben ist durch

$$\frac{1}{\theta} := \eta'_\varepsilon(\vec{\varrho}, \varepsilon) \,.$$

Hinweis: Für eine einzelne Substanz hatten wir in 3.2 schon eine Definition gegeben. In 11.9 werden wir noch eine allgemeinere Definition machen für den Fall, dass die konstitutiven Funktionen noch von den Gradienten der Massendichten abhängen.

Gas in a porous Medium

Wir betrachten die Diffusion eines Gases in einem festen Körper. Es sei ein fester Körper mit Dichte ϱ_s und eine Flüssigkeit oder ein Gas mit der Dichte ϱ_g gegeben. Die Diffusionsgleichungen lauten (mit $m = 2$, $\varrho_1 = \varrho_g$ die Dichte des Gases, $\varrho_2 = \varrho_s$ die Dichte des festen Körpers)

$$\begin{aligned}
&\partial_t \varrho_g + \mathrm{div}(\varrho_g v + \mathbf{J}_g) = 0 \,, \\
&\partial_t \varrho_s + \mathrm{div}(\varrho_s v) = 0 \,, \\
&\varrho = \varrho_s + \varrho_g \,, \\
&\varrho(\partial_t v + v \bullet \nabla v) + \mathrm{div}\Pi = \mathbf{f} \,, \\
&\partial_t \varepsilon + \mathrm{div}(\varepsilon v + q) = -\mathrm{D}v \bullet \Pi \,.
\end{aligned} \tag{IV11.8}$$

Here we have written the momentum equation and the energy equation as proved in in III.2.3 (with $\mathbf{J} = \mathbf{J}_g$ and $g = 0$). Further, Π is assumed to be symmetric and $\mathbf{f}$ is the classical force. We suppose that the entropy principle holds for solutions of (IV11.8).

11.2 Entropieprinzip (Gas im festen Körper). Das Entropieprinzip gilt für das System (IV11.8), falls für die Entropiedichte und den Entropiefluss

$$\eta = \widehat{\eta}(\varrho_g, \varrho_s, \varepsilon)\,, \quad \psi = \eta v + \eta'_{\varrho_g}\mathbf{J}_g + \eta'_{\varepsilon} q\,,$$

und in der Impuls- und Energiegleichung (IV11.8)

$$\Pi = p\mathrm{Id}\,, \quad p = \theta(\eta - \varrho_g \eta'_{\varrho_g} - \varrho_s \eta'_{\varrho_s}) - \varepsilon\,, \quad g = 0\,,$$

und wenn die Residualungleichung

$$\sigma = \nabla\eta'_{\varrho_g} \bullet \mathbf{J}_g + \nabla\eta'_{\varepsilon} \bullet q \geq 0$$

erfüllt ist. Mit den Identitäten (siehe 11.1)

$$\frac{1}{\theta} = \eta'_{\varepsilon}(\vec{\varrho}, \varepsilon)\,, \quad -\frac{\mu_g}{\theta} = \eta'_{\varrho_g}(\vec{\varrho}, \varepsilon)$$

muss

$$\sigma = -\nabla\big(\frac{\mu_g}{\theta}\big) \bullet \mathbf{J}_g + \nabla\big(\frac{1}{\theta}\big) \bullet q \geq 0$$

sein. Das ist zum Beispiel erfüllt, wenn skalare Funktionen $d \geq 0$ und $k \geq 0$ existieren mit

$$\mathbf{J}_g = -d\,\nabla\big(\frac{\mu_g}{\theta}\big)\,, \quad q = k\nabla\big(\frac{1}{\theta}\big) = -\frac{k}{\theta^2}\nabla\theta\,.$$

Proof. Die Massengleichungen und die Energiegleichung im System können geschrieben werden als

$$\begin{aligned}
\overset{\circ}{\varrho}_g + \varrho_g \,\mathrm{div} v &= -\,\mathrm{div}\mathbf{J}_g\,,\\
\overset{\circ}{\varrho}_s + \varrho_s \,\mathrm{div} v &= 0\,,\\
\overset{\circ}{\varepsilon} + \varepsilon \,\mathrm{div} v &= -\mathrm{D}v \mathbin{:} \Pi - \,\mathrm{div} q + g\,.
\end{aligned}$$

Da $\eta = \widehat{\eta}(\varrho_g, \varrho_s, \varepsilon)$ sagt das Entropieprinzip, dass

$$\begin{aligned}
0 \leq \sigma &= \partial_t \eta + \,\mathrm{div}\psi = \overset{\circ}{\eta} + \eta\,\mathrm{div} v + \,\mathrm{div}(\psi - \eta v)\\
&= \eta'_{\varrho_g}\overset{\circ}{\varrho}_g + \eta'_{\varrho_s}\overset{\circ}{\varrho}_s + \eta'_{\varepsilon}\overset{\circ}{\varepsilon} + \eta\,\mathrm{div} v + \,\mathrm{div}(\psi - \eta v)\\
&= (\eta - \varrho_g\eta'_{\varrho_g} - \varrho_s\eta'_{\varrho_s} - \varepsilon\eta'_{\varepsilon})\,\mathrm{div} v - \eta'_{\varepsilon}\mathrm{D}v \mathbin{:} \Pi\\
&\qquad -\eta'_{\varrho_g}\,\mathrm{div}\mathbf{J}_g - \eta'_{\varepsilon}\,\mathrm{div} q + \eta'_{\varepsilon} g + \,\mathrm{div}(\psi - \eta v)\\
&= \mathrm{D}v \mathbin{:} \Big((\eta - \varrho_g\eta'_{\varrho_g} - \varrho_s\eta'_{\varrho_s} - \varepsilon\eta'_{\varepsilon})\mathrm{Id} - \eta'_{\varepsilon}\Pi\Big)\\
&\qquad\qquad + \nabla\eta'_{\varrho_g} \bullet \mathbf{J}_g + \nabla\eta'_{\varepsilon} \bullet q + \eta'_{\varepsilon} g\\
&\qquad\qquad + \,\mathrm{div}(\psi - \eta v - \eta'_{\varrho_g}\mathbf{J}_g - \eta'_{\varepsilon} q)\,.
\end{aligned}$$

The representation of Π and ψ in the statement give that the first term and the last term vanish. Thus

$$\sigma = \nabla\eta'_{\varrho_g} \bullet \mathbf{J}_g + \nabla\eta'_{\varepsilon} \bullet q + \eta'_{\varepsilon} g\,.$$

Finally, we choose $g = 0$ in accordance with the energy conservation. □

Wenn wir nun zum isothermen Fall $\theta = \text{const}$ übergehen, so wird die Energiegleichung zusammen mit der Entropieungleichung ersetzt durch die freie Energieungleichung (siehe dazu III.5.4 und III.5.5), also bleiben zur Beschreibung des Systems nur die Massenerhaltungen und die Impulserhaltung übrig, d.h. mit $\varrho := \varrho_g + \varrho_s$

$$\begin{aligned} &\partial_t \varrho_g + \operatorname{div}(\varrho_g v + \mathbf{J}_g) = 0\,, \\ &\partial_t \varrho_s + \operatorname{div}(\varrho_s v) = 0\,, \\ &\varrho(\partial_t v + v\bullet\nabla v) + \operatorname{div}\Pi = \mathbf{f}\,. \end{aligned} \tag{IV11.9}$$

Zusätzlich haben wir die freie Energieungleichung III.5.4 zu berücksichtigen, die wir nun unabhängig diskutieren.

11.3 Freie Energieungleichung. Für das System (IV11.9) ist die freie Energieungleichung III.5.5 erfüllt, falls

$$f = \widehat{f}(\varrho_g, \varrho_s)\,, \quad \varphi = fv + f'_{\varrho_g}\mathbf{J}_g\,,$$

und in der Impulsgleichung (IV11.9)

$$\Pi = p\mathrm{Id}\,, \quad p = \varrho_g f'_{\varrho_g} + \varrho_s f'_{\varrho_s} - f\,,$$

und wenn die Residualungleichung

$$\sigma_f = \nabla f'_{\varrho_g}\bullet\mathbf{J}_g \le 0$$

erfüllt ist. *Beispiel:* Mit $d \ge 0$ sei etwa $\mathbf{J}_g = -d\,\nabla f'_{\varrho_g}(\varrho_g, \varrho_s)$.

Proof. Für $\varrho = \varrho_g + \varrho_s$ ist $\partial_t\varrho + \operatorname{div}(\varrho v + \mathbf{J}_g) = 0$ die Massengleichung für die gesamte Masse, und die gesamte Impulserhaltung ist in (IV11.9) enthalten. Daraus ergibt sich für die innere freie Energie $f = \widehat{f}(\varrho_g, \varrho_s)$, nach III.5.5 mit $g = 0$, die Ungleichung

$$\begin{aligned} 0 \ge \sigma_f &= \partial_t f + \operatorname{div}\varphi + \mathrm{D}v\!:\!\Pi = \partial_t f + \operatorname{div}(fv + f'_{\varrho_g}\mathbf{J}_g) + \mathrm{D}v\!:\!\Pi \\ &= \overset{\circ}{f} + \operatorname{div}(f'_{\varrho_g}\mathbf{J}_g) + \mathrm{D}v\!:\!(f\mathrm{Id} + \Pi) \\ &= f'_{\varrho_g}\overset{\circ}{\varrho}_g + f'_{\varrho_s}\overset{\circ}{\varrho}_s + \operatorname{div}(f'_{\varrho_g}\mathbf{J}_g) + \mathrm{D}v\!:\!(f\mathrm{Id} + \Pi)\,. \end{aligned}$$

Nun gilt

$$\begin{aligned} &\overset{\circ}{\varrho}_g + \varrho_g \operatorname{div}v + \operatorname{div}\mathbf{J}_g = 0\,, \\ &\overset{\circ}{\varrho}_s + \varrho_s \operatorname{div}v = 0\,, \end{aligned}$$

und daher ist

$$\begin{aligned} \sigma_f &= f'_{\varrho_g}(\overset{\circ}{\varrho}_g + \operatorname{div}\mathbf{J}_g) + f'_{\varrho_s}\overset{\circ}{\varrho}_s + \nabla f'_{\varrho_g}\bullet\mathbf{J}_g + \mathrm{D}v\!:\!(f\mathrm{Id} + \Pi) \\ &= \nabla f'_{\varrho_g}\bullet\mathbf{J}_g + \mathrm{D}v\!:\!\big((f - \varrho_g f'_{\varrho_g} - \varrho_s f'_{\varrho_s})\mathrm{Id} + \Pi\big)\,. \end{aligned}$$

If we choose Π so that the second term vanishes, we are left with

$$0 \geq \sigma_f = \nabla f'_{\varrho_g} \bullet \mathbf{J}_g \,.$$

This proves the theorem. □

Wir hatten noch nichts über die Geschwindigkeit v gesagt. Bei beliebigen Massendichten ϱ_g und ϱ_s gehen wir von einem baryzentrischen Mittel aus. Wenn jedoch ϱ_g sehr viel kleiner als ϱ_s ist, ist v nicht von der Geschwindigkeit des festen Körper zu unterscheiden, also ist v die Geschwindigkeit der dominierenden Komponente des festen Körpers. Wie dem auch sei, wenn wir nun davon ausgehen, dass wir einen starren Körper haben, also $\varrho_s = \text{const}$ ist, und wenn der Beobachter, in den wir uns versetzen, von diesem starren Körper aus die Diffusion beobachtet, also wenn $v = 0$ ist, so bleiben nur noch zwei Gleichungen übrig, die Massenerhaltung für das Gas

$$\partial_t \varrho_g + \text{div}\mathbf{J}_g = 0 \tag{IV11.10}$$

und die davon entkoppelte Impulserhaltung $\text{div}\Pi = \mathbf{f}$. Eine spezielle Lösung für diese erste Gasgleichung wurde von Barenblatt in [97] angegeben und zwar für den nichtlinearen Diffusionsfall. Die Lösung ist von der Gestalt $\varrho_g = u$ und $\mathbf{J}_g = -a(u, \nabla u)$, wobei im einfachsten Fall $f = \widehat{f}(\varrho_g) = \frac{1}{2}|\varrho_g|^2$ die Elliptizität $0 \geq \nabla f'_{\varrho_g} \bullet \mathbf{J}_g$ der Gleichung aus dem Entropieprinzip 11.2 bzw. der freien Energieungleichung 11.3 herrührt. Also gilt die Massenerhaltung, dies falls u stetig differenzierbar ist,

$$\partial_t u - \text{div}\, a(u, \nabla u) = 0\,, \tag{IV11.11}$$

was in 11.6 gezeigt wird. Der Beweis benutzt

11.4 Ähnlichkeitslösungen. Wir suchen Lösungen $(t, x) \mapsto u(t, x)$ von (IV11.11), wobei die Funktion a stetig sei, so dass $u(t, \bullet)$ für alle t "dieselbe Gestalt" hat, d.h. es gibt Funktionen $t \mapsto s(t)$ und $t \mapsto r(t) > 0$, so dass

$$u(t, x) = s(t)\,\widetilde{u}\big(r(t)x\big) \tag{IV11.12}$$

mit einer Funktion $y \mapsto \widetilde{u}(y)$. Hier ist gemeint, dass die Funktionen r und s auf einem Zeitintervall stetig sind, und dass $\widetilde{u}$ eine stetige Funktion auf $\mathbb{R}^n$ ist.

11.5 Theorem. Consider the equation (IV11.11) with the property that there exist $m \geq 1$ and $l > 0$ with

$$a(\sigma w, \sigma\tau q) = \sigma^m \tau^l a(w, q) \text{ for all } \sigma > 0,\ \tau > 0,\ w \in \mathbb{R},\ q \in \mathbb{R}^n.$$

Beachte: Für $\tau \to 0$ folgt $a(w, 0) = 0$ für alle $w \in \mathbb{R}$.
Weiter sei u nicht negativ und selbstähnlich zu $\widetilde{u}$ wie in (IV11.12) und $\nabla_y \widetilde{u}(y)$ existiere klassisch in fast allen Punkten $y \in \mathbb{R}^n$. Darüberhinaus habe $\widetilde{u}$ ein geeignetes Abklingverhalten im Unendlichen. Dann sind äquivalent:

(1) It is u a weak solution of (IV11.11) and for some $t = t_0$, e.g. the initial time, there exists the nonzero "total mass" $M(t) := \int_{\mathbb{R}^n} u(t,x)\,\mathrm{d}x$.

(2) Für ein $\lambda \in \mathbb{R}$ gilt die Differentialgleichung

$$\mathrm{div}_y\big[a(\widetilde{u}, \nabla_y \widetilde{u}) + \lambda \widetilde{u} y\big] = 0$$

wobei $\widetilde{M} := \int_{\mathbb{R}^n} \widetilde{u}(y)\,\mathrm{d}y \neq 0$ existiert und

$$r(t) = c_0 t^{-\frac{1}{\alpha}}\,, \quad s(t) = c_1 r(t)^n\,,$$
$$\alpha = n(m-1) + l + 1\,, \quad \lambda \alpha c_0^\alpha c_1^{m-1} = 1\,.$$

Bemerkung: Für $r = \mathrm{const}$ ist dies die Separation in den Variablen t und x.

Proof. For the "total mass" we have

$$M(t) = \int_{\mathbb{R}^n} s(t)\,\widetilde{u}(r(t)x)\,\mathrm{d}x = \frac{s(t)}{r^n(t)} \int_{\mathbb{R}^n} \widetilde{u}(y)\,\mathrm{d}y = \frac{s(t)}{r^n(t)} \widetilde{M}$$

so that $M(t_0) \neq 0$ is equivalent to $\widetilde{M} \neq 0$. We consider test functions

$$\zeta(t,x) = r(t)^n \widetilde{\zeta}(t,y) \text{ for } y = r(t)x\,.$$

Then (without arguments)

$$u = s\widetilde{u}\,, \quad \nabla u = rs\nabla\widetilde{u}\,, \quad a(u,\nabla u) = s^m r^l a(\widetilde{u}, \nabla\widetilde{u})\,,$$
$$\zeta = r^n\widetilde{\zeta}\,, \quad \nabla\zeta = r^{n+1}\nabla\widetilde{\zeta}\,, \quad \partial_t\zeta = r^{n-1}\big(nr'\widetilde{\zeta} + r\partial_t\widetilde{\zeta} + r'y\bullet\nabla\widetilde{\zeta}\big)\,,$$

and therefore

$$\begin{aligned}
&\langle\,\zeta\,,\,\partial_t[u] - \mathrm{div}[a(u,\nabla u)]\,\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \big(-\partial_t\zeta\cdot u + \nabla\zeta\bullet a(u,\nabla u)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Big(-\frac{s}{r}\big(nr'\widetilde{\zeta} + r\partial_t\widetilde{\zeta} + r'y\bullet\nabla\widetilde{\zeta}\big)\widetilde{u} + r^{l+1}s^m\nabla\widetilde{\zeta}\bullet a(\widetilde{u},\nabla\widetilde{u})\Big)\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Big(\widetilde{\zeta}\cdot(s' - \frac{sn}{r}r')\widetilde{u} + \nabla\widetilde{\zeta}\bullet\big(r^{l+1}s^m a(\widetilde{u},\nabla\widetilde{u}) - \frac{s}{r}r'\widetilde{u}y\big)\Big)\,\mathrm{d}y\,\mathrm{d}t\,.
\end{aligned}$$

We use this calculation to derive two things. First we conclude that (2) implies (1). In fact, the definition of s and r implies

$$\frac{s'}{s} = n\frac{r'}{r}\,, \quad r^{l+1}s^m = c_0^{n+\alpha}c_1^m t^{-\frac{n+\alpha}{\alpha}}\,, \quad -s\frac{r'}{r} = \lambda c_0^{n+\alpha}c_1^m t^{-\frac{n+\alpha}{\alpha}}\,,$$

therefore the term with $\widetilde{\zeta}$ vanishes and the integral equals

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \nabla\widetilde{\zeta} \bullet \big(r^{l+1} s^m a(\widetilde{u}, \nabla\widetilde{u}) - \frac{s}{r} r' \widetilde{u} y\big) \,\mathrm{d}y \,\mathrm{d}t$$
$$= c_0^{n+\alpha} c_1^m \int_{\mathbb{R}} t^{-\frac{n+\alpha}{\alpha}} \Big(\int_{\mathbb{R}^n} \nabla\widetilde{\zeta} \bullet \big(a(\widetilde{u}, \nabla\widetilde{u}) + \lambda \widetilde{u} y\big) \,\mathrm{d}y \Big) \,\mathrm{d}t = 0\,.$$

Second we prove that (1) implies (2). Then (1) says that u is a weak solution of (IV11.11) and we derive writing $\widetilde{\zeta}(t,y) = \xi(t)\varphi(y)$ with $\xi \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$0 = \langle\, \zeta\,,\, \partial_t[u] - \operatorname{div}[a(u, \nabla u)] \,\rangle_{\mathscr{D}(\mathbb{R}\times\mathbb{R}^n)}$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \Big(\widetilde{\zeta} \cdot (s' - \frac{sn}{r} r') \widetilde{u} + \nabla\widetilde{\zeta} \bullet \big(r^{l+1} s^m a(\widetilde{u}, \nabla\widetilde{u}) - \frac{s}{r} r' \widetilde{u} y\big) \Big) \,\mathrm{d}y \,\mathrm{d}t$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \Big(\xi\varphi \cdot (s' - \frac{sn}{r} r') \widetilde{u} + \xi \nabla\varphi \bullet \big(r^{l+1} s^m a(\widetilde{u}, \nabla\widetilde{u}) - \frac{s}{r} r' \widetilde{u} y\big) \Big) \,\mathrm{d}y \,\mathrm{d}t$$
$$= \int_{\mathbb{R}} \xi \cdot \Big(s' - \frac{sn}{r} r', -\frac{s}{r} r', r^{l+1} s^m \Big) \,\mathrm{d}t$$
$$\bullet \int_{\mathbb{R}^n} \Big(\varphi\widetilde{u}, \nabla\varphi \bullet (\widetilde{u} y), \nabla\varphi \bullet a(\widetilde{u}, \nabla\widetilde{u}) \Big) \,\mathrm{d}y\,.$$

If we denote by

$$I_1(\xi) := \int_{\mathbb{R}} \xi \cdot \Big(s' - \frac{sn}{r} r', -\frac{s}{r} r', r^{l+1} s^m \Big) \,\mathrm{d}t \in \mathbb{R}^3\,,$$
$$I_2(\varphi) := \int_{\mathbb{R}^n} \Big(\varphi\widetilde{u}, \nabla\varphi \bullet (\widetilde{u} y), \nabla\varphi \bullet a(\widetilde{u}, \nabla\widetilde{u}) \Big) \,\mathrm{d}y \in \mathbb{R}^3$$

the two integrals, we have proved that for every ξ and φ

$$0 = I_1(\xi) \bullet I_2(\varphi) \tag{IV11.13}$$

We conclude that there exist two subspaces U_1 and U_2 for which $U_1 \perp U_2$ is a subset of $\mathbb{R}^3$ such that $I_1(\xi) \in U_1$ and $I_2(\varphi) \in U_2$ for all ξ and φ.

From (IV11.13) it follows that for every ξ and φ

$$0 = \int_{\mathbb{R}} \xi(t) \Big(s' - \frac{sn}{r} r', -\frac{s}{r} r', r^{l+1} s^m \Big)(t) \bullet I_2(\varphi) \,\mathrm{d}t$$

and therefore that for almost all t (hence for t outside a null set which depends on φ), say for all $t \in \mathbb{R} \setminus \mathcal{N}_\varphi$ with a zero set $\mathcal{N}_\varphi$,

$$0 = \Big(s' - \frac{sn}{r} r', -\frac{s}{r} r', r^{l+1} s^m \Big)(t) \bullet I_2(\varphi)$$
$$= \int_{\mathbb{R}^n} \Big(\mu(t)\varphi\widetilde{u} + \nu(t) \nabla\varphi \bullet (\widetilde{u} y) + \kappa(t) \nabla\varphi \bullet a(\widetilde{u}, \nabla\widetilde{u}) \Big) \,\mathrm{d}y \tag{IV11.14}$$
$$\text{where} \quad \mu := s' - \frac{sn}{r} r'\,, \quad \nu := -\frac{s}{r} r'\,, \quad \kappa := r^{l+1} s^m\,.$$

If we choose a denumerable dense set $\{\varphi_i\,;\ i \in \mathbb{N}\}$ of test functions, we conclude that (IV11.14) holds for all $\varphi = \varphi_i$, $i \in \mathbb{N}$, and all

$$t \in \mathbb{R} \setminus \mathcal{N}\,, \text{ where } \mathcal{N} := \bigcup_{i\in\mathbb{N}} \mathcal{N}_{\varphi_i}$$

is also a null set. Now since the set $\{\varphi_i\,;\ i \in \mathbb{N}\}$ is a dense set in the space of test functions one can approach all characteristic functions $\mathcal{X}_{\mathrm{B}_R(0)}$ with $R > 0$ (in the weak sense of BV-functions) by this numerable subset of test functions. Therefore one obtains as limit of (IV11.14) for $t \in \mathbb{R} \setminus \mathcal{N}$

$$0 = \int_{\mathrm{B}_R(0)} \mu(t)\widetilde{u}\,\mathrm{dL}^n - \int_{\partial\mathrm{B}_R(0)} \nu_{\mathrm{B}_R(0)}\bullet\Big(\nu(t)\widetilde{u}y + \kappa(t)\,a(\widetilde{u},\nabla\widetilde{u})\Big)\,\mathrm{dH}^{n-1}\,.$$

Hence, since by assumption $\widetilde{u}(y)$ decays fast enough as $|y| \to \infty$,

$$\begin{aligned}\mu(t)\widetilde{M} &= \lim_{R\to\infty}\int_{\mathrm{B}_R(0)} \mu(t)\widetilde{u}\,\mathrm{dL}^n\\ &= \lim_{R\to\infty}\int_{\partial\mathrm{B}_R(0)} \nu_{\mathrm{B}_R(0)}\bullet\Big(\nu(t)\widetilde{u}y + \kappa(t)\,a(\widetilde{u},\nabla\widetilde{u})\Big)\,\mathrm{dH}^{n-1} = 0\,.\end{aligned}$$

Therefore, since $\widetilde{M} \neq 0$, one concludes $\mu(t) = 0$ for $t \in \mathbb{R} \setminus \mathcal{N}$. Since

$$\lambda := \frac{\nu}{\kappa}$$

is continuous it follows that for $\varphi = \varphi_i$, $i \in \mathbb{N}$, equation (IV11.14) reads

$$\int_{\mathbb{R}^n} \nabla\varphi\bullet\big(\lambda(t)\widetilde{u}y + a(\widetilde{u},\nabla\widetilde{u})\big)\,\mathrm{d}y = 0\,, \tag{IV11.15}$$

and therefore also for all test functions φ. Now, if the subset U_1 is more than one-dimensional, there would exist two different values $\lambda(t_1)$ and $\lambda(t_2)$ satisfying the differential equation (IV11.15). Then $\mathrm{div}_y\big(\widetilde{u}y\big) = 0$, hence $y\bullet\nabla_y\widetilde{u} + n\widetilde{u} = 0$ and thus $\widetilde{u}(y) = u_0|y|^{-n}$ is unbounded, a contradiction to the assumed continuity of $\widetilde{u}$. Therefore U_1 is one-dimensional and λ independent of time, that is $U_1 = \mathrm{span}\,(0,\lambda,1)$. And for this λ the equation (IV11.15) holds, that is $\mathrm{div}_y[a(\widetilde{u},\nabla_y\widetilde{u}) + \lambda\widetilde{u}y] = 0$.

And we have the properties from above

$$\mu(t) = 0\,, \quad \nu(t) = \lambda\,\kappa(t)\,,$$

or

$$s' - \frac{sn}{r}r' = 0\,, \quad -\frac{s}{r}r' = \lambda\,r^{l+1}s^m\,.$$

The first equation says $s = c_1r^n$, c_1 a constant, and than the second implies, if $c_1 \neq 0$,

$$-r' = \lambda c_1^{m-1}r^{\alpha+1}\,, \quad \alpha = n(m-1) + l + 1\,,$$

that is $(r^{-\alpha})' = \alpha\lambda c_1^{m-1}$ and therefore $r(t) = c_0 t^{-\frac{1}{\alpha}}$, with time shifted so that 0 becomes the singularity. Here c_0 is as in the formulation of the statement. □

Now we choose the elliptic term as $a(u, \nabla u) := d(u)\nabla u$ with a scalar $d(u)$. Hence the condition in 11.4 it satisfied, if $l = 1$ and

$$d(u) := \begin{cases} 1 \text{ for } m = 1, \\ mu^{m-1} \text{ for } m > 1. \end{cases}$$

It is $u = r^n \widetilde{u}(rx)$, where $\widetilde{u}$ is by 11.4 the solution of

$$\operatorname{div}_y\big(d(\widetilde{u})\nabla_y \widetilde{u} + \lambda \widetilde{u} y\big) = 0.$$

If $m = 1$ (and as above $d = 1$) the differential equation is the linear diffusion equation $\partial_t u - \operatorname{div}\nabla u = 0$. It is $u = s\widetilde{u}(rx)$, where $r(t) = c_0 t^{-\frac{1}{2}}$, $s(t) = c_1 t^{-\frac{n}{2}}$, and $2\lambda c_0^2 = 1$, and where $\widetilde{u}(y) = \exp\big(-\frac{\lambda}{2}|y|^2\big)$, hence

$$u(t,x) = \frac{c_1}{t^{\frac{n}{2}}} \exp\Big(-\Big|\frac{x}{2\sqrt{t}}\Big|^2\Big), \tag{IV11.16}$$

wich is, up to the constant c_1, the fundamental solution of the heat operator.

If $m > 1$ (and as above $d(u) = mu^{m-1}$) the differential equation is a nonlinear equation $\partial_t u - \operatorname{div}\nabla(u^m) = 0$. There is the Barenblatt solution, which has compact support and for $m \geq 2$ is a distributional solution.

11.6 Barenblatt solution. Let $m \in \mathbb{R}$ and $m > 1$. A solution of the dis-tributional equation

$$\partial_t[u] + \operatorname{div}(-[\nabla(u^m)]) = 0 \text{ in } \mathscr{D}'(]0,\infty[\times\mathbb{R}^n)$$

is given for

$$u(t,x) = Ct^{-\frac{n}{\alpha}} \max\big(0, w(t,x)\big)^{\frac{1}{m-1}},$$
$$w(t,x) = R^2 - \Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2,$$

where $C \in \mathbb{R}$ and $R > 0$ are constants. The solution is radially symmetric and has compact support in space. The function u^m is continuously differentiable, hence $\nabla(u^m)$ is continuous. For $1 < m < 2$ the solution is a classical solution, that is

$$\partial_t u - \Delta(u^m) = 0 \text{ in }]0,\infty[\times\mathbb{R}^n.$$

Proof in $\{u > 0\}$. It is $m > 1$ and $d(u) = mu^{m-1}$. By 11.4 we have the equation for $\widetilde{u}$

$$\operatorname{div}_y(\nabla_y \widetilde{u}^m + \lambda \widetilde{u} y) = 0,$$

which is satisfied, if $\lambda\widetilde{u}y = -\nabla_y\widetilde{u}^m$. This is true for

$$\widetilde{u}(y) := \big(a - b|y|^2\big)^{\frac{1}{m-1}} \text{ for } a - b|y|^2 > 0 \text{ where } a, b \in \mathbb{R}\,,$$

if $\lambda = \frac{2mb}{m-1}$. Indeed, $\widetilde{u}(y)^m = \big(a - b|y|^2\big)^{\frac{m}{m-1}}$ and

$$\begin{aligned}\nabla_y\widetilde{u}^m &= \frac{m}{m-1}\big(a - b|y|^2\big)^{\frac{1}{m-1}-1}2by \\ &= \frac{2bm}{m-1}\widetilde{u}(y)y = \lambda\widetilde{u}(y)y \text{ if } \lambda = \frac{2mb}{m-1}\,.\end{aligned}$$

And $r(t) = c_0t^{-\frac{1}{\alpha}}$, $s(t) = c_1t^{-\frac{n}{\alpha}}$ (if the time is shifted) with $\alpha\lambda c_1^{m-1}c_0^{\alpha+1} = 1$ imply that

$$\begin{aligned}u(t,x) &= r(t)\widetilde{u}(s(t)x) = c_1t^{-\frac{n}{\alpha}}\Big(a - b\Big|c_0\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\Big)^{\frac{1}{m-1}} \\ &= c_1t^{-\frac{n}{\alpha}}\Big(a - bc_0^2\Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\Big)^{\frac{1}{m-1}} \\ &= (c_1(bc_0^2)^{\frac{1}{m-1}})\,t^{-\frac{n}{\alpha}}\Big(\frac{a}{bc_0^2} - \Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\Big)^{\frac{1}{m-1}} \\ &= C\,t^{-\frac{n}{\alpha}}\Big(R^2 - \Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\Big)^{\frac{1}{m-1}}\end{aligned}$$

with

$$C = c_1(bc_0^2)^{\frac{1}{m-1}}\,,\quad R = \Big(\frac{a}{bc_0^2}\Big)^{\frac{1}{2}}$$

$$b = \frac{(m-1)\lambda}{2m}\,,\quad \alpha\lambda c_1^{m-1}c_0^{\alpha+1} = 1\,.$$

Here C, R, and λ are three independent constants. □

Proof in whole domain. Auf jeden Fall ist die Lösung u stetig im ganzen Raum. Für $m < 2$, d.h. $\frac{1}{m-1} > 1$, ist die Lösung am Rande von $D := \{(t,x)\,;\ |x| < Rt^{\frac{1}{\alpha}}\}$ stetig differenzierbar, also eine klassische Lösung in $\mathbb{R}\times\mathbb{R}^n$. Im allgemeinen ist in D

$$u^m = C^mt^{-\frac{nm}{\alpha}}\big(R^2 - \Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\big)^{\frac{m}{m-1}}\,,$$

und da $\frac{m}{m-1} > 1$ ist, ist u^m auf dem Rand von D stetig differenzierbar. Also ist $\nabla(u^m)$ punktweise auf ganz $\mathbb{R}\times\mathbb{R}^n$ definiert und dort eine stetige Funktion. Daher ist es eine Distribution. Man kann dies auch einsehen, indem man $\nabla(u^m)$ auf D berechnet

$$\begin{aligned}\nabla(u^m) &= -C^m\frac{1}{t^{\frac{nm}{\alpha}}}\frac{m}{m-1}\big(R^2 - \Big|\frac{x}{t^{\frac{1}{\alpha}}}\Big|^2\big)^{\frac{1}{m-1}}\frac{2}{t^{\frac{2}{\alpha}}}x \\ &= -C^m\frac{1}{t^{\frac{nm+2}{\alpha}}}\frac{m}{m-1}\frac{t^{\frac{n}{\alpha}}}{C}2ux = -C^{m-1}\frac{1}{t^{\frac{n(m-1)+2}{\alpha}}}\frac{m}{m-1}2ux\,.\end{aligned}$$

□

Baryzentric velocity

Beim baryzentrischen Mittel in III.3.1(1) werden die Konzentrationen der einzelnen Phasen

$$c_\alpha := \frac{\varrho_\alpha}{\varrho} \text{ für die Phase } \alpha, \text{ mit } \sum_\alpha c_\alpha = 1, \tag{IV11.17}$$

betrachtet. Daraus ergeben sich die folgenden Gleichungen.

11.7 Lemma. Die Massenbilanzen sind äquivalent zu

$$\varrho(\partial_t c_\alpha + v\bullet\nabla c_\alpha) + \mathrm{div}\mathbf{J}_\alpha - c_\alpha\,\mathrm{div}\mathbf{J} = \mathbf{r}_\alpha - c_\alpha\mathbf{r}$$

für $\alpha = 1, \dots, m$ und der Gasamtmassenbilanz in (IV11.6). Dabei sind $\mathbf{J}$ and $\mathbf{r}$ wie in (IV11.4) definiert. *Häufiger Standardfall:* Sind die Gesamtgrößen $\mathbf{J} = 0$ und $\mathbf{r} = 0$, so gelten für die Konzentration c_α

$$\varrho(\partial_t c_\alpha + v\bullet\nabla c_\alpha) + \mathrm{div}\mathbf{J}_\alpha = \mathbf{r}_\alpha\,. \tag{IV11.18}$$

Proof. Die Massenerhaltung für die Phase α ist

$$\begin{aligned}\mathbf{r}_\alpha &= \partial_t(\varrho c_\alpha) + \mathrm{div}(\varrho c_\alpha v + \mathbf{J}_\alpha)\\ &= \varrho(\partial_t c_\alpha + v\bullet\nabla c_\alpha) + \mathrm{div}\mathbf{J}_\alpha + c_\alpha(\underbrace{\partial_t\varrho + \mathrm{div}(\varrho v)}_{=\mathbf{r}-\mathrm{div}\mathbf{J}}),\end{aligned}$$

which is the assertion. Remember that the m equations for the concentrations are $m-1$ independent equations. □

Wichtig bei der Berechnung von σ wird folgende Definition sein. Hierbei wird g ein "Funktional" sein, d.h. es hängt von Funktionen, hier $\vec{\varrho}$ und $\vec{u}$, ab. Dies kann dadurch gegeben sein, dass $g(\vec{\varrho},\vec{u})(y) = \widehat{g}(\vec{\varrho}(y),\vec{u}(y))$ oder dass zum Beispiel $g(\vec{\varrho},\vec{u})(y) = \widehat{g}(\vec{\varrho}(y),\nabla\vec{\varrho}(y),\vec{u}(y))$, also eine "lokale Abhängigkeit".

11.8 Definition (First variation). Let g be a ***functional*** depending on vector valued functions $\vec{\varrho} = (\varrho_\alpha)_\alpha$ and $\vec{u} = (u_k)_k$ in $\mathcal{U} \subset \mathbb{R}^N$, that is

$$g = g(\vec{\varrho},\vec{u}) \text{ im Funktionenraum.}$$

Then there exists the ***first variation*** $\frac{\delta g}{\delta\vec{\varrho}}$ of g with respect to $\vec{\varrho}$, if $\frac{\delta g}{\delta\vec{\varrho}}$ is a $\mathcal{L}^1$-function, which for $\zeta = (\zeta_\alpha)_\alpha$ with $\zeta_\alpha \in \mathscr{D}(\mathcal{U})$ satisfies

$$\int_{\mathcal{U}} \zeta\bullet\frac{\delta g}{\delta\vec{\varrho}}\,\mathrm{dL}^N = \lim_{\delta\to 0}\int_{\mathcal{U}} \frac{1}{\delta}\big(g(\vec{\varrho}+\delta\zeta,\vec{u}) - g(\vec{\varrho},\vec{u})\big)\,\mathrm{dL}^N\,.$$

The following statements hold:

(1) If g is a constitutive function $y \mapsto g(\vec{\varrho}, \vec{u})(y) = \widehat{g}(\vec{\varrho}(y), \vec{u}(y))$, then

$$\frac{\delta g}{\delta \vec{\varrho}} = \left(\frac{\delta g}{\delta \varrho_\alpha}\right)_\alpha , \quad \frac{\delta g}{\delta \varrho_\alpha} = \widehat{g}\,'_{\varrho_\alpha} .$$

(2) If g is a constitutive function $y \mapsto g(\vec{\varrho}, \vec{u})(y) = \widehat{g}(\vec{\varrho}(y), \nabla\vec{\varrho}(y), \vec{u}(y))$, then

$$\frac{\delta g}{\delta \vec{\varrho}} = \left(\frac{\delta g}{\delta \varrho_\alpha}\right)_\alpha , \quad \frac{\delta g}{\delta \varrho_\alpha} = \widehat{g}\,'_{\varrho_\alpha} - \mathrm{div}(\widehat{g}\,'_{\nabla\varrho_\alpha}) .$$

Hinweis: Die Darstellung in (1) bedeutet $g(\vec{\varrho}, \vec{u}) = \widehat{g}(\vec{\varrho}(\bullet), \vec{u}(\bullet))$, in (2) entsprechend $g(\vec{\varrho}, \vec{u}) = \widehat{g}(\vec{\varrho}(\bullet), \nabla\vec{\varrho}(\bullet), \vec{u}(\bullet))$.

Proof (1).

$$\int_{\mathcal{U}} \zeta \bullet \frac{\delta g}{\delta \vec{\varrho}} \, \mathrm{dL}^N = \lim_{\delta \to 0} \int_{\mathcal{U}} \frac{1}{\delta}\big(g(\vec{\varrho} + \delta\zeta, \vec{u}) - g(\vec{\varrho}, \vec{u})\big) \, \mathrm{dL}^N$$
$$= \lim_{\delta \to 0} \int_{\mathcal{U}} \frac{1}{\delta}\big(\widehat{g}(\vec{\varrho} + \delta\zeta, \vec{u}) - \widehat{g}(\vec{\varrho}, \vec{u})\big) \, \mathrm{dL}^N$$
$$= \lim_{\delta \to 0} \int_{\mathcal{U}} \int_0^1 (\nabla_{\vec{\varrho}}\widehat{g})(\vec{\varrho} + s\delta\zeta, \vec{u}) \bullet \zeta \, \mathrm{d}s \, \mathrm{dL}^N = \int_{\mathcal{U}} (\nabla_{\vec{\varrho}}\widehat{g})(\vec{\varrho}, \vec{u}) \bullet \zeta \, \mathrm{dL}^N .$$

This holds for all test functions ζ, hence

$$\frac{\delta g}{\delta \vec{\varrho}} = \nabla_{\vec{\varrho}}\, g = g\,'_{\vec{\varrho}} .$$

□

Proof (2). Wir nehmen an, dass $g(\vec{\varrho}, \vec{u}) = \widehat{g}\big((\partial^\beta \vec{\varrho})_{|\beta| \le 1}, \vec{u}\big)$.

$$\int_{\mathcal{U}} \zeta \bullet \frac{\delta g}{\delta \vec{\varrho}} \, \mathrm{dL}^N = \lim_{\delta \to 0} \int_{\mathcal{U}} \frac{1}{\delta}\big(g(\vec{\varrho} + \delta\zeta, \vec{u}) - g(\vec{\varrho}, \vec{u})\big) \, \mathrm{dL}^N$$
$$= \lim_{\delta \to 0} \int_{\mathcal{U}} \frac{1}{\delta}\Big(\widehat{g}\Big((\partial^\beta \vec{\varrho} + \delta\partial^\beta\zeta)_{|\beta| \le 1}, \vec{u}\Big) - \widehat{g}\Big((\partial^\beta \vec{\varrho})_{|\beta| \le 1}, \vec{u}\Big)\Big) \, \mathrm{dL}^N$$
$$= \lim_{\delta \to 0} \int_{\mathcal{U}} \int_0^1 \sum_{\alpha : |\alpha| \le 1} \widehat{g}\,'_{\partial^\alpha \vec{\varrho}}\Big((\partial^\beta \vec{\varrho} + s\delta\partial^\beta\zeta)_{|\beta| \le 1}, \vec{u}\Big) \bullet \partial^\alpha \zeta \, \mathrm{d}s \, \mathrm{dL}^N$$
$$= \int_{\mathcal{U}} \sum_{\alpha : |\alpha| \le 1} \widehat{g}\,'_{\partial^\alpha \vec{\varrho}}\big((\partial^\beta \vec{\varrho})_{|\beta| \le 1}, \vec{u}\big) \bullet \partial^\alpha \zeta \, \mathrm{dL}^N$$
$$= \int_{\mathcal{U}} \sum_{\alpha : |\alpha| \le 1} (-1)^{|\alpha|} \partial^\alpha \Big(\widehat{g}\,'_{\partial^\alpha \vec{\varrho}}\big((\partial^\beta \vec{\varrho})_{|\beta| \le 1}, \vec{u}\big)\Big) \bullet \zeta \, \mathrm{dL}^N .$$

This holds for all test funktions ζ, hence

$$\frac{\delta g}{\delta \vec{\varrho}} = \sum_{\alpha : |\alpha| \le 1} (-1)^{|\alpha|} \partial^\alpha \Big(\widehat{g}\,'_{\partial^\alpha \vec{\varrho}}\big((\partial^\beta \vec{\varrho})_{|\beta| \le 1}, \vec{u}\big)\Big) = g\,'_{\vec{\varrho}} - \mathrm{div}(g\,'_{\nabla\vec{\varrho}}) .$$

□

Dass die Gleichungen in 11.8(2) eine Bedeutung haben, werden wir bei der Allen-Cahn und der Cahn-Hilliard Gleichung sehen, in der Mathematik sind die Euler-Lagrange Gleichungen ein Beispiel. Wir benutzen die erste Variation zur Definition des

11.9 Chemical potential. Sei η die Entropie und es sei mit $\vec{\varrho} = (\varrho_\alpha)_\alpha$ entweder $\eta = \widehat{\eta}(\vec{\varrho}, \varepsilon)$ oder $\eta = \widehat{\eta}(\vec{\varrho}, \nabla\vec{\varrho}, \varepsilon)$. Wir setzen wieder

$$f = \varepsilon - \theta\eta\,, \quad \theta = \frac{1}{\eta'_\varepsilon}\,,$$

wobei $\eta'_\varepsilon := \widehat{\eta}'_\varepsilon$, sowie ε die innere Energie, θ die Temperatur und f die innere freie Energie ist. Dann ist das ***chemische Potential*** bzgl. der Komponente ϱ_α einerseits definiert durch die Entropie η

$$\boxed{\mu_\alpha^s := -\theta \frac{\delta\eta}{\delta\varrho_\alpha}}$$

und andererseits durch die innere freie Energie f

$$\boxed{\mu_\alpha^f := \frac{\delta f}{\delta\varrho_\alpha}\,,}$$

was im Allgemeinen ein anderes Potential ist.

(1) Im Falle dass $\eta = \widehat{\eta}(\vec{\varrho}, \varepsilon)$, entsprechend $f = \widehat{f}(\vec{\varrho}, \theta)$, gilt, sind die chemischen Potentiale $\mu_\alpha := \mu_\alpha^s = \mu_\alpha^f$ beide gleich, also

$$\boxed{\mu_\alpha := -\theta\eta'_{\varrho_\alpha} = f'_{\varrho_\alpha}\,.}$$

(2) Gilt $\eta = \widehat{\eta}(\vec{\varrho}, \nabla\vec{\varrho}, \varepsilon)$, entsprechend $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho}, \theta)$, so ist

$$\mu_\alpha^f = f'_{\varrho_\alpha} - \mathrm{div}(f'_{\nabla\varrho_\alpha})\,, \quad \mu_\alpha^s = -\theta\eta'_{\varrho_\alpha} + \theta\,\mathrm{div}(\eta'_{\nabla\varrho_\alpha})\,,$$

und die Differenz zwischen μ_α^f und μ_α^s ist

$$\mu_\alpha^f - \mu_\alpha^s = \eta'_{\nabla\varrho_\alpha} \bullet \nabla\theta\,.$$

Bemerkung: Eine einheitliche Definition von μ_α findet sich auch in 12.3.

(3) ***Falls die freie Energieungleichung benutzt wird.*** Ist $\theta = \mathrm{const}$, so wird nur f gebraucht und falls $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho}, \theta)$ gilt

$$\boxed{\mu_\alpha := \mu_\alpha^f = f'_{\varrho_\alpha} - \mathrm{div} f'_{\nabla\varrho_\alpha}\,.}$$

Historie: Zur Definition des chemischen Potentials $\mu_\alpha = -\theta\eta{}'_{\varrho_\alpha}$ siehe die Formel [87, (6.47) S. 185] im Buch von I. Müller. Zur Historie sei bemerkt, dass der Begriff des "chemischen Potentials" von Gibbs stammt, siehe dazu [Wikipedia: Chemical potential – History].

In diesem Zusammenhang sei auch auf die Stetigkeit des chemischen Potentials an der Grenze verschiedener Medien in 12.2 Bezug genommen.

Referenzen: Es sei auf I. Müller [87, 6.3.2.5 Chemical potentials] verwiesen. Siehe auch [Wikipedia: Chemical potential], wo chemische Potentiale bzgl. der N_α (siehe III.3.1(2)) betrachtet werden.

Proof. Wenn wir konstitutive Funktionen annehmen, also

$$\eta = \widehat{\eta}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \varepsilon\big)\,, \quad f = \widehat{f}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \theta\big)$$
$$\theta = \widehat{\theta}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \varepsilon\big)\,,$$

dann gelten die Gleichungen

$$f = \varepsilon - \theta\eta\,, \quad \theta\eta{}'_\varepsilon = 1\,, \tag{IV11.19}$$

wobei bemerkt sei, dass wie immer die Ableitung $\eta{}'_\varepsilon := \widehat{\eta}{}'_\varepsilon$ ist. Also können wir schreiben

$$\widehat{f}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \widehat{\theta}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \varepsilon\big)\big) = \varepsilon - \widehat{\theta}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \varepsilon\big)\,\widehat{\eta}\big((\partial^\beta \vec{\varrho})_{|\beta|\le 1}, \varepsilon\big)\,.$$

Wenn wir dies nach ε ableiten, erhalten wir (ohne Argumente)

$$\widehat{f}{}'_\theta \widehat{\theta}{}'_\varepsilon = \underbrace{1 - \widehat{\theta}\widehat{\eta}{}'_\varepsilon}_{=0} - \widehat{\theta}{}'_\varepsilon \widehat{\eta}\,,$$

also $\widehat{f}{}'_\theta = -\widehat{\eta}$ (falls $\widehat{\theta}{}'_\varepsilon \neq 0$, or $\eta{}'_{\varepsilon\varepsilon} \neq 0$). Nun betrachten wir die Ableitung nach $\partial^\gamma \vec{\varrho}$ für $|\gamma| \le 1$. Sie ist gleich

$$\widehat{f}{}'_{\partial^\gamma \vec{\varrho}} + \widehat{f}{}'_\theta \widehat{\theta}{}'_{\partial^\gamma \vec{\varrho}} = -\widehat{\theta}\widehat{\eta}{}'_{\partial^\gamma \vec{\varrho}} - \widehat{\theta}{}'_{\partial^\gamma \vec{\varrho}}\widehat{\eta}\,.$$

Da schon $\widehat{f}{}'_\theta = -\widehat{\eta}$ gezeigt war, ist also $\widehat{f}{}'_{\partial^\gamma \vec{\varrho}} - \widehat{\theta}\widehat{\eta}{}'_{\partial^\gamma \vec{\varrho}}$. Wir haben also gezeigt, dass

$$\widehat{f}{}'_\theta = -\widehat{\eta}\,,$$
$$\widehat{f}{}'_{\partial^\gamma \vec{\varrho}} = -\widehat{\theta}\,\widehat{\eta}{}'_{\partial^\gamma \vec{\varrho}} \text{ für } |\gamma| \le 1\,.$$

Oder anders geschrieben

$$\widehat{f}{}'_\theta = -\widehat{\eta}\,,$$
$$f{}'_{\varrho_\alpha} = -\theta\,\eta{}'_{\varrho_\alpha}\,,$$
$$f{}'_{\nabla\varrho_\alpha} = -\theta\,\eta{}'_{\nabla\varrho_\alpha}\,.$$

Zum Beweis von (2) gilt daher

$$\mu_\alpha^f - \mu_\alpha^s = f_{'\varrho_\alpha} + \theta\eta_{'\varrho_\alpha} - \operatorname{div}(f_{'\nabla\varrho_\alpha}) - \theta\operatorname{div}(\eta_{'\nabla\varrho_\alpha})$$
$$= \operatorname{div}(\theta\eta_{'\nabla\varrho_\alpha}) - \theta\operatorname{div}(\eta_{'\nabla\varrho_\alpha}) = \eta_{'\nabla\varrho_\alpha}\bullet\nabla\theta\,.$$

Die Aussage (1) ergibt sich als Spezialfall, da dann

$$\mu_\alpha^s = -\theta\frac{\delta\eta}{\delta\varrho_\alpha} = -\theta\eta_{'\varrho_\alpha} = f_{'\varrho_\alpha} = \frac{\delta f}{\delta\varrho_\alpha} = \mu_\alpha^f$$

da ja $f_{'\varrho_\alpha} = -\theta\eta_{'\varrho_\alpha}$. □

Isothermal case

Häufig werden Reaktions-Diffusionsgleichungen in der isothermen Situation $\theta = \theta_0 = \text{const}$ behandelt, wobei wir hier der Einfachheit halber nur den Fall

$$\mathbf{r} = \textstyle\sum_\alpha \mathbf{r}_\alpha = 0\,, \quad \mathbf{J} = \textstyle\sum_\alpha \mathbf{J}_\alpha = 0 \qquad \text{(IV11.20)}$$

betrachten. In dieser Situation ist das Entropieprinzip durch die freie Energiegleichung zu ersetzen, siehe dazu Abschnitt III.5. Das System der Erhaltungsgleichungen besteht in diesem Fall nur aus den Massenerhaltungen und der gemeinsamen Impulserhaltung (die Kraft ist die klassische Kraft, da $\mathbf{r} = 0$ und $\mathbf{J} = 0$ ist)

Isothermes Reaktions-Diffusions-System:

$$\partial_t\varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1,\dots,m\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,$$

$\varrho := \sum_\alpha \varrho_\alpha$ Gesamtmasse, $\sum_\alpha \mathbf{r}_\alpha = 0$,

Π symmetrischer Drucktensor, $\sum_\alpha \mathbf{J}_\alpha = 0$,

$\mathbf{f}$ klassische Kraft (siehe II.3.8). (IV11.21)

Es wird die freie Energieungleichung (siehe im Abschnitt III.5 das Axiom III.5.4)

$$\sigma_f := \partial_t f^{tot} + \operatorname{div}\varphi^{tot} - v\bullet\mathbf{f} \le 0$$

gefordert, also mit

$$f^{tot} = f + \frac{\varrho}{2}|v|^2\,, \quad f \text{ die innere freie Energie,}$$

ergeben sich folgenden Gleichungen.

11.10 Lemma. Wenn

$$f^{tot} = f + f^{kin}\,, \quad \varphi^{tot} = f^{tot} v + \Pi^{\mathrm{T}}\, v + \varphi\,. \tag{IV11.22}$$

so gilt für die kinetische Energie $f^{kin} := \frac{\varrho}{2}|v|^2$

$$\partial_t f^{kin} + \operatorname{div}\bigl(f^{kin} v + \Pi^{\mathrm{T}}\, v\bigr) = v \bullet \mathbf{f} + \mathrm{D}v \mathbin{:} \Pi\,.$$

und für die freie Energieproduktion

$$\sigma_f = \partial_t f + \operatorname{div}(f v + \varphi) + \mathrm{D}v \mathbin{:} \Pi\,.$$

Hinweis: Im Allgemeinen muss $\sigma_f = \partial_t f^{tot} + \operatorname{div}\varphi^{tot} + g^{tot}$ als objektiver Skalar gewählt werden, also etwa $g^{tot} = -\frac{|v|^2}{2}\mathbf{r} - v\bullet \mathrm{D}v\mathbf{J} - v\bullet\mathbf{f}$. Hier ist aber $\mathbf{r} = 0$ und $\mathbf{J} = 0$.

Proof. Die Gleichung für die Gesamtmasse und den Impuls ist

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,,\\
&\varrho(\partial_t v + (v\bullet\nabla)v) + \operatorname{div}\Pi = \mathbf{f}\,.
\end{aligned}$$

Dann ist (vgl. III.2.2)

$$\begin{aligned}
&\partial_t\Bigl(\frac{\varrho}{2}|v|^2\Bigr) + \operatorname{div}\Bigl(\frac{\varrho}{2}|v|^2 v\Bigr)\\
&= \frac{|v|^2}{2}(\partial_t\varrho + \operatorname{div}(\varrho v)) + \varrho v\bullet(\partial_t v + (v\bullet\nabla)v)\\
&\qquad\qquad \Bigl(\text{wegen } \nabla\frac{|v|^2}{2} = v\bullet(\mathrm{D}v)^{\mathrm{T}} = (v\bullet\nabla)v\Bigr)\\
&= v\bullet(\mathbf{f} - \operatorname{div}\Pi) = -\operatorname{div}(\Pi^{\mathrm{T}}\, v) + v\bullet\mathbf{f} + \mathrm{D}v\mathbin{:}\Pi\,.
\end{aligned}$$

Das ist die erste Gleichung und sie weist das gleiche Transformationsverhalten wie die Energiegleichung auf. Die zweite Gleichung folgt wegen

$$\begin{aligned}
&\sigma_f = \partial_t f^{tot} + \operatorname{div}\varphi^{tot} - v\bullet\mathbf{f}\\
&= \partial_t(f + f^{kin}) + \operatorname{div}(f v + f^{kin} v + \Pi^{\mathrm{T}}\, v + \varphi) - v\bullet\mathbf{f}\\
&= \partial_t f + \operatorname{div}(f v + \varphi) + \mathrm{D}v\mathbin{:}\Pi\,.
\end{aligned}$$

Es ist φ eine objektiver Vektor. □

Somit lautet die freie Energieungleichung

$$0 \geq \sigma_f = \partial_t f + \operatorname{div}(f v + \varphi) + \mathrm{D}v\mathbin{:}\Pi\,. \tag{IV11.23}$$

Wir können die folgenden Fälle unterscheiden.

11.11 Spezialfälle. Wir nehmen an, dass für die innere Energie (IV11.22) gilt und dass mit $\Pi = P - S$ der Spannungstensor S

$$\mathrm{D}v\mathbin{:}S \geq 0\,, \quad S = \widehat{S}(\vec{\varrho},(\mathrm{D}v)^{\mathrm{S}})\,, \quad \widehat{S}(\vec{\varrho},0) = 0$$

erfüllt und symmetrisch ist. In allen drei Fällen gilt

$$\sigma_f = -\mathrm{D}v\mathbin{:}S + \sum_\alpha(\mu_\alpha \mathbf{r}_\alpha + \nabla\mu_\alpha \bullet \mathbf{J}_\alpha)\,,$$

wobei in den Fällen folgende Voraussetzungen gelten:

(1) ***Reaktions-Diffusions-Modelle.*** Es ist $f = \widehat{f}(\vec{\varrho})$, also $\mu_\alpha = f'_{\varrho_\alpha}$. Es sind $\mathbf{J}_\alpha$ die Diffusionsterme und $\mathbf{r}_\alpha$ die Reaktionsraten. Die Massenerhaltungen lauten

$$\partial_t\varrho_\alpha + \mathrm{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1,\ldots,m.$$

Die freie Energieungleichung ist erfüllt, wenn (es ist $\varphi = \sum_\alpha \mu_\alpha \mathbf{J}_\alpha$)

$$P = \big(\sum_\alpha \varrho_\alpha f'_{\varrho_\alpha} - f\big)\mathrm{Id} \quad \text{und} \quad \sum_\alpha \big(\nabla\mu_\alpha \bullet \mathbf{J}_\alpha + \mu_\alpha \cdot \mathbf{r}_\alpha\big) \leq 0\,.$$

Wähle die Diffusionsterme $\mathbf{J}_\alpha = \widehat{\mathbf{J}}_\alpha(\vec{\varrho},\nabla\vec{\varrho})$ und die Reaktionsraten $\mathbf{r}_\alpha = \widehat{\mathbf{r}}_\alpha(\vec{\varrho})$ so, dass die Ungleichung

$$\sum_\alpha \big(\nabla(f'_{\varrho_\alpha}(\vec{\varrho}))\bullet\widehat{\mathbf{J}}_\alpha(\vec{\varrho},\nabla\vec{\varrho}) + f'_{\varrho_\alpha}(\vec{\varrho}) \cdot \widehat{\mathbf{r}}_\alpha(\vec{\varrho})\big) \leq 0$$

erfüllt ist.

(2) ***Allen-Cahn Gleichung.*** Es ist $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho})$ (beachte 11.12) und es ist $\mathbf{J}_\alpha = 0$ für alle α. Dann ist $\mu_\alpha = \frac{\delta f}{\delta \varrho_\alpha}$ und die Massengleichungen lauten

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \ldots, m.$$

Die freie Energieungleichung ist erfüllt, wenn (es ist $\varphi = -\sum_\alpha \overset{\circ}{\varrho}_\alpha f_{'\nabla\varrho_\alpha}$)

$$P = \big(\sum_\alpha \varrho_\alpha \mu_\alpha - f\big)\mathrm{Id} + \sum_\alpha \nabla\varrho_\alpha (f_{'\nabla\varrho_\alpha})^{\mathrm{T}} \quad \text{und} \quad \sum_\alpha \mu_\alpha \mathbf{r}_\alpha \leq 0\,.$$

Also wähle etwa $\mathbf{r}_\alpha = \widehat{\mathbf{r}}_\alpha(\vec{\varrho}, (\mu_\beta)_\beta)$ mit der Ungleichung

$$\sum_\alpha \mu_\alpha \widehat{\mathbf{r}}_\alpha(\vec{\varrho}, (\mu_\beta)_\beta) \leq 0\,.$$

Beispiel: $\mathbf{r}_\alpha = -\sum_\beta c_{\alpha\beta}(\vec{\varrho})\mu_\beta$ mit positiv semidefinitem $(c_{\alpha\beta}(\vec{\varrho}))_{\alpha,\beta}$.

(3) ***Cahn-Hilliard Gleichung.*** Es ist $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho})$ (beachte 11.12) und es ist $\mathbf{r}_\alpha = 0$ für alle α. Also sind $\mu_\alpha = \frac{\delta f}{\delta \varrho_\alpha}$ und die Massengleichungen lauten

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = 0 \text{ für } \alpha = 1, \ldots, m.$$

Die freie Energieungleichung ist erfüllt, wenn ($\varphi = \sum_\alpha \mu_\alpha \mathbf{J}_\alpha - \sum_\alpha \overset{\circ}{\varrho}_\alpha f_{'\nabla\varrho_\alpha}$)

$$P = \big(\sum_\alpha \varrho_\alpha \mu_\alpha - f\big)\mathrm{Id} + \sum_\alpha \nabla\varrho_\alpha (f_{'\nabla\varrho_\alpha})^{\mathrm{T}} \quad \text{und} \quad \sum_\alpha \nabla\mu_\alpha \bullet \mathbf{J}_\alpha \leq 0\,.$$

Also wähle etwa $\mathbf{J}_\alpha = \widehat{\mathbf{J}}_\alpha(\vec{\varrho}, \vec{\mu}, (\nabla\mu_\beta)_\beta)$, wobei $\vec{\mu} := (\mu_\beta)_\beta$, mit der Ungleichung

$$\sum_\alpha \nabla\mu_\alpha \bullet \widehat{\mathbf{J}}_\alpha(\vec{\varrho}, \vec{\mu}, (\nabla\mu_\beta)_\beta) \leq 0\,.$$

Beispiel: $\mathbf{J}_\alpha = -\sum_\beta c_{\alpha\beta}(\vec{\varrho}, \vec{\mu})\nabla\mu_\beta$ mit positiv semidefinitem $(c_{\alpha\beta}(\vec{\varrho}, \vec{\mu}))_{\alpha,\beta}$.

Man muss hierbei die Nebenbedingung (IV11.20) und die Energieungleichung berücksichtigen. Man kann diese Gleichungen natürlich auch in den Konzentrationen schreiben. Oft werden diese Gleichungen auch mit $\varrho = \sum_\alpha \varrho_\alpha = \text{const}$ und auch, falls die Impulsgleichung dies erlaubt, mit Beobachtern betrachtet, für die $v = 0$ ist.

Proof (1). Es ist $f = \widehat{f}(\vec{\varrho})$, also $\mu_\alpha = f_{'\varrho_\alpha}$ und

$$\overset{\circ}{f} = \sum_\alpha f_{'\varrho_\alpha} \overset{\circ}{\varrho}_\alpha = \sum_\alpha \mu_\alpha \overset{\circ}{\varrho}_\alpha\,, \quad \overset{\circ}{\varrho}_\alpha + \varrho_\alpha \operatorname{div} v = \mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha\,.$$

Damit folgt mit (IV11.23)

$$\begin{aligned}
\sigma_f &= \overset{\circ}{f} + f \operatorname{div} v + \operatorname{div}\varphi + \mathrm{D}v \mathbin{:} \Pi = \sum_\alpha f_{'\varrho_\alpha} \overset{\circ}{\varrho}_\alpha + f \operatorname{div} v + \mathrm{D}v \mathbin{:} \Pi + \operatorname{div}\varphi \\
&= \big(f - \sum_\alpha \varrho_\alpha f_{'\varrho_\alpha}\big) \operatorname{div} v + \mathrm{D}v \mathbin{:} \Pi + \sum_\alpha \mu_\alpha (\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \operatorname{div}\varphi \\
&= \mathrm{D}v \mathbin{:} \Big(\big(f - \sum_\alpha \varrho_\alpha f_{'\varrho_\alpha}\big)\mathrm{Id} + \Pi\Big) + \sum_\alpha (\mu_\alpha \mathbf{r}_\alpha + \nabla\mu_\alpha \bullet \mathbf{J}_\alpha)\,,
\end{aligned}$$

wenn $\varphi = \sum_\alpha \mu_\alpha \mathbf{J}_\alpha$ gesetzt wird. Wenn wir nun S so wählen, dass $\Pi = \big(\sum_\alpha \varrho_\alpha f'_{\varrho_\alpha} - f\big)\mathrm{Id} - S$, bleibt also

$$\sigma_f = -\mathrm{D}v\mathbin{:}S + \sum_\alpha (\mu_\alpha \mathbf{r}_\alpha + \nabla\mu_\alpha \bullet \mathbf{J}_\alpha)$$

übrig. Hier $\mathrm{D}v\mathbin{:}S \geq 0$ aus den allgemeinen Voraussetzungen. □

Proof (2) *und* (3). Es ist $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho})$, wobei dies als innere Energie ein objektiver Skalar sein soll. Daher ist

$$\mu_\alpha = \frac{\delta f}{\delta \varrho_\alpha} = f'_{\varrho_\alpha} - \mathrm{div} f'_{\nabla\varrho_\alpha} .$$

Wir haben jetzt

$$\overset{\circ}{f} = \sum_\alpha f'_{\varrho_\alpha} \overset{\circ}{\varrho}_\alpha + \sum_\alpha f'_{\nabla\varrho_\alpha} \bullet (\nabla\varrho_\alpha)^\circ ,$$

wobei für die totale Zeitableitung von $\nabla\varrho_\alpha$ gilt

$$(\nabla\varrho_\alpha)^\circ = \nabla\overset{\circ}{\varrho}_\alpha - \mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha , \qquad \text{(IV11.24)}$$

was sich aus der folgenden Rechnung ergibt:

$$\begin{aligned}
(\partial_i\varrho_\alpha)^\circ &= \partial_t(\partial_i\varrho_\alpha) + (v\bullet\nabla)(\partial_i\varrho_\alpha) \\
&= \underbrace{\partial_t\partial_i}_{\partial_i\partial_t}\,\varrho_\alpha + \sum_{j=1}^n v_j\,\underbrace{\partial_j\partial_i}_{\partial_i\partial_j}\,\varrho_\alpha \\
&= \partial_i\partial_t\varrho_\alpha + \sum_{j=1}^n \big(\partial_i(v_j\partial_j\varrho_\alpha) - (\partial_i v_j)(\partial_j\varrho_\alpha)\big) \\
&= \partial_i\Big(\partial_t\varrho_\alpha + \sum_{j=1}^n v_j\partial_j\varrho_\alpha\Big) - \sum_{j=1}^n (\partial_i v_j)(\partial_j\varrho_\alpha) \\
&= \partial_i\overset{\circ}{\varrho}_\alpha - \big(\mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha\big)_i .
\end{aligned}$$

Damit ist

$$\begin{aligned}
\overset{\circ}{f} &= \sum_\alpha f'_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \sum_\alpha f'_{\nabla\varrho_\alpha}\bullet\nabla\overset{\circ}{\varrho}_\alpha - \sum_\alpha \underbrace{f'_{\nabla\varrho_\alpha}\bullet\mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha}_{=\mathrm{D}v\mathbin{:}(\nabla\varrho_\alpha(f'_{\nabla\varrho_\alpha})^{\mathrm{T}})} \\
&= \sum_\alpha \mu_\alpha\overset{\circ}{\varrho}_\alpha + \mathrm{div}\Big(\sum_\alpha \overset{\circ}{\varrho}_\alpha f'_{\nabla\varrho_\alpha}\Big) - \mathrm{D}v\mathbin{:}\Big(\sum_\alpha \nabla\varrho_\alpha(f'_{\nabla\varrho_\alpha})^{\mathrm{T}}\Big) .
\end{aligned}$$

Also folgt wegen $\overset{\circ}{\varrho}_\alpha = -\varrho_\alpha \,\mathrm{div}v + \mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha$

$$\begin{aligned}
\sigma_f &= \overset{\circ}{f} + f\,\mathrm{div}v + \mathrm{div}\varphi + \mathrm{D}v{:}\Pi \\
&= \big(f - \sum_\alpha \varrho_\alpha\mu_\alpha\big)\,\mathrm{div}v + \sum_\alpha \mu_\alpha(\mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha) \\
&+ \mathrm{div}\Big(\varphi + \sum_\alpha \overset{\circ}{\varrho}_\alpha f'_{\nabla\varrho_\alpha}\Big) + \mathrm{D}v{:}\Big(\Pi - \sum_\alpha \nabla\varrho_\alpha\,(f'_{\nabla\varrho_\alpha})^{\mathrm{T}}\Big) \\
&= \mathrm{D}v{:}\Big(\Pi - \sum_\alpha \nabla\varrho_\alpha\,(f'_{\nabla\varrho_\alpha})^{\mathrm{T}} - \big(\sum_\alpha \varrho_\alpha\mu_\alpha - f\big)\mathrm{Id}\Big) \\
&+ \sum_\alpha(\mu_\alpha\mathbf{r}_\alpha + \nabla\mu_\alpha\bullet\mathbf{J}_\alpha) + \mathrm{div}\Big(\varphi + \sum_\alpha \overset{\circ}{\varrho}_\alpha f'_{\nabla\varrho_\alpha} - \sum_\alpha \mu_\alpha\mathbf{J}_\alpha\Big)\,.
\end{aligned}$$

Wenn nun

$$\begin{aligned}
\Pi &= \big(\sum_\alpha \varrho_\alpha\mu_\alpha - f\big)\mathrm{Id} + \sum_\alpha \nabla\varrho_\alpha\,(f'_{\nabla\varrho_\alpha})^{\mathrm{T}} - S\,, \\
\varphi &= \sum_\alpha \mu_\alpha\mathbf{J}_\alpha - \sum_\alpha \overset{\circ}{\varrho}_\alpha f'_{\nabla\varrho_\alpha}
\end{aligned}$$

gesetzt wird, erhalten wir

$$\sigma_f = -\mathrm{D}v{:}S + \sum_\alpha(\mu_\alpha\mathbf{r}_\alpha + \nabla\mu_\alpha\bullet\mathbf{J}_\alpha)\,.$$

Hier $\mathrm{D}v{:}S \geq 0$ aus den allgemeinen Voraussetzungen. □

Zu Beginn des Abschnittes hatten wir angenommen, dass Π symmetrisch ist, und unter dieser Voraussetzung das allgemeine Diffusionssystem (IV11.2) aufgestellt. Wir müssen also zeigen, dass dies erfüllt ist. Nach 11.11 haben wir dazu zu zeigen:

11.12 Symmetrie von Π. Nach dem Entropieprinzip ist η ein objektiver Skalar, also auch $f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho})$. Daraus folgt, dass

$$\sum_\alpha \nabla\varrho_\alpha \otimes f'_{\nabla\varrho_\alpha}$$

eine symmetrische Matrix ist, auf jeden Fall, wenn $f = \widetilde{f}(\vec{\varrho}, (d_{\beta\gamma})_{\beta\leq\gamma})$, wobei

$$d_{\beta\gamma} := (\widetilde{f}'_{\nabla\varrho_\beta})\bullet(\widetilde{f}'_{\nabla\varrho_\gamma})\,.$$

Proof für die spezielle Darstellung. Es ist

$$f'_{\nabla\varrho_\alpha} = 2\widetilde{f}'_{d_{\alpha\alpha}}\nabla\varrho_\alpha + \sum_{\beta<\gamma} \widetilde{f}'_{d_{\beta\gamma}} \cdot (\delta_{\beta,\alpha}\nabla\varrho_\gamma + \delta_{\gamma,\alpha}\nabla\varrho_\beta)\,,$$

und daraus folgt

$$\begin{aligned}
\sum_\alpha \nabla\varrho_\alpha \otimes f'_{\nabla\varrho_\alpha} &= 2\sum_\alpha \widetilde{f}'_{d_{\alpha\alpha}} \nabla\varrho_\alpha \otimes \nabla\varrho_\alpha \\
&\quad + \sum_{\beta<\gamma}\sum_\alpha \widetilde{f}'_{d_{\beta\gamma}} \cdot (\delta_{\beta,\alpha}\nabla\varrho_\alpha \otimes \nabla\varrho_\gamma + \delta_{\gamma,\alpha}\nabla\varrho_\alpha \otimes \nabla\varrho_\beta) \\
&= 2\widetilde{f}'_{d_{\alpha\alpha}} \nabla\varrho_\alpha \otimes \nabla\varrho_\alpha + \sum_{\beta<\gamma} \widetilde{f}'_{d_{\beta\gamma}} \cdot (\nabla\varrho_\beta \otimes \nabla\varrho_\gamma + \nabla\varrho_\gamma \otimes \nabla\varrho_\beta) \\
&= \sum_{\beta\le\gamma} \widetilde{f}'_{d_{\beta\gamma}} \cdot (\nabla\varrho_\beta \otimes \nabla\varrho_\gamma + \nabla\varrho_\gamma \otimes \nabla\varrho_\beta)
\end{aligned}$$

which is a symmetric matrix. □

Proof in der allgemeinen Situation. Da η ein objektiver Skalar ist, ist dies auch $f = \varepsilon - \theta\eta$. Es ist $f = \widehat{f}((\varrho_\beta)_\beta, (\nabla\varrho_\beta)_\beta)$, wobei wir die Variable von $\nabla\varrho_\beta$ mit z_β bezeichnen, also $\widehat{f}((\varrho_\beta)_\beta, (z_\beta)_\beta)$. Nun gilt mit $\varrho_\alpha \circ Y = \varrho_\alpha^*$ und daraus folgend $(\nabla\varrho_\alpha)\circ Y = Q\nabla\varrho_\alpha^*$

$$\widehat{f}((\varrho_\beta^*)_\beta, (Q\nabla\varrho_\beta^*)_\beta) = f\circ Y = f^* = \widehat{f}((\varrho_\beta^*)_\beta, (\nabla\varrho_\beta^*)_\beta)\,.$$

Indem wir dies nun bei gegebenem Punkt (t^+, x^*) als Gleichung für alle Transformationen auffassen, erhalten wir, wenn wir $Q = Q_s$ wählen, für die Ableitung nach s

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\widehat{f}((\varrho_\beta^*)_\beta, (Q_s\nabla\varrho_\beta^*)_\beta) = \sum_\alpha \widehat{f}'_{z_\alpha}(\ldots)\bullet(\frac{\mathrm{d}}{\mathrm{d}s}Q_s\nabla\varrho_\alpha^*) \\
&= \sum_\alpha \widehat{f}'_{z_\alpha}(\ldots)\bullet A_s Q\nabla\varrho_\alpha^* \qquad \Big(\text{wobei } A_s := (\frac{\mathrm{d}}{\mathrm{d}s}Q_s)\,{Q_s}^{\mathrm{T}}\Big) \\
&= \sum_\alpha \big(\widehat{f}'_{z_\alpha}((\varrho_\beta)_\beta, (\nabla\varrho_\beta)_\beta)\circ Y\big)\bullet\big(A_s\nabla\varrho_\alpha\circ Y\big) \qquad \Big(\text{wobei } \varrho_\alpha \text{ wie oben}\Big) \\
&= A_s \mathbin{:} \Big(\sum_\alpha \widehat{f}'_{z_\alpha}((\varrho_\beta)_\beta, (\nabla\varrho_\beta)_\beta)\otimes\nabla\varrho_\alpha\Big)\circ Y
\end{aligned}$$

Da A_s antisymmetrisch ist für alle s, können wir dies für ein gegebenes s als eine beliebige antisymmetrische Matrix $A_s = A$ wählen, so dass also für dieses s

$$0 = A \mathbin{:} \sum_\alpha \widehat{f}'_{z_\alpha}((\varrho_\beta)_\beta, (\nabla\varrho_\beta)_\beta)\otimes\nabla\varrho_\alpha\,,$$

das heißt, da ϱ_β und $\nabla\varrho_\beta$ beliebig gewählt werden können,

$$\sum_\alpha \widehat{f}'_{z_\alpha}((\varrho_\beta)_\beta, (\nabla\varrho_\beta)_\beta)\otimes\nabla\varrho_\alpha$$

ist eine symmetrische Matrix, was zu zeigen war. (Die Beweismethode findet sich auch in II.4.14 beim Satz über konstante objektive Tensoren.) □

Nachdem wir diese Gleichungen mit Gradientenabhängigkeit studiert haben, ist jetzt natürlich die temperaturabhängige Version der Diffusionsgleichungen von besonderem Interesse.

12 Temperature dependent diffusion

Wir behandeln nun Diffusionsgleichungen mit thermischer Ausdehnung, was eine wesentliche Eigenschaft bei Reaktionen ist. Wir betrachten hier der Einfachheit halber wieder nur den Fall (IV11.20), d.h. $\mathbf{J} = 0$ und $\mathbf{r} = 0$ für die Gesamtgrößen, weshalb die (gesamte) Kraft gleich der klassischen Kraft ist. Das Differentialgleichungssystem (IV11.2) ist dann

Reaktions-Diffusions-System:

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \dots, m\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}} v + q) = v \bullet \mathbf{f} + g\,,$$

$$\varrho := \sum_\alpha \varrho_\alpha \text{ Gesamtmasse}, \qquad \sum_\alpha \mathbf{r}_\alpha = 0$$
$$\Pi \text{ symmetrischer Drucktensor}, \qquad \sum_\alpha \mathbf{J}_\alpha = 0$$
$$e = \varepsilon + \frac{\varrho}{2}|v|^2 \text{ Energie}, \tag{IV12.1}$$

Das Entropieprinzip besagt das Folgende, wobei wir hier die Voraussetzung machen, dass die Entropie nicht von Ableitungen der Dichten abhängt.

12.1 Entropieprinzip. Wenn für die Entropie und den Entropiefluss gilt

$$\eta = \widehat{\eta}((\varrho_\alpha)_\alpha, \varepsilon)\,, \qquad \psi = \eta v + \psi_0\,,$$
$$\psi_0 = \frac{1}{\theta} q^\epsilon - \sum_\alpha \mu_\alpha^\eta \mathbf{J}_\alpha = \frac{1}{\theta} q - \frac{1}{\theta} \sum_\alpha \mu_\alpha \mathbf{J}_\alpha\,,$$

so ist das Entropieprinzip erfüllt, falls in den Gleichungen (IV11.2) (bzw. in (IV11.1) und (IV11.6)) gilt

$$\Pi = p\mathrm{Id} - S\,, \qquad p = \sum_\alpha \varrho_\alpha \mu_\alpha - f\,,$$
$$q = q^\epsilon + \sum_\alpha \mu_\alpha^\epsilon \mathbf{J}_\alpha\,, \qquad \mu_\alpha = \mu_\alpha^\epsilon + \theta \mu_\alpha^\eta\,, \qquad g = 0$$

und die Residualungleichung

$$0 \leq \sigma = \frac{1}{\theta} \mathrm{D}v \bullet S + \nabla\Big(\frac{1}{\theta}\Big) \bullet q^\epsilon - \sum_\alpha \frac{1}{\theta} \mu_\alpha \mathbf{r}_\alpha - \sum_\alpha \Big(\nabla \mu_\alpha^\eta + \frac{1}{\theta} \nabla \mu_\alpha^\epsilon\Big) \bullet \mathbf{J}_\alpha$$

erfüllt ist, wobei $f = \widehat{f}((\varrho_\alpha)_\alpha, \theta)$ und

$$\mu_\alpha = -\theta \eta'_{\varrho_\alpha} = f'_{\varrho_\alpha}\,.$$

Zum Zusammenheng zwischen f und η siehe (IV12.5).
Definition: Die Gleichung $\mu_\alpha = \mu_\alpha^\epsilon + \theta \mu_\alpha^\eta$ “zerlegt” das chemische Potential in einen Energieanteil μ_α^ϵ und einen Entropieanteil μ_α^η.

Man kann also zum Beispiel $\mu_\alpha^\epsilon = \mu_\alpha$ setzen, so dass dann $\mu_\alpha^\eta = 0$, also hat der Entropiefluss die Darstellung wie bei Clausius-Duhem, oder man setzt $\mu_\alpha^\eta = \frac{1}{\theta}\mu_\alpha = -\eta\prime_{\varrho_\alpha}$ und dann $\mu_\alpha^\epsilon = 0$. In diesem Falle besteht der Energiefluss q nur aus dem Wärmeanteil, dafür hat der Entropiefluss zusätzliche $\mathbf{J}_\alpha$-Terme. Diese Alternative wurde zuerst in Alt & Pawlow [18] benutzt, und zwar in dem Fall, der in 12.4 behandelt wird.

Proof. We repeat the proof of III.2.4 but for the entropy in this mixture case

$$\eta = \widehat{\eta}((\varrho_\alpha)_\alpha, \varepsilon)\,,$$

that is, now the entropy is a function of each mass density ϱ_α of the mixture. Define $\overset{\circ}{h} = \partial_t h + v\bullet\nabla h$ for each function h. This gives

$$\overset{\circ}{\eta} = \sum_\alpha \eta\prime_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \eta\prime_\varepsilon\overset{\circ}{\varepsilon}$$

and $\overset{\circ}{\varrho}_\alpha$ and $\overset{\circ}{\varepsilon}$ can be computed by the equations (set $\Pi = p\mathrm{Id} - S$, we always can do this without meaning of terms)

$$\begin{aligned}&\overset{\circ}{\varrho}_\alpha + \varrho_\alpha\,\mathrm{div}v = \mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha\,,\\ &\overset{\circ}{\varepsilon} + (\varepsilon + p)\,\mathrm{div}v + \mathrm{div}q = \mathrm{D}v\bullet S + g\,.\end{aligned} \tag{IV12.2}$$

Hence

$$\begin{aligned}0 \le \sigma &= \partial_t\eta + \mathrm{div}\psi\\ &= \overset{\circ}{\eta} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v)\\ &= \sum_\alpha \eta\prime_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \eta\prime_\varepsilon\overset{\circ}{\varepsilon} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v)\,.\end{aligned}$$

Now

$$\begin{aligned}&\sum_\alpha \eta\prime_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \eta\prime_\varepsilon\overset{\circ}{\varepsilon} + \eta\,\mathrm{div}v\\ &= \sum_\alpha \eta\prime_{\varrho_\alpha}(-\varrho_\alpha\,\mathrm{div}v + \mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha)\\ &+\eta\prime_\varepsilon(-(\varepsilon+p)\,\mathrm{div}v - \mathrm{div}q + \mathrm{D}v\bullet S + g) + \eta\,\mathrm{div}v\\ &= \mathrm{div}v\cdot\big(\eta - \sum_\alpha \varrho_\alpha\eta\prime_{\varrho_\alpha} - (\varepsilon+p)\eta\prime_\varepsilon\big) + \eta\prime_\varepsilon\mathrm{D}v\bullet S\\ &+ \sum_\alpha \eta\prime_{\varrho_\alpha}(\mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha) + \eta\prime_\varepsilon(g - \mathrm{div}q)\,.\end{aligned}$$

Gehe jetzt zum Beweis des klassischen Falles oder zum Beweis des allgemeinen Falles über. □

Proof (klassisch). Wir nehmen an, dass wir $\mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha$ wie Raten behandeln können (womit also der Fall abgedeckt ist, dass die Flüsse $\mathbf{J}_\alpha$ verschwinden).

Dann ist

$$\begin{aligned}
&\sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \eta'_\varepsilon(g - \operatorname{div}q) \\
&= \sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \eta'_\varepsilon g - \eta'_\varepsilon \operatorname{div}q \\
&= \sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \eta'_\varepsilon g + \nabla\eta'_\varepsilon \bullet q + \operatorname{div}(-\eta'_\varepsilon q)\,,
\end{aligned} \tag{IV12.3}$$

wobei die letzte Gleichung die Standardmanipulation

$$\eta'_\varepsilon \operatorname{div}q = \operatorname{div}(\eta'_\varepsilon q) - \nabla\eta'_\varepsilon \bullet q$$

darstellt. Also erhalten wir

$$\begin{aligned}
&0 \le \sigma = \operatorname{div}(\psi - \eta v - \eta'_\varepsilon q) \\
&+ \operatorname{div} v \cdot \big(\eta - \sum_\alpha \varrho_\alpha \eta'_{\varrho_\alpha} - (\varepsilon + p)\eta'_\varepsilon\big) + \eta'_\varepsilon \mathrm{D}v \bullet S \\
&+ \sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \eta'_\varepsilon g + \nabla\eta'_\varepsilon \bullet q\,.
\end{aligned}$$

Dies ist, falls $\mathbf{J}_\alpha$ verschwindet, die klassische Darstellung der Entropieproduktion. Hierbei hat der Entropiefluss nur den Term $\eta'_\varepsilon q$, der ja schon in der Clausius-Duhem Ungleichung auftrat. Bei allgemeinem $\mathbf{J}_\alpha$ ist

$$q = q^\epsilon + \sum_\alpha \mu_\alpha^\epsilon \mathbf{J}_\alpha\,,$$

und obige Darstellung von σ geht in die Darstellung in der Fortführung des Beweises über. Gehe jetzt zur Fortführung des Beweises. □

Proof (allgemein). Für allgemeine $\mathbf{J}_\alpha$ erhalten wir, wenn wir $q = q^\epsilon + \mathbf{J}^\epsilon$ schreiben,

$$\begin{aligned}
&\sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \operatorname{div}\mathbf{J}_\alpha) + \eta'_\varepsilon(g - \operatorname{div}q) \\
&= \sum_\alpha \eta'_{\varrho_\alpha}\mathbf{r}_\alpha + \eta'_\varepsilon g - \eta'_\varepsilon \operatorname{div}q^\epsilon \\
&- \sum_\alpha \eta'_{\varrho_\alpha} \operatorname{div}\mathbf{J}_\alpha - \eta'_\varepsilon \operatorname{div}\mathbf{J}^\epsilon\,.
\end{aligned}$$

Den q^ϵ-Term formen wir wie bei Clausius-Duhem um

$$\eta'_\varepsilon \operatorname{div}q^\epsilon = \operatorname{div}(\eta'_\varepsilon q^\epsilon) - \nabla\eta'_\varepsilon \bullet q^\epsilon$$

und die restlichen Flussterme sind, wenn wir setzen

$$\mathbf{J}^\epsilon = \sum_\alpha \mu_\alpha^\epsilon \mathbf{J}_\alpha\,, \quad \mathbf{J}^\eta := \sum_\alpha \mu_\alpha^\eta \mathbf{J}_\alpha\,,$$

gleich

$$\begin{aligned}
&-\sum_\alpha \eta'_{\varrho_\alpha}\,\mathrm{div}\mathbf{J}_\alpha - \eta'_\varepsilon\,\mathrm{div}\mathbf{J}^\epsilon \\
&= \sum_\alpha(-\eta'_{\varrho_\alpha}\,\mathrm{div}\mathbf{J}_\alpha - \eta'_\varepsilon\,\mathrm{div}(\mu^\epsilon_\alpha\mathbf{J}_\alpha) - \mathrm{div}(\mu^\eta_\alpha\mathbf{J}_\alpha)) + \mathrm{div}\mathbf{J}^\eta \\
&= \sum_\alpha(-\eta'_{\varrho_\alpha} - \eta'_\varepsilon\mu^\epsilon_\alpha - \mu^\eta_\alpha)\,\mathrm{div}\mathbf{J}_\alpha + \mathrm{div}\mathbf{J}^\eta \\
&+ \sum_\alpha(-\eta'_\varepsilon\nabla\mu^\epsilon_\alpha - \nabla\mu^\eta_\alpha)\bullet\mathbf{J}_\alpha\,,
\end{aligned}$$

somit

$$\begin{aligned}
&\sum_\alpha \eta'_{\varrho_\alpha}(\mathbf{r}_\alpha - \mathrm{div}\mathbf{J}_\alpha) + \eta'_\varepsilon(g - \mathrm{div}q) \\
&= \sum_\alpha \eta'_{\varrho_\alpha}\mathbf{r}_\alpha + \eta'_\varepsilon g - \mathrm{div}(\eta'_\varepsilon q^\epsilon - \mathbf{J}^\eta) + \nabla\eta'_\varepsilon\bullet q^\epsilon \\
&+ \sum_\alpha(-\eta'_{\varrho_\alpha} - \eta'_\varepsilon\mu^\epsilon_\alpha - \mu^\eta_\alpha)\,\mathrm{div}\mathbf{J}_\alpha \\
&+ \sum_\alpha(-\eta'_\varepsilon\nabla\mu^\epsilon_\alpha - \nabla\mu^\eta_\alpha)\bullet\mathbf{J}_\alpha\,.
\end{aligned} \tag{IV12.4}$$

Gehe jetzt zur Fortführung des Beweises. □

Proof (Fortführung). Also erhalten wir

$$\begin{aligned}
&0 \le \sigma = \mathrm{div}\big(\psi - \eta v - \eta'_\varepsilon q^\epsilon + \sum_\alpha \mu^\eta_\alpha\mathbf{J}_\alpha\big) \\
&+ \mathrm{div}\,v\cdot\big(\eta - \sum_\alpha \varrho_\alpha\eta'_{\varrho_\alpha} - (\varepsilon+p)\eta'_\varepsilon\big) + \eta'_\varepsilon\mathrm{D}v\bullet S \\
&+ \sum_\alpha \eta'_{\varrho_\alpha}\mathbf{r}_\alpha + \eta'_\varepsilon g + \nabla\eta'_\varepsilon\bullet q^\epsilon + \sum_\alpha(-\eta'_\varepsilon\nabla\mu^\epsilon_\alpha - \nabla\mu^\eta_\alpha)\bullet\mathbf{J}_\alpha \\
&+ \sum_\alpha(-\eta'_{\varrho_\alpha} - \eta'_\varepsilon\mu^\epsilon_\alpha - \mu^\eta_\alpha)\,\mathrm{div}\mathbf{J}_\alpha\,.
\end{aligned}$$

Setzen wir nun die vier Gleichungen

$$\begin{aligned}
&\psi - \eta v - \eta'_\varepsilon q^\epsilon + \sum_\alpha \mu^\eta_\alpha\mathbf{J}_\alpha = 0 && \text{(Entropiefluss)}, \\
&\eta - \sum_\alpha \varrho_\alpha\eta'_{\varrho_\alpha} - (\varepsilon+p)\eta'_\varepsilon = 0 && \text{(Gibbs-Relation)}, \\
&g = 0 && \text{(Energieerhaltung)}, \\
&\eta'_{\varrho_\alpha} + \eta'_\varepsilon\mu^\epsilon_\alpha + \mu^\eta_\alpha = 0 && \text{(Verteilung der } \mu_\alpha\text{-Terme)},
\end{aligned}$$

voraus, so folgt

$$0 \le \sigma = \eta'_\varepsilon\mathrm{D}v\bullet S + \nabla\eta'_\varepsilon\bullet q^\epsilon + \sum_\alpha\big(\eta'_{\varrho_\alpha}\mathbf{r}_\alpha - (\nabla\mu^\eta_\alpha + \eta'_\varepsilon\nabla\mu^\epsilon_\alpha)\bullet\mathbf{J}_\alpha\big)\,,$$

was äquivalent zur behaupteten Residualungleichung ist. □

Proof (Innere freie Energie). Wir führen die innere freie Energie ein

$$f((\varrho_\alpha)_\alpha, \theta) := \varepsilon - \theta\eta((\varrho_\alpha)_\alpha, \varepsilon)$$
$$\text{with } \theta = \widehat{\theta}((\varrho_\alpha)_\alpha, \varepsilon) := \frac{1}{\eta'_\varepsilon((\varrho_\alpha)_\alpha, \varepsilon)} \,,$$

und berechnen die Ableitung nach ε

$$f'_\theta \theta'_\varepsilon = 1 - \theta\eta'_\varepsilon - \theta'_\varepsilon \eta = -\theta'_\varepsilon \eta \,,$$

also $f'_\theta = -\eta$, und ϱ_α

$$f'_{\varrho_\alpha} + f'_\theta \theta'_{\varrho_\alpha} = -\theta\eta'_{\varrho_\alpha} - \theta'_{\varrho_\alpha}\eta \,,$$

also $f'_{\varrho_\alpha} = -\theta\eta'_{\varrho_\alpha}$. Es gilt somit

$$f'_\theta = -\eta \text{ und } f'_{\varrho_\alpha} = -\theta\eta'_{\varrho_\alpha} \,.$$

Also ist

$$\begin{aligned} 0 &= \theta\big(\eta - \sum_\alpha \varrho_\alpha \eta'_{\varrho_\alpha} - (\varepsilon + p)\eta'_\varepsilon\big) \\ &= \theta\eta - \sum_\alpha \varrho_\alpha \theta\eta'_{\varrho_\alpha} - \varepsilon - p \\ &= \sum_\alpha \varrho_\alpha f'_{\varrho_\alpha} - f - p \,, \end{aligned} \tag{IV12.5}$$

und

$$\begin{aligned} 0 &= \theta\big(-\eta'_{\varrho_\alpha} - \eta'_\varepsilon \mu^\epsilon_\alpha - \mu^\eta_\alpha\big) \\ &= \mu_\alpha - \mu^\epsilon_\alpha - \theta\mu^\eta_\alpha \,, \end{aligned}$$

und daher wird die Entropieproduktion zu

$$0 \le \sigma = \frac{1}{\theta}\mathrm{D}v \bullet S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q^\epsilon - \sum_\alpha \frac{1}{\theta} f'_{\varrho_\alpha} \mathbf{r}_\alpha - \sum_\alpha \Big(\nabla\mu^\eta_\alpha + \frac{1}{\theta}\nabla\mu^\epsilon_\alpha\Big)\bullet \mathbf{J}_\alpha \,.$$

□

Das Reaktion-Diffusions-System schreibt sich also unter der Annahme, dass (IV11.20) erfüllt ist, mit den Konzentrationen wie folgt

Reaktions-Diffusions-System:

$$\begin{aligned} &\partial_t(\varrho c_\alpha) + \operatorname{div}(\varrho c_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \dots, m, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathbf{f} \,, \\ &\partial_t e + \operatorname{div}((e+p)v - Sv + q^\epsilon + \sum_\alpha \mu^\epsilon_\alpha \mathbf{J}_\alpha) = v\bullet\mathbf{f} \,, \\ \hline &e = \varepsilon + \frac{\varrho}{2}|v|^2 \,, \quad p = (\sum_\alpha \varrho_\alpha f'_{\varrho_\alpha}) - f \,, \\ &\mu_\alpha = \mu^\epsilon_\alpha + \theta\mu^\eta_\alpha \,, \quad \sum_\alpha \mathbf{r}_\alpha = 0 \,, \quad \sum_\alpha \mathbf{J}_\alpha = 0 \,, \end{aligned} \tag{IV12.6}$$

wobei S, q^ϵ, $\mathbf{r}_\alpha$, and $\mathbf{J}_\alpha$ die Residualungleichung erfüllen müssen. Hierbei sind die Entropie und der Entropiefluss wie in 12.1 gewählt. Siehe jetzt insbesondere den Abschnitt 13, wo temperaturabhängige Anwendungsprobleme behandelt werden.

Stetigkeit des chemischen Potentials

Wir vergegenwärtigen uns nocheinmal die Definition der Temperatur und die Definition des chemischen Potentials in 11.1 (und 11.9, wenn man $\mu_\alpha = \mu_\alpha^s$ setzt)

$$\frac{1}{\theta} := \eta'_{\varepsilon} \Big(= \frac{\delta\eta}{\delta\varepsilon}\Big), \quad -\frac{\mu_\alpha}{\theta} := \eta'_{\varrho_\alpha} \Big(= \frac{\delta\eta}{\delta\varrho_\alpha}\Big),$$

wobei wir annehmen wollen, dass $\eta = \widehat{\eta}((\varrho_\alpha)_\alpha, \varepsilon)$ ist (in Klammern ist der Fall angegeben, dass η vom Gradienten der Dichten abhängt). Wir sehen also, dass es sich sowohl bei der Temperatur als auch beim chemischen Potential um erste Ableitungen der Entropie handelt. In III.6.1 hatten wir schon gezeigt, dass daraus an der Grenze zwischen zwei Medien die Stetigkeit der Temperatur folgt. Wir zeigen nun, dass das chemische Potential am Grenzübergang stetig ist. Voraussetzung dafür ist, dass es keinen Produktionsterm auf der Grenze gibt.

12.2 Stetigkeit des chemischen Potentials. Es gelten die Massen-, Impuls- und die Energieerhaltung im Distributionssinn im stationären Sinn in $U \subset \mathbb{R}^n$ und es gelte (für einen Beobachter)

$$U = D^{(1)} \cup \Gamma \cup D^{(2)}, \quad \Gamma \text{ eine glatte Fläche, } .$$

Dann folgt für die chemischen Potentiale

$$\mu^{(1)} = \mu^{(2)} \text{ auf } \Gamma,$$

wenn im Distributionssinne Massen-, Impuls- und Energieerhaltung sowie das Entropieprinzip gelten.

Proof. Wir fassen die Erhaltungsgleichungen als Distributionsgleichungen auf, also gilt für die stationären Gleichungen, dass in $\mathscr{D}'(U)$

$$\begin{aligned} &\operatorname{div}[\varrho v + \mathbf{J}] = [\mathbf{r}], \\ &\operatorname{div}[\varrho v\, v^{\mathrm{T}} + \Pi] = [\mathbf{f}], \\ &\operatorname{div}[ev + \Pi^{\mathrm{T}} v + q] = [g], \end{aligned}$$

wobei $\varrho = \varrho^{(k)}$ in $D^{(k)}$, $k = 1, 2$, entsprechend für die anderen Größen, und $v^{(1)} \bullet \nu = 0$, $v^{(2)} \bullet \nu = 0$ auf Γ, wobei ν eine Einheitsnormale auf Γ ist. Für

Testfunktionen $\zeta \in \mathscr{D}(U;\mathbb{R})$ heißt dies für die Massenerhaltung

$$\begin{aligned}
0 &= \int_{\mathcal{U}} \big(\nabla\zeta \bullet (\varrho v + \mathbf{J}) + \zeta \mathbf{r}\big)\,\mathrm{d}x \\
&= \sum_k \int_{D^{(k)}} \big(\nabla\zeta \bullet (\varrho^{(k)} v^{(k)} + \mathbf{J}^{(k)}) + \zeta \mathbf{r}^{(k)}\big)\,\mathrm{d}x \\
&= \sum_k \int_{D^{(k)}} \zeta\big(-\operatorname{div}(\varrho^{(k)} v^{(k)} + \mathbf{J}^{(k)}) + \mathbf{r}^{(k)}\big)\,\mathrm{d}x \\
&+ \int_{\mathbb{R}} \int_{\Gamma} \zeta\nu \bullet (\varrho^{(2)} v^{(2)} + \mathbf{J}^{(2)} - \varrho^{(1)} v^{(1)} - \mathbf{J}^{(1)})\,\mathrm{dH}^{n-1}(x)\,,
\end{aligned}$$

wobei $\nu = \nu_{D^{(2)}} = -\nu_{D^{(1)}}$. Daraus folgt

$$-\operatorname{div}(\varrho^{(k)} v^{(k)} + \mathbf{J}^{(k)}) + \mathbf{r}^{(k)} = 0 \text{ in } D^{(k)}\,,$$

das heißt die Differentialgleichung in den einzelnen Phasen, und am Grenzübergang Γ

$$\mathbf{J}^{(2)} \bullet \nu = \mathbf{J}^{(1)} \bullet \nu\,.$$

Entsprechend folgt aus der distributionellen Impulserhaltung neben den Differentialgleichungen in $D^{(k)}$

$$\Pi^{(2)}\nu = \Pi^{(1)}\nu \text{ auf } \Gamma$$

und die distributionelle Energieerhaltung ergibt dann

$$q^{(2)} \bullet \nu = q^{(1)} \bullet \nu \text{ auf } \Gamma\,.$$

Jetzt schauen wir uns noch die Entropieungleichung an, die in der distributionellen Version lautet

$$\partial_t[\eta] + \operatorname{div}[\psi] \geq 0 \text{ in } \mathscr{D}'(U)$$

mit $\eta = \eta^{(k)}$ in $D^{(k)}$ und entsprechend für ψ, das heißt, wir nehmen an, es treten keine distributionellen Terme auf Γ auf. Die Entropieungleichung sagt für nichtnegative Testfunktionen ζ

$$\begin{aligned}
0 &\geq \int_U \big(\partial_t\zeta \cdot \eta + \nabla\zeta \bullet \psi\big)\,\mathrm{d}(t,x) \\
&= \sum_k \int_{\mathbb{R}} \int_{D^{(k)}} \big(\partial_t\zeta \cdot \eta^{(k)} + \nabla\zeta \bullet \psi^{(k)}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \sum_k \int_{\mathbb{R}} \int_{D^{(k)}} \zeta\big(-\partial_t\eta^{(k)} - \operatorname{div}\psi^{(k)}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \sum_k \int_{\mathbb{R}} \int_{\Gamma} \zeta\nu \bullet (\psi^{(2)} - \psi^{(1)})\,\mathrm{dH}^{n-1}(x)\,\mathrm{d}t\,.
\end{aligned}$$

Die Entropieungleichung in U enthält nun einerseits

$$0 \geq -\partial_t \eta^{(k)} - \operatorname{div}\psi^{(k)} \text{ in } D^{(k)} ,$$

die Entropieungleichung in $D^{(k)}$, und zum anderen

$$0 \geq \nu \bullet (\psi^{(2)} - \psi^{(1)}) .$$

Da der Entropiefluss $\psi^{(k)}$ in $D^{(k)}$ erfüllt

$$\begin{aligned}\psi^{(k)} &= \eta^{(k)} v^{(k)} + \frac{1}{\theta^{(k)}} q^{(k)h} + \mu^{(k)\eta} \mathbf{J}^{(k)} \\ &\qquad \Big(\text{es ist } \mu^{(k)} = \mu^{(k)h} - \theta^{(k)} \mu^{(k)\eta}\Big) \\ &= \eta^{(k)} v^{(k)} + \frac{1}{\theta^{(k)}} q^{(k)} - \frac{1}{\theta^{(k)}} \mu^{(k)} \mathbf{J}^{(k)} ,\end{aligned}$$

ist

$$0 \geq \nu \bullet (\psi^{(2)} - \psi^{(1)}) = \Big(\frac{1}{\theta^{(2)}} - \frac{1}{\theta^{(1)}}\Big) \nu \bullet q - \Big(\frac{\mu^{(2)}}{\theta^{(2)}} - \frac{\mu^{(1)}}{\theta^{(1)}}\Big) \nu \bullet \mathbf{J} ,$$

wobei $\nu \bullet q = \nu \bullet q^{(1)} = \nu \bullet q^{(2)}$ und $\nu \bullet \mathbf{J} = \nu \bullet \mathbf{J}^{(1)} = \nu \bullet \mathbf{J}^{(2)}$, wie oben gezeigt. Da $\nu \bullet q$ und $\nu \bullet \mathbf{J}$ beliebige Werte haben können, folgt

$$\frac{1}{\theta^{(2)}} = \frac{1}{\theta^{(1)}} , \quad \frac{\mu^{(2)}}{\theta^{(2)}} = \frac{\mu^{(1)}}{\theta^{(1)}} .$$

□

Die Differentialgleichungen mit freiem Rand erhält man auch durch eine Phasenfeldapproximation (siehe z.B. [22]).

Gradient dependence

Wir leiten nun noch das Entropieprinzip her, falls die frei Energie auch von den Gradienten der Dichten abhängt.

12.3 Chemical potential (Gradient dependence). Assume

$$\eta = \widehat{\eta}(\vec{\varrho}, \nabla\vec{\varrho}, \varepsilon) \quad \text{resp.} \quad f = \widehat{f}(\vec{\varrho}, \nabla\vec{\varrho}, \theta) .$$

Seien Vektorfelder $\vec{h}^{\epsilon}_{\alpha}$ and $\vec{h}^{\eta}_{\alpha}$ gegeben mit

$$\eta'_{\nabla \varrho_\alpha} = \vec{h}^{\eta}_{\alpha} + \frac{1}{\theta} \vec{h}^{\epsilon}_{\alpha} \tag{IV12.7}$$

und definiere

$$\boxed{\overline{\mu}_\alpha := \frac{\delta f}{\delta \varrho_\alpha} - \nabla\theta \bullet \vec{h}^{\eta}_{\alpha} = -\theta\Big(\frac{\delta\eta}{\delta\varrho_\alpha} + \nabla\Big(\frac{1}{\theta}\Big) \bullet \vec{h}^{\epsilon}_{\alpha}\Big) ,}$$

so dass also gilt

$$\begin{aligned}\overline{\mu}_\alpha &= f'_{\varrho_\alpha} + \theta \operatorname{div}\vec{h}^{\eta}_{\alpha} + \operatorname{div}\vec{h}^{\epsilon}_{\alpha} \\ &= -\theta\big(\eta'_{\varrho_\alpha} - \operatorname{div}\vec{h}^{\eta}_{\alpha} - \eta'_{\varepsilon} \operatorname{div}\vec{h}^{\epsilon}_{\alpha}\big) .\end{aligned}$$

Erste Variation: Wir habe die Definition der ersten Variation in 11.8 benutzt.
Speziell: Es gilt im Zusammenhang mit 11.9

$$\overline{\mu}_\alpha = \begin{cases} -\theta \dfrac{\delta\eta}{\delta\varrho_\alpha} = \mu_\alpha^s \text{ wenn } \vec{h}_\alpha^\epsilon = 0\,, \\ \dfrac{\delta f}{\delta\varrho_\alpha} = \mu_\alpha^f \text{ wenn } \vec{h}_\alpha^\eta = 0\,. \end{cases}$$

Außerdem, wenn $\vec{h}_\alpha^\eta = 0$ und $\vec{h}_\alpha^\epsilon = 0$, also η und f vom Gradienten unabhängig, so gilt die alte Definition $\overline{\mu}_\alpha = \mu_\alpha = -\theta\eta'_{\varrho_\alpha} = f'_{\varrho_\alpha}$ in 11.9(3).

Proof. Es gilt (immer mit den richtigen Argumenten) $f'_{\nabla\varrho_\alpha} = -\theta\eta'_{\nabla\varrho_\alpha}$, also ist (IV12.7) äquivalent zu

$$f'_{\nabla\varrho_\alpha} = -\theta\eta'_{\nabla\varrho_\alpha} = -\theta\vec{h}_\alpha^\eta - \vec{h}_\alpha^\epsilon\,, \tag{IV12.8}$$

also ist

$$\begin{aligned} &\frac{\delta f}{\delta\varrho_\alpha} - \nabla\theta\bullet\vec{h}_\alpha^\eta = f'_{\varrho_\alpha} + \theta\,\mathrm{div}\vec{h}_\alpha^\eta + \mathrm{div}\vec{h}_\alpha^\epsilon \\ &= -\theta\big(\eta'_{\varrho_\alpha} - \mathrm{div}\vec{h}_\alpha^\eta - \eta'_\varepsilon\,\mathrm{div}\vec{h}_\alpha^\epsilon\big) \\ &= -\theta\Big(\frac{\delta\eta}{\delta\varrho_\alpha} + \nabla\Big(\frac{1}{\theta}\Big)\bullet\vec{h}_\alpha^\epsilon\Big)\,. \end{aligned}$$

□

12.4 Entropy principle with gradient dependence. Wenn für die Entropie und den Entropiefluss gilt, dass

$$\begin{aligned} &\eta = \widehat{\eta}((\varrho_\alpha)_\alpha, (\nabla\varrho_\alpha)_\alpha, \varepsilon) \quad \text{ein objektiver Skalar ist,} \\ &\psi = \eta v + \frac{1}{\theta}q^\epsilon - \sum_\alpha(\mu_\alpha^\eta\mathbf{J}_\alpha + \mathring{\varrho}_\alpha\vec{h}_\alpha^\eta)\,, \end{aligned}$$

so ist das Entropieprinzip erfüllt, falls in den Gleichungen

$$\begin{aligned} &\Pi = P - S\,, \\ &P = \Big(\theta(\eta - \varepsilon\eta'_\varepsilon) + \sum_\alpha\varrho_\alpha\mu_\alpha\Big)\mathrm{Id} - \theta\sum_\alpha\nabla\varrho_\alpha\otimes\eta'_{\nabla\varrho_\alpha}\,, \\ &q = q^\epsilon + \sum_\alpha(\mu_\alpha^\epsilon\mathbf{J}_\alpha + \mathring{\varrho}_\alpha\vec{h}_\alpha^\epsilon)\,, \quad g = 0 \end{aligned}$$

gilt und die Residualungleichung

$$\boxed{0 \le \sigma = \frac{1}{\theta}\mathrm{D}v\bullet S + \nabla\Big(\frac{1}{\theta}\Big)\bullet q^\epsilon - \sum_\alpha\frac{1}{\theta}\mu_\alpha\mathbf{r}_\alpha - \sum_\alpha\Big(\nabla\mu_\alpha^\eta + \frac{1}{\theta}\nabla\mu_\alpha^\epsilon\Big)\bullet\mathbf{J}_\alpha}$$

erfüllt ist, wobei $f = \widehat{f}((\varrho_\alpha)_\alpha, (\nabla\varrho_\alpha)_\alpha, \theta)$ und

$$\mu_\alpha = \mu_\alpha^\epsilon + \theta\mu_\alpha^\eta\,, \quad \eta'_{\nabla\varrho_\alpha} = \vec{h}_\alpha^\eta + \tfrac{1}{\theta}\vec{h}_\alpha^\epsilon\,. \tag{IV12.9}$$

Hierbei ist das chemische Potential $\mu_\alpha := \overline{\mu}_\alpha$ wie oben in 12.3 definiert.
Wichtig: Die Zerlegung der Terme in (IV12.9) bestimmt, welcher Anteil in den Gleichungen der physikalischen Prozesse auftritt.

Proof. Wir beginnen wie im klassischen Fall III.2.4, wobei jetzt für die Entropie $\eta = \widehat{\eta}(\vec{\varrho}, \nabla\vec{\varrho}, \varepsilon)$ gilt, weshalb

$$\begin{aligned} &0 \le \sigma = \partial_t\eta + \mathrm{div}\psi \\ &= \mathring{\eta} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v) \\ &= \sum_\alpha(\eta'_{\varrho_\alpha}\mathring{\varrho}_\alpha + \eta'_{\nabla\varrho_\alpha}\bullet(\nabla\varrho_\alpha)^\circ) + \eta'_\varepsilon\mathring{\varepsilon} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v)\,, \end{aligned}$$

wobei wir hier $\overset{\circ}{\varrho}_\alpha$ und $\overset{\circ}{\varepsilon}$ durch die Gleichungen

$$\begin{aligned} &\overset{\circ}{\varrho}_\alpha + \varrho_\alpha \,\mathrm{div} v = \mathbf{r}_\alpha - \,\mathrm{div}\mathbf{J}_\alpha \,, \\ &\overset{\circ}{\varepsilon} + \varepsilon \,\mathrm{div} v + \mathrm{div} q = -\mathrm{D}v \bullet \Pi + g \end{aligned} \tag{IV12.10}$$

ersetzen wollen, also bleibt $(\nabla\varrho_\alpha)^\circ$ zu betrachten. Die totale Zeitableitung $(\nabla\varrho_\alpha)^\circ$ erfüllt:

$$(\nabla\varrho_\alpha)^\circ = \nabla\overset{\circ}{\varrho}_\alpha - \mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha\,, \tag{IV12.11}$$

was sich aus der folgenden Rechnung ergibt:

$$\begin{aligned} &(\partial_i\varrho_\alpha)^\circ = \partial_t(\partial_i\varrho_\alpha) + (v\bullet\nabla)(\partial_i\varrho_\alpha) \\ &= \underbrace{\partial_t\partial_i}_{\partial_i\partial_t}\,\varrho_\alpha + \sum_{j=1}^n v_j\,\underbrace{\partial_j\partial_i}_{\partial_i\partial_j}\,\varrho_\alpha \\ &= \partial_i\partial_t\varrho_\alpha + \sum_{j=1}^n \big(\partial_i(v_j\partial_j\varrho_\alpha) - (\partial_i v_j)(\partial_j\varrho_\alpha)\big) \\ &= \partial_i\Big(\partial_t\varrho_\alpha + \sum_{j=1}^n v_j\partial_j\varrho_\alpha\Big) - \sum_{j=1}^n(\partial_i v_j)(\partial_j\varrho_\alpha) \\ &= \partial_i\overset{\circ}{\varrho}_\alpha - \big(\mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha\big)_i\,. \end{aligned}$$

Indem wir nun (IV12.11) benutzen und $\overset{\circ}{\varepsilon}$ von (IV12.10) einsetzen, erhalten wir

$$\begin{aligned} &0 \le \sigma = \,\mathrm{div}(\psi - \eta v) + \eta\,\mathrm{div} v \\ &+ \sum_\alpha \eta'_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \sum_\alpha \eta'_{\nabla\varrho_\alpha}\bullet\nabla\overset{\circ}{\varrho}_\alpha - \sum_\alpha \eta'_{\nabla\varrho_\alpha}\bullet(\mathrm{D}v^{\mathrm{T}}\,\nabla\varrho_\alpha) \\ &-\varepsilon\eta'_\varepsilon\,\mathrm{div} v + \mathrm{D}v\bullet(-\eta'_\varepsilon\Pi) - \eta'_\varepsilon\,\mathrm{div} q + \eta'_\varepsilon g \\ &= \,\mathrm{div}(\psi - \eta v) + \mathrm{D}v\bullet\Big((\eta - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \sum_\alpha \nabla\varrho_\alpha\,(\eta'_{\nabla\varrho_\alpha})^{\mathrm{T}} - \eta'_\varepsilon\Pi\Big) \\ &+ \sum_\alpha \eta'_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \sum_\alpha \eta'_{\nabla\varrho_\alpha}\bullet\nabla\overset{\circ}{\varrho}_\alpha - \eta'_\varepsilon\,\mathrm{div} q + \eta'_\varepsilon g\,. \end{aligned}$$

In der letzten Zeile nutze nun

$$\begin{aligned} &\eta'_{\nabla\varrho_\alpha} = \vec{h}^\eta_\alpha + \eta_\varepsilon\vec{h}^\epsilon_\alpha\,, \\ &q = q^\epsilon + \sum_\alpha \mu^\epsilon_\alpha\mathbf{J}_\alpha + \sum_\alpha \overset{\circ}{\varrho}_\alpha\vec{h}^\epsilon_\alpha \end{aligned}$$

aus, um zu erhalten

$$\begin{aligned} &\sum_\alpha \eta'_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \sum_\alpha \eta'_{\nabla\varrho_\alpha}\bullet\nabla\overset{\circ}{\varrho}_\alpha - \eta'_\varepsilon\,\mathrm{div} q + \eta'_\varepsilon g \\ &= \sum_\alpha \eta'_{\varrho_\alpha}\overset{\circ}{\varrho}_\alpha + \sum_\alpha(\vec{h}^\eta_\alpha + \eta_\varepsilon\vec{h}^\epsilon_\alpha)\bullet\nabla\overset{\circ}{\varrho}_\alpha \\ &- \sum_\alpha \eta'_\varepsilon\,\mathrm{div}(\overset{\circ}{\varrho}_\alpha\vec{h}^\epsilon_\alpha + \mu^\epsilon_\alpha\mathbf{J}_\alpha) - \eta'_\varepsilon\,\mathrm{div} q^\epsilon + \eta'_\varepsilon g \\ &= \sum_\alpha\,\mathrm{div}(\overset{\circ}{\varrho}_\alpha\vec{h}^\eta_\alpha) - \eta'_\varepsilon\,\mathrm{div} q^\epsilon + \eta'_\varepsilon g \\ &+ \sum_\alpha(\eta'_{\varrho_\alpha} - \,\mathrm{div}\vec{h}^\eta_\alpha - \eta'_\varepsilon\,\mathrm{div}\vec{h}^\epsilon_\alpha)\overset{\circ}{\varrho}_\alpha - \sum_\alpha \eta'_\varepsilon\,\mathrm{div}(\mu^\epsilon_\alpha\mathbf{J}_\alpha)\,. \end{aligned}$$

Indem in der letzten Zeile

$$-\eta'_\varepsilon\mu_\alpha = \eta'_{\varrho_\alpha} - \,\mathrm{div}\vec{h}^\eta_\alpha - \eta'_\varepsilon\,\mathrm{div}\vec{h}^\epsilon_\alpha$$

ausgenutzt wird, wird dies, wenn wir $\overset{\circ}{\varrho}$ von (IV12.10) einsetzen,

$$
\begin{aligned}
&= \sum_\alpha \operatorname{div}(\overset{\circ}{\varrho}_\alpha \vec{h}^\eta_\alpha) - \eta'_\varepsilon \operatorname{div} q^\epsilon + \eta'_\varepsilon g \\
&- \sum_\alpha \eta'_\varepsilon \mu_\alpha \overset{\circ}{\varrho}_\alpha - \sum_\alpha \eta'_\varepsilon \operatorname{div}(\mu^\epsilon_\alpha \mathbf{J}_\alpha) \\
&= \sum_\alpha \operatorname{div}(\overset{\circ}{\varrho}_\alpha \vec{h}^\eta_\alpha) - \eta'_\varepsilon \operatorname{div} q^\epsilon + \eta'_\varepsilon g \\
&- \sum_\alpha \eta'_\varepsilon \mu_\alpha (-\varrho_\alpha \operatorname{div} v + \mathbf{r}_\alpha - \operatorname{div} \mathbf{J}_\alpha) - \sum_\alpha \eta'_\varepsilon \operatorname{div}(\mu^\epsilon_\alpha \mathbf{J}_\alpha) \\
&= \operatorname{div}\Big(\sum_\alpha \overset{\circ}{\varrho}_\alpha \vec{h}^\eta_\alpha + \sum_\alpha \mu^\eta_\alpha \mathbf{J}_\alpha\Big) - \eta'_\varepsilon \operatorname{div} q^\epsilon + \eta'_\varepsilon g \\
&+ \mathrm{D}v \bullet \Big(\sum_\alpha \varrho_\alpha \eta'_\varepsilon \mu_\alpha \mathrm{Id}\Big) - \sum_\alpha \eta'_\varepsilon \mu_\alpha \mathbf{r}_\alpha \\
&+ \sum_\alpha \eta'_\varepsilon \mu_\alpha \operatorname{div} \mathbf{J}_\alpha - \sum_\alpha \operatorname{div}(\mu^\eta_\alpha \mathbf{J}_\alpha) - \sum_\alpha \eta'_\varepsilon \operatorname{div}(\mu^\epsilon_\alpha \mathbf{J}_\alpha) \,.
\end{aligned}
$$

And since $\eta'_\varepsilon \mu_\alpha - \mu^\eta_\alpha - \eta'_\varepsilon \mu^\epsilon_\alpha = 0$ we see that the last line is

$$
-\sum_\alpha (\nabla \mu^\eta_\alpha + \eta'_\varepsilon \nabla \mu^\epsilon_\alpha) \mathbf{J}_\alpha \,.
$$

Altogether

$$
\begin{aligned}
0 \le \sigma = \;& \operatorname{div}(\psi - \eta v) \\
&+ \mathrm{D}v \bullet \Big((\eta - \varepsilon \eta'_\varepsilon)\mathrm{Id} - \sum_\alpha \nabla \varrho_\alpha \, (\eta'_{\nabla \varrho_\alpha})^{\mathrm{T}} - \eta'_\varepsilon \Pi\Big) \\
&+ \operatorname{div}\Big(\sum_\alpha \overset{\circ}{\varrho}_\alpha \vec{h}^\eta_\alpha + \sum_\alpha \mu^\eta_\alpha \mathbf{J}_\alpha\Big) - \eta'_\varepsilon \operatorname{div} q^\epsilon + \eta'_\varepsilon g \\
&+ \mathrm{D}v \bullet \Big(\sum_\alpha \varrho_\alpha \eta'_\varepsilon \mu_\alpha \mathrm{Id}\Big) - \sum_\alpha \eta'_\varepsilon \mu_\alpha \mathbf{r}_\alpha \\
&- \sum_\alpha (\nabla \mu^\eta_\alpha + \eta'_\varepsilon \nabla \mu^\epsilon_\alpha) \mathbf{J}_\alpha \,.
\end{aligned}
$$

This proves the assertion. □

13 Chemical reactions

We describe here some examples for the reaction term $\mathbf{r}_\alpha$ in (IV11.2), as they occur in chemistry. We base the computation on an internal free energy depending on the denisties ϱ_α:

$$f = \widehat{f}((\varrho_\alpha)_\alpha, \theta)\,, \quad \vec{\varrho} = (\varrho_\alpha)_\alpha\,. \tag{IV13.1}$$

Bei chemischen Reaktionen ist von dem Einfluss der Temperatur θ auszugehen. Wir kommen darauf zurück und konstatieren zunächst nur, dass für jede Masse der Reaktion der Erhaltungssatz

$$\partial_t \varrho_\alpha + \operatorname{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha\,. \tag{IV13.2}$$

erfüllt ist. In diesem Abschnitt betrachten wir die Konzentrationen:

Densities:	$\varrho_\alpha\,,\ \vec{\varrho} = (\varrho_\alpha)_\alpha\,,\ \varrho := \sum_\alpha \varrho_\alpha\,,$
Concentrations:	$c_\alpha := \dfrac{\varrho_\alpha}{\varrho}\,,\ \vec{c} = (c_\alpha)_\alpha\,,\ \sum_\alpha c_\alpha = 1\,.$

Also können die Massenerhaltungen (IV13.2) auch geschrieben werden als

$$\partial_t(\varrho c_\alpha) + \operatorname{div}(\varrho c_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha = \varrho \mathbf{r}_\alpha^{\mathrm{sp}}\,, \quad \mathbf{r}_\alpha^{\mathrm{sp}} := \tfrac{1}{\varrho}\mathbf{r}_\alpha\,, \tag{IV13.3}$$

wobei $\mathbf{r}_\alpha^{\mathrm{sp}}$ die ***spezifischen Reaktionsraten*** sind. Falls alle $\mathbf{J}_\alpha = 0$ und die Gesamtrate $\mathbf{r} = \sum_\alpha \mathbf{r}_\alpha = 0$ können wir (IV13.3) auch schreiben als, siehe den Standardfall in (IV11.18),

$$\boxed{\overset{\circ}{c}_\alpha = \mathbf{r}_\alpha^{\mathrm{sp}}\,,} \tag{IV13.4}$$

wobei $\overset{\circ}{h} := \partial_t h + v \bullet \nabla h$ für jede Funktion h.

13.1 Spezifische freie Energie. Die freie Energie f habe die Eigenschaft wie in (IV13.1) und Entsprechendes gelte für die Entropie. Die ***spezifische freie Energie*** ist

$$f^{\mathrm{sp}} := \frac{1}{\varrho} f\,. \tag{IV13.5}$$

In 11.1 waren die ***chemischen Potentiale*** μ_α schon eingeführt, es gilt

$$\mu_\alpha = -\theta \eta'_{\varrho_\alpha}(\vec{\varrho}, \theta) = f'_{\varrho_\alpha}(\vec{\varrho}, \theta)\,.$$

(1) Ist $f^{\mathrm{sp}} = \widehat{f}^{\mathrm{sp}}(\vec{\varrho}, \theta)$ so gilt, wenn $\vec{\varrho} = \varrho \vec{c}$,

$$\mu_\alpha = f'_{\varrho_\alpha}(\vec{\varrho}, \theta) = \frac{\partial}{\partial c_\alpha}\big(\widehat{f}^{\mathrm{sp}}(\varrho \vec{c}, \theta)\big)\,. \tag{IV13.6}$$

(2) Ist $f^{\mathrm{sp}} = \widetilde{f}^{\mathrm{sp}}(\vec{c}, \varrho, \theta)$, so gilt mit einer von α unabhängigen Funktion d

$$\mu_\alpha = f_{,\varrho_\alpha} = \widetilde{f}^{\mathrm{sp}}_{,c_\alpha}(\vec{c}, \varrho, \theta) + d(\vec{c}, \varrho, \theta)\,. \tag{IV13.7}$$

(3) Mit $\widetilde{f}^{\mathrm{sp}}$ wie in (2) ist

$$\widetilde{f}^{\mathrm{sp}}_{,c_\alpha} = \widetilde{f}^{\mathrm{sp}}_{,\tau_\alpha} + \widetilde{f}^{\mathrm{sp}}_{,\nu}\,,$$

wobei $\tau_\alpha := \sum_{\beta:\beta\neq\alpha}\frac{1}{N}(\mathbf{e}_\alpha - \mathbf{e}_\beta)$ Tangentialvektoren an die Bedingung an $\vec{c}$ sind und wobei ein Normalenvektor durch $\nu := \frac{1}{N}(1,\ldots,1)$ gegeben ist.

Also auch wenn $\widetilde{f}^{\mathrm{sp}}$ für Vektoren $\vec{c}$, welche nicht die Nebenbedingung erfüllen, gebraucht wird, ändern sich dadurch die Darstellung der chemischen Potentiale μ_α nur durch eine konstante Funktion, d.h. durch einen von α unabhängigen Term.

Proof (1).

$$\big(f^{\mathrm{sp}}(\varrho c, \theta)\big)_{,c_\alpha} = \Big(\frac{1}{\varrho} f(\varrho c, \theta)\Big)_{,c_\alpha} = \frac{1}{\varrho}\big(f(\varrho c, \theta)\big)_{,c_\alpha} = \frac{\varrho}{\varrho} f_{,\varrho_\alpha}(\varrho c, \theta) = \mu_\alpha\,.$$

Die Nebenbedingung $\sum_\alpha c_\alpha = 1$ für die Konzentrationen wird hier nicht benutzt. □

Proof (2). Es ist $f(\vec{\varrho}, \theta) = \varrho \widetilde{f}^{\mathrm{sp}}(\frac{\vec{\varrho}}{\varrho}, \varrho, \theta)$ mit $\varrho = \sum_\alpha \varrho_\alpha$, also

$$\begin{aligned}\mu_\alpha &= f_{,\varrho_\alpha} = f^{\mathrm{sp}} + \varrho \frac{\partial}{\partial \varrho_\alpha} f^{\mathrm{sp}} = f^{\mathrm{sp}} + \varrho \widetilde{f}^{\mathrm{sp}}_{,\varrho} + \varrho \sum_\beta \widetilde{f}^{\mathrm{sp}}_{,c_\beta}\big(\frac{\delta_{\beta,\alpha}}{\varrho} - \frac{\varrho_\beta}{\varrho^2}\big)\\ &= \widetilde{f}^{\mathrm{sp}} + \varrho \widetilde{f}^{\mathrm{sp}}_{,\varrho} + \widetilde{f}^{\mathrm{sp}}_{,c_\alpha} - \sum_\beta c_\beta \widetilde{f}^{\mathrm{sp}}_{,c_\beta} = \widetilde{f}^{\mathrm{sp}}_{,c_\alpha} + \mathrm{const}(\vec{c}, \varrho, \theta)\end{aligned}$$

mit $\mathrm{const}(\vec{c}, \varrho, \theta) = \widetilde{f}^{\mathrm{sp}} + \varrho \widetilde{f}^{\mathrm{sp}}_{,\varrho} - \sum_\beta c_\beta \widetilde{f}^{\mathrm{sp}}_{,c_\beta}$. □

Proof (3). Es ist $\mathbf{e}_\alpha = \tau_\alpha + \nu$ und $\mathbf{e}_\alpha - \mathbf{e}_\beta$ sind Tangentialvektoren. □

References: See DeGroot & Mazur [6, Ch. X §3 Coupled Chemical Reactions] and I. Müller [87, 3.2.2.3. Seite 68, 6.5 Seite 196 Chemical Equilibrium], and the book of G.R. Gavalas [41, Nonlinear Differential Equations of Chemically Reacting Systems]. Also visit [Wikipedia: Chemical reaction] and see the references below. For an overview see [Wikipedia: Stöchiometrie] and [Wikipedia: Stoichiometry] and [Wikipedia: Stoeichiometrie].

Stöchiometrie

Betrachte Moleküle M_k, $k = 1, \dots, N$, in einer Mischung mit J Reaktionen, und zwar für $j = 1, \dots, J$, welche wir schreiben als

Stöchiometrische Gleichungen:

$$\sum_k \nu_k^j \mathrm{M}_k \rightleftarrows \sum_k \overline{\nu}_k^j \mathrm{M}_k$$

M_k Partikel (Molekül, Atom, Ion, Elektron, ...) (IV13.8)

$\nu_k^j,\ \overline{\nu}_k^j \in \mathbb{N} \cup \{0\}\,,$

$\gamma_k^j := \overline{\nu}_k^j - \nu_k^j$ stöchiometrische Koeffizienten,

(stoicheion = Grundstoff, metron = Maß).

Als Beispiel geben wir die Bildung von Wasser an.

13.2 Beispiel: Bildung von Wasser ($N = 6$, $J = 4$). (Nach I. Müller [87, 3.2.2.3])

(1) Die Reaktion lautet vereinfacht

$$2H_2 + O_2 \rightleftarrows 2H_2O\,. \tag{IV13.9}$$

Allerdings hat man dabei die Reaktion von 3 Molekülen, nämlich H_2, H_2 und O_2, was ein seltenes Ereignis darstellt.

(2) Genauer haben wir die folgenden Reaktionen. Es sind die beteiligten Molekülarten

$$H,\quad O,\quad OH,\quad H_2,\quad O_2,\quad H_2O$$

und die Reaktionen schreiben sich als

$$\begin{aligned} H + O &\rightleftarrows OH \\ H_2 &\rightleftarrows 2H \\ O_2 &\rightleftarrows 2O \\ H + OH &\rightleftarrows H_2O \end{aligned} \tag{IV13.10}$$

Die stöchiometrische Matrix ist

$$\gamma := (\gamma_k^j)_{kj} = \begin{bmatrix} -1 & +2 & 0 & -1 \\ -1 & 0 & +2 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}.$$

Es ist $\operatorname{rank}\gamma = J = 4$, d.h. die Reaktionen sind unabhängig voneinander.

"Addiert" man die einzelnen Reaktionen in (IV13.10), so erhält man (IV13.9).

Proof (2). Es sind

$$\nu := (\nu_k^j)_{kj} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad \overline{\nu} := (\overline{\nu}_k^j)_{kj} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

also hat $\gamma = \overline{\nu} - \nu$ die angegebenen Einträge. □

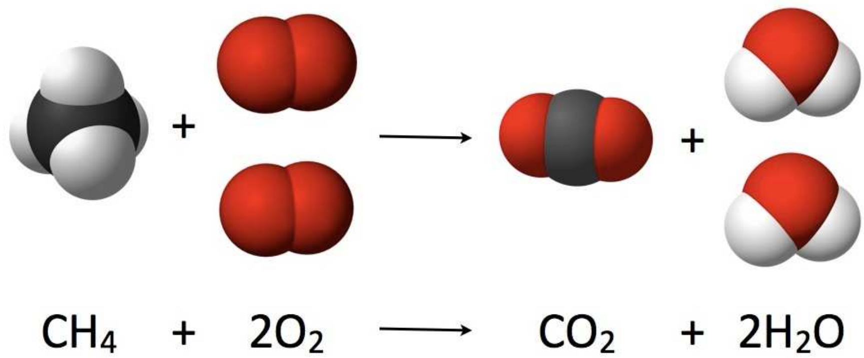

Fig. 30: "As seen from the equation .... , a coefficient of 2 must be placed before the oxygen gas on the reactants side and before the water on the products side in order for, as per the law of conservation of mass, the quantity of each element does not change during the reaction" from Wikipedia

13.3 Ansatz für Reaktionsraten. Mit den stöchiometrischen Gleichungen in (IV13.8) machen wir für die Reaktionsrate des k-ten Moleküls den Ansatz

$$\mathbf{r}_k^{\mathrm{sp}} := \sum_j \iota_k \gamma_k^j \lambda^j ,$$

wobei λ^j die chemische Reaktionsrate der j-ten Reaktion ist, und ι_k noch zu bestimmende Konstanten sind. Die Reaktionsrate λ^j ist eine Funktion der chemischen Potentiale μ_k, und wie folgt gegeben. Definiere

$$\lambda^j := \lambda^j \Big(\vec{c}, \varrho, \theta, \underbrace{\sum_k \iota_k \nu_k^j \mu_k}_{=:\, \xi^j} , \underbrace{\sum_k \iota_k \overline{\nu}_k^j \mu_k}_{=:\, \overline{\xi}^j} \Big) .$$

Dann ist also

$$\boxed{\mathbf{r}_k^{\mathrm{sp}}(\vec{c},\varrho,\theta) = \sum_j \iota_k \gamma_k^j \lambda^j(\vec{c},\varrho,\theta,\xi^j,\overline{\xi}^j)\,.}$$

Mit den so definierten Reaktionsraten gilt

$$\sum_k \mu_k \mathbf{r}_k = \varrho \sum_k \mu_k \mathbf{r}_k^{\mathrm{sp}} = \varrho \sum_{kj} \mu_k \underbrace{\iota_k \gamma_k^j}_{= \iota_k(\overline{\nu}_k^j - \nu_k^j)} \lambda^j(\vec{c},\varrho,\theta,\xi^j,\overline{\xi^j})$$
$$= \varrho \sum_j \Big(\underbrace{\sum_k \iota_k(\overline{\nu}_k^j - \nu_k^j)\mu_k}_{= \overline{\xi}^j - \xi^j} \Big) \lambda^j(\vec{c},\varrho,\theta,\xi^j,\overline{\xi^j})$$
$$= \varrho \sum_j \underbrace{(\overline{\xi}^j - \xi^j)\lambda^j(\vec{c},\varrho,\theta,\xi^j,\overline{\xi^j})}_{\le 0 \text{ nach der Voraussetzung}} \le 0\,,$$

also hat dieser Teil der Entropieproduktion das richtige Vorzeichen.

13.4 Voraussetzung. Für jedes j gelte, dass

$$\lambda^j : \mathbb{R}^{N+2} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\,,$$
$$(\xi - \overline{\xi})\lambda^j(z,\xi,\overline{\xi}) \ge 0 \text{ für alle } z,\xi,\overline{\xi}.$$

Wir verlangen darüberhinaus noch eine Nichttrivialität, die besagt

$$(\xi - \overline{\xi})\lambda^j(z,\xi,\overline{\xi}) > 0 \text{ für } \xi \ne \overline{\xi}\,. \tag{IV13.11}$$

Wenn λ^j stetig ist, folgt dann $\lambda^j(z,\xi,\xi) = 0$ für alle z und ξ.

Kehrt man eine chemische Reaktion um, so vertauschen sich ν_k^j mit $\overline{\nu}_k^j$, also wird γ^j durch $-\gamma^j$ ersetzt und ξ^j vertauscht sich mit $\overline{\xi}^j$. Die Bedingung (IV13.11) ist invariant bezüglich dieser Vertauschung.

Free energy

Die freie Energie für ideale Mischungen von Gasen wurde in 9.11 angegeben mit

$$\boxed{f = \widehat{f}(\vec{\varrho},\theta) = \sum_k \big(R^k\theta\varrho_k \log\varrho_k - c_V^k\varrho_k\theta\log\theta + d^k\theta\varrho_k\big)\,.} \tag{IV13.12}$$

Die zugehörige spezifische freie Energie ist

$$f^{\mathrm{sp}} = \frac{\widehat{f}(\varrho\vec{c},\theta)}{\varrho} = \sum_k \big(R^k\theta c_k \log(c_k\varrho) - c_V^k c_k\theta\log\theta + d^k\theta c_k\big)$$
$$= \sum_k \big(R^k\theta c_k(\log c_k - 1) + a_k(\varrho,\theta)\,c_k\big)\,,$$
$$a_k(\varrho,\theta) := R^k\theta(\log\varrho + 1) - c_V^k\theta\log\theta + d^k\theta\,,$$

and therefore by 13.1(1)

$$\mu_k = \widehat{f}'_{\varrho_k}(\vec{\varrho},\theta) = \frac{\partial}{\partial c_k}\Big(\frac{\widehat{f}(\varrho\vec{c},\theta)}{\varrho}\Big)\,,$$

also

$$\boxed{\mu_k = R^k\theta\log c_k + a_k(\varrho,\theta)\,.}$$

Und daher ist mit den Definitionen in 13.3

$$\begin{aligned}\xi^j &= \sum_k \iota_k\nu_k^j\mu_k = \sum_k \iota_k R^k\theta\log\big(c_k^{\nu_k^j}\big) + a^j(\varrho,\theta)\,,\ a^j(\varrho,\theta) := \sum_k \iota_k\nu_k^j a_k(\varrho,\theta)\,,\\ \overline{\xi}^j &= \sum_k \iota_k\overline{\nu}_k^j\mu_k = \sum_k \iota_k R^k\theta\log\big(c_k^{\overline{\nu}_k^j}\big) + \overline{a}^j(\varrho,\theta)\,,\ \overline{a}^j(\varrho,\theta) := \sum_k \iota_k\overline{\nu}_k^j a_k(\varrho,\theta)\,.\end{aligned}$$

Dies ist in die Gleichungen einzusetzen

$$\overset{\circ}{c}_k = \mathbf{r}_k^{\mathrm{sp}} = \sum_j \iota_k\gamma_k^j\lambda^j(\vec{c},\varrho,\theta,\xi^j,\overline{\xi}^j)\,,$$

wobei wir ξ^j und $\overline{\xi}^j$, wie gerade berechnet, einzusetzen haben. Folgende Version wird für die chemischen Reaktionsraten λ^j meistens verwendet.

13.5 Chemische Reaktionsraten. Das klassische Modell ist

$$\boxed{\lambda^j(\vec{c},\varrho,\theta,\xi,\overline{\xi}) := \sigma^j(\vec{c},\varrho,\theta)\cdot\Big(\exp\Big(\frac{\xi}{\bar{R}\theta}\Big) - \exp\Big(\frac{\overline{\xi}}{\bar{R}\theta}\Big)\Big)\,,} \tag{IV13.13}$$

wobei $\sigma^j > 0$ und $\bar{R} > 0$ ist. *Bemerkung:* Dieses Modell stammt aus der statistischen Mechanik für Gase, wobei $\bar{R} = k_B$ die Boltzmann-Konstante ist. *Linearer Fall:* Ein lineares Modell nahe dem Equilibuium ist mit $\widetilde{\sigma}^j > 0$ gegeben durch $\lambda^j(\vec{c},\varrho,\theta,\xi,\overline{\xi}) := \widetilde{\sigma}^j(\vec{c},\varrho,\theta)\cdot(\xi - \overline{\xi})$.

Wir bestimmen nun die Werte ι_k so dass für einen Wert $\bar{R}$ gilt

$$\boxed{\begin{gathered}\iota_k R^k = \bar{R}\,,\\ \text{in detail } R^k = \frac{\mathcal{R}}{M^k}\,,\ \iota_k := \frac{M^k}{M}\,,\ \bar{R} := \frac{\mathcal{R}}{M}\,,\end{gathered}} \tag{IV13.14}$$

wobei M frei gesetzt werden kann. Dann wird

$$\begin{aligned}\frac{\xi^j}{\bar{R}\theta} &= \frac{1}{\bar{R}\theta}\sum_k \iota_k\nu_k^j\mu_k = \sum_k \frac{\iota_k R^k}{\bar{R}}\log\big(c_k^{\nu_k^j}\big) + \frac{a^j(\varrho,\theta)}{\bar{R}\theta}\\ &= \sum_k \log\big(c_k^{\nu_k^j}\big) + \frac{a^j(\varrho,\theta)}{\bar{R}\theta} = \log\Big(\prod_k c_k^{\nu_k^j}\Big) + \frac{a^j(\varrho,\theta)}{\bar{R}\theta}\end{aligned}$$

und analoges für $\overline{\xi}^j$. Damit gilt

13.6 Lemma. Falls die freie Energie wie in (IV13.12) gegeben ist, lautet die Annahme (IV13.13), dass für alle Reaktionen j

$$\lambda^j = \sigma^j(\vec{c}, \varrho, \theta)\Big(\alpha^j(\varrho, \theta)\prod_k c_k^{\nu_k^j} - \overline{\alpha}^j(\varrho, \theta)\prod_k c_k^{\overline{\nu}_k^j}\Big) \tag{IV13.15}$$

wenn

$$\alpha^j(\varrho, \theta) = \exp\Big(\frac{a^j(\varrho, \theta)}{\bar{R}\theta}\Big), \quad \overline{\alpha}^j(\varrho, \theta) = \exp\Big(\frac{\overline{a}^j(\varrho, \theta)}{\bar{R}\theta}\Big).$$

Wir bekommen also die Differentialgleichungen zu den Konzentrationen

$$\boxed{\begin{gathered}\overset{\circ}{c}_k = \sum_j \iota_k \gamma_k^j \sigma^j(\vec{c}, \varrho, \theta)\Big(\alpha^j(\varrho, \theta)\prod_l c_l^{\nu_l^j} - \overline{\alpha}^j(\varrho, \theta)\prod_l c_l^{\overline{\nu}_l^j}\Big) \\ \text{für } k = 1, \dots, N.\end{gathered}} \tag{IV13.16}$$

Was sind die stationären Lösungen dieses Systems? Lösungen mit rechter Seite gleich 0, oder sind das Lösungen mit $\xi^j = \overline{\xi}^j$ für jede Reaktion j? Antwort auf diese Fragen gibt das folgende Statement.

13.7 Stationäre Lösungen. Es mögen die Voraussetzungen in 13.4 erfüllt sein. Wir führen nur die Abhängigkeit von den Konzentrationen $\vec{c}$ an und definieren $\vec{r} := (\mathbf{r}_k^{\mathrm{sp}})_k$, entsprechend $\vec{\mu} := (\mu_k)_k$. Dann gilt:

(1) Falls (IV13.11): $\vec{r}(\vec{c}) \neq 0 \iff \vec{\mu}(\vec{c}) \bullet \vec{r}(\vec{c}) < 0$.

(2) Falls (IV13.11): $\vec{r}(\vec{c}) = 0 \iff$ Für alle $j = 1, \dots, J$ gilt

$$\sum_k \iota_k \gamma_k^j \mu_k(\vec{c}) = 0 \qquad (\textbf{\textit{General mass action law}}).$$

(3) Falls (IV13.15): $\vec{r}(\vec{c}) = 0 \iff$ Für alle $j = 1, \dots, J$ gilt

$$\boxed{\prod_k c_k^{\gamma_k^j} = \frac{\alpha^j(\varrho, \theta)}{\overline{\alpha}^j(\varrho, \theta)} \qquad (\textbf{\textit{Mass action law}}),}$$

wobei gilt

$$\frac{\alpha^j(\varrho, \theta)}{\overline{\alpha}^j(\varrho, \theta)} = \exp\Big(-\frac{1}{\bar{R}\theta}\sum_k \iota_k \gamma_k^j a_k(\varrho, \theta)\Big).$$

Proof (1). Unter der Voraussetzung (IV13.11) gilt nach der obigen Identität

$$\vec{\mu}(\vec{c}) \bullet \vec{r}(\vec{c}) = \sum_k \mu_k \mathbf{r}_k^{\mathrm{sp}} = -\sum_j (\xi^j(\vec{c}) - \overline{\xi}^j(\vec{c}))\lambda^j(\vec{c}, \xi^j(\vec{c}), \overline{\xi}^j(\vec{c}))$$
$$\leq 0 \text{ und } = 0 \text{ genau dann,wenn } \xi^j(\vec{c}) = \overline{\xi}^j(\vec{c}) \text{ für alle } j.$$

Ist also $\vec{\mu}(\vec{c}) \bullet \vec{r}(\vec{c}) = 0$, so folgt $\xi^j(\vec{c}) = \overline{\xi}^j(\vec{c})$ für alle j, nach 13.4 ist also $\lambda^j(\vec{c}, \xi^j(\vec{c}), \overline{\xi}^j(\vec{c})) = 0$ und daher $\vec{r}(\vec{c}) = 0$. Damit ist (1) bewiesen. □

Proof (2). Es ist $\vec{r}(\vec{c}) = 0$ nach (1) äquivalent dazu, dass $\xi^j(\vec{c}) = \overline{\xi}^j(\vec{c})$ für alle j, was nach der Definition in 13.3 bedeutet, dass

$$\sum_k \iota_k \gamma_k^j \mu_k(\vec{c}) = 0 \text{ für alle } j.$$

□

Proof (3). Wegen $\xi^j = \overline{\xi}^j$ für alle j ist $\lambda^j = 0$ für alle j nach (IV13.13), also

$$\alpha^j \prod_k c_k^{\nu_k^j} = \overline{\alpha}^j \prod_k c_k^{\overline{\nu}_k^j},$$

das heißt

$$\prod_k c_k^{\gamma_k^j} = \frac{\alpha^j}{\overline{\alpha}^j}.$$

□

Wir behandeln nun einige konkrete Standardbeispiele.

13.8 Water. (Nach I. Müller [87, 3.2.2.3], siehe 13.2.) Für die Reaktion von Wasser in 13.2 ist $J = 4$, $N = 6$ und

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} c_{\mathrm{H}} \\ c_{\mathrm{O}} \\ c_{\mathrm{OH}} \\ c_{\mathrm{H_2}} \\ c_{\mathrm{O_2}} \\ c_{\mathrm{H_2O}} \end{bmatrix}, \quad (\gamma_k^j)_{kj} = \begin{bmatrix} -1 & +2 & 0 & -1 \\ -1 & 0 & +2 & 0 \\ +1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}.$$

Dann lauten die gewöhnlichen Differentialgleichungen

$$\overset{\circ}{c}_k = \sum_{j=1}^{J} \iota_k \gamma_k^j \lambda^j \qquad \text{für } k = 1, \dots, N$$

mit (es seien $\sigma^j = 1$)

$$\begin{aligned} \lambda^1 &= \alpha^1 c_1 c_2 - \overline{\alpha}^1 c_3, \\ \lambda^2 &= \alpha^2 c_4 - \overline{\alpha}^2 c_1^2, \\ \lambda^3 &= \alpha^3 c_5 - \overline{\alpha}^3 c_2^2, \\ \lambda^4 &= \alpha^4 c_1 c_3 - \overline{\alpha}^4 c_6. \end{aligned}$$

Zeige, dass die Summe der Konzentrationen gleich 1 ist.

Da die Summe der Konzentrationen gleich 1 ist, muß immer gelten

$$0 = \Big(\sum_k c_k\Big)^{\circ} = \sum_k \overset{\circ}{c}_k = \sum_{j,k} \iota_k \gamma_k^j \lambda^j = \sum_j \Big(\sum_k \iota_k \gamma_k^j\Big) \lambda^j, \tag{IV13.17}$$

und aus den Eigenschaften der λ^j (dass sie beliebige Werte annehmen können) folgt daraus, dass für alle j

$$\boxed{\sum_k \iota_k \gamma_k^j = 0} \qquad \text{(IV13.18)}$$

sein muss. Die Größen sind konstante Materialgrößen.

Proof. Wir haben (IV13.18) zu zeigen, wobei $M_1 = M_H$ und $M_2 = M_O$ gegeben sind und

$$\begin{aligned} M_3 &= M_{OH} = M_O + M_H \\ M_4 &= M_{H_2} = 2M_H \\ M_5 &= M_{O_2} = 2M_O \\ M_6 &= M_{H_2O} = 2M_H + M_O\,. \end{aligned}$$

Nun gilt

$$\Big(\sum_k M_k \gamma_k^j\Big)_j = (\gamma_k^j)_{jk}(M_k)_k = \begin{bmatrix} -1 & -1 & +1 & 0 & 0 & 0 \\ +2 & 0 & 0 & -1 & 0 & 0 \\ 0 & +2 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{bmatrix}$$

$$= \begin{bmatrix} -M_1 - M_2 + M_3 \\ +2M_1 - M_4 \\ +2M_2 - M_5 \\ -M_1 - M_3 + M_6 \end{bmatrix} = \begin{bmatrix} -M_H - M_O + M_{OH} \\ +2M_H - M_{H_2} \\ +2M_O - M_{O_2} \\ -M_H - M_{OH} + M_{H_2O} \end{bmatrix} = 0\,,$$

woraus die Behauptung folgt. □

13.9 Haber-Bosch process. (See I.Müller [87, 6.5.1.4 Seite 198], see also [Wikipedia: Haber process].) Bildung von Ammoniak aus Stickstoff und Wasserstoff. Die Reaktion lautet

$$\mathrm{N_2 + 3H_2 \rightleftarrows 2NH_3}$$

mit (es ist $J = 1$, $N = 3$)

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_{\mathrm{N_2}} \\ c_{\mathrm{H_2}} \\ c_{\mathrm{NH_3}} \end{bmatrix}, \quad \nu = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \overline{\nu} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \gamma = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

Das ideale Gas-Modell nach (IV13.15) ist daher (wenn $\sigma := 1$) mit Koeffizientenfunktionen $\alpha > 0$ und $\overline{\alpha} > 0$

$$\lambda = \alpha(\theta) c_1 c_2^3 - \overline{\alpha}(\theta) c_3^2\,.$$

Also ist $\vec{c}$ eine Lösung der gewöhnlichen Differentialgleichung (IV13.16) mit nur einer Reaktion

$$\overset{\circ}{c}_1 = -\iota_1\lambda\,, \quad \overset{\circ}{c}_2 = -3\iota_2\lambda\,, \quad \overset{\circ}{c}_3 = 2\iota_3\lambda\,.$$

Löse im Fall, dass $\alpha, \overline{\alpha} = \text{const}$ ist, das Gleichungssystem und zeige, dass die Summe der Konzentrationen gleich 1 ist.

Proof. Es ist $(c_1 + c_2 + c_3)^\circ = (-\iota_1 - 3\iota_2 + 2\iota_3)\lambda = 0$, wenn

$$\iota_1 + 3\iota_2 = 2\iota_3\,.$$

Dies ist erfüllt, da

$$\begin{aligned}\iota_1 &= \iota_{\mathrm{N}_2} = 2\iota_{\mathrm{N}}\,,\\ \iota_2 &= \iota_{\mathrm{H}_2} = 2\iota_{\mathrm{H}}\,,\\ \iota_3 &= \iota_{\mathrm{NH}_3} = \iota_{\mathrm{N}} + 3\iota_{\mathrm{H}}\,.\end{aligned}$$

Nun zur Existenz. Für Konstanten K_1 und K_2 ist

$$\frac{c_1}{\iota_1} + \frac{c_3}{2\iota_3} = \text{const} =: K_1 \quad \text{und} \quad \frac{c_2}{3\iota_2} + \frac{c_3}{2\iota_3} = \text{const} =: \frac{K_2}{3}\,,$$

also

$$c_1 = \iota_1\big(K_1 - \frac{c_3}{2\iota_3}\big)\,, \quad c_2 = \iota_2\big(K_2 - \frac{3c_3}{2\iota_3}\big)\,,$$

das heißt, dass K_1 und K_2 durch die Anfangsbedingung gegeben ist. Und es folgt

$$\begin{aligned}\overset{\circ}{c}_3 &= 2\iota_3\lambda = 2\iota_3(\alpha c_1 c_2^3 - \overline{\alpha} c_3^2)\\ &= 2\iota_3\Big(\alpha\iota_1\iota_2^3\big(K_1 - \frac{c_3}{2\iota_3}\big)\big(K_2 - \frac{3c_3}{2\iota_3}\big)^3 - \overline{\alpha}c_3^2\Big)\,.\end{aligned}$$

Also ist c_3 eine Lösung dieser Gleichung 4.Ordnung. (Siehe Lösungsformel in I.Müller [87, 6.5.1.4 Seite 199].) □

13.10 Saha equation. (From the book of I. Müller [87, 6.5.1.5 Seite 199], see also [Wikipedia: Saha ionization equation]) (Dies ist wichtig in der Astrophysik, siehe [Oskar von der Lühe: Einführung in die Astronomie und Astrophysik, 2.9 Physik der Sternatmosphären]. Spektroskopische Messungen eines Sternenlichtes erlaubt Berechnung der Oberflächentemperatur.) Wir betrachten hier nur den Fall, dass ein Atom ein Elektron verliert. Die Reaktion ist

$$\mathrm{Na} \rightleftarrows \mathrm{Na}^+ + \mathrm{e}\,.$$

Sei (es ist $J = 1$, $N = 3$)

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_{\mathrm{Na}} \\ c_{\mathrm{Na}^+} \\ c_{\mathrm{e}} \end{bmatrix}, \quad \nu = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{\nu} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Nach (IV13.15) ist daher (wenn $\sigma := 1$) mit Koeffizientenfunktionen $\alpha > 0$ und $\overline{\alpha} > 0$

$$\lambda = \alpha(\theta)c_1 - \overline{\alpha}(\theta)c_2c_3\,,$$

also ist $t \mapsto c(t)$ eine Lösung der Differentialgleichung (IV13.16), falls

$$\begin{aligned}
\overset{\circ}{c}_1 &= -\iota_1(\alpha(\theta)c_1 - \overline{\alpha}(\theta)c_2c_3)\,,\\
\overset{\circ}{c}_2 &= +\iota_2(\alpha(\theta)c_1 - \overline{\alpha}(\theta)c_2c_3)\,,\\
\overset{\circ}{c}_3 &= +\iota_3(\alpha(\theta)c_1 - \overline{\alpha}(\theta)c_2c_3)
\end{aligned}$$

Die Molgewichte sind

$$M_{\mathrm{Na}} = 23\,, \quad M_{\mathrm{e}} = \frac{1}{1840}\,, \quad M_{\mathrm{Na}^+} = M_{\mathrm{Na}} - M_{\mathrm{e}}\,.$$

Wenn die ι-Werte dementsprechend gewählt werden, ist die Summe der Konzentrationen gleich 1. Der stationäre Punkt erfüllt $\alpha\, c_1 = \overline{\alpha}\, c_2c_3$, d.h. $\alpha\, c_{\mathrm{Na}} = \overline{\alpha}\, c_{\mathrm{Na}^+}c_{\mathrm{e}}$, oder

$$\frac{c_{\mathrm{Na}^+}}{c_{\mathrm{Na}}} \cdot c_{\mathrm{e}} = \frac{\alpha(\theta)}{\overline{\alpha}(\theta)} \quad (\textbf{\textit{Saha-Gleichung}}). \tag{IV13.19}$$

Die allgemeine Saha-Gleichung gilt für ein Atom, welches mehrere Elektronen freisetzen kann. *Bemerkung:* (Siehe ...Exercises..)

Temperaturabhängige Reaktionen

Solche Phänomene treten insbesondere bei der mathematischen Behandlung von Verbrennungsvorgängen (*en:* Combustion) auf.

Referenzen: Für das folgende Beispiel siehe [B. Larrouturou: Modelisation Physique, Numerique et Mathematique des Phenomenes de Propagation de Flammes, pp. 65-119] in Larrouturou [52].

Als Beispiel betrachten wir eine simple exotherme Reaktion

$$\mathrm{R}\ \ (\mathit{Reactant}) \quad \rightleftarrows \quad \mathrm{P}\ \ (\mathit{Product})$$

und dazu nach dem Massenwirkungsgesetz die Funktion

$$\begin{aligned} &\lambda = \sigma(\varrho,\theta)(\alpha_1(\theta)\,c_1 - \alpha_2(\theta)\,c_2)\,, \quad \varrho = \varrho_1 + \varrho_2\,, \\ &\varrho_1 = \varrho c_1\,, \quad \varrho_2 = \varrho c_2\,, \quad c_1 = c\,, \quad c_2 = 1 - c\,, \\ &\alpha_1(\theta) := \exp\Big(\frac{h_{\mathrm{R}}^0}{R\theta}\Big)\,, \quad \alpha_2(\theta) := \exp\Big(\frac{h_{\mathrm{P}}^0}{R\theta}\Big)\,. \end{aligned} \tag{IV13.20}$$

Mit dieser Reaktion gilt das Folgende.

13.11 Beispiel (Nach B. Larrouturou). Betrachte für die Dichte ϱ und die Konzentration c das Differentialgleichungssystem

$$\begin{aligned} &\partial_t(\varrho c) + \operatorname{div}(\varrho c v - \varrho D\nabla c) = -\varrho\lambda\,, \\ &\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = \mathbf{f}\,, \\ &\partial_t e + \operatorname{div}((e+p)v - a\nabla\theta - \varrho Q D\nabla c) = v\bullet\mathbf{f}\,. \end{aligned}$$

Hierbei sind D und Q und a positive Konstanten, und λ ist in (IV13.20) gegeben. Dieses Problem erfüllt das Entropieprinzip in 12.1. Die Energie und der Druck sind gegeben durch

$$\begin{aligned} &\varepsilon = \varrho c(h_{\mathrm{R}}^0 + c_V\theta) + \varrho(1-c)(h_{\mathrm{P}}^0 + c_V\theta)\,, \quad e = \varepsilon + \frac{\varrho}{2}|v|^2\,, \\ &p = R\varrho\theta \quad (\text{vgl. } 2.4)\,. \end{aligned}$$

Daraus bestimmen sich innere freie Energie und die Entropie und es folgt, dass

$$Q = h_{\mathrm{R}}^0 - h_{\mathrm{P}}^0\,.$$

Wenn $Q = h_{\mathrm{R}}^0 - h_{\mathrm{P}}^0 > 0$ ist, ist die Reaktion $\mathrm{R} \rightleftarrows \mathrm{P}$ wie behauptet exotherm. Es sei bemerkt, dass wenn $h_{\mathrm{P}}^0 \to -\infty$, dann konvergiert $\alpha_2(\theta) \to 0$, also $\lambda \to \sigma(\theta)\alpha_1(\theta)\,c$, d.h. im Limes ist $c = 0$ die stationäre Lösung.

Proof freie Energie. Es sind zwei Phasen der Gesamtmasse ϱ vorhanden,

$$c_1 := c \quad \text{und} \quad c_2 := 1 - c, \quad c_1 + c_2 = 1,$$
$$\varrho_k := \varrho c_k, \quad \varrho = \varrho_1 + \varrho_2, \quad \vec{\varrho} = (\varrho_1, \varrho_2).$$

Die innere Energie ist gegeben durch

$$\begin{aligned}\varepsilon &= \varrho c(h_R^0 + c_V\theta) + \varrho(1-c)(h_P^0 + c_V\theta) \\ &= \varrho_1(h_R^0 + c_V\theta) + \varrho_2(h_P^0 + c_V\theta)\,.\end{aligned}$$

Wir sehen dass daraus die freie Energie berechnet werden kann. Es gilt nämlich $f = \varepsilon - \theta\eta$, und da $f_{'\theta} = -\eta$ ist, folgt

$$f - \theta f_{'\theta} = \varepsilon\,,$$

wobei sowohl f als auch ε Funktionen von $(\vec{\varrho}, \theta)$ sind. Also ist

$$\begin{aligned}\Big(\frac{-f}{\theta}\Big)_{'\theta} &= \frac{f - \theta f_{'\theta}}{\theta^2} = \frac{\varepsilon}{\theta^2} \\ &= \varrho_1\big(\frac{h_R^0}{\theta^2} + \frac{c_V}{\theta}\big) + \varrho_2\big(\frac{h_P^0}{\theta^2} + \frac{c_V}{\theta}\big) \\ &= \frac{\varrho_1 h_R^0 + \varrho_2 h_P^0}{\theta^2} + \frac{\varrho c_V}{\theta}\,.\end{aligned}$$

und daher mit einer Funktion $\vec{\varrho} \mapsto g_0(\vec{\varrho})$

$$\frac{-f}{\theta} = -g_0(\vec{\varrho}) - \frac{\varrho_1 h_R^0 + \varrho_2 h_P^0}{\theta} + \varrho c_V \log\theta$$

oder

$$f = g_0(\vec{\varrho})\theta + \varrho_1 h_R^0 + \varrho_2 h_P^0 - \varrho c_V \theta \log\theta\,.$$

Daraus ergibt sich die Entropie als

$$\eta = -f_{'\theta} = -g_0(\vec{\varrho}) + \varrho c_V(\log\theta + 1)\,,$$

und für den Druck $p = R\varrho\theta$ (vgl. 2.4) erhält man

$$R\varrho\theta = p = \sum_k \varrho_k f_{'\varrho_k} - f = \big(\sum_k \varrho_k g_{0\,'\varrho_k} - g_0\big)\theta\,,$$

und damit

$$\sum_k \varrho_k g_{0\,'\varrho_k} - g_0 = R\varrho\,.$$

Eine Lösung dieser Differentiagleichung ist (siehe Aufgabe 18.1)

$$g_0(\vec{\varrho}) = R\sum_k \varrho_k(\log\varrho_k - 1)\,,$$

also ergibt sich

$$f(\vec{\varrho},\theta) = R\theta \sum_k \varrho_k(\log \varrho_k - 1) + \varrho_1 h^0_{\mathrm{R}} + \varrho_2 h^0_{\mathrm{P}} - \varrho c_V \theta \log \theta\,.$$

Daraus folgt für die chemischen Potentiale mit $h^0_1 = h^0_{\mathrm{R}}$ und $h^0_2 = h^0_{\mathrm{P}}$, wobei wir $\log \varrho_k = \log \varrho + \log c_k$ schreiben,

$$\mu_k = f_{,\varrho_k} = R\theta \log c_k + a_k(\varrho,\theta)\,, \quad a_k(\varrho,\theta) := h^0_k - d(\varrho,\theta)\,, \tag{IV13.21}$$

wobei $d(\varrho,\theta) = R\theta \log \varrho + c_V \theta \log \theta$. Weiter unten werden wir μ_1 und μ_2 noch zerlegen. □

Proof Entropieprinzip. Die beiden Massenerhaltungen sind äquivalent zu (die zweite Gleichung ist die Gleichung für die gesamte Masse minus der Gleichung für den ersten Massenanteil)

$$\begin{aligned}\partial_t \varrho_1 + \mathrm{div}(\varrho_1 v - \varrho D \nabla c) &= -\varrho\lambda\,,\\ \partial_t \varrho_2 + \mathrm{div}(\varrho_2 v + \varrho D \nabla c) &= +\varrho\lambda\,,\end{aligned}$$

also ist

$$\begin{aligned}&\mathbf{J}_1 = -\varrho D \nabla c\,, \quad \mathbf{J}_2 = +\varrho D \nabla c\,, \quad \mathbf{J}_1 + \mathbf{J}_2 = 0\,,\\ &\mathbf{r}_1 = -\varrho\lambda\,, \quad \mathbf{r}_2 = +\varrho\lambda\,, \quad \mathbf{r}_1 + \mathbf{r}_2 = 0\,.\end{aligned}$$

The momentum equation has as pressure tensor $\Pi = p\mathrm{Id}$, and the energy equation has as flux

$$\begin{aligned}&q = q^\epsilon + \mathbf{J}^\epsilon\,, \quad q^\epsilon := -a\nabla\theta\,,\\ &\mathbf{J}^\epsilon := -\varrho Q D \nabla c = \mu^\epsilon_1 \mathbf{J}_1 + \mu^\epsilon_2 \mathbf{J}_2\,, \quad \mu^\epsilon_1 - \mu^\epsilon_2 = Q\,.\end{aligned}$$

Wir müssen noch die Entropieungleichung verifizieren. Für den Entropiefluss setzen wir $\psi = \eta v + \frac{1}{\theta} q^\epsilon - \mathbf{J}^\eta$ an, wobei wir gemäß der Entropieungleichung in 12.1 für den zuzätzlichen Fluss $\mathbf{J}^\eta = \sum_k \mu^\eta_k \mathbf{J}_k$ annehmen, wobei

$$\mu_k = \mu^\epsilon_k + \theta \mu^\eta_k\,, \quad \mu^\epsilon_k \text{ with } \mu^\epsilon_1 - \mu^\epsilon_2 = Q\,. \tag{IV13.22}$$

Die Entropieungleichung lautet dann (es ist $S = 0$)

$$0 \le \sigma = \nabla\Big(\frac{1}{\theta}\Big)\bullet q^\epsilon - \sum_\alpha \frac{1}{\theta}\mu_\alpha \mathbf{r}_\alpha - \sum_k (\nabla \mu^\eta_k + \frac{1}{\theta}\nabla\mu^\epsilon_k)\bullet \mathbf{J}_k\,.$$

Es werden in diesem Fall alle drei Terme nichtnegativ sein. Nun gilt für den Wärmefluss q^ϵ

$$\nabla\Big(\frac{1}{\theta}\Big)\bullet q^\epsilon = -\frac{1}{\theta^2}\nabla\theta\bullet q^\epsilon = +\frac{a}{\theta^2}|\nabla\theta|^2 \ge 0 \quad \text{wenn } a > 0.$$

Wegen (IV13.21), d.h. $\mu_k = R\theta \log c_k + h^0_k - d(\varrho,\theta)$, können wir setzen

$$\mu^\epsilon_k = h^0_k - d^\epsilon \quad \text{und} \quad \mu^\eta_k = R \log c_k - d^\eta\,,$$

wobei $d^\epsilon + \theta d^\eta = d(\varrho, \theta)$ (as you wish), so dass also die erste Gleichung in (IV13.22) erfüllt ist. Also ist

$$Q = \mu_1^\epsilon - \mu_2^\epsilon = h_{\mathrm{R}}^0 - h_{\mathrm{P}}^0 \,.$$

Aus diesen Darstellungen folgt für den letzten Term der Entropieproduktion, da $h_{\mathrm{R}}^0 = \mathrm{const}$ und $h_{\mathrm{P}}^0 = \mathrm{const}$ konstant sind, also $\mu_2^\epsilon - \mu_1^\epsilon = \mathrm{const}$,

$$\begin{aligned}
&-\sum_k \big(\nabla \mu_k^\eta + \frac{1}{\theta}\nabla \mu_k^\epsilon\big)\bullet \mathbf{J}_k \\
&= -\big(\nabla(\mu_2^\eta - \mu_1^\eta) + \frac{1}{\theta}\nabla(\mu_2^\epsilon - \mu_1^\epsilon)\big)\bullet(\varrho D \nabla c) \\
&= \nabla(\mu_1^\eta - \mu_2^\eta)\bullet(\varrho D\nabla c) = R\nabla(\log c_1 - \log c_2)\bullet(\varrho D \nabla c) \\
&= \varrho D R \cdot \Big(\frac{1}{c_1} + \frac{1}{c_2}\Big)|\nabla c|^2 \geq 0 \qquad \text{wenn } D > 0.
\end{aligned}$$

Es bleibt der zweite Term der Entropieproduktion zu behandeln, aber dass der Reaktionsterm einen nichtnegativen Anteil der Entropieproduktion ergibt, folgt aus der allgemeinen Theorie. □

Proof Massenwirkungsgesetz. Die Reaktion R $\rightleftarrows$ P ergibt

$$\begin{bmatrix}\gamma_R \\ \gamma_P\end{bmatrix} = \begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} = \begin{bmatrix}-1 \\ 1\end{bmatrix}, \quad \begin{bmatrix}\nu_1 \\ \nu_2\end{bmatrix} = \begin{bmatrix}1 \\ 0\end{bmatrix}, \quad \begin{bmatrix}\overline{\nu}_1 \\ \overline{\nu}_2\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix}.$$

Der allgemeinen Theorie folgend haben wir in (IV13.15) für die Reaktionsrate λ angesetzt, wobei die Darstellung der chemischen Potentiale in (IV13.21) benutzt wird, wobei wir nicht die ϱ-Abhängigkeit nennen,

$$\lambda = \widetilde{\sigma}(\theta)\cdot\big(\alpha(\theta)\prod_k c_k^{\nu_k} - \overline{\alpha}(\theta)\prod_k c_k^{\overline{\nu}_k}\big) = \widetilde{\sigma}(\theta)\cdot(\alpha(\theta)\, c_1 - \overline{\alpha}(\theta)\, c_2)\,,$$

wobei, wegen $\iota_k = 1$ für $k = 1, 2$ und $\bar{R} = R$,

$$\begin{aligned}
\alpha(\theta) &= \exp\big(\frac{a(\theta)}{R\theta}\big)\,, \quad \overline{\alpha}(\theta) = \exp\big(\frac{\overline{a}(\theta)}{R\theta}\big)\,, \\
a(\theta) &= \sum_k \nu_k a_k(\theta) = a_1(\theta)\,, \quad \overline{a}(\theta) = \sum_k \overline{\nu}_k a_k(\theta) = a_2(\theta)\,.
\end{aligned}$$

In (IV13.21) wurde $a_k(\theta) = h_k^0 - d(\theta)$ gezeigt, also

$$\alpha(\theta) = \alpha_1(\theta)\exp\big(-\frac{d(\theta)}{R\theta}\big)\,, \quad \overline{\alpha}(\theta) = \alpha_2(\theta)\exp\big(-\frac{d(\theta)}{R\theta}\big)$$

und damit folgt die Darstellung in (IV13.20). □

Es sei noch das Massenwirkungsgesetz genannt, welches hier besagt, dass

$$\frac{c}{1-c} = \frac{c_1}{c_2} = \frac{\overline{\alpha}}{\alpha} = \frac{\alpha_2}{\alpha_1} = \exp\big(\frac{h_{\mathrm{P}}^0 - h_{\mathrm{R}}^0}{R\theta}\big) = \exp\big(-\frac{Q}{R\theta}\big).$$

Das bestätigt obige Aussage, dass $Q >> 1$ zur Folge hat, dass im stationären Fall c fast 0 ist. Es sei noch bemerkt, dass das Differentialgleichungssystem in der Aussage 13.11 äquivalent geschrieben werden kann als

$$\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho v) = 0\,, \\
&\varrho \overset{\circ}{c} - \operatorname{div}(\varrho D \nabla c) = -\varrho\lambda\,, \\
&\varrho \overset{\circ}{v} + \nabla p = \mathbf{f}\,, \\
&\overset{\circ}{\varepsilon} + (\varepsilon + p)\operatorname{div} v - \operatorname{div}(a\nabla\theta + \varrho Q D \nabla c) = 0\,,
\end{aligned}$$

wobei λ wir in (IV13.20), $a \geq 0$ und $D \geq 0$, und

$$p = R\theta\varrho \quad \text{and} \quad \varepsilon = \varrho\big(c h_{\mathrm{R}}^0 + (1-c)h_{\mathrm{P}}^0 + c_V\theta\big)\,.$$

IN BEARBEITUNG

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14 Reactions in biology

We describe here some examples for the reaction term $\mathbf{r}_\alpha$ in (IV11.2), as they occur in theoretical biology. We consider the isothermal case and the examples are based on an internal free energy depending on the densities ϱ_α:

$$f = \widehat{f}((\varrho_\alpha)_\alpha)\,, \quad \vec{\varrho} = (\varrho_\alpha)_\alpha\,,$$

so that by 11.9 the chemical potential is

$$\mu_\alpha = f'_{\varrho_\alpha}(\vec{\varrho})\,.$$

The differential system of the mass equations is

$$\partial_t \varrho_\alpha + \mathrm{div}(\varrho_\alpha v + \mathbf{J}_\alpha) = \mathbf{r}_\alpha \text{ für } \alpha = 1, \dots, m\,,$$

where we assume that

$$\varrho := \sum_\alpha \varrho_\alpha\,, \quad \mathbf{J} := \sum_\alpha \mathbf{J}_\alpha = 0\,, \quad \mathbf{r} := \sum_\alpha \mathbf{r}_\alpha = 0\,, \tag{IV14.1}$$

References: See the examples in [H.W. Alt & W.Alt: Fluid mixtures and applications to biological systems. In M. Griebel: Singular Phenomena and Scaling in Mathematical Models (SFB 611), pp. 191-220. Springer 2014]. Also visit [Wikipedia: Lotka-Volterra equation]. We refer also to [93, Chap. 14 §F Applications to the Equations of Mathematical Ecology].

We treat here only the case, that the masses ϱ_α are spatially homogeneous (hence $\mathbf{J}_\alpha = 0$) and that $v = 0$ (this is mainly done for simplicity). Then the reaction equations become

$$\boxed{\dot{\varrho}_\alpha = \mathbf{r}_\alpha \text{ for all } \alpha.}$$

14.1 Remark. In this situation the entropy principle becomes

$$\dot{f} = \sum_\alpha \mu_\alpha \mathbf{r}_\alpha \le 0\,.$$

Proof. This is also true because

$$\dot{f} = \sum_\alpha f'_{\varrho_\alpha} \dot{\varrho}_\alpha = \sum_\alpha \mu_\alpha \mathbf{r}_\alpha\,.$$

□

There is one example, a gradient flow, for which the $\mathbf{r}_\alpha$ are proportinal to μ_α, say, $\dot{\varrho}_\alpha = -\lambda\mu_\alpha$, where $\lambda = \widehat{\lambda}(\varrho) > 0$. In this case the entropy inequality is trivially satisfied.

As first example we consider cyclic reactions, which are important cases and often the basis for biological processes.

14.2 Cyclic processes. The system is

$$\begin{aligned} \dot\varrho_\alpha &= \mathbf{r}_\alpha(\overline{\varrho}) \text{ for } \alpha = 1,\dots,N, \\ \mathbf{r}_\alpha(\overline{\varrho}) &:= \eta_{\alpha+1}\varrho_{\alpha+1} - \eta_\alpha\varrho_\alpha\,, \end{aligned} \tag{IV14.2}$$

with cyclic repetition, $\varrho_{N+1} := \varrho_1$, $\eta_{N+1} := \eta_1$. Here $\overline{\varrho} = (\varrho_\alpha)_\alpha$ and η_α are positive constants. This system satisfies the entropy inequality in 14.1, if

$$f = \widehat{f}(\overline{\varrho}) = f_0(\varrho) + b(\varrho)\sum_{\alpha=1}^{N}\eta_\alpha\varrho_\alpha^2\,,$$

where $\varrho = \sum_{\alpha=1}^{N}\varrho_\alpha$ and with a positive functions $b(\varrho) > 0$.

The stationary solutions, if they exist, are values $\overline{\varrho}^s = (\varrho_\alpha^s)_\alpha$ with

$$\eta_{\alpha+1}\varrho_{\alpha+1}^s = \eta_\alpha\varrho_\alpha^s =: \eta^s\,.$$

This $\overline{\varrho}^s \in \mathbb{R}^N$ is a unique point, if the value of $\varrho^s = \sum_{\alpha=1}^{N}\varrho_\alpha^s$ is considered to be given. For general solutions $\overline{\varrho}$ is rotating around the stationary line and converging to a value $\overline{\varrho}^s$, what can be seen from the free energy. We mention that the sum in the free energy can be written as

$$\sum_\alpha \eta_\alpha\varrho_\alpha^2 = \sum_\alpha \eta_\alpha(\varrho_\alpha - \varrho_\alpha^s)^2 + 2\eta^s\varrho - \sum_\alpha \eta_\alpha(\varrho_\alpha^s)^2\,.$$

Moreover, we obtain overall mass conservation, that is $\mathbf{r} = \sum_\alpha \mathbf{r}_\alpha = 0$ and therefore $\dot\varrho = 0$.

Proof. We take a free energy

$$f = \widehat{f}(\overline{\varrho}) = \frac{1}{2}\sum_\alpha b_\alpha\varrho_\alpha^2$$

with $b_\alpha \in \mathbb{R}$, so that

$$\mu_\alpha = f'_{\varrho_\alpha} = b_\alpha\varrho_\alpha\,.$$

Then, with $b_\alpha = \eta_\alpha\widetilde{b}_\alpha$ and assuming $\widetilde{b}_\alpha > 0$,

$$\begin{aligned} \sum_\alpha \mu_\alpha\mathbf{r}_\alpha &= \sum_\alpha b_\alpha\varrho_\alpha(\eta_{\alpha+1}\varrho_{\alpha+1} - \eta_\alpha\varrho_\alpha) \\ &= \sum_\alpha \big(\widetilde{b}_\alpha(\eta_{\alpha+1}\varrho_{\alpha+1})(\eta_\alpha\varrho_\alpha) - \widetilde{b}_\alpha(\eta_\alpha\varrho_\alpha)^2\big) \\ &= \sum_\alpha \left(\sqrt{\frac{\widetilde{b}_\alpha}{\widetilde{b}_{\alpha+1}}}\cdot\xi_{\alpha+1}\xi_\alpha - \xi_\alpha^2\right), \end{aligned}$$

where $\xi_\alpha := \eta_\alpha\varrho_\alpha\sqrt{\widetilde{b}_\alpha}$. Letting

$$c_\alpha := \sqrt{\frac{\widetilde{b}_\alpha}{\widetilde{b}_{\alpha+1}}}$$

and using $\xi_\alpha\xi_{\alpha+1} \le \frac{1}{2}(\xi_\alpha^2 + \xi_{\alpha+1}^2)$, this is

$$= \sum_\alpha \big(c_\alpha \xi_{\alpha+1}\xi_\alpha - \xi_\alpha^2\big) \le \sum_\alpha \big(\frac{c_\alpha}{2}\xi_{\alpha+1}^2 + \frac{c_\alpha}{2}\xi_\alpha^2 - \xi_\alpha^2\big)$$
$$= \sum_\alpha \big(\frac{c_{\alpha-1}}{2} + \frac{c_\alpha}{2} - 1\big)\xi_\alpha^2 = 0 \quad \text{if } c_\alpha = 1 \text{ for all } \alpha,$$

that is
$$b = \widetilde{b}_\alpha = \widetilde{b}_{\alpha+1} > 0 \text{ for all } \alpha,$$

or $b_\alpha = \eta_\alpha b$. This shows the theorem in this special case.

If we take a free energy $f = f(\overline{\varrho}, \varrho)$ as in the assertion, the derivative with respect to ϱ has no effect, since the total mass production $\mathbf{r}$ is zero. Therefore the theorem is proved. □

As next example we treat the case which you will find in the literature, its a system consisting of the second and third equation below, the classical Lotka-Volterra system.

14.3 Lotka-Volterra system. For the predator-prey model (*de:* Räuber-Beute Modell) we let $x > 0$ be the number of prey (*de:* Beute) and $y > 0$ the number of predator (*de:* Räuber) and consider the system

$$\dot{b} = -\lambda x\,,$$
$$\dot{x} = x \cdot (\alpha - \beta y)\,,$$
$$\dot{y} = -y \cdot (\gamma - \delta x)\,,$$
$$\dot{z} = \eta x y\,,$$
$$\dot{d} = \kappa y\,.$$

The additional variables are a the quantities b, d, and z. They are modelled by $\dot{b}$ proportional to birth of prey, $\dot{d}$ proportional to death of predator, and $\dot{z}$ proportional to interactions between predator and prey. This system satisfies the inequality in 14.1, which reduces to

$$\varepsilon\lambda x + \zeta\kappa y + \xi\eta x y \ge 0\,,$$

if the free energy is given by

$$f = \widehat{f}(b, x, y, z, d) = -\gamma \log x - \alpha \log y + \delta x + \beta y + \varepsilon b - \zeta d - \xi z\,,$$

which is a convex function for constants $\gamma > 0$ and $\alpha > 0$. The inequality in 14.1 holds, if in addition the constants ε, ζ, η, λ, κ and ξ satisfy $\varepsilon\lambda > 0$, $\zeta\kappa > 0$, and $\xi\eta > 0$. The remaining quantities β and δ are positive because of biological reasons.

The variables transform into (bio)mass densities by

$$\varrho_b = bm_b,\ \varrho_x = xm_x,\ \varrho_y = ym_y,\ \varrho_d = dm_d,\ \varrho_z = zm_z$$

with positive mass constants satisfying

$$\begin{gathered}\lambda = \frac{\alpha m_x}{m_b}\,, \quad \kappa = \frac{\gamma m_y}{m_d}\,, \\ \eta = \frac{\beta m_x - \delta m_y}{m_z}\,, \end{gathered} \tag{IV14.3}$$

which implies that the sum of the mass production terms are 0. The parameter η is positive if and only if biomass is lost during transfer from prey to predator.

Proof. It is $f = \widetilde{f}(b,x,y,z,d) = -\log K + \varepsilon b - \zeta d - \xi z$ with

$$K \equiv \widehat{K}(x,y) = \frac{x^\gamma y^\alpha}{\mathrm{e}^{\delta x}\mathrm{e}^{\beta y}}$$

and one computes for solutions of the system that $\dot{K} = 0$, that is, this convex part of f is constant for solutions, and moreover, we see that solutions rotate around the equilibrium

$$x = \frac{\gamma}{\delta}\,, \quad y = \frac{\alpha}{\beta}\,.$$

This is the basis for the entire result: For the mass densities the system is

$$\begin{aligned}
\dot{\varrho}_b &= \mathbf{r}_b = m_b \mathbf{r}'_b\,, \quad \mathbf{r}'_b := -\lambda x\,,\\
\dot{\varrho}_x &= \mathbf{r}_x = m_x \mathbf{r}'_x\,, \quad \mathbf{r}'_x = x\cdot(\alpha - \beta y)\,,\\
\dot{\varrho}_y &= \mathbf{r}_y = m_y \mathbf{r}'_y\,, \quad \mathbf{r}'_y = -y\cdot(\gamma - \delta x)\,,\\
\dot{\varrho}_z &= \mathbf{r}_z = m_z \mathbf{r}'_z\,, \quad \mathbf{r}'_z := \eta xy\,,\\
\dot{\varrho}_d &= \mathbf{r}_d = m_d \mathbf{r}'_d\,, \quad \mathbf{r}'_d := \kappa y\,,
\end{aligned}$$

and, using the identities (IV14.3), that is

$$\beta m_x = \delta m_y + \eta m_z\,, \quad \lambda m_b = \alpha m_x\,, \quad \kappa m_d = \gamma m_y\,, \tag{IV14.4}$$

we obtain

$$\begin{aligned}
&\mathbf{r}_b = -\tau_b\,, \quad \tau_b := \lambda m_b x\,,\\
&\mathbf{r}_x = \tau_b - \tau_{xy}\,, \quad \tau_{xy} := cxy\,, \quad c := \beta m_x\,,\\
&\mathbf{r}_y = -\tau_d + (1-\omega)\tau_{xy}\,, \quad \omega := \frac{\eta m_z}{c}\,,\\
&\mathbf{r}_z = \omega\tau xy\,, \quad \big(\omega c = \eta m_z,\ (1-\omega)c = \delta m_y\big)\\
&\mathbf{r}_d = \tau_d\,, \quad \tau_d := \kappa m_d y\,,
\end{aligned}$$

hence $\mathbf{r} = 0$. Then one easily computes

$$\mathbf{r}'_x \widetilde{f}_{\prime x} + \mathbf{r}'_y \widetilde{f}_{\prime y} = -\frac{1}{K}(\mathbf{r}'_x K_{\prime x} + \mathbf{r}'_y K_{\prime y})$$
$$= -\frac{1}{K}(\dot{x} K_{\prime x} + \dot{y} K_{\prime y}) = -\frac{1}{K}\dot{K} = 0\,,$$

and therefore

$$\sum_\beta \mathbf{r}_\beta \mu_\beta = \mathbf{r}'_b \widetilde{f}_{\prime b} + \mathbf{r}'_x \widetilde{f}_{\prime x} + \mathbf{r}'_y \widetilde{f}_{\prime y} + \mathbf{r}'_z \widetilde{f}_{\prime z} + \mathbf{r}'_d \widetilde{f}_{\prime d}$$
$$= \mathbf{r}'_b \widetilde{f}_{\prime b} + \mathbf{r}'_z \widetilde{f}_{\prime z} + \mathbf{r}'_d \widetilde{f}_{\prime d} = -\varepsilon\lambda x - \zeta\kappa y - \xi\eta x y \leq 0\,.$$

□

Jetzt noch einige Beispiele aus dem Buch von [93, §14 C]. Diese Beispiele sind für die Konzentrationen c_k geschrieben, die Gleichungen lauten

$$\varrho\dot{c}_k + \operatorname{div}\mathbf{J}_k = \mathbf{r}_K$$

für alle k, wobei (IV14.1) gelte. Es werden hier nur die zwei wesentlichen Gleichungen aufgeführt. Ähnlich wie bei 14.3 hat man dies noch zu einem vollständigen System (biologisch abgeschlossen) zu erweitern, so dass dann die Entropieungleichung gilt. Es ist auch noch nicht klar, wie die zugehörige freie Energie aussieht. Es ist $v = 0$.

14.4 Ökologische Interaktionen. Es werden Differentialgleichungen von der Form

$$\partial_t c_k - \operatorname{div}\Big(\sum_l a_{kl}(c)\nabla c_l\Big) = g_k(c) \text{ für } k = 1, 2,$$
$$g_k(c) = c_k h_k(c)\,,$$

behandelt.

(1) ***Räuber-Beute-Modell (Predator-Prey).***

c_1 Räuber, c_2 Beute,

$\partial_1 h_2 < 0$ (mehr Räuber erniedrigt Beute-Wachstumsrate),

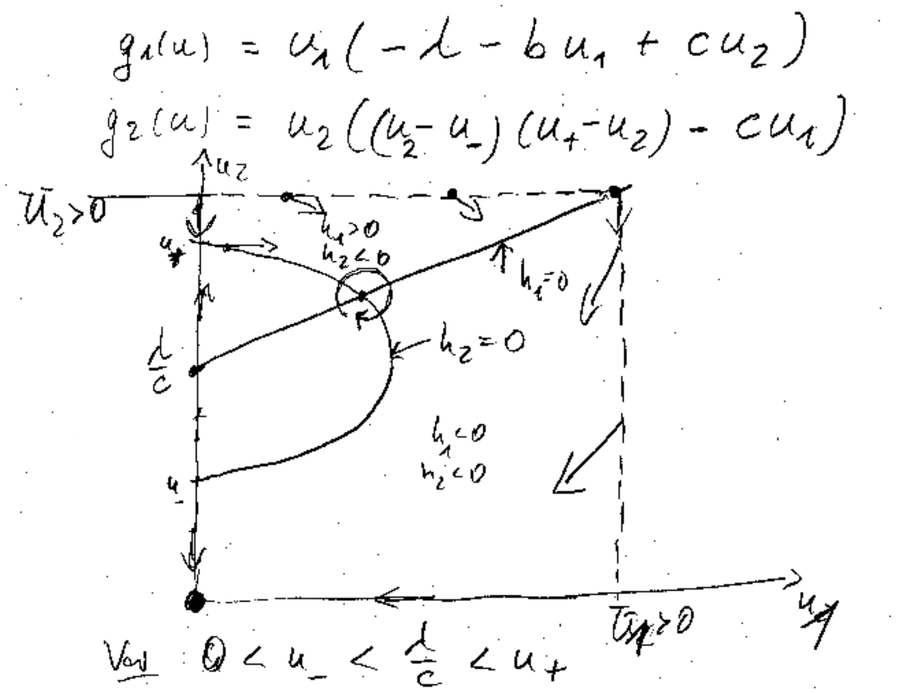
$\partial_2 h_1 > 0$ (mehr Beute erhöht Räuber-Wachstumsrate).

Das Standardbeispiel ist

$g_1(c) := c_1(-\lambda + \nu c_2)$ ($\mu > 0$ Geburtenrate Beute),

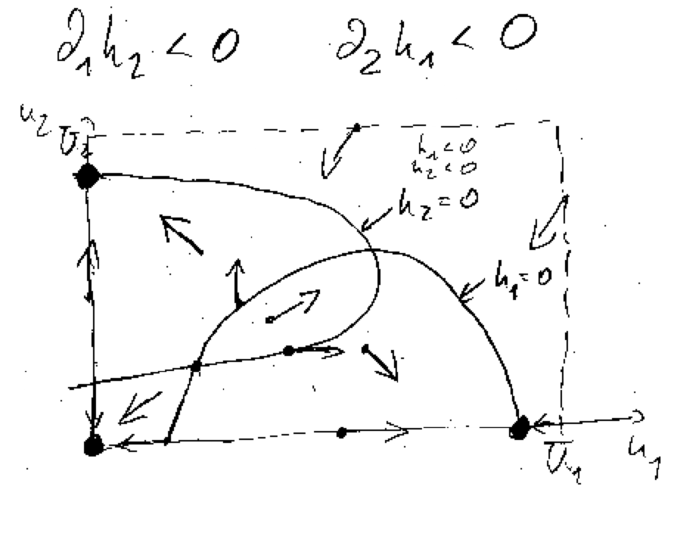
$g_2(c) := c_2(\mu - \nu c_1)$ ($\lambda > 0$ Todesrate Räuber),

($\nu > 0$ Fressrate).



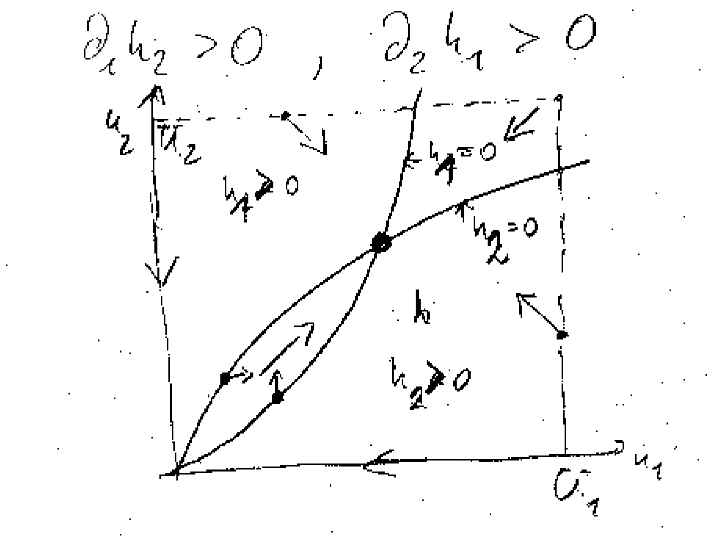
(2) ***Rivalisierende Spezies (Competing species).***

$$\partial_2 h_1 < 0\,, \quad \partial_1 h_2 < 0\,.$$



(3) ***Symbiose.***

$$\partial_2 h_1 > 0\,, \quad \partial_1 h_2 > 0\,.$$



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15 Prandtl's boundary layer

In this section we consider liquids with low viscosity or high speed in an area Ω, for which the boundary condition $v = 0$ is given on $\partial\Omega$. In the case that this very high velocities occur in the liquid, the transition to zero velocity at the boundary is realized only on a boundary layer, an area which can be so narrow that it is very well approximated by a surface term on $\partial\Omega$. Then there is no longer a stationary incompressible Poiseuille flow I.3.7 the physical solution, but rather an unsteady solution will occur in many cases. The transition from laminar flow to turbulent flow can happen quickly (see the top of the wing in Fig. 31).

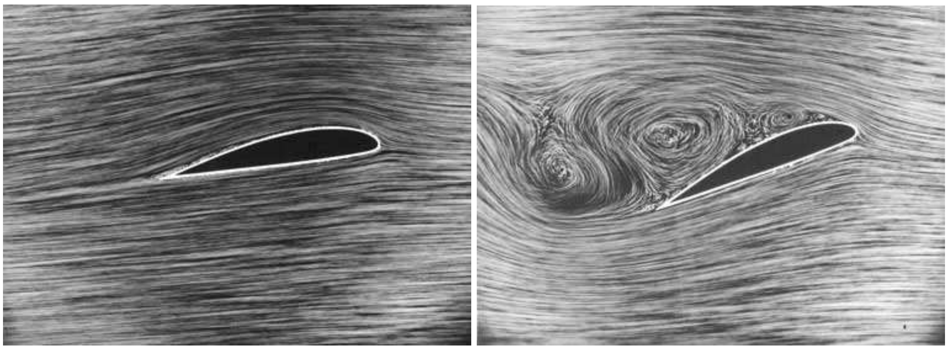

Fig. 31: *Left:* Laminar flow. *Right:* Turbulent flow.

Let us now consider this boundary layer in a model for liquids in an open domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$ (the observer sits on Ω, therefore Ω is independent of time). In this situation let us take the incompressible Navier-Stokes equation in Ω

$$\begin{aligned} &\mathrm{div}_x v = 0\,,\\ &\partial_t(\varrho_0 v) + \mathrm{div}_x(\varrho_0 v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,\\ &\Pi = p\mathrm{Id} - S\,,\quad S = 2a\,(\mathrm{D}v)^{\mathrm{S}}\,,\\ &v = v(t,x)\,,\quad p = p(t,x)\,, \end{aligned} \tag{IV15.1}$$

and consider $\mathbb{R}^3\backslash\Omega$ as rigid body, that is, we assume the boundary conditions

$$v = 0 \text{ on } \Gamma \tag{IV15.2}$$

(other non-trivial boundary conditions for the mass and momentum conservation are not considered here).

We now introduce the boundary layer in a three-dimensional model with flat boundary. We use the notation of "Matched asymptotic expansion" [Wikipedia: Matched asymptotic expansion]. Let

$$\Omega := \{x \in \mathbb{R}^3\,;\ x_3 > 0\}\,,\quad \Gamma := \partial\Omega = \{x \in \mathbb{R}^3\,;\ x_3 = 0\}\,.$$

For $x \in \Omega$ we introduce the variables $y = (y_1, y_2, y_3)$ near the boundary Γ by

$$y_1 = x_1\,, \quad y_2 = x_2\,, \quad y_3 = \frac{x_3}{\delta} \quad \text{as} \quad \delta > 0\,, \tag{IV15.3}$$

and we define to a quantity $(t,x) \mapsto u(t,x)$ a quantity $(t,y) \mapsto U(t,y)$ by the relation $U(t,y) = u(t,x)$, if x and y as in (IV15.3). Let us denote by

$\mathbb{R}^2 \times]\varepsilon_\delta, \infty[$ the ***outer region***,

$\mathbb{R}^2 \times [0, 2\varepsilon_\delta[$ the ***inner region***,

where $\varepsilon_\delta \to 0$ and $\dfrac{\varepsilon_\delta}{\delta} \to \infty$ as $\delta \to 0$.

For example we take $\varepsilon_\delta := \sqrt{\delta}$. We imagine that in the outer region the function $(t,x) \mapsto u(t,x)$ is relevant and in the inner region the function $(t,y) \mapsto U(t,y)$. Where the outer region coincides with the inner region, we consider the equation

$$U(t, y_1, y_2, y_3) = u(t, y_1, y_2, \delta y_3) \text{ für } \frac{\varepsilon_\delta}{\delta} \leq y_3 \leq \frac{2\varepsilon_\delta}{\delta}\,, \tag{IV15.4}$$

so the relation between the original function u to the newly defined function U. We now ask for the convergence

$$\begin{aligned} &\left\| u - u^{(0)} \right\|_{C^0(K^a_\delta)} \to 0 \text{ as } \delta \to 0\,, \quad K^a_\delta := K^a \cap (\mathbb{R}_+ \times \mathbb{R}^2 \times]\varepsilon_\delta, \infty[)\,, \\ &\left\| U - U^{(0)} \right\|_{C^0(K^i_\delta)} \to 0 \text{ as } \delta \to 0\,, \quad K^i_\delta := \big(K^i \cap (\mathbb{R}_+ \times \mathbb{R}^2)\big) \times [0, 2\varepsilon_\delta[\end{aligned}$$

for compact sets $K^a \subset \mathbb{R} \times \mathbb{R}^3$. If this consequence is true, we call $U^{(0)}$ an ***inner solution*** and $u^{(0)}$ an ***outer solution***, where the connection of both is given by the ***matching condition*** which is derived from (IV15.4).

Usually the expansion for the inner solution U is $U = U^{(0)} + \delta U^{(1)} + \ldots$ with different matching conditions for the higher terms.

Prandtl boundary layer equation

We apply this convergence to a flow with small viscosity, that is, with a viscosity coefficient which goes to 0 (see for example the picture in Fig. 32)

$$a = \delta^2 .$$

The flow equations with boundary condition are then

$$\left.\begin{aligned} &\mathrm{div}_x v = 0\,, \\ &\varrho_0(\partial_t v + (v \bullet \nabla_x) v) = -\nabla_x p + \delta^2 \Delta_x v + \mathbf{f} \end{aligned}\right\} \text{ in } \mathbb{R}\times\Omega, \qquad v = 0 \text{ on } \mathbb{R}\times\Gamma. \tag{IV15.5}$$

Let (v, p) be a solution and

$$\begin{aligned} V(t, y_1, y_2, y_3) &= v(t, y_1, y_2, \delta y_3)\,, \\ P(t, y_1, y_2, y_3) &= p(t, y_1, y_2, \delta y_3)\,. \end{aligned} \tag{IV15.6}$$

Then we get the following statements as δ converges to 0.

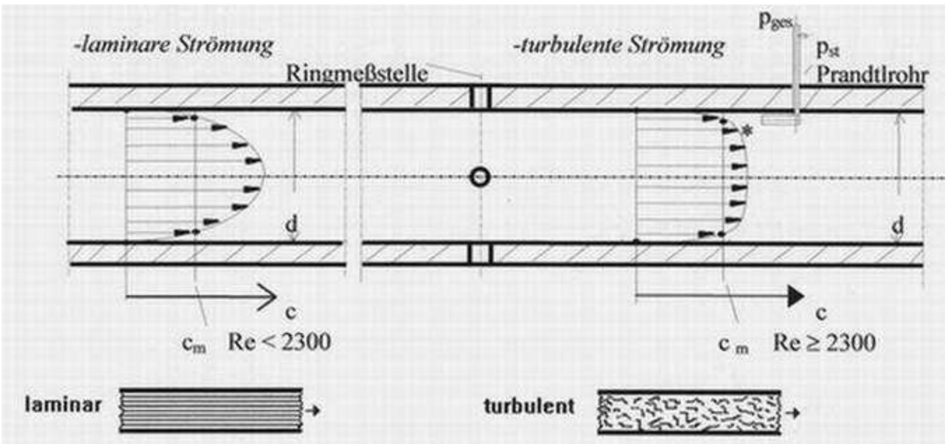

Fig. 32: Flow through a pipe

On the subinterval $\mathbb{R}_+ \times \mathbb{R}^2 \times [\varepsilon_\delta, \infty[$ **.** If $v \to v^{(0)}$, $p \to p^{(0)}$ (we assume that $\mathbf{f}$ is independent of δ) we get the limit equation

$$\begin{aligned} &\mathrm{div}_x v^{(0)} = 0\,, \\ &\varrho_0\big(\partial_t v^{(0)} + (v^{(0)} \cdot \nabla_x) v^{(0)}\big) = -\nabla_x p^{(0)} + \mathbf{f} \end{aligned} \tag{IV15.7}$$

in $\mathbb{R}_+ \times \mathbb{R}^2 \times]0, \infty[$.[8]

[8] These equations are sometimes called incompressible isothermal Euler equations.

On the subinterval $\mathbb{R}_+ \times \mathbb{R}^2 \times [\varepsilon_\delta, 2\varepsilon_\delta]$ **.** We have the "boundary conditions"

$$\left.\begin{aligned} V(t,y_1,y_2,y_3) &= v(t,y_1,y_2,\delta y_3) \\ P(t,y_1,y_2,y_3) &= p(t,y_1,y_2,\delta y_3) \end{aligned}\right\} \text{ for } \frac{\varepsilon_\delta}{\delta} \le y_3 \le \frac{2\varepsilon_\delta}{\delta}\,.$$

If we set $\varepsilon_\delta := \sqrt{\delta}$ and if

$$y_3 = \frac{\varepsilon_\delta}{\delta} = \frac{1}{\sqrt{\delta}} \to \infty\,, \quad \delta y_3 \to 0\,,$$

we get, as $\delta \to 0$,

$$V^{(0)}(t,y_1,y_2,\infty) \leftarrow V(t,y_1,y_2,\frac{1}{\sqrt{\delta}}) = v(t,y_1,y_2,\sqrt{\delta}) \to v^{(0)}(t,y_1,y_2,0)\,,$$

$$P^{(0)}(t,y_1,y_2,\infty) \leftarrow P(t,y_1,y_2,\frac{1}{\sqrt{\delta}}) = p(t,y_1,y_2,\sqrt{\delta}) \to p^{(0)}(t,y_1,y_2,0)\,,$$

where we assume $v \to v^{(0)}$, $p \to p^{(0)}$ and $V \to V^{(0)}$, $P \to P^{(0)}$. Thus we have the matching conditions

$$\begin{aligned} V^{(0)}(t,y_1,y_2,\infty) &= v^{(0)}(t,y_1,y_2,0)\,, \\ P^{(0)}(t,y_1,y_2,\infty) &= p^{(0)}(t,y_1,y_2,0)\,. \end{aligned} \tag{IV15.8}$$

On the subinterval $\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 2\varepsilon_\delta[$ **.** From the identity

$$V(t,y_1,y_2,y_3) = v(t,y_1,y_2,\delta y_3)\,, \tag{IV15.9}$$

the boundary condition $v(t,y_1,y_2,0) = 0$, and $V \to V^{(0)}$, it follows that

$$V^{(0)}(t,y_1,y_2,0) = 0\,. \tag{IV15.10}$$

Equation (IV15.9) yields the following formulas

$$\begin{aligned} &\partial_t v = \partial_t V\,, \quad \partial_{x_i} v = \partial_{y_i} V\,, \quad \partial_{x_i}^2 v = \partial_{y_i}^2 V \quad \text{for } i = 1,2, \\ &\partial_{x_3} v = \frac{1}{\delta}\partial_{y_3} V\,, \quad \partial_{x_3}^2 v = \frac{1}{\delta^2}\partial_{y_3}^2 V\,, \\ &\partial_{x_i} p = \partial_{y_i} P \quad \text{for } i = 1,2\,, \\ &\partial_{x_3} p = \frac{1}{\delta}\partial_{y_3} P\,. \end{aligned}$$

Since the Navier-Stokes equations (IV15.1) are

$$\begin{aligned} &\partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3 = 0\,, \\ &\varrho_0 \partial_t v_i + \varrho_0 (v_1 \partial_{x_1} + v_2 \partial_{x_2} + v_3 \partial_{x_3}) v_i = -\partial_{x_i} p \\ &+\delta^2 (\partial_{x_1}^2 v_i + \partial_{x_2}^2 v_i + \partial_{x_3}^2 v_i) + \mathbf{f}_i \quad \text{for } i = 1,2\,, \\ &\varrho_0 \partial_t v_3 + \varrho_0 (v_1 \partial_{x_1} + v_2 \partial_{x_2} + v_3 \partial_{x_3}) v_3 = -\partial_{x_3} p \\ &\qquad +\delta^2 (\partial_{x_1}^2 v_3 + \partial_{x_2}^2 v_3 + \partial_{x_3}^2 v_3) + \mathbf{f}_3\,, \end{aligned}$$

the transformed equations for (V,P) are

$$\begin{aligned}
&\partial_{y_1}V_1+\partial_{y_2}V_2+\frac{1}{\delta}\partial_{y_3}V_3=0\,,\\
&\varrho_0\partial_tV_i+\varrho_0(V_1\partial_{y_1}+V_2\partial_{y_2}+\frac{1}{\delta}V_3\partial_{y_3})V_i=-\partial_{y_i}P\\
&\qquad+\delta^2(\partial_{y_1}^2V_i+\partial_{y_2}^2V_i)+\partial_{y_3}^2V_i+\mathbf{f}_i\quad \text{für } i=1,2\,,\\
&\varrho_0\partial_tV_3+\varrho_0(V_1\partial_{y_1}+V_2\partial_{y_2}+\frac{1}{\delta}V_3\partial_{y_3})V_3=-\frac{1}{\delta}\partial_{y_3}P\\
&\qquad\qquad+\delta^2(\partial_{y_1}^2V_3+\partial_{y_2}^2V_3)+\partial_{y_3}^2V_3+\mathbf{f}_3\,.
\end{aligned}$$

If it is

$$\begin{aligned}
V&=V^{(0)}+\delta V^{(1)}+\mathcal{O}(\delta^2)\,,\\
P&=P^{(0)}+\delta P^{(1)}+\mathcal{O}(\delta^2)\,,
\end{aligned}$$

then the conservation of mass reads

$$\partial_{y_1}V_1^{(0)}+\partial_{y_2}V_2^{(0)}+\frac{1}{\delta}\partial_{y_3}V_3^{(0)}+\partial_{y_3}V_3^{(1)}=\mathcal{O}(\delta)\,,$$

and from this it follows

$$\begin{aligned}
&\partial_{y_3}V_3^{(0)}=0\,,\\
&\partial_{y_1}V_1^{(0)}+\partial_{y_2}V_2^{(0)}+\partial_{y_3}V_3^{(1)}=0\,.
\end{aligned}$$

The first equation states that $V_3^{(0)}$ is a function of (t,y_1,y_2), and due to the boundary conditions $V^{(0)}(t,y_1,y_2,0)=0$ follows $V_3^{(0)}=0$. Because of the matching condition $V^{(0)}(t,y_1,y_2,\infty)=v^{(0)}(t,y_1,y_2,0)$ we get

$$v_3^{(0)}(t,y_1,y_2,0)=0\,.$$

Thus, the expansion of (V,P) reduces to

$$\begin{aligned}
&V_i=V_i^{(0)}+\mathcal{O}(\delta)\quad \text{for } i=1,2\,,\\
&\frac{1}{\delta}V_3=V_3^{(1)}+\mathcal{O}(\delta)\,,\\
&P=P^{(0)}+\delta P^{(1)}+\mathcal{O}(\delta^2)\,,
\end{aligned}$$

and the momentum conservation becomes

$$\begin{aligned}
&\varrho_0\partial_tV_i^{(0)}+\varrho_0(V_1^{(0)}\partial_{y_1}+V_2^{(0)}\partial_{y_2}+V_3^{(1)}\partial_{y_3})V_i^{(0)}=-\partial_{y_i}P^{(0)}\\
&\qquad\qquad+\partial_{y_3}^2V_i^{(0)}+\mathbf{f}_i+\mathcal{O}(\delta)\quad \text{for } i=1,2\,,\\
&0=-\frac{1}{\delta}\partial_{y_3}P^{(0)}-\partial_{y_3}P^{(1)}+\mathbf{f}_3+\mathcal{O}(\delta)\,.
\end{aligned}$$

Thus this gives the equations

$$\begin{aligned}
&\varrho_0\partial_t V_i^{(0)} + \varrho_0(V_1^{(0)}\partial_{y_1} + V_2^{(0)}\partial_{y_2} + V_3^{(1)}\partial_{y_3})V_i^{(0)} \\
&\qquad = -\partial_{y_i}P^{(0)} + \partial_{y_3}^2 V_i^{(0)} + \mathbf{f}_i^{(0)} \quad \text{for } i=1,2\,, \\
&0 = \partial_{y_3}P^{(0)}\,, \\
&0 = \partial_{y_3}P^{(1)} - \mathbf{f}_3^{(0)}\,.
\end{aligned} \tag{IV15.11}$$

Here we have used that

$$\mathbf{f}(t,y_1,y_2,\delta y_3) = \underbrace{\mathbf{f}(t,y_1,y_2,0)}_{=:\,\mathbf{f}^{(0)}(t,y_1,y_2)} + \mathcal{O}(\delta)\,.$$

Result on $\mathbb{R}_+ \times \mathbb{R}^2 \times [0,\infty[$ ***.*** With the functions (not to be confused with the initial functions)

$$\begin{aligned}
&v := v^{(0)}, \quad V_i := V_i^{(0)} \quad \text{for } i = 1,2, \\
&V_3 := V_3^{(1)}, \quad P := P^{(0)},
\end{aligned}$$

the following is true.

15.1 Prandtl boundary layer. The following equations hold, the equations in (t,x)

$$\begin{aligned}
&\mathrm{div}_x v = 0\,, \\
&\varrho_0\big(\partial_t v + (v\cdot\nabla_x)v\big) = -\nabla_x p + \mathbf{f}\,, \\
&v_3(t,x_1,x_2,0) = 0\,,
\end{aligned} \tag{IV15.12}$$

and the equations in (t,y)

$$\begin{aligned}
&\partial_{y_1}V_1 + \partial_{y_2}V_2 + \partial_{y_3}V_3 = 0\,, \\
&\varrho_0\partial_t V_i + \varrho_0(V_1\partial_{y_1} + V_2\partial_{y_2} + V_3\partial_{y_3})V_i \\
&\qquad = -\partial_{y_i}P + \partial_{y_3}^2 V_i + \mathbf{f}_i^{(0)} \quad \text{for } i=1,2\,, \\
&0 = \partial_{y_3}P\,, \\
&V(t,y_1,y_2,0) = 0\,,
\end{aligned} \tag{IV15.13}$$

with the matching conditions

$$\begin{aligned}
&V_i(t,y_1,y_2,+\infty) = v_i(t,y_1,y_2,0) \quad \text{for } i=1,2, \\
&P(t,y_1,y_2,+\infty) = p(t,y_1,y_2,0)\,.
\end{aligned} \tag{IV15.14}$$

Reference: For the Prandtl boundary layer theory see Stemmer [65] and White [70, 4-2 Laminar boundary layer equations].

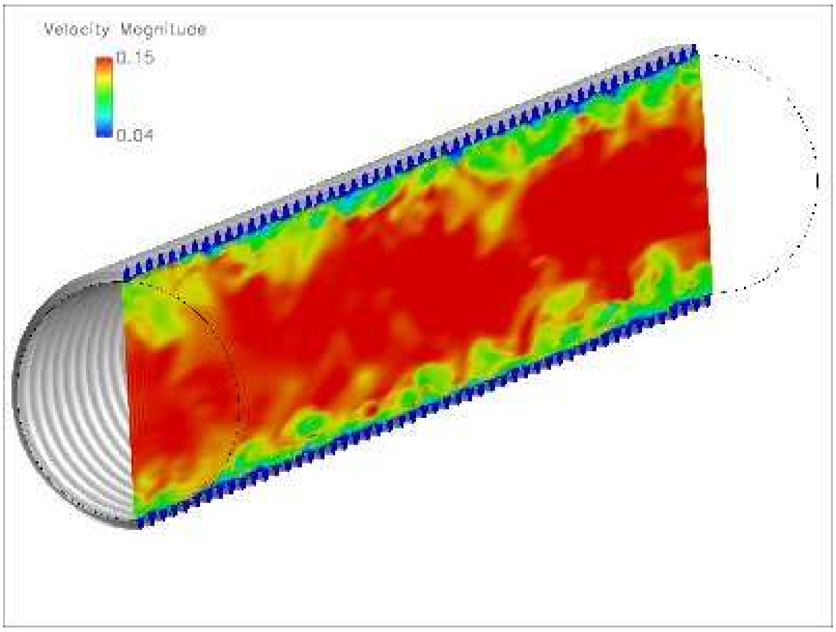

Fig. 33: Turbulent flow in a pipe

The only equations for (V, P) are to be solved are the momentum conservations for $i = 1, 2$ with the matching boundary conditions for V_1, V_2, since it is

$$P(t, y_1, y_2, y_3) = P(t, y_1, y_2, \infty) = p(t, y_1, y_2, 0)$$

und

$$V_3(t, y_1, y_2, y_3) = -\int_0^{y_3} (\partial_{y_1} V_1 + \partial_{y_2} V_2)(t, y_1, y_2, s)\, \mathrm{d}s\,,$$

which enclose the boundary condition for V_3.

The only boundary condition requiring the boundary layer at the remaining flow is $v_3(t, x_1, x_2, 0) = 0$. It is easy to perform a generalization for

- boundaries Γ which are not planar,
- the compressible case.

Naturally, the convergence of the asymptotic expansion are restrictive. Here they don't allow to describe the flow at the "stagnation points" [Wikipedia: Stagnation point] (*de:* [Wikipedia: Staupunkt]).

16 Self-gravitation

Physical objects generate a gravitation, and this gravitation is distributed in the entire space. We call it self-gravity if we want to describe the result of this gravitation on the object itself. Stars are particular objects which are formed by their self-gravity. The same effect is the gravitational force of the earth, and we feel it in our daily life.

Let us assume for a moment that all quantities are represented by smooth functions throughout the entire spacetime. With this the gravity equation together with the conservation of mass, momentum and energy reads as

$$\begin{aligned}
&-\Delta\boldsymbol{\phi} = \varrho\,, \qquad \boldsymbol{\phi}(t,x) \to 0 \text{ if } |x| \to \infty,\\
&\partial_t\varrho + \operatorname{div}(\varrho v) = 0 \qquad (\varrho \text{ total mass}),\\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f} \qquad (\Pi \text{ pressure tensor}),\\
&\qquad \mathbf{f} = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} + \mathbf{f}_0 \qquad (\mathbf{f} \text{ classical force}),\\
&\partial_t e + \operatorname{div}\left(ev + \Pi^{\mathrm{T}}\, v + q\right) = v \bullet \mathbf{f}\,,\\
&\qquad e = \varepsilon + \frac{\varrho}{2}|v|^2 \qquad (\varepsilon \text{ internal energy}).
\end{aligned} \tag{IV16.1}$$

The first equation is the gravitational equation where $\boldsymbol{\phi}$ denotes the gravitational potential. In the momentum equation $\mathfrak{g}\varrho\nabla\boldsymbol{\phi}$ is an objective vector and the remaining force $\mathbf{f}_0$ is transformed like a classical force (see section II.3). Under the assumption that there is no force besides the one caused by gravity, the remaining force $\mathbf{f}_0$ vanishes for observers, which realize that they are an inertial frame. In reality the functions, in particular ϱ, are not so smooth to ensure the equations as presented. Instead we make use of the distributional version.

We consider the equation for the gravitational potential $\boldsymbol{\phi}$, which is generated by the mass density ϱ. This equation we have already seen in (I2.10), and it is

$$\operatorname{div}(-\nabla[\boldsymbol{\phi}]) = [\varrho] \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n), \tag{IV16.2}$$

where ϱ is the total mass density in $\mathbb{R}\times\mathbb{R}^n$, here a function $\varrho \in L^1_{loc}(\mathbb{R}\times\mathbb{R}^n)$, where physically $n = 3$. The Newtonian gravitational potential $\boldsymbol{\phi}$ is an objective scalar, and the map $x \mapsto \boldsymbol{\phi}(t,x)$ is in the space $C^1(\mathbb{R}^n)$ (see I.2.15), if ϱ has, what we assume, only jumps on C^1-surfaces. Then $\nabla[\boldsymbol{\phi}] = [\nabla\boldsymbol{\phi}]$ holds in (IV16.2). In the momentum conservation occurs as a force term

$$\boxed{\mathfrak{g}\varrho\nabla\boldsymbol{\phi} \qquad \textit{Newton's force density},} \tag{IV16.3}$$

which is a product and therefore requires a certain explanation. In our case, ϱ and $\nabla\boldsymbol{\phi}$ are functions, and thus, the product of L^∞-functions is well-defined as a distribution. The equation (IV16.2) is normalized such that $\phi_{Literatur} = -4\pi G\boldsymbol{\phi}$ (for $n = 3$) is the usual potential in the physics literature, where

$$G = 6.67384 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$$

is the gravitational constant, see (I3.13). And $\mathfrak{g} = 4\pi G$ (for $n = 3$).

Now consider a planet or a star (or a collection of these objects) occupied by $\Omega_t = \{x \in \mathbb{R}^n\,;\ (t,x) \in \Omega\}$ where $\Omega \subset \mathbb{R} \times \mathbb{R}^n$. Further we assume that the exterior domain $\Omega' := (\mathbb{R} \times \mathbb{R}^n) \setminus \overline{\Omega}$ is a rare gas and no mass transition occurs between the gas and the planet. We model this situation by

$\boldsymbol{\phi}$ global defined potential,

$\varrho\,,\ v\,,\ \Pi = p\mathrm{Id} - S$ quantities in the star Ω,

$\varrho' = 0$ (v' undefined), $\Pi' = p'\mathrm{Id}$ quantities of the external gas in Ω'.

With this notation we have the following non-stationary distributional equations, where the gravitational equation as well as the mass, momentum and energy conservation are written in their original distributional version:

Self-gravitation in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$:

$$\begin{gathered}
\mathrm{div}[-\nabla\boldsymbol{\phi}] = [\varrho\mathcal{X}_\Omega]\,,\\
\partial_t[\varrho\mathcal{X}_\Omega] + \mathrm{div}[\varrho\mathcal{X}_\Omega v] = 0\,,\\
\partial_t[\varrho\mathcal{X}_\Omega v] + \mathrm{div}[\varrho\mathcal{X}_\Omega v\, v^{\mathrm{T}} + \mathcal{X}_\Omega(p\mathrm{Id} - S) + \mathcal{X}_{\Omega'}p'\mathrm{Id}] = [\mathbf{f}]\,,\\
\partial_t[e^{tot}] + \mathrm{div}\,[e^{tot}v + \Pi^{\mathrm{T}}\,\mathcal{X}_\Omega v + \mathcal{X}_\Omega q + \mathcal{X}_{\Omega'}\widetilde{q}'] = [v \bullet \mathbf{f}]\,,\\
e^{tot} := e\mathcal{X}_\Omega + e'\mathcal{X}_{\Omega'}
\end{gathered} \tag{IV16.4}$$

$\boldsymbol{\phi}$ gravitational potential, $\boldsymbol{\phi}(t,x) \to 0$ for $|x| \to \infty$,

$\mathbf{f} = \mathfrak{g}\varrho\mathcal{X}_\Omega\nabla\boldsymbol{\phi} + \mathbf{f}_0$ ($\mathbf{f}_0$ contains fictitious forces),

$\Omega \subset \mathbb{R} \times \mathbb{R}^n$ region of planet or star,

$e = \varepsilon + \frac{\varrho}{2}|v|^2\,,\quad \varepsilon = \widehat{\varepsilon}(\varrho,\theta)\,,$

(ϱ, v, θ) quantities of planet or star.

This section is divided into several parts. First we consider stars from a certain distance so that they can be considered as mass points. Next we study the incompressible case under the assumption of rotation, which is the classical result of Newton. Then in this section we bring as results of Caratheodory the self-gravitation of gaseous planets or stars which are assumed to be radially symmetric, therefore they do not rotate. Finally we give an interpretation of this situation for gaseous stars. Their energy is kept inside the gas with a nonconstant temperature.

Star as point

Wir betrachten einen Stern aus einiger Entfernung, bei der er uns schon wie ein Punkt erscheint, d.h. es gibt eine Trajektorie $t \mapsto \xi(t)$ so dass sich der Stern zur Zeit t in

$$\Omega_t^\delta \subset \mathrm{B}_\delta(\xi(t))$$

befindet, wobei δ eine kleine Zahl sei. Der Stern habe die Massendichte ϱ^δ in Ω^δ und die Gesamtmasse ist dann

$$m^\delta(t) := \int_{\Omega_t^\delta} \varrho^\delta(t,x)\,\mathrm{d}x\,.$$

16.1 Star as point. If $m^\delta \to m$ uniformly (the standard case is $m^\delta := m$) as $\delta \to 0$ it follows that

$$[\varrho^\delta \mu_{\Omega^\delta}] \longrightarrow m\boldsymbol{\mu}_\xi \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)\,.$$

Hence the gravity potential $\boldsymbol{\phi}^\delta$ induced by the star, that is

$$\operatorname{div}(-\nabla[\boldsymbol{\phi}^\delta]) = [\varrho^\delta \mu_{\Omega^\delta}]\,,$$

converges in L^1 to the limit potential $\boldsymbol{\phi}$ satisfying

$$\operatorname{div}(-\nabla[\boldsymbol{\phi}]) = m\boldsymbol{\mu}_\xi\,.$$

Note that $\boldsymbol{\phi}$ is a fundamental solution of $-\Delta$, the negative Laplace operator. And if the limit $m^\delta \to m$ exists, this implies that $\sup \varrho^\delta \to \infty$.

Proof. For $\zeta \in \mathscr{D}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$ we have as $\delta \to 0$

$$\begin{aligned}
&\Big\langle \zeta\,,\, [\varrho^\delta \mu_{\Omega^\delta}] \Big\rangle = \int_{\Omega^\delta} (\varrho^\delta\zeta)(t,x)\,\mathrm{d}(t,x) \\
&= \int_{\mathbb{R}}\int_{\Omega_t^\delta} \zeta(t,\xi(t))\varrho^\delta(t,x)\,\mathrm{dL}^4(t,x) + \mathcal{O}(\delta) \\
&= \int_{\mathbb{R}} \zeta(t,\xi(t)) \int_{\Omega_t^\delta} \varrho^\delta(t,x)\,\mathrm{dL}^3(x)\,\mathrm{dL}^1(t) + \mathcal{O}(\delta) \\
&\longrightarrow \int_{\mathbb{R}} \zeta(t,\xi(t)) m(t)\,\mathrm{dL}^1(t) = \big\langle \zeta\,,\, m\boldsymbol{\mu}_\xi \big\rangle\,,
\end{aligned}$$

hence $[\varrho^\delta \mu_{\Omega^\delta}] \to m\boldsymbol{\mu}_\xi$ in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$. In the same way it follows that for all times t for all $\eta \in \mathscr{D}(\mathbb{R}^n;\mathbb{R})$

$$\Big\langle \eta\,,\, \varrho^\delta(t,\bullet)\mathcal{X}_{\Omega_t^\delta}\mathrm{L}^n \Big\rangle \longrightarrow \big\langle \eta\,,\, m(t)\boldsymbol{\delta}_{\xi(t)} \big\rangle\,.$$

Since

$$\boldsymbol{\phi}(t,x) = \frac{m(t)}{\sigma_n|x-\xi(t)|}\,, \quad \boldsymbol{\phi}^\delta(t,x) = \int_{\mathbb{R}^n} \frac{\varrho^\delta(t,y)\mathcal{X}_{\Omega_t^\delta}(y)}{\sigma_n|x-y|}\,\mathrm{d}y\,,$$

this implies

$$\begin{aligned}\boldsymbol{\phi}(t,x) &= \frac{m(t)}{\sigma_n|x-\xi(t)|} = \int_{\mathbb{R}^n} \frac{m(t)}{\sigma_n|x-y|}\,\mathrm{d}\boldsymbol{\delta}_{\xi(t)}(y) \\ &\longleftarrow \int_{\mathbb{R}^n} \frac{\varrho^\delta(t,y)\mathcal{X}_{\Omega_t^\delta}(y)}{\sigma_n|x-y|}\,\mathrm{dL}^n(y) = \boldsymbol{\phi}^\delta(t,x)\,.\end{aligned}$$

Thus we have shown that $\boldsymbol{\phi}^\delta \to \boldsymbol{\phi}$ pointwise. From this it follows immediately that $\boldsymbol{\phi}^\delta \to \boldsymbol{\phi}$ in $L^1(\mathbb{R}\times\mathbb{R}^n)$ since $\boldsymbol{\phi}^\delta$ is estimated uniformly by $C\boldsymbol{\phi}$ where $C = \text{const}$. □

What about the mass-momentum equations in (IV16.4)? Let us neglect for a moment the superscript δ. If Ω'_t is connected for all t this yields $p' = \text{const}$ in Ω'. Hence we subtract the constant $p'\mathrm{Id}$ in the momentum flux, and we assume $S = 0$. Then we obtain for the mass and momentum equation in (IV16.4), including the gravity,

$$\begin{aligned}\operatorname{div}[-\nabla\boldsymbol{\phi}] &= [\varrho\mathcal{X}_\Omega]\,,\\ \partial_t[\varrho\mathcal{X}_\Omega] + \operatorname{div}[\varrho\mathcal{X}_\Omega v] &= 0\,,\\ \partial_t[\varrho\mathcal{X}_\Omega v] + \operatorname{div}[\varrho\mathcal{X}_\Omega v\,v^{\mathrm{T}} + \mathcal{X}_\Omega(p-p')\mathrm{Id}] &= [\mathfrak{g}\varrho\mathcal{X}_\Omega\nabla\boldsymbol{\phi} + \mathbf{f}_0]\,,\end{aligned} \tag{IV16.5}$$

that is, only the self-gravitation acts on the object Ω in consideration, and everything else, gravity from other stellar objects included, is contained in the smooth force $\mathbf{f}_0$, which we assume is of the form $\varrho\mathcal{X}_\Omega \mathrm{g}_0$. We impose now for the velocity the identity $v = \overline{v} + u$ with a “global in time” velocity $\overline{v}$, which in detail reads

$$\begin{aligned}(m\overline{v})(t) := \int_{\Omega_t}(\varrho v)(t,x)\,\mathrm{d}x \quad \text{where} \quad m(t) &:= \int_{\Omega_t}\varrho(t,x)\,\mathrm{d}x\,,\\ u(t,x) := v(t,x) - \overline{v}(t)&\end{aligned} \tag{IV16.6}$$

(compare the definition of mean velocity in III.3.1(1)).

16.2 Lemma. It follows that for all t

$$\int_{\Omega_t}(\varrho u)(t,x)\,\mathrm{d}x = 0\,.$$

And on the boundary there is the condition $(v - v_{\partial\Omega})\bullet\nu_\Omega = 0$ (compare with (IV16.9)) where $v = \overline{v} + u$.

Proof. It follows

$$\int_{\Omega_t} (\varrho u)(t,x)\,\mathrm{d}x = \int_{\Omega_t} \big((\varrho v)(t,x) - \varrho(t,x)\overline{v}(t)\big)\,\mathrm{d}x$$
$$= \int_{\Omega_t} \varrho v(t,x)\,\mathrm{d}x - m(t)\overline{v}(t) = 0$$

by the definitions. On the boundary there is a condition only on v, not on $\overline{v}$ and u, since the mass equation in (IV16.5) says for all $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ (this is as in the proof of 3.8)

$$0 = -\langle \zeta\,,\, \partial_t[\varrho\mathcal{X}_\Omega] + \mathrm{div}[\varrho\mathcal{X}_\Omega v]\,\rangle = \int_\Omega \big(\partial_t\zeta\cdot\varrho + \nabla\zeta\bullet(\varrho v)\big)\,\mathrm{d}(t,x)$$
$$= \int_\Omega \nabla_{(t,x)}\zeta\bullet(\varrho,\varrho v)\,\mathrm{dL}^{1+n}(t,x) \qquad \big(\nabla_{(t,x)} := (\partial_t,\nabla_x)\big)$$
$$= \int_{\partial\Omega} \zeta n_\Omega\bullet(\varrho,\varrho v)\,\mathrm{dH}^n - \int_\Omega \zeta\,\mathrm{div}_{(t,x)}(\varrho,\varrho v)\,\mathrm{dL}^{1+n}\,,$$

where n_Ω is the outer unit normal to $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ (see the Definition in 3.8). Therefore

$$n_\Omega\bullet(1,v) = 0 \text{ on } \partial\Omega\,,$$
$$\partial_t\varrho + \mathrm{div}_x(\varrho v) = 0 \text{ in } \Omega\,.$$

The first equation is equivalent to $(v - v_{\partial\Omega})\bullet\nu_\Omega = 0$, see (IV3.10). □

With this definition the equations in (IV16.5) read, it is $\mathbf{f}_0 = \varrho\mathcal{X}_\Omega \mathrm{g}_0$,

$$\mathrm{div}[-\nabla\boldsymbol{\phi}] = [\varrho\mathcal{X}_\Omega]\,,$$
$$\partial_t[\varrho\mathcal{X}_\Omega] + \mathrm{div}[\varrho\mathcal{X}_\Omega\overline{v}] + \boxed{\mathrm{div}[\varrho\mathcal{X}_\Omega u]} = 0\,,$$
$$\partial_t[\varrho\mathcal{X}_\Omega\overline{v}] + \mathrm{div}[\varrho\mathcal{X}_\Omega\overline{v}\,\overline{v}^{\mathrm{T}}] + \boxed{\partial_t[\varrho\mathcal{X}_\Omega u] + \mathrm{div}[\varrho\mathcal{X}_\Omega(\overline{v}u^{\mathrm{T}} + u\overline{v}^{\mathrm{T}})]}$$
$$+\boxed{\mathrm{div}[\varrho\mathcal{X}_\Omega u u^{\mathrm{T}} + \mathcal{X}_\Omega(p-p')\mathrm{Id}] - [\mathfrak{g}\varrho\mathcal{X}_\Omega\nabla\boldsymbol{\phi}]} - \varrho\mathcal{X}_\Omega\mathrm{g}_0 = 0\,.$$

(On the boundary there is only the equation $(\overline{v}+u-v_{\partial\Omega})\bullet\nu_\Omega = 0$. Therefore this is only to indicate what we are going to do.) If now we apply this to the above sequence we obtain the following result.

16.3 Theorem. We now consider the sequence $[\varrho^\delta\mathcal{X}_{\Omega^\delta}]$, we use again the superscript δ, and we make the assumptions of 16.1. In addition we assume that $\mathbf{f}_0^\delta := \varrho^\delta\mathcal{X}_{\Omega^\delta}\mathrm{g}_0$ and

$$m^\delta,\ |\overline{v}^\delta|,\ |u^\delta| \le \mathrm{const}\,,\quad m^\delta\to m,\ \overline{v}^\delta\to\overline{v}.$$[9]

If in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$

$$\lim_{\delta\to0}\big(\mathrm{div}[\varrho^\delta\mathcal{X}_{\Omega^\delta}u^\delta u^{\delta\mathrm{T}} + \mathcal{X}_{\Omega^\delta}(p^\delta - p')\mathrm{Id}] - [\mathfrak{g}\varrho^\delta\mathcal{X}_{\Omega^\delta}\nabla\boldsymbol{\phi}^\delta]\big) = m\mathrm{g}\boldsymbol{\mu}_\xi\,,$$

[9] we mean the pointwise convergence with respect to t.

then as limit we obtain a mass point $m\boldsymbol{\mu}_\xi$ satisfying the properties of I.3.1, that is,

$$\begin{aligned} &\overline{v}(t,\xi(t)) = \dot{\xi}(t)\,, \quad \dot{m} = 0\,, \\ &m\ddot{\xi} = \mathbf{f}\,, \quad \mathbf{f} = m(\mathrm{g}_0 - \mathrm{g})\,. \end{aligned} \tag{IV16.7}$$

Special case: Usually $\mathrm{g} = 0$. This is so in the case I.4.5 for $u = 0$. And we will show this also in 16.5 as a result of Newton. This holds if $\mathrm{g}_0 = 0$.

Fig. 34: "2014 MU69, aufgenommen am 1. Januar 2019 von New Horizons, 7 Min. vor der engsten Annäherung." Von NASA/Johns Hopkins University Applied Physics Laboratory [Wikipedia: (486958) 2014 MU69]

Proof. As shown in 16.1 the δ-gravitational formula $\mathrm{div}[-\nabla\boldsymbol{\phi}^\delta] = [\varrho^\delta\mathcal{X}_{\Omega^\delta}]$ converges to the limit equation

$$\mathrm{div}[-\nabla\boldsymbol{\phi}] = m\boldsymbol{\mu}_\xi\,.$$

Next we look at the mass conservation $\partial_t[\varrho^\delta\mathcal{X}_{\Omega^\delta}] + \mathrm{div}[\varrho^\delta\mathcal{X}_{\Omega^\delta}v^\delta] = 0$ in (IV16.5), which in distributional formulation reads for $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$

$$\begin{aligned} 0 &= \int_{\Omega^\delta} \big(\partial_t\zeta\cdot\varrho^\delta + \nabla_x\zeta\bullet(\varrho^\delta v^\delta)\big)\,\mathrm{d}(t,x) \qquad \big(v^\delta = \overline{v}^\delta + u^\delta\big) \\ &= \int_{\Omega^\delta} \big(\partial_t\zeta + \overline{v}^\delta\bullet\nabla_x\zeta\big)\varrho^\delta\,\mathrm{d}(t,x) + \int_{\Omega^\delta} \nabla_x\zeta\bullet(\varrho^\delta u^\delta)\,\mathrm{d}(t,x)\,. \end{aligned}$$

The first term is

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\Omega_t^\delta} (\partial_t + \overline{v}^\delta(t) \bullet \nabla_x)\zeta(t,x)\varrho^\delta(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} ((\partial_t + \overline{v}^\delta(t)\bullet\nabla_x)\zeta)(t,\xi(t)) \int_{\Omega_t^\delta} \varrho^\delta(t,x)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{O}(\delta) \\
&\longrightarrow \int_{\mathbb{R}} ((\partial_t + \overline{v}(t)\bullet\nabla_x)\zeta)(t,\xi(t))m(t)\,\mathrm{d}t \\
&= \big\langle \zeta\,,\, -\partial_t(m\boldsymbol{\mu}_\xi) - \operatorname{div}_x(m\overline{v}\boldsymbol{\mu}_\xi) \big\rangle\ .
\end{aligned}
$$

since $\overline{v}^\delta \to \overline{v}$ pointwise. The second term is

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\Omega_t^\delta} \nabla_x\zeta(t,x)\bullet(\varrho^\delta u^\delta)(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} (\nabla_x\zeta)(t,\xi(t))\bullet\Big(\int_{\Omega_t^\delta} (\varrho^\delta u^\delta)(t,x)\,\mathrm{d}x\Big)\,\mathrm{d}t + \mathcal{O}(\delta) = \mathcal{O}(\delta)
\end{aligned}
$$

by 16.2, which tends to 0. Thus the mass equation in the limit is

$$
\partial_t(m\boldsymbol{\mu}_\xi) + \operatorname{div}_x(m\overline{v}\boldsymbol{\mu}_\xi) = 0 \text{ in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n).
$$

Next we have to consider the momentum equation. From (IV16.5) we get

$$
\begin{aligned}
\partial_t[\varrho^\delta \mathcal{X}_{\Omega^\delta}(\overline{v}^\delta + u^\delta)] + \operatorname{div}_x[\varrho^\delta \mathcal{X}_{\Omega^\delta}(\overline{v}^\delta + u^\delta)\,(\overline{v}^\delta + u^\delta)^{\mathrm{T}} + \mathcal{X}_{\Omega^\delta}(p^\delta - p')\mathrm{Id}] \\
= [\mathfrak{g}\varrho^\delta \mathcal{X}_{\Omega^\delta}\nabla\boldsymbol{\phi}^\delta + \varrho^\delta \mathcal{X}_{\Omega^\delta}\mathrm{g}_0]\,.
\end{aligned}
$$

The velocity terms read with test functions $\zeta \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$

$$
\begin{aligned}
&\int_{\Omega^\delta} \Big(\partial_t\zeta\bullet(\varrho^\delta(\overline{v}^\delta + u^\delta)) + \mathrm{D}_x\zeta\,\boldsymbol{:}\,(\varrho^\delta(\overline{v}^\delta + u^\delta)\,(\overline{v}^\delta + u^\delta)^{\mathrm{T}}\,)\Big)\,\mathrm{d}(t,x) \\
&= \int_{\Omega^\delta} \partial_t\zeta\bullet(\varrho^\delta\overline{v}^\delta)\,\mathrm{d}(t,x) + \int_{\Omega^\delta} \partial_t\zeta\bullet(\varrho^\delta u^\delta)\,\mathrm{d}(t,x) \\
&+ \int_{\Omega^\delta} \mathrm{D}_x\zeta\,\boldsymbol{:}\,(\varrho^\delta\overline{v}^\delta\overline{v}^{\delta\mathrm{T}}\,)\,\mathrm{d}(t,x) + \int_{\Omega^\delta} \mathrm{D}_x\zeta\,\boldsymbol{:}\,(\overline{v}^\delta\,(\varrho^\delta u^\delta)^{\mathrm{T}} + (\varrho^\delta u^\delta)\overline{v}^{\delta\mathrm{T}}\,)\,\mathrm{d}(t,x) \\
&\qquad + \int_{\Omega^\delta} \mathrm{D}_x\zeta\,\boldsymbol{:}\,(\varrho^\delta u^\delta u^{\delta\mathrm{T}}\,)\,\mathrm{d}(t,x)\,.
\end{aligned}
$$

Except the last term the terms on the right side have a limit

$$
\begin{aligned}
&= \int_{\mathbb{R}} \Big((\partial_t\zeta)(t,\xi(t))\bullet\overline{v}^\delta(t) + (\mathrm{D}_x\zeta)(t,\xi(t))\,\boldsymbol{:}\,(\overline{v}^\delta(t)\,\overline{v}^\delta(t)^{\mathrm{T}}\,)\Big)\Big(\int_{\Omega_t^\delta} \varrho^\delta\,\mathrm{d}x\Big)\,\mathrm{d}t \\
&+ \int_{\mathbb{R}} \Big((\partial_t\zeta)(t,\xi(t)) + 2\,\big((\mathrm{D}_x\zeta)(t,\xi(t))\overline{v}^\delta(t)\big)^{\mathrm{S}}\Big)\bullet\Big(\int_{\Omega_t^\delta} (\varrho^\delta u^\delta)(t,x)\,\mathrm{d}x\Big)\,\mathrm{d}t \\
&\qquad + \mathcal{O}(\delta) \\
&\longrightarrow \int_{\mathbb{R}} \Big((\partial_t\zeta)(t,\xi(t))\bullet\overline{v}(t) + (\mathrm{D}_x\zeta)(t,\xi(t))\,\boldsymbol{:}\,(\overline{v}(t)\,\overline{v}(t)^{\mathrm{T}}\,)m(t)\,\mathrm{d}t\Big) \\
&= \big\langle \zeta\,,\, -\partial_t(m\overline{v}\,\boldsymbol{\mu}_\xi) - \operatorname{div}_x(m\overline{v}\,\overline{v}^{\mathrm{T}}\,\boldsymbol{\mu}_\xi) \big\rangle\ .
\end{aligned}
$$

Therefore if we define

$$W^\delta := \operatorname{div}[\varrho^\delta \mathcal{X}_{\Omega^\delta} u^\delta u^{\delta\mathrm{T}} + \mathcal{X}_{\Omega^\delta}(p^\delta - p')\mathrm{Id}] - [\mathfrak{g}\varrho^\delta \mathcal{X}_{\Omega^\delta} \nabla \boldsymbol{\phi}^\delta]$$

we have shown that the limt equation is

$$\partial_t(m\overline{v}\,\boldsymbol{\mu}_\xi) + \operatorname{div}_x(m\overline{v}\,\overline{v}^{\mathrm{T}}\,\boldsymbol{\mu}_\xi) + \lim_{\delta\to 0} W^\delta = \lim_{\delta\to 0}[\varrho^\delta \mathcal{X}_{\Omega^\delta} \mathrm{g}_0] = m\mathrm{g}_0 \boldsymbol{\mu}_\xi \,.$$

Finally, if we apply theorem III.6.4 or I.4.5 with $\mathbf{f} = m(\mathrm{g}_0 - \mathrm{g})$, we get the assertion. □

The convergence of a star to a point is not due to an observer transformation, it is an approximation on long distance. If one introduces the variable $y = \frac{x}{\delta}$ in order to have the star in a handable unit ball, the tansformation $(t,x) = \tau(t,y) = (t,\delta y)$ is like the transformation from reference coordinates.

Stationary case

Let us assume that the outer forces $\mathbf{f}_0$ are zero, that is, the observer assumes to have an inertial frame. Therefore the only force to consider is the gravitational force. In the very thin gas, that is in Ω', we assume $\varrho' = 0$, so the surrounding of our stars do not contribute to the gravity. In the gas the following equations hold

$$\operatorname{div}\nabla\boldsymbol{\phi} = 0 \quad \text{in } \Omega' \,,$$
$$\nabla p' = 0 \,, \quad \partial_t e' + \operatorname{div} q' = 0 \quad \text{in } \Omega' \,.$$

Hence, if Ω'_t is connected for all t this yields $p' = \text{const}$ in Ω'.

We now assume that the spacetime domain Ω represents a single object and that we have waited long enough, so eventually this object will approach a periodic movement (we mean for example an asterorid turning around itself like "Ultimate Thule" in Fig. 34). If the object is a planet or star, after such a long time the compression due to self-gravitation will make that it is a stationary solution of the equations (it means that $v = v(x)$ and that $\Omega = \mathbb{R} \times D$). With this assumption we have to consider the stationary mass and momentum equations alone, might be for an incompressible or compressible object. Therefore

$$\Omega := \mathbb{R} \times D \,, \quad \Omega' := \mathbb{R} \times D' \,, \quad D' = \mathbb{R}^3 \setminus \overline{D} \,,$$

where D is a fixed bounded domain in $\mathbb{R}^3$. In this case one has to solve the following special statements:

Stationary gravitation:

$$\operatorname{div}[-\nabla\boldsymbol{\phi}] = [\varrho\mathcal{X}_D]\,,$$
$$\operatorname{div}[\varrho v\mathcal{X}_D] = 0\,,$$
$$\operatorname{div}[\mathcal{X}_D(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) + \mathcal{X}_{D'}p'\mathrm{Id}] = \mathfrak{g}[\varrho\mathcal{X}_D\nabla\boldsymbol{\phi}]$$

$\boldsymbol{\phi}$ gravitational potential, $\boldsymbol{\phi}(t,x) \to 0$ for $|x| \to \infty$,
$D \subset \mathbb{R}^3$ the star or planet having gravity,
ϱ the mass density, v the velocity,
p the pressure:

$$\begin{cases} p = \widehat{p}(\varrho) \text{ the pressure in the compressible case,} \\ p \text{ variable, } \varrho = \text{const, in the incompressible case.} \end{cases} \qquad \text{(IV16.8)}$$

Here the conservation laws are written in the original version, i.e. they have to be understood in the distributional sense. This implies the following.

16.4 Theorem. Assume that the complement D' is connected. Then the equations in (IV16.8) are equivalent to $p' = \text{const}$ in D', the boundary conditions

$$v\bullet\nu_D = 0 \quad \text{and} \quad (p - p')\nu_D = S\nu_D \quad \text{on } \partial D \qquad \text{(IV16.9)}$$

and the differential equations

$$\begin{aligned} &\operatorname{div}[-\nabla\boldsymbol{\phi}] = [\varrho\mathcal{X}_D] \text{ in } \mathscr{D}(\mathbb{R}^3)\,, \\ &\operatorname{div}(\varrho v) = 0 \text{ in } D\,, \\ &\operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} \text{ in } D\,. \end{aligned} \qquad \text{(IV16.10)}$$

Remark: If $S = 0$ the last equation can be written as

$$\varrho v\bullet\nabla v + \nabla p = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} \text{ in } D\,. \qquad \text{(IV16.11)}$$

Proof. The momentum equation in D' says that $\nabla p' = \operatorname{div}(p'\mathrm{Id}) = 0$. Since D' is assumed to be connected, we conclude $p' = \text{const}$ in D'. The mass equation contains the boundary condition $v\bullet\nu_D = 0$ and $\operatorname{div}(\varrho v) = 0$ is the remaining differential equation in D. It remains to write down the momentum equation which is

$$\operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id} - S) = \mathfrak{g}\varrho\nabla\boldsymbol{\phi}$$

in D together with the boundary condition

$$\varrho v \bullet \nu v + p\nu - S\nu = p'\nu$$

for a normal ν on ∂D, where $v \bullet \nu = 0$ is known. The remark follows as usual since $\operatorname{div}(\varrho v v^{\mathrm{T}}) = v \bullet \operatorname{div}(\varrho v) + \varrho v \bullet \nabla v$. □

Newton's spheroid

Here we study a rotating planet under self-gravitation. In general the planet has to satisfy (IV16.4). But after millions of years it came possibly to a stationary solution. If we assume this, then the planet has the stationary form $\Omega = \mathbb{R} \times D$ and its velocity v is time independent. We further assume that the planet is incompressible with constant density $\varrho = \varrho_0 = \text{const}$. Hence we have to solve (IV16.8) and the speed v is that of a rotation.

16.5 Isaac Newton: Rotating planet. We consider a rotating incompressible planet modelled by constant mass density $\varrho = \varrho_0$ and a vanishing stress tensor $S = 0$. We treat solutions of the stationary equations (IV16.8), where the rotation axis is given by the x_3-axis

$$\{x \in \mathbb{R}^3 ;\ x_1 = 0,\ x_2 = 0\} \quad \text{with} \quad v(x) = \omega(-x_2, x_1, 0)\,,$$

i.e. ω is the constant angular speed. Then it holds:

(1) If D is convex, then $p' = \text{const}$ and the choice of v implies that equations (IV16.8) are equivalent to the rotational symmetry of D and

$$p(x) - p' = 0 \text{ for } x \in \partial D\,,$$

$$p(x) - p' = \frac{\varrho_0 \omega^2}{2}(x_1^2 + x_2^2) + \mathfrak{g}\varrho_0 \boldsymbol{\phi}(x) + c_0 \text{ for } x \in \overline{D}$$

with a constant $c_0 = \text{const}$, and

$$\boldsymbol{\phi}(x) = \frac{\varrho_0}{4\pi} \int_D \frac{\mathrm{d}y}{|x-y|} \text{ for } x \in \mathbb{R}^3\,.$$

(2) Suppose D be an ***oblate spheroid***, that means, there are constants a and c with $0 < c < a$ such that the surface ∂D consists of those points $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ which satisfy

$$\frac{x_1^2 + x_2^2}{a} + \frac{x_3^2}{c} = 1\,, \quad 0 < c < a\,.$$

Then D is a rotating planet as in (IV16.8) with angular velocity ω around the x_3-axis if

$$\omega^2 a = G\varrho_0 B\big(\frac{c}{a}\big)\,, \quad B(z) = \int_{\mathbb{S}^2} \frac{2\xi_3^2 - z(\xi_1^2 + \xi_2^2)}{\xi_3^2 + z(\xi_1^2 + \xi_2^2)}\, \mathrm{dH}^2(\xi)\,.$$

The proof of 16.5(2) shows that the spheroid is a solution of the interface problem at ∂D. In D it is made use of the fact, that the planet is assumed to be incompressible. Für den Schluss, dass das Interfaceproblem notwendigerweise einen Spheroid als Lösung hat, siehe ????? ggf. EXERCISES.

References: The fact that an oblate spheroid D is a solution has been proved by Newton in his "Principia Mathematica" in the year 1687. You find it there in "Book III Proposition XIX", see the translation in [119] und [120]. We mention also Fig. 35 and the historical remarks in [newtonreception6.pdf]. Besides this original literature we refer to the article of Solonnikov [64], where he proves the stabillity of a rotating incompressible body.

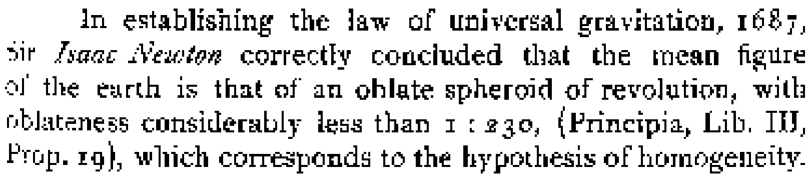
In establishing the law of universal gravitation, 1687, Sir *Isaac Newton* correctly concluded that the mean figure of the earth is that of an oblate spheroid of revolution, with oblateness considerably less than 1 : 230, (Principia, Lib. III, Prop. 19), which corresponds to the hypothesis of homogeneity.

Fig. 35: From Astronomische Nachrichten Band 213 (1921)

Proof (1). Equation (IV16.8) in D' reads only

$$\operatorname{div}(p'\mathrm{Id}) = 0 \text{ in } D' ,$$

and from this it follows that $p' = \text{const}$ locally. This also applies globally on D' since D is assumed to be convex and therefore D' is connected. Then the term $p'\mathrm{Id}$ can be subtracted. Thus the equations (IV16.8) are equivalent to

$$\begin{aligned} &\operatorname{div}[-\nabla\phi] = [\varrho_0\mathcal{X}_D] , \\ &\operatorname{div}[v\mathcal{X}_D] = 0 , \\ &\operatorname{div}[\mathcal{X}_D(\varrho_0 v\, v^{\mathrm{T}} + (p-p')\mathrm{Id})] = \mathfrak{g}[\varrho_0\mathcal{X}_D\nabla\phi] \end{aligned} \tag{IV16.12}$$

The second equation of (IV16.12) is equivalent to

$$\operatorname{div} v = 0 \text{ in } D , \quad v\bullet\nu_D = 0 \text{ auf } \partial D .$$

The differential equation is satified by the given velocity v. The boundary condition $v\bullet\nu_D = 0$ yields the rotational symmetry of D due to the chosen velocity v. The third equation of (IV16.12) is equivalent to an boundary condition and a differential equation

$$\begin{aligned} &p - p' = 0 \text{ on } \partial D , \\ &\operatorname{div}(\varrho_0 v\, v^{\mathrm{T}} + (p-p')\mathrm{Id}) = \mathfrak{g}\varrho_0\nabla\phi \text{ in } D , \end{aligned}$$

where the boundary condition is simplified by $v\bullet\nu_D = 0$. Since it is (we had this already in I.4.5)

$$\mathfrak{g}\varrho_0\nabla\phi = \nabla(\mathfrak{g}\varrho_0\phi) ,$$

the differential equation reads

$$\mathrm{div}(\varrho_0 v\, v^{\mathrm{T}} + (p - p' - \mathfrak{g}\varrho_0 \boldsymbol{\phi})\mathrm{Id}) = 0$$

or

$$\begin{aligned}\nabla(p - p' - \mathfrak{g}\varrho_0\boldsymbol{\phi}) &= -\,\mathrm{div}(\varrho_0 v\, v^{\mathrm{T}}) = -\varrho_0 v \bullet \nabla v \\ &= -\varrho_0\omega^2\Big(-x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_1\begin{bmatrix}-1\\0\\0\end{bmatrix}\Big) = \varrho_0\omega^2\begin{bmatrix}x_1\\x_2\\0\end{bmatrix} \\ &= \nabla\Big(\varrho_0\omega^2\frac{x_1^2 + x_2^2}{2}\Big)\end{aligned}$$

(we had this already in I.3.5), thus (since D is convex and therefore simply connected) for $x \in D$

$$p(x) - p' - \mathfrak{g}\varrho_0\boldsymbol{\phi}(x) = \frac{\varrho_0\omega^2}{2}(x_1^1 + x_2^2) + c_0\,, \quad c_0 = \mathrm{const}\,,$$

quod erat demonstrandum. The solution of the first equation of (IV16.12), that is, the gravity equation is given by

$$\boldsymbol{\phi}(x) = \frac{\varrho_0}{4\pi}\int_D \frac{\mathrm{d}y}{|x - y|}\,. \tag{IV16.13}$$

(See [64, equation (3) and following].) □

Proof (2). Since p is continuous on $\overline{D}$, we have to determine the planet D such that, we use the result of (1),

$$\frac{\varrho_0\omega^2}{2}(x_1^2 + x_2^2) + \mathfrak{g}\varrho_0\boldsymbol{\phi}(x) + c_0 = p(x) - p' = 0 \text{ für } x \in \partial D,$$

or, if $c_0 + \mathfrak{g}\varrho_0\boldsymbol{\phi}_0 = 0$,

$$\boldsymbol{\phi}(x) = \boldsymbol{\phi}_0 - \frac{\omega^2}{2\mathfrak{g}}(x_1^2 + x_2^2) \text{ für } x \in \partial D \tag{IV16.14}$$

($\boldsymbol{\phi}_0$ is the value of the potential for points $x \in \partial D$ that are on the axis). So we have to choose D such that the quadratic representation of $\boldsymbol{\phi}$ on ∂D is satisfied, where $\boldsymbol{\phi}$ is given in (IV16.13).

We introduce polar coordinates around a point x, where now first for simplicity assume that $x \in D$. Thus we have

$$y = x + r\xi \text{ with } r > 0 \text{ and } \xi \in \mathbb{S}^2.$$

If we assume that D is convex, then for given $x \in D$ there is a radius $r_x(\xi) > 0$ with

$$D = \{x + r\xi\,;\ \xi \in \mathbb{S}^2,\, 0 \le r < r_x(\xi)\}$$

and, due to $|x-y| = r$ and $\mathrm{d}y = r^2\,\mathrm{d}r\,\mathrm{dH}^2(\xi)$, it is

$$\begin{aligned}\boldsymbol{\phi}(x) &= \frac{\varrho_0}{4\pi}\int_D \frac{\mathrm{d}y}{|x-y|} = \frac{\varrho_0}{4\pi}\int_{\mathbb{S}^2}\int_0^{r_x(\xi)} r\,\mathrm{d}r\,\mathrm{dH}^2(\xi)\\ &= \frac{\varrho_0}{8\pi}\int_{\mathbb{S}^2} r_x(\xi)^2\,\mathrm{dH}^2(\xi)\,.\end{aligned}$$

Let be now D an axis-parallel ellipsoid, that is

$$D = \{y\in\mathbb{R}^3\,;\ \sum_{i=1}^3 a_i|y_i|^2 < 1\}$$

with positive $a_i > 0$. If we set $y_i = x_i + r\xi_i$ with $r = r_x(\xi)$, i.e. $y\in\partial D$, then it is

$$\begin{aligned}1 &= \sum_{i=1}^3 a_i|y_i|^2 = \sum_{i=1}^3 a_i|x_i + r\xi_i|^2\\ &= \sum_{i=1}^3 a_i|x_i|^2 + \sum_{i=1}^3 a_i\big(2r\xi_i x_i + r^2|\xi_i|^2\big)\,,\end{aligned}$$

hence

$$r_x(\xi)^2\underbrace{\sum_{i=1}^3 a_i|\xi_i|^2}_{=:\,a_0(\xi)} + 2r_x(\xi)\sum_{i=1}^3 a_i\xi_i x_i = 1 - \sum_{i=1}^3 a_i|x_i|^2 > 0\,.$$

We obtain

$$r_x(\xi) = -\frac{\sum_{i=1}^3 a_i\xi_i x_i}{a_0(\xi)} + \sqrt{\frac{1-\sum_{i=1}^3 a_i|x_i|^2}{a_0(\xi)} + \left(\frac{\sum_{i=1}^3 a_i\xi_i x_i}{a_0(\xi)}\right)^2}\,.$$

Now, if x approches a boundary point, that it in the limit $x\in\partial D$ and hence $\sum_{i=1}^3 a_i|x_i|^2 = 1$, then it follows

$$r_x(\xi) = \begin{cases} -\dfrac{2\sum_{i=1}^3 a_i\xi_i x_i}{a_0(\xi)} & \text{for } \sum_{i=1}^3 a_i\xi_i x_i < 0,\\ 0 & \text{elsewhere}\,.\end{cases}$$

We now plug $r_x(\xi)$ into the above integral, then we obtain the desired quadratic representation

$$\begin{aligned}\boldsymbol{\phi}(x) &= \frac{\varrho_0}{8\pi}\int_{\mathbb{S}^2} r_x(\xi)^2\,\mathrm{dH}^2(\xi)\\ &= \frac{\varrho_0}{2\pi}\sum_{i,j=1}^3 x_i x_j\int_{\{\xi\in\mathbb{S}^2\,;\ \sum_i a_i\xi_i x_i<0\}}\frac{a_i a_j\xi_i\xi_j}{a_0(\xi)^2}\,\mathrm{dH}^2(\xi)\\ &= \frac{\varrho_0}{4\pi}\sum_{i,j=1}^3 x_i x_j\int_{\mathbb{S}^2}\frac{a_i a_j\xi_i\xi_j}{a_0(\xi)^2}\,\mathrm{dH}^2(\xi)\end{aligned}$$

(we split the integration area into two halves $\{\xi\,;\,\sum_i a_i\xi_i x_i < 0\}$ and $\{\xi\,;\,\sum_i a_i\xi_i x_i > 0\}$ and we map the first to the second with $\xi \rightsquigarrow -\xi$, and since $(-\xi_i)(-\xi_j) = \xi_i\xi_j$ the integrand stays the same). Since $a_0(\xi)$ depends only on $|\xi_i|^2$, $i = 1, 2, 3$, it is

$$\int_{\mathbb{S}^2} \frac{a_i a_j \xi_i \xi_j}{a_0(\xi)^2}\,\mathrm{dH}^2(\xi) = \delta_{ij} a_i \underbrace{\int_{\mathbb{S}^2} \frac{a_i \xi_i^2}{a_0(\xi)^2}\,\mathrm{dH}^2(\xi)}_{=:\, b_i > 0}$$

(we split the integration area for $i \neq j$ into two halves and these we map to each other such that $\xi_i \rightsquigarrow -\xi_i$, that is, is mapped into its negative, while ξ_j remains the same value.) Thus we obtain

$$\phi(x) = \frac{\varrho_0}{4\pi} \sum_{i,j=1}^{3} x_i x_j \underbrace{\int_{\mathbb{S}^2} \frac{a_i a_j \xi_i \xi_j}{a_0(\xi)^2}\,\mathrm{dH}^2(\xi)}_{=\,\delta_{ij} a_i b_i} = \frac{\varrho_0}{4\pi} \sum_{i=1}^{3} b_i a_i x_i^2 .$$

Since x is a boundary point, i.e.

$$a_3 x_3^2 = 1 - \sum_{i=1,2} a_i x_i^2 ,$$

this representation becomes

$$\begin{aligned}\phi(x) &= \frac{\varrho_0}{4\pi}\Big(b_3\Big(1 - \sum_{i=1,2} a_i x_i^2\Big) + \sum_{i=1,2} b_i a_i x_i^2\Big) \\ &= \frac{\varrho_0}{4\pi}\Big(b_3 - \sum_{i=1,2}(b_3 - b_i) a_i x_i^2\Big)\end{aligned}$$

for $x \in \partial D$. Now we observe that if $a_1 = a_2 < a_3$ then

$$b_1 = b_2 < b_3 .$$

(For the proof of $b_1 = b_2$ consider in the integrals the map $(\xi_1, \xi_2) \rightsquigarrow (\xi_2, \xi_1)$. For the proof of $b_1 < b_3$ consider in the integrals the map $(\xi_1, \xi_3) \rightsquigarrow (\xi_3, \xi_1)$ and use the inequalities $\frac{a_3}{a_1} > 1 > \frac{a_1}{a_3}$.) We thus obtain

$$\phi(x) = \frac{\varrho_0 b_3}{4\pi} - \frac{\varrho_0 b}{4\pi} \sum_{i=1,2} x_i^2 , \quad b := (b_3 - b_1) a_1 = (b_3 - b_2) a_2 ,$$

which is the formula (IV16.14) if

$$\phi_0 = \frac{\varrho_0 b_3}{4\pi} \quad \text{and} \quad \frac{\omega^2}{2\mathfrak{g}} = \frac{\varrho_0 b}{4\pi} .$$

And

$$\begin{aligned}&2ba = B\big(\frac{c}{a}\big) , \quad a_3 = \frac{1}{c} > \frac{1}{a} = a_1 = a_2 , \quad z = \frac{c}{a} , \\ &B(z) = 2(b_3 - b_1) = 2(b_3 - b_2) = 2b_3 - (b_1 + b_2) \\ &= \int_{\mathbb{S}^2} \Big(\frac{2\xi_3^2}{z(\xi_1^2 + \xi_2^2) + \xi_3^2} - \frac{z(\xi_1^2 + \xi_2^2)}{z(\xi_1^2 + \xi_2^2) + \xi_3^2}\Big)\,\mathrm{dH}^2(\xi) ,\end{aligned}$$

which gives the assertion, since $\mathfrak{g} = 4\pi G$. □

If the force in the momentum equation in (IV16.8), which is self-gravitation, would change a little bit then this results in a different shape of the planet. This is because, for example, other objects through their gravity field lead to a perturbation of the situation, and therefore the physics is different. For example, this is definitely true for the moon Io, which is perturbed by the gravity of Jupiter. If however one assumes that one is not in an inertial frame, the situation is different. One can imagine a rotating system around the x_3-axis. This would leave the above gravity term unchanged, but introduce an additional Coriolis force, and therefore the equations in (IV16.8) would have an additional force term, which is a fictitious force. The physics would of course be the same.

Chandrasekhar's compressible stars

We treat again the stationary case (IV16.8) where the trace of the star is $\Omega = \mathbb{R} \times D$. But now we restrict ourselves to the case that D is a ball and $v = 0$. As in (IV16.8), now ϱ is arbitrary, but it is again $S = 0$.

Reference: We refer to the book by Chandrasekhar [28], which is a thorough study of the self-gravity of stars. In particular, for this section [28, IV. Polytropic and isothermal gas spheres] is relevant. We also refer to the original book [105] of Robert Emden. Also look at the exercise [21, 7 Gravitation of a rotationally symmetric star].

As before, if the outer region $D' = \mathbb{R}^3 \setminus \overline{D}$ is connected, one can conclude that $p' = \text{const}$. Therefore one can subtract the constant $p'\text{Id}$ in the momentum flux. Then equations (IV16.8) read

$$\begin{aligned} &\operatorname{div}[-\nabla\boldsymbol{\phi}] = [\varrho\mathcal{X}_D] \text{ with } \boldsymbol{\phi}(t,x) \to 0 \text{ for } |x| \to \infty, \\ &\operatorname{div}[\mathcal{X}_D(p - p')\text{Id}] = \mathfrak{g}[\varrho\mathcal{X}_D\nabla\boldsymbol{\phi}]\,. \end{aligned} \tag{IV16.15}$$

The system (IV16.15) consists of two equations, the gravity equation and the stationary momentum equation. The gravity equation has the solution

$$\boldsymbol{\phi}(x) = \frac{1}{4\pi}\int_D \frac{\varrho(y)\,\mathrm{d}y}{|x-y|}\,, \tag{IV16.16}$$

and the momentum equation in (IV16.15) is equivalent to

$$\begin{aligned} &p = p' \text{ on } \partial D\,, \\ &\nabla p = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} \text{ in } D\,. \end{aligned} \tag{IV16.17}$$

We assume the following.

16.6 Assumptions and lemma.

(1) We assume with $\widehat{p}\colon [0,\infty[\to \mathbb{R}$ that the constitutive equation $p = \widehat{p}(\varrho)$ is satisfied, and that $p'_{\varrho} > 0$. Since $p' = \text{const}$ in D' this implies that there exists a unique $\varrho' \in \mathbb{R}$ with $\varrho' \geq 0$ that $\widehat{p}(\varrho') = p'$.

(2) From the momentum equation it follows $\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(\varrho)$, see (IV16.18), and therefore $\boldsymbol{\phi} = \boldsymbol{\phi}' := \widehat{\boldsymbol{\phi}}(\varrho')$ on ∂D. *Attention:* This follows since $v = 0$ and since besides self-gravitation there are no forces or fictitious forces.

Proof of (2). It follows from the momentum equation (IV16.17) that inside D

$$\nabla\boldsymbol{\phi} = \frac{1}{\mathfrak{g}\varrho}\nabla p(\varrho) = \frac{p'_{\varrho}(\varrho)}{\mathfrak{g}\varrho}\nabla\varrho = \nabla\Big(\frac{1}{\mathfrak{g}}\int_{\varrho_0}^{\varrho}\frac{p'_{\varrho}(s)\,\mathrm{d}s}{s}\Big)$$

for any positive constant ϱ_0. Since D is connected there is a constant c_0 with

$$\boldsymbol{\phi} = c_0 + \frac{1}{\mathfrak{g}}\int_{\varrho_0}^{\varrho}\frac{p'_{\varrho}(s)\,\mathrm{d}s}{s} =: \widehat{\boldsymbol{\phi}}(\varrho) \text{ in } D \tag{IV16.18}$$

Consequently $\boldsymbol{\phi} = \widehat{\boldsymbol{\phi}}(\varrho') =: \boldsymbol{\phi}'$ on ∂D. □

In this case, the gravity potential can be determined using the momentum equation (IV16.17) in D, and the gravity equation $\operatorname{div}(-\nabla\boldsymbol{\phi}) = \varrho$ in the domain D, which is a ball, is an elliptic equation in the variable ϱ.

16.7 Elliptic problem. Let $D = \mathrm{B}_R(0)$. Then the isothermal stationary gravity problem is an elliptic boundary value problem: Find a solution ϱ of

$$\begin{aligned} -\operatorname{div}\Big(\frac{1}{\varrho}\nabla\big(p(\varrho)\big)\Big) &= \mathfrak{g}\varrho \text{ in } D, \\ \varrho &= \varrho' \text{ on } \partial D. \end{aligned} \tag{IV16.19}$$

Such a solution does not have to exist. If a rotationally symmetric solution ϱ exists, the desired gravity potential $\boldsymbol{\phi}$ is given by the equation (IV16.18).

Proof. By (IV16.17) the momentum equation implies

$$\nabla\boldsymbol{\phi} = \tfrac{1}{\mathfrak{g}\varrho}\nabla(p(\varrho)) \text{ in } D, \tag{IV16.20}$$

Inserting this in the gravity equation of (IV16.8) gives

$$\varrho = -\operatorname{div}(\nabla\boldsymbol{\phi}) = -\operatorname{div}\Big(\frac{1}{\mathfrak{g}\varrho}\nabla\big(p(\varrho)\big)\Big) \text{ in } D\,,$$

which is the differential equation (IV16.19) in ϱ. □

Proof of ellipticity. The differential equation is of the form

$$-\operatorname{div}\big(a(\varrho)\nabla\varrho\big) + f(\varrho) = 0\,,$$

where

$$a(\varrho) = \frac{p'_{\varrho}(\varrho)}{\varrho} > 0$$

by assumption. Since

$$f(\varrho) = -\mathfrak{g}\varrho$$

is monotonically decreasing in ϱ, the existence is not assured. (The theory of monotone operators[10] apply only if f would be monotonically nondecreasing or would satisfy a one-sided smallness condition.) $\square$

Hence the right-hand side $\mathfrak{g}\varrho$ has a sign which does not imply a priori the existence of a solution. This is also physically clear, since the problem involves a self-gravitation, and this self-gravitation may cause the fact that the star compresses itself to a point (with a non-stationary solution). We first examine several models with different pressure functions

$$p = K\varrho^{\gamma}\,, \quad K = \text{const} \tag{IV16.21}$$

for which we search a solution with bounded ϱ in the domain D. To get an impression of this constitutive equation, see definition 16.13 and (IV16.33), and the statement 16.14 on polytropic stars.

16.8 Lane-Emden Gleichung. Es sei

$$\widehat{p}(\varrho) = K\varrho^{\gamma} \text{ mit Konstanten } \gamma > 1 \text{ und } K > 0.$$

(1) Dann lautet das elliptische Problem in (IV16.19)

$$-\operatorname{div}\Big(\frac{K\gamma}{\varrho^{2-\gamma}}\nabla\varrho\Big) = \mathfrak{g}\varrho \text{ in } D,$$
$$\varrho = \varrho' \text{ auf } \partial D.$$

(2) Mit $u := \varrho^{\gamma-1}$ ist dies

$$-\Delta u = c_{\gamma}u^{\frac{1}{\gamma-1}} \text{ in } D, \quad c_{\gamma} := \big(1 - \frac{1}{\gamma}\big)\frac{\mathfrak{g}}{K}\,,$$
$$u = u_0 := (\varrho')^{\gamma-1} \text{ auf } \partial D.$$

Proof (1). $\frac{1}{\varrho}\nabla\big(p(\varrho)\big) = \frac{K}{\varrho}\nabla(\varrho^{\gamma}) = \frac{K\gamma}{\varrho^{2-\gamma}}\nabla\varrho.$ $\square$

Proof (2). $\frac{\gamma}{\varrho^{2-\gamma}}\nabla\varrho = \frac{\gamma}{\gamma-1}\nabla(\varrho^{\gamma-1}) = \frac{\gamma}{\gamma-1}\nabla u.$ $\square$

[10] See e.g. [H. Brézis: *Operateur maximaux monotones*, Theoreme 2.3]

Dieser Fall wird ausführlich in dem Kapitel des Buches [28, IV. Polytropic and isothermal gas spheres] von Chandrasekhar behandelt. Dabei ist $n \in \mathbb{R}$ mit $0 < n < \infty$ definiert durch

$$n := \frac{1}{\gamma - 1} \text{ also } \gamma = 1 + \frac{1}{n} \text{ so dass}$$
$$\gamma \leq 2 \Longleftrightarrow n \geq 1 \quad \text{und} \quad n \leq 5 \Longleftrightarrow \gamma \geq \frac{6}{5}\,.$$

16.9 Lane-Emden function. A solution s of the nonlinear equation

$$-\Delta s = 3as^n \quad \text{with} \quad s(0) = 1\,,$$

it is called ***Lane-Emden function of index n*** $\in \mathbb{R}$ with $a = \text{const} > 0$. We call it $s_n := s$.

If u solves 16.8(2) then the Lane-Emden function with index n is

$$s_n := \frac{u}{u(0)} \quad \text{and} \quad 3a = c_\gamma u(0)^{n-1}\,, \quad n(\gamma - 1) = 1\,.$$

The normalization $s_n = 1$ only changes the coefficient of the differential equation. If one determines s_n numerically one sees that for $n < 5$ the value R at which s_n attains it's first zero is finite, whereas s_5 has an unbounded support (see [28, IV.5 The Lane-Emden function for general n] and Fig. 36). The function s_5 is known explicitely.

16.10 Schuster-Emden Lösung. Für $a > 0$ ist

$$s_5(x) := \frac{1}{(1 + a|x|^2)^{\frac{1}{2}}}$$

die Lösung der Lane-Emden Gleichung für den Index $n = 5$. Die Lösung verschwindet im Unendlichen, siehe dazu Fig. 36.

Proof. Ausrechnen. □

Mit dieser Schuster-Emden Lösung lässt sich jetzt mit Hilfe von Variationsungleichungen die allgemeine Lösung konstruieren, und zwar im Falle $1 \leq n < 5$.

16.11 Theorem. Let $1 \leq n < 5$ and $D = \mathrm{B}_R(0)$. We assume that in 16.8(2) the boundary data u_0 on ∂D satisfy

$$u_0^{n-1} \leq \frac{3(n+1)K}{4\mathfrak{g}R^2}\,. \tag{IV16.22}$$

Then there is a solution u of the Lane-Emden equation 16.8.

TABLE 4

THE CONSTANTS OF THE LANE-EMDEN FUNCTIONS*

n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar\rho$	$\omega_n = -\xi_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n	W_n	$-\frac{1}{(n+1)\xi_1\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}}$
0.	2.4494	4.8988	1.0000	0.33333	....	0.119366	0.5
0.5	2.7528	3.7871	1.8361	0.02156	2.270	0.26227	0.53847
1.0	3.14159	3.14159	3.28987	....	0.63662	0.392699	0.5
1.5	3.65375	2.71406	5.99071	132.3843	0.42422	0.770140	0.53849
2.0	4.35287	2.41105	11.40254	10.4950	0.36475	1.63818	0.60180
2.5	5.35528	2.18720	23.40646	3.82662	0.35150	3.90906	0.69956
3.0	6.89685	2.01824	54.1825	2.01824	0.36394	11.05066	0.85432
3.25	8.01894	1.94980	88.153	1.54716	0.37898	20.365	0.96769
3.5	9.53581	1.89056	152.884	1.20426	0.40104	40.9098	1.12087
4.0	14.97155	1.79723	622.408	0.729202	0.47720	247.558	1.66606
4.5	31.83646	1.73780	6189.47	0.394356	0.65798	4922.125	3.33100
4.9	169.47	1.7355	934800	0.14239	1.340	3.693×10^6	16.550
5.0	∞	1.73205	∞	0	∞	∞	∞

* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

Fig. 36: The radius of the Lane-Emden functions [28]

Hinweis: Für die Randwerte $\varrho' \geq 0$ der Dichte ϱ heißt dies

$$0 \leq (\varrho')^{2-\gamma} \leq \frac{3(n+1)K}{4\mathfrak{g}R^2}, \quad \gamma \leq 2.$$

Proof for $u_0 > 0$. Let u_0 be the Dirichlet condition of u on $\partial\mathrm{B}_R(0)$ and assume this to be positive. Then

$$\overline{u} := \frac{u}{u_0}$$

satisfies

$$-\Delta\overline{u} = c_\gamma {u_0}^{-1} u^n = c_\gamma {u_0}^{n-1}\overline{u}^n = \overline{c}\,\overline{u}^n, \quad \overline{c} = c_\gamma {u_0}^{n-1}.$$

Hence $\overline{u}$ solves the boundary value problem

$$\begin{aligned} -\Delta\overline{u} &= \overline{c}\,\overline{u}^n \text{ in } \mathrm{B}_R(0), \\ \overline{u} &:= 1 \text{ on } \partial\mathrm{B}_R(0). \end{aligned} \tag{IV16.23}$$

We compare it with the function

$$s := \left(1 + aR^2\right)^{\frac{1}{2}} s_5 \geq 1 \text{ on } \overline{\mathrm{B}_R(0)},$$

which satisfies

$$-\Delta s = \frac{3a}{\left(1 + aR^2\right)^2} s^5$$

with the same boundary data $s = 1$ on $\partial B_R(0)$. We now make the assumption that

$$\overline{c} < \frac{3a}{\left(1+aR^2\right)^2} \,. \tag{IV16.24}$$

If we further have, purely formal, a solution $\overline{u}$ of (IV16.23) which satisfies $1 \le \overline{u} \le s$ then

$$s^5 \ge s^n \ge \overline{u}^n \tag{IV16.25}$$

and with (IV16.24)

$$-\Delta s = \frac{3a}{\left(1+aR^2\right)^2} s^5 > \overline{c}s^5 \ge \overline{c}\,\overline{u}^n = -\Delta\overline{u}\,.$$

Therefore $\Delta(\overline{u} - s) > 0$, which means that $\overline{u} - s$ is nonpositive and subharmonic, therefore it cannot reach the value 0 in the domain D, hence $\overline{u}$ does not touch the upper obstacle.

To make this rigorous let $\overline{u}$ be the solution of the obstacle problem[11]

$$\begin{aligned}
&\overline{u} \in \mathscr{K} \text{ and } \mathscr{E}(\overline{u}) \le \mathscr{E}(v) \text{ for all } v \in \mathscr{K}\,, \text{ where}\\
&\mathscr{E}(v) := \int_D \Big(\frac{1}{2}|\nabla v|^2 - \frac{\overline{c}}{n+1} v^{n+1}\Big)\,\mathrm{dL}^3\,,\\
&\mathscr{K} := \{v \in W^{1,2}(D)\,;\ 0 \le v \le s \text{ in } D,\ v = 1 \text{ on } \partial D\}\,.
\end{aligned}$$

The solution of this minimum problem is radially symmetric, positive and continuous and satisfies for all $v \in \mathscr{K}$

$$\int_D \big(\nabla(\overline{u} - v)\bullet\nabla\overline{u} - \overline{c}\,\overline{u}^n(\overline{u} - v)\big)\,\mathrm{dL}^3 \le 0\,. \tag{IV16.26}$$

We show that $\overline{u}$ cannot touch the upper obstacle. Let $\zeta \in C_0^\infty(D)$ be a nonnegative test function, then we can take $v = \overline{u} - \zeta$ and obtain

$$\begin{aligned}
0 &\ge \int_D \big(\nabla\zeta\bullet\nabla\overline{u} - \overline{c}\,\overline{u}^n\zeta\big)\,\mathrm{dL}^3\\
&= \int_D \big(\nabla\zeta\bullet\nabla(\overline{u} - s) + \zeta(\underbrace{-\Delta s - \overline{c}\,\overline{u}^n}_{=:\,\lambda})\big)\,\mathrm{dL}^3\\
&= \int_D \big(\nabla\zeta\bullet\nabla(\overline{u} - s) + \lambda\zeta\big)\,\mathrm{dL}^3\,.
\end{aligned}$$

Since $s \ge 1$ and $\overline{u} \le s$ the inequalities (IV16.25), that is $s^5 \ge s^n \ge \overline{u}^n$, hold and therefore because of the assumption (IV16.24)

$$\lambda = -\Delta s - \overline{c}\,\overline{u}^n = \frac{3a}{\left(1+aR^2\right)^2} s^5 - \overline{c}\,\overline{u}^n > \overline{c}(s^5 - \overline{u}^n) \ge 0$$

[11] see Theorem 2.1 of [D. Kinderlehrer, G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications.* SIAM's Classics in Applied Mathematics 31 (2000)]

that is $\lambda > 0$ is a positive function. Hence the nonpositve function $\overline{u} - s$ is a weak solution of

$$-\Delta(\overline{u} - s) + \lambda \leq 0 \quad \text{therefore} \quad \Delta(\overline{u} - s) \geq \lambda > 0\,.$$

Then the mean value property implies that pointwise $\overline{u}(x) < s(x)$. With this it follows from the variational inequality (IV16.26) that $\overline{u}$ is a solution of (IV16.23), that is, $u = \overline{u}u_0$ is a solution of 16.8(2).

If we choose in (IV16.24) the optimal $a = R^{-2}$ we obtain

$$c_\gamma u_0^{n+1} < \frac{3}{4R^2}$$

as stated in the theorem. □

Hence the planet allows a stationary solution if it has a constitutive relation $p = K\varrho^\gamma$, $\gamma = 1 + \frac{1}{n}$, with $n < 5$, i.e. $\gamma > \frac{6}{5}$, and the mass density at the boundary of the planet is 0.

Polytropic gas clouds

We mention that this solution does not carry heat to the outside provided the star is a gas cloud, that is, in the section about fluids and gases the properties in 2.5 are satisfied, in particular for the internal energy

$$\varepsilon = c_V \theta \varrho\,. \tag{IV16.27}$$

Why is this true? In the general instationary case the equations (IV16.4) in the gas cloud Ω are (see 16.12)

$$\begin{aligned}
&-\Delta\boldsymbol{\phi} = \varrho\,,\\
&\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,,\\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = \varrho\nabla\boldsymbol{\phi}\,,\\
&\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) + p\operatorname{div}v = 0\,.
\end{aligned} \tag{IV16.28}$$

The entropy principle in III.2.4 or 2.2 says $\eta = \widehat{\eta}(\varrho, \varepsilon)$ with

$$\partial_t\eta + \operatorname{div}(\eta v + \frac{1}{\theta}q) = \sigma = \nabla\Big(\frac{1}{\theta}\Big)\bullet q \geq 0\,,$$

and that the Gibbs relation for p is satisfied, that is,

$$\begin{aligned}
&0 = \varrho\eta'_\varrho - \eta + \frac{1}{\theta}(\varepsilon + p) \quad \text{or}\\
&p + \varepsilon + \theta\varrho^2\eta^{\mathrm{sp}}{}'_\varrho(\varrho, \varepsilon) = 0 \quad \text{or}\\
&p = \varrho f'_\varrho(\varrho, \theta) - f(\varrho, \theta)\,, \quad f = \varepsilon - \theta\eta\,.
\end{aligned}$$

Now, we can replace (see 16.12) the energy equation in (IV16.28) in the domain Ω, that is, inside the gaseous star, by the entropy equation and obtain the system

$$\begin{aligned}
&-\Delta\boldsymbol{\phi} = \varrho\,, \\
&\partial_t\varrho + \mathrm{div}(\varrho v) = 0\,, \\
&\partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = \varrho\nabla\boldsymbol{\phi}\,, \\
&\partial_t\eta + \mathrm{div}(\eta v) + \tfrac{1}{\theta}\,\mathrm{div}q = 0\,,
\end{aligned} \tag{IV16.29}$$

again with the Gibbs relation and with q satisfying $\nabla\big(\frac{1}{\theta}\big){\bullet}q \geq 0$.

16.12 Theorem. In the general case the system (IV16.4) inside Ω is equivalent to (IV16.28) and also equivalent to (IV16.29).

Proof of (IV16.28). The energy equation inside Ω is

$$\partial_t e + \mathrm{div}\,(ev + pv + q) = v{\bullet}\mathbf{f}\,,$$

and this is modulo the mass and momentum equation as usual, using III.2.2, equivalent to the equation for ε. □

Proof of (IV16.29). We perform the method of Liu & Müller explained in section III.4 and use the fact that we can write $\eta = \widehat{\eta}(\varrho,\varepsilon)$. We obtain the following for all functions satisfying the Gibbs relation

$$\begin{aligned}
&\partial_t\eta + \mathrm{div}(\eta v + \psi_0) - \sigma \qquad \big(\ \psi_0 = \tfrac{1}{\theta}q \text{ and } \sigma = \nabla\big(\tfrac{1}{\theta}\big){\bullet}q\ \big) \\
&= (\partial_t + v{\bullet}\nabla)\widehat{\eta}(\varrho,\varepsilon) + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 - \sigma \\
&= \eta'_{\varrho}(\partial_t + v{\bullet}\nabla)\varrho + \eta'_{\varepsilon}(\partial_t + v{\bullet}\nabla)\varepsilon + \eta\,\mathrm{div}v + \mathrm{div}\psi_0 - \sigma \\
&= \eta'_{\varrho}\big(\partial_t\varrho + v{\bullet}\nabla\varrho + \varrho\,\mathrm{div}v\big) + \eta'_{\varepsilon}\big(\partial_t\varepsilon + v{\bullet}\nabla\varepsilon + \varepsilon\,\mathrm{div}v + \mathrm{div}q + p\,\mathrm{div}v\big) \\
&\qquad\qquad + \mathrm{div}v\cdot\big(\eta - \varrho\eta'_{\varrho} - \varepsilon\eta'_{\varepsilon} - \eta'_{\varepsilon}p\big) - \eta'_{\varepsilon}\,\mathrm{div}q + \mathrm{div}\psi_0 - \sigma \\
&= \eta'_{\varrho}\big(\partial_t\varrho + \mathrm{div}(\varrho v)\big) + \eta'_{\varepsilon}\big(\partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) + p\,\mathrm{div}v\big) \\
&\qquad + \mathrm{div}v\cdot\underbrace{\big(\eta - \varrho\eta'_{\varrho} - (\varepsilon + p)\eta'_{\varepsilon}\big)}_{=0\ \text{(Gibbs relation)}} + \underbrace{\nabla\eta'_{\varepsilon}{\bullet}q - \sigma}_{=0} + \mathrm{div}\,\underbrace{(\psi_0 - \eta'_{\varepsilon}q)}_{=0} \\
&= \eta'_{\varrho}\big(\partial_t\varrho + \mathrm{div}(\varrho v)\big) + \eta'_{\varepsilon}\big(\partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) + p\,\mathrm{div}v\big)\,,
\end{aligned}$$

if we define (siehe Abschnitt 11)

$$\eta'_{\varepsilon} = \frac{1}{\theta}\,, \quad \eta'_{\varrho} = \frac{\mu}{\theta}\,.$$

Thus we have just repeated the calculation of the proof of the entropy inequality, but now for a larger class. It gives that

$$\begin{aligned}
&\partial_t\varepsilon + \mathrm{div}(\varepsilon v + q) + p\,\mathrm{div}v \\
&= \theta\big(\partial_t\eta + \mathrm{div}(\eta v + \psi_0) - \sigma\big) - \mu\big(\partial_t\varrho + \mathrm{div}(\varrho v)\big)
\end{aligned}$$

for all functions satisfying the Gibbs relation. Thus, since $\theta > 0$, we can replace in system (IV16.28) the equation $\partial_t\varepsilon + \operatorname{div}(\varepsilon v + q) + p\operatorname{div}v = 0$ by

$$0 = \partial_t\eta + \operatorname{div}(\eta v + \psi_0) - \sigma = \partial_t\eta + \operatorname{div}(\eta v) + \frac{1}{\theta}\operatorname{div}q$$

and obtain (IV16.29). □

16.13 Adiabatic processes (Definition). A physical process is called ***adiabatic***, if the heat flux $q = 0$. *Addendum:* If $q = -k(\varrho,\theta)\nabla\theta$ (then the entropy principle is satisfied, if $k \geq 0$) this is the case, if the heat capacity $k = 0$.

$\varepsilon = c_V\theta\varrho$!

Therefore, if we have an adiabatic process then $q = 0$ and the entropy equation in (IV16.29) becomes

$$\begin{aligned} 0 &= \partial_t\eta + \operatorname{div}(\eta v) = \partial_t(\varrho\eta^{\mathrm{sp}}) + \operatorname{div}(\varrho\eta^{\mathrm{sp}}v) \\ &= \eta^{\mathrm{sp}}\big(\partial_t\varrho + \operatorname{div}(\varrho v)\big) + \varrho\big(\partial_t\eta^{\mathrm{sp}} + v\bullet\nabla\eta^{\mathrm{sp}}\big) = \varrho(\partial_t + v\bullet\nabla)\eta^{\mathrm{sp}}\,, \end{aligned}$$

hence $(\partial_t + v\bullet\nabla)\eta^{\mathrm{sp}} = 0$. Since η^{sp} does not depend on v it is plausible to assume the stronger condition that $(\partial_t, \nabla)\eta^{\mathrm{sp}} = 0$ and therefore, since Ω is connected,

$$\eta^{\mathrm{sp}} = \text{const.} \tag{IV16.30}$$

Therefore this condition implies the entropy equality. Thus

$$\begin{aligned} &-\Delta\boldsymbol{\phi} = \varrho\,, \\ &\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,, \\ &\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + p\mathrm{Id}) = \varrho\nabla\boldsymbol{\phi}\,, \\ &\eta^{\mathrm{sp}} = \text{const} \end{aligned} \tag{IV16.31}$$

are the equations we assume. And that η^{sp} is constant means by 2.5(3)

$$\begin{aligned} \text{const} &= \eta^{\mathrm{sp}} = c_V\log\varepsilon - c_P\log\varrho + \text{const} \\ &= c_V\big(\log\varepsilon - \gamma\log\varrho\big) + \text{const} = c_V\log\frac{\varepsilon}{\varrho^\gamma} + \text{const}\,, \end{aligned}$$

that is, there exists a constant $c_\varepsilon \in \mathbb{R}$ with $\varepsilon = c_\varepsilon\varrho^\gamma$ hence $c_V\theta\varrho = \varepsilon = c_\varepsilon\varrho^\gamma$ or

$$\theta = \theta_\gamma(\varrho) := \frac{c_\varepsilon}{c_V}\varrho^{\gamma-1}\,, \quad \gamma = \frac{c_P}{c_V}\,. \tag{IV16.32}$$

Then 2.5(2) says, it is $R = c_p - c_v$,

$$p = R\theta\varrho = \tfrac{Rc_\varepsilon}{c_V}\varrho^\gamma = c_\varepsilon(\gamma-1)\varrho^\gamma \tag{IV16.33}$$

a constitutive equation, which was assumed also in (IV16.21). Hence, the solutions which were found with this assumption on the pressure apply for ideal gases and they are adiabatic solutions.

16.14 Polytropic processes (Definition). A process is defined ***polytropic***, see [28, II Physical principles] specially [28, II.3 Polytropic changes], if

$$\eta^{\text{sp}} = c \log \theta \,.$$

Then there exists a constant c_ε such that

$$p = c_\varepsilon(\gamma' - 1)\varrho^{\gamma'} \quad \text{with} \quad \gamma' := \frac{c_P - c}{c_V - c} \quad \Big(\text{instead of } \gamma = \frac{c_P}{c_V}\Big) \,.$$

$\varepsilon = c_V \theta \varrho$!

Proof. It is by assumption, since $\varepsilon = c_V \theta \varrho$,

$$\eta^{\text{sp}} = c \log \theta = c(\log \varepsilon - \log \varrho) + \text{const}$$

and by 2.5(3), since $p = R\theta\varrho$,

$$\eta^{\text{sp}} = c_V \log \varepsilon - c_P \log \varrho + \text{const}$$

hence

$$\text{const} = (c_V - c) \log \varepsilon - (c_P - c) \log \varrho = (c_V - c) \log \frac{\varepsilon}{\varrho^{\gamma'}}$$

which gives the assertion. □

The question arises whether the stationary solutions presented so far are stable or unstable with respect to the general system (IV16.29). This question can be answered if one discretizes the instationary system and proceeds with numerical computation.

If one wants to study more about stars under self-gravitation, one has to look at other physical principles. This is because the material is compressed so much that chemical reactions will occur and this will be the source of radiation.

17 LiquidCrystals

– Dieser Abschnitt folgt noch —

18 Exercises

18.1 Spezifische freie Energie. Sei

$$g(\vec{c}) := \sum_{k=1}^{M} c_k(\log c_k - 1)$$

für $\vec{c} = (c_k)_k \in \mathbb{R}^M$ with $c_k > 0$ for $k = 1, \ldots, M$. Zeige:

$$(\sum_{k=1}^{M} c_k g_{\,'c_k}) - g = \sum_{k=1}^{M} c_k \,.$$

18.2 Diffusionskoeffizienten. Sei

$$\sum_k \mathbf{J}_k = 0 \quad \text{and} \quad \sum_k c_k = 1\,.$$

Zeige: Sind für beliebige Funktionen c_k mit der angegebenen Nebenbedingung $\mathbf{J}_k = -\sum_l d_{kl} \nabla c_l$, so folgt

$$\sum_k d_{kl} = D \text{ für alle } l$$

mit einer Funktion D.

V Higher moments

Für manche Prozesse ist die Angabe eines Systems für die Masse, den Impuls und die Energie nicht hinreichend, um die Dynamik genau genug zu beschreiben. Dies gilt insbesondere für die in der zweiten Hälfte des 20. Jahrhunderts durchgeführten Experimente, siehe den Text der Fig. 1 und Fig. 2.

However, the recent decade has witnessed a surge of interest in going beyond the classical formulation. There are several reasons for this. One of them is the development of experimental methods able to deal with the response of systems to high-frequency and short-wavelength perturbations, such as ultrasound propagation and light and neutron scattering. The observed results have led to generalizations of the classical hydrodynamical theories, by including memory functions or generalized transport coefficients depending on the frequency and the wavevector. This field has generated impressive progress in non-equilibrium statistical mechanics, but for the moment it has not brought about a parallel development in non-equilibrium thermodynamics. An extension of thermodynamics compatible with generalized hydrodynamics therefore appears to be a natural subject of research.

Fig. 1: Aus dem Buch von Jou & Casas-Vázquez & Lebon

Es sind also Modelle von mehr als den klassischen 5 Momenten erforderlich, welche die Dichte ϱ, die Geschwindigkeit v und die innere Energie ε (beziehungsweise für die Temperatur θ) darstellen. Basis für diese Theorie sind die Boltzmann-Gleichungen für Materialien aus einzelnen Molekülen, die einen gewissen auf Impuls- und Energiebilanz basierten Kollisionsmechanismus haben, also für Gase (siehe Abschnitt 2). Aus diesen Gleichungen lassen sich die üblichen Erhaltungsgleichungen extrahieren. Man erhält darüber hinaus eine ganze Hierarchie von Differentialgleichungen für höhere Momente (siehe Abschnitt 3). Ein Teil dieser Gleichungen wird als Grundlage für die Theorie genommen.

Es bleibt aber trotzdem das Entropieprinzip als grundlegende Eigenschaft bestehen. Daher wurde die Theorie auch “Extended Thermodynamics” genannt, obwohl ja nur das Entropieprinzip auf eine andere Klasse von Modellen angewandt wird (in unserer Sprache wird eine neue Matrix Z von nöten sein). Von dieser Theorie handeln die Abschnitte.

Our aim is to propose a theory which goes beyond the classical formulation of thermodynamics. This is achieved by enlarging the space of basic independent variables, through the introduction of non-equilibrium variables, such as the dissipative fluxes appearing in the balance equations. The next step is to find evolution equations for the dissipative fluxes. Whereas the evolution equations for the classical variables are given by the usual balance laws, no general criteria exist concerning the evolution equations of the dissipative fluxes, with the exception of the restrictions imposed on them by the second law of thermodynamics.

The independent character of the fluxes is made evident in high-frequency phenomena. In general, they are *fast* variables that decay to their local-equilibrium values after a short relaxation time. Whereas many authors have studied the elimination of such *fast* variables in order to obtain a description of the system in terms of slow variables, our aim is the opposite one. We want to describe phenomena at frequencies comparable to the inverse of the relaxation times of the fluxes. Therefore, at such time scales, they must be included in the set of basic independent variables.

Fig. 2: Aus dem gleichen Buch

Referenzen: Im Zusammenhang der "Extended Thermodynamics" verweisen wir auf die Bücher von Müller [88] und Jou & Casas-Vázquez & Lebon [82]. Wir verweisen weiter auf den Originalartikel von Grad [108]. Zur Boltzmann Gleichung siehe den Ausgangsartikel von Boltzmann in [98] und die Darstellungen in [Wikipedia: Boltzmann Gleichung] und andererseits in [Wikipedia: H-Theorem].

1 Cattaneo's 8-Momente Gleichung

Für manche Prozesse ist die Angabe einer Gleichung für die Masse, den Impuls und die Energie nicht hinreichend, um die Dynamik zu beschreiben. Es bleibt aber trotzdem das Entropieprinzip als grundlegende Eigenschaft bestehen. Die Entropie muss nur noch von weiteren Variablen abhängen. Wir geben hier das Beispiel von Cattaneo (1948) an. Es besteht aus der Energiebilanz, zusammen mit den Erhaltungsgleichungen für Masse und impuls, aber mit dem Unterschied, dass für den Wärmefluss keine konstitutive Gleichung wie üblich vorausgesetzt wird, sondern eine Differentialgleichung.

Referenzen: Zum Cattaneo Modell siehe den Ausgangsartikel von Cattaneo [101], den Artikel von Herrera & Falcon [45, Speziell die Gleichung (6)], and the general statements in Müller [87, 2.1 The Cattaneo Equation] and Dou & Casas-Vázquez & Lebon [82, 6.6 Heat Conduction in a Rotating Rigid Cylinder]. Recently, in [66] Straughan has applied this to porous media and used the article of Christov [30] who gave an frame indifferent version of the Cattaneo model.

Wir haben also Massen-, Impuls- und Energieerhaltung (siehe (III2.5))

$$\begin{aligned}\partial_t\varrho + \mathrm{div}(\varrho v) &= 0\,,\\ \partial_t(\varrho v) + \mathrm{div}(\varrho v\, v^{\mathrm{T}} + \Pi) &= \mathbf{f}\,,\\ \partial_t e + \mathrm{div}(ev + \Pi^{\mathrm{T}}\, v + q) &= v\bullet\mathbf{f}\,,\\ e &= \varepsilon + \tfrac{\varrho}{2}|v|^2\,,\end{aligned} \tag{V1.1}$$

und versuchen, ein Entropieprinzip herzuleiten, wobei die Entropie η nun im Gegensatz zu Abschnitt III.1 noch vom Wärmefluss q abhängt, der als unabhängige Variable betrachtet wird. Das heißt wir nehmen an, dass

$$\eta = \widehat{\eta}(\varrho, \varepsilon, q)\,. \tag{V1.2}$$

1.1 Lemma. Da η ein objektiver Skalar sein muss, hängt η in Wirklichkeit nur vom Modulus von q, d.h. von $|q|^2$, ab. Es gibt daher einen objektiven Skalar $\tau = \widehat{\tau}(\varrho, \varepsilon, |q|)$ mit

$$\eta'_q = -\tau q\,.$$

Proof. Es sind ϱ und ε objektive Skalare. Da η ein objektiver Skalar ist, muss daher die konstitutiven Funktion in (V1.2) für alle Werte Qq^* denselben Wert wie für q^* haben. Also folgt die Behauptung. □

Es soll das Entropieprinzip

$$\sigma := \partial_t\eta + \mathrm{div}\psi \geq 0$$

erfüllt sein, wobei der Entropiefluss noch zu bestimmen ist. Um die Entropieproduktion σ zu berechnen, definieren wir wie immer $\dot{g} = \partial_t g + v\bullet\nabla g$ für jede Funktion g, und schreiben die Gleichungen in (V1.1) entsprechend um und erhalten das äquivalente System

$$\begin{aligned}&\dot{\varrho} + \varrho\,\mathrm{div}v = 0\,,\\ &\varrho\dot{v} + \varrho\,\mathrm{div}v + \mathrm{div}\Pi = \mathbf{f}\,,\\ &\dot{\varepsilon} + \varepsilon\,\mathrm{div}v + \mathrm{div}q = -(\mathrm{D}v)\bullet\Pi\,.\end{aligned}$$

Nun ist in Analogie zum Beweis von III.2.4

$$\begin{aligned}\sigma &= \dot{\eta} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v)\\ &= \eta'_\varrho\dot{\varrho} + \eta'_\varepsilon\dot{\varepsilon} + \eta'_q\bullet\dot{q} + \eta\,\mathrm{div}v + \mathrm{div}(\psi - \eta v)\\ &= \mathrm{D}v\bullet\big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi\big)\\ &\quad+\eta'_q\bullet\dot{q} - \eta'_\varepsilon\,\mathrm{div}q + \mathrm{div}(\psi - \eta v)\\ &= \mathrm{D}v\bullet\big((\eta - \varrho\eta'_\varrho - \varepsilon\eta'_\varepsilon)\mathrm{Id} - \eta'_\varepsilon\Pi\big)\\ &\quad+q\bullet(\nabla\eta'_\varepsilon - \tau\dot{q}) + \mathrm{div}(\psi - \eta v - \eta'_\varepsilon q)\,.\end{aligned}$$

If we define as usual the temperature θ and the pressure p by

$$\frac{1}{\theta} = \eta'_{\varepsilon}\,, \quad \eta = \varrho\eta'_{\varrho} + (\varepsilon + p)\eta'_{\varepsilon}\,, \tag{V1.3}$$

and if we let the entropy flux

$$\psi = \eta v + \eta'_{q} q\,, \tag{V1.4}$$

then the entropy production becomes

$$\sigma = \frac{1}{\theta}\mathrm{D}v\bullet(p\mathrm{Id} - \Pi) + q\bullet\big(\nabla\big(\frac{1}{\theta}\big) - \tau\dot{q}\big)\,.$$

Das Ziel ist es, dass die Entropieproduktion nichtnegativ ist, d.h. $\sigma \geq 0$. Indem wir ausnutzen, dass

$$q\bullet(\mathrm{D}v\, q) = \sum_{ij} q_i q_j \partial_j v_i = \mathrm{D}v\bullet(q\, q^{\mathrm{T}})\,,$$

können wir die Entropieungleichung schreiben als

$$\sigma = \frac{1}{\theta}\mathrm{D}v\bullet(p\mathrm{Id} - \lambda\theta q\, q^{\mathrm{T}} - \Pi) + q\bullet\big(\nabla\big(\frac{1}{\theta}\big) + \lambda\mathrm{D}v\, q - \tau\dot{q}\big) \geq 0 \tag{V1.5}$$

mit einer beliebigen Funktion λ, was ein objektiver Skalar sein soll. Wir wollen jetzt wie üblicherweise annehmen, dass beide Terme nichtnegativ sind, d.h. es soll gelten

$$\frac{1}{\theta}\mathrm{D}v\bullet(p\mathrm{Id} - \lambda\theta q\, q^{\mathrm{T}} - \Pi) \geq 0\,,$$
$$q\bullet\big(\nabla\big(\frac{1}{\theta}\big) + \lambda\mathrm{D}v\, q - \tau\dot{q}\big) \geq 0\,.$$

Die erste Ungleichung schreiben wir wie gewohnt als

$$\mathrm{D}v\bullet S \geq 0\,, \quad \Pi := p\mathrm{Id} - S - \lambda\theta q\, q^{\mathrm{T}} \tag{V1.6}$$

mit einer objektiven Matrix S, so dass also Π wie gewünscht eine objektive Matrix ist. Hier hat Π die Standardform, falls der λ-Term verschwindet. Die zweite Ungleichung ist erfüllt, wenn

$$q\bullet Mq \geq 0\,, \quad \tau\dot{q} - \lambda\mathrm{D}v\, q = \nabla\big(\frac{1}{\theta}\big) - Mq\,. \tag{V1.7}$$

Here we have to make clear that the equalities and inequalities are objective. Therefore let us show the following.

1.2 Lemma. Die Gleichung und die Ungleichung in (V1.7) sind beobachterunabhängig, falls M die Transformationseigenschaft

$$M\circ Y = (\lambda^* - \tau^*)\dot{Q}\, Q^{\mathrm{T}} + QM^*\, Q^{\mathrm{T}}$$

We now write energy balances for the solid and fluid parts of the porous medium separately, and likewise write Cattaneo–Christov heat flux laws for each of the solid and fluid components (cf. Straughan 2008, pp. 14–15), which only deals with the classical Fourier theory. The solid and fluid parts are denoted by subscript s, f. Then we have the energy balance and Cattaneo law for the solid,

$$(\rho_0 c)_s T_{,t} = -\tilde{Q}_{i,i} \tag{2.6}$$

and

$$\tau_s \tilde{Q}_{i,t} = -\tilde{Q}_i - k_s T_{,i}, \tag{2.7}$$

where c is the specific heat, τ_s is the relaxation time, k_s is thermal conductivity and $\tilde{Q}_i$ the heat flux. For the fluid, the energy balance and Christov (2009) heat flux laws take the form

$$(\rho_0 c_p)_f (T_{,t} + V_i T_{,i}) = -\tilde{Q}_{i,i} \tag{2.8}$$

and

$$\tau_f (\tilde{Q}_{i,t} + V_j \tilde{Q}_{i,j} - \tilde{Q}_j V_{i,j}) = -\tilde{Q}_i - k_f T_{,i} \tag{2.9}$$

where c_p is the specific heat at constant pressure, k_f is the thermal conductivity and τ_f is the relaxation time. (The classical Cattaneo law is found from equation (2.9) by putting $\mathbf{V} \equiv \mathbf{0}$. If then $\tau_f = 0$, one recovers Fourier's law $\tilde{Q}_i = -k_f T_{,i}$. Equation (2.9) is an invariant form of Cattaneo law valid in a moving body and proposed by Christov (2009)).

Fig. 3: Aus dem Paper von Straughan

erfüllt, d.h. der antisymmetrische Anteil M^{A} diese Eigenschaft hat und der symmetrische Anteil M^{S} eine objektive Matrix ist:

$$\begin{aligned} M^{\mathrm{S}} \circ Y &= Q\, M^{*\mathrm{S}} Q^{\mathrm{T}} \,, \\ M^{\mathrm{A}} \circ Y &= (\lambda^* - \tau^*) \dot{Q} Q^{\mathrm{T}} + Q\, M^{*\mathrm{A}} Q^{\mathrm{T}} \,. \end{aligned}$$

Proof. Es ist v ist eine Geschwindigkeit, also $v \circ Y = \dot{X} + Qv^*$, und q ein objektiver Vektor, d.h. $q \circ Y = Qq^*$. Indem wir diese Identität ableiten, erhalten wir

$$\begin{aligned} \partial_{x_j^*}(Qq^*) &= \partial_{x_j^*}(q \circ Y) = \sum_i (\partial_{x_i} q) \circ Y Q_{ij} \,, \\ \partial_{t^*}(Qq^*) &= (\partial_t q) \circ Y + \sum_i (\partial_{x_i} q) \circ Y \dot{X}_i \,, \end{aligned}$$

also

$$\begin{aligned} \dot{q} \circ Y &= (\partial_t q) \circ Y + \sum_i v_i \circ Y (\partial_{x_i} q) \circ Y \\ &= (\partial_t q) \circ Y + \sum_i (\partial_{x_i} q) \circ Y \dot{X}_i + \sum_i (v_i \circ Y - \dot{X}_i)(\partial_{x_i} q) \circ Y \\ &= \partial_{t^*}(Qq^*) + \sum_{ij} Q_{ij} v_j^* (\partial_{x_i} q) \circ Y \\ &= \partial_{t^*}(Qq^*) + \sum_j v_j^* \partial_{x_j^*}(Qq^*) \\ &= Q\big(\partial_{t^*} q^* + \sum_j v_j^* \partial_{x_j^*} q^*\big) + \dot{Q} q^* \\ &= Q((q^*)^{\cdot}) + \dot{Q} q^* \,. \end{aligned}$$

Somit haben wir gezeigt, dass die Transformationsregel

$$\dot{q}\circ Y = Q\,(q^*)^{\cdot} + \dot{Q}q^*$$

gilt. Hier ist also $\dot{q} = (\partial_t + v\bullet\nabla_x)q$ und $\dot{q}^* = (q^*)^{\cdot} = (\partial_{t^*} + v^*\bullet\nabla_{x^*})q^*$. Da $v\circ Y = \dot{X} + Qv^*$, ist (indem wir diese Gleichung nach x^* ableiten)

$$\mathrm{D}v\circ Y = \dot{Q}\,Q^{\mathrm{T}} + Q\mathrm{D}v^*\,Q^{\mathrm{T}}\ ,$$

folglich wird (es ist τ und λ ein objektiver Skalar)

$$\begin{aligned}&(\tau\dot{q} - \lambda\mathrm{D}v\,q + Mq)\circ Y\\ &= \tau^*(Q\dot{q}^* + \dot{Q}q^*) + (M\circ Y - \lambda^*(\dot{Q}\,Q^{\mathrm{T}} + Q\mathrm{D}v^*\,Q^{\mathrm{T}}))Qq^*\\ &= Q(\tau^*\dot{q}^* - \lambda^*\mathrm{D}v^*\,q^*) + (M\circ Y + (\tau^* - \lambda^*)\dot{Q}\,Q^{\mathrm{T}})Qq^*\\ &= Q(\tau^*\dot{q}^* - \lambda^*\mathrm{D}v^*\,q^* + M^*q^*)\,,\end{aligned}$$

falls

$$M\circ Y + (\tau^* - \lambda^*)\dot{Q}\,Q^{\mathrm{T}} = QM^*\,Q^{\mathrm{T}}\ .$$

□

Insgesamt ist damit den beiden Gleichungen und Ungleichungen (V1.6) und (V1.7), die sich aus dem Entropieprinzip ergaben, Genüge getan.

1.3 Theorem. Wenn die Temperatur und der Druck wie in (V1.3) definiert sind und für die Lösungen von (V1.1) die Gleichungen

$$\begin{aligned}\Pi &= p\mathrm{Id} - S - \lambda\theta q\,q^{\mathrm{T}}\ ,\\ \tau\dot{q} - \lambda\mathrm{D}v\,q &= \nabla\Big(\frac{1}{\theta}\Big) - Mq\end{aligned}$$

gelten, wobei M die Bedingung in 1.2 hat, so ist das Entropieprinzip erfüllt, falls

$$\mathrm{D}v\bullet S \geq 0\,,\quad q\bullet Mq \geq 0\,.$$

Wenn wir nun $\lambda = \tau$ setzen, sind die Bedingungen von Cattaneo-Christov erfüllt.

1.4 Cattaneo-Christov Modell. Es ist $\lambda = \tau$ in 1.3 und M symmetrisch. Dann ist das Entropieprinzip erfüllt, falls zusätzlich zu (V1.1)

$$\begin{aligned}\Pi &= p\mathrm{Id} - S - \tau\theta q\,q^{\mathrm{T}}\\ \tau\big(\partial_t q + \textstyle\sum_i v_i\partial_i q - \sum_i q_i\partial_i v\big) &= \nabla\big(\tfrac{1}{\theta}\big) - Mq\end{aligned}\tag{V1.8}$$

und

$$\mathrm{D}v\bullet S \geq 0\,,\quad q\bullet Mq \geq 0\,.$$

Relaxationszeit: The function τ is related to the "relaxation time". Geht $\tau \to 0$, so konvergiert das Modell zu dem gehabten klassischen Modell.

Fassen wir also zusammen.

Cattaneo-Christov Modell:

$$\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,,$$
$$\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,$$
$$\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}}\, v + q) = v{\bullet}\mathbf{f}\,,$$
$$\tau\big(\partial_t q + v{\bullet}\nabla q - q{\bullet}\nabla v\big) = \nabla\big(\frac{1}{\theta}\big) - Mq$$

$\Pi = p\mathrm{Id} - S - \tau\theta q\, q^{\mathrm{T}}$,

$e = \varepsilon + \frac{\varrho}{2}|v|^2\,,\quad \varepsilon$ die innere Energie,

$\mathrm{D}v{\bullet}S \geq 0\,,\quad M$ symmetrisch positiv definit,

θ die absolute Temperatur,

q der Wärmefluss,

τ proportional zur Relaxationszeit. (V1.9)

Siehe auch die Gleichungen in der Fig. 3 ((2.8) und (2.9), bei der Tempera-

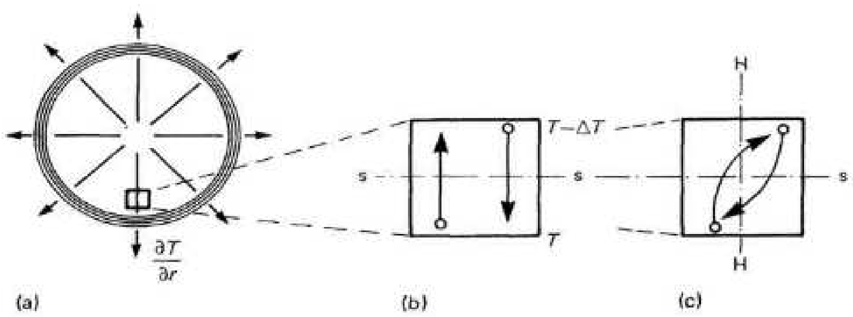

Fig. 4: Paradoxon (aus dem Buch von Müller)

turgleichung fehlt der Term mit div$v$). Es gibt aber auch andere Modelle, und zwar für $\lambda = 0$, d.h. der Drucktensor Π ist gegenüber dem normalen Drucktensor nicht geändert. Der Grund ist, dass $q{\bullet}(\mathrm{D}v)^{\mathrm{A}}\, q = 0$ ist.

1.5 Beispiel. Es ist $\lambda = 0$ in 1.3 und $M^{\mathrm{A}} = -\tau\,\mathrm{D}v^{\mathrm{A}}$ und $M_0 := M^{\mathrm{S}}$ positiv definit. Dann ist das Entropieprinzip erfüllt, falls zusätzlich zu (V1.1)

$$\Pi = p\mathrm{Id} - S$$
$$\tau(\dot q - (\mathrm{D}v)^{\mathrm{A}}\, q) = \nabla\big(\tfrac{1}{\theta}\big) - M_0 q \qquad \text{(V1.10)}$$

mit $\mathrm{D}v{\bullet}S \geq 0$.

Ist v die Geschwindigkeit eines sich drehenden Zylinders, so sind die beiden Gleichungen für q, d.h. die zweiten Gleichungen in (V1.8) und (V1.10), identisch, die beiden Gleichungen für Π jedoch nicht. Wir verweisen in diesem Zusammenhang auf das Paradoxon in Fig. 4.

2 Boltzmann Gleichung

Die Boltzmann Gleichung ist ein Erhaltungssatz für eine Wahrscheinlichkeit

$$(t,x,c) \mapsto f(t,x,c) \in \mathbb{R}\,,$$

die angibt, wie wahrscheinlich es ist, dass in einem Gas ein Partikel an einem Ort x zu einer Zeit t sich mit einer gewissen Geschwindigkeit c fortbewegt. Die Differentialgleichung für f lautet

$$\begin{aligned} \partial_t f + \sum_{i=1}^n c_i \partial_{x_i} f + \sum_{i=1}^n \mathbf{g}_i \partial_{c_i} f = \mathbf{r}\,, \\ \sum_{i=1}^n \partial_{c_i} \mathbf{g}_i = 0\,. \end{aligned} \tag{V2.1}$$

The quantity f is the density of atoms at (t,x) with velocity c, and $\mathbf{g}$ denotes the external acceleration and is a function of (t,x,c), and $\mathbf{r}$ is the "collision product", which we explain later in this section in (V2.8), and is also a function of (t,x,c). First let us clarify frame indifference. The variables (t,x,c) and the probability f and acceleration $\mathbf{g}$ and the collision rate $\mathbf{r}$ have the following transformation behaviour under an observer change.

2.1 Observer change. With the usual notations we have for the coordinates the transformation

$$\begin{bmatrix} t \\ x \\ c \end{bmatrix} = Y_B\left(\begin{bmatrix} t^* \\ x^* \\ c^* \end{bmatrix}\right) = \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \\ \dot{X}(t^*,x^*) + Q(t^*)c^* \end{bmatrix}, \tag{V2.2}$$

where T, X, and Q are given by the classical Newton transformation

$$Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} T(t^*) \\ X(t^*,x^*) \end{bmatrix} = \begin{bmatrix} t^* + \mathrm{a} \\ Q(t^*)x^* + \mathrm{b}(t^*) \end{bmatrix}.$$

So we can say, the variable (t,x) transforms like usual and c satisfies $c = \dot{X}(t^*,x^*) + Q(t^*)c^*$ like a velocity. Moreover, the probability f and the collision product $\mathbf{r}$ are objective scalars, that is

$$f(t,x,c) = f^*(t^*,x^*,c^*)\,, \quad \mathbf{r}(t,x,c) = \mathbf{r}^*(t^*,x^*,c^*)\,, \tag{V2.3}$$

if coordinates transform as above. The ***acceleration*** satisfies

$$\mathbf{g}(t,x,c) = \ddot{X}(t^*,x^*) + 2\dot{Q}(t^*)c^* + Q(t^*)\mathbf{g}^*(t^*,x^*,c^*)\,. \tag{V2.4}$$

The transformation rule for the acceleration is the same as for the force in II.3.7 except that the velocity is taken as the individual velocity and the equation is divided by the mass density. We mention that this rule is mandatory if one wants to achieve an equation like (V3.1) for the moments (see the statement in 3.5). With this transformation rules it is true, that

2.2 Lemma. The Boltzmann equations (V2.1) are objective.

Proof. The transformation of the variables (t, x, c) is given in (V2.2). Then (V2.3) implies that

$$f^*(t^*, x^*, c^*) = f(t^* + \mathrm{a}, Q(t^*)x^* + \mathrm{b}(t^*), \dot{X}(t^*, x^*) + Q(t^*)c^*)$$

and from there we obtain (we omit arguments)

$$\begin{aligned}
&\nabla_{c^*} f^* = Q^{\mathrm{T}} \nabla_c f, \\
&\nabla_{x^*} f^* = Q^{\mathrm{T}} \nabla_x f + (\mathrm{D}_{x^*} \dot{X})^{\mathrm{T}} \nabla_c f = Q^{\mathrm{T}} \nabla_x f + \dot{Q}^{\mathrm{T}} \nabla_c f, \\
&\partial_{t^*} f^* = \partial_t f + \dot{X} \bullet \nabla_x f + (\ddot{X} + \dot{Q} c^*) \bullet \nabla_c f.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\partial_{t^*} f^* + c^* \bullet \nabla_{x^*} f^* + \mathbf{g}^* \bullet \nabla_{c^*} f^* \\
&= \partial_t f + (\dot{X} + Q c^*) \bullet \nabla_x f + (\ddot{X} + 2\dot{Q} c^* + Q \mathbf{g}^*) \bullet \nabla_c f.
\end{aligned}$$

Since $c = \dot{X} + Qc^*$ the result follows, if $\mathbf{g} = \ddot{X} + 2\dot{Q}c^* + Q\mathbf{g}^*$, that is (V2.4) is assumed. Here we have used that $\mathbf{r}$ is an objective scalar.
We also have to show that the condition $\mathrm{div}_c \mathbf{g} = 0$ is objective. To prove this we compute the derivative of (V2.4) with respect to c^*, that is

$$\mathrm{D}_c \mathbf{g}\, Q = 2\dot{Q} + Q \mathrm{D}_{c^*} \mathbf{g}^* \quad \text{or} \quad \mathrm{D}_c \mathbf{g} = 2\dot{Q} Q^{\mathrm{T}} + Q \mathrm{D}_{c^*} \mathbf{g}^* Q^{\mathrm{T}} \; .$$

From this it follows that $\mathrm{trace}\, \mathrm{D}_c \mathbf{g} = \mathrm{trace}\, \mathrm{D}_{c^*} \mathbf{g}^*$, since $\dot{Q} Q^{\mathrm{T}}$ is antisymmetric. □

Referenzen: Zur Boltzmann Gleichung siehe den Ausgangsartikel von Boltzmann in [98] und die Darstellungen in [Wikipedia: H-Theorem] sowie [Wikipedia: Boltzmann Gleichung]. Darüberhinaus siehe die Ausführungen in I.Müller [87, 5.2.1 The Boltzmann equation] und für Systeme in DeGroot & Mazur [6, Ch. IX §2-4 The Boltzmann equation]. Als mathematische Arbeit siehe DiPerna & Lions [35].

The entropy principle for the Boltzmann equation has the following form.

2.3 *H*-Theorem. If $a, b \in \mathbb{R}$ and $a > 0$ then

$$\begin{aligned}
\eta(t, x) &:= -\int a \ln(b f(t, x, c))\, f(t, x, c)\, \mathrm{d}c, \\
\psi_i(t, x) &:= -\int a\, c_i \ln(b f(t, x, c))\, f(t, x, c)\, \mathrm{d}c
\end{aligned}$$

satisfies

$$\partial_t \eta(t, x) + \mathrm{div} \psi(t, x) = -\int a \ln(b f(t, x, c))\, \mathbf{r}(t, x, c)\, \mathrm{d}c \geq 0 \; ,$$

if $\mathbf{r}$ is given as in (V2.8) below.

Proof. We compute

$$\frac{\mathrm{d}}{\mathrm{d}f}(\ln(bf)\cdot f) = \ln(bf) + 1$$

and therefore with $\varphi := -\ln(bf) - 1$ we obtain

$$\begin{aligned}
&\partial_t\Big(-\int \log(bf)\, f\,\mathrm{d}c\Big) + \sum_i \partial_{x_i}\Big(-\int c_i \log(bf)\, f\,\mathrm{d}c\Big)\\
&= -\int \partial_t(\ln(bf)\, f)\,\mathrm{d}c - \sum_i \int c_i\partial_{x_i}(\ln(bf)\, f)\,\mathrm{d}c\\
&= \int \varphi(\partial_t f + \sum_i c_i\partial_{x_i} f)\,\mathrm{d}c = \int (\varphi\mathbf{r} - \sum_i \varphi\mathbf{g}_i\partial_{c_i} f)\,\mathrm{d}c\\
&= \int \varphi\mathbf{r}\,\mathrm{d}c + \sum_i \int \partial_{c_i}(\mathbf{g}_i \log(bf)\, f)\,\mathrm{d}c = \int \varphi\mathbf{r}\,\mathrm{d}c,
\end{aligned}$$

if f vanishes fast enough for $|c| \to \infty$. That the last term is nonnegative, is a consequence of 2.5(2). In fact, there the integral over $\mathbf{r}$ is an integral of a probability times (the quantities φ_k, φ'_k, f_k, f'_k are defined as there)

$$\begin{aligned}
&(\varphi_1 + \varphi_2 - \varphi'_1 - \varphi'_2)(f'_1 f'_2 - f_1 f_2)\\
&= \big(-\ln(bf_1) - \ln(bf_2) + \ln(bf'_1) + \ln(bf'_2)\big)(f'_1 f'_2 - f_1 f_2)\\
&= \big(\ln(f'_1 f'_2) - \ln(f_1 f_2)\big)(f'_1 f'_2 - f_1 f_2) \ge 0
\end{aligned}$$

since the logarithm is a monotone increasing function. □

Hence the H-Theorem plays the role of the entropy principle and it is due to the special form of the collision term, which will be introduced now.

2.4 Collision of mass points. Es seien zwei Massepunkte gegeben mit Massen m_k für $k = 1, 2$, die mit Geschwindigkeiten c_k aufeinandertreffen. Nach der Kollision sind die Massen unverändert und die Geschwindigkeiten sind c'_k. Bei der Kollision bleiben die Massen, der Impuls und die Energie erhalten (siehe I.3.2 für die Massen und den Impuls), d.h. es gilt

$$\begin{aligned}
m_1 c_1 + m_2 c_2 &= m_1 c'_1 + m_2 c'_2\,,\\
\frac{m_1}{2}|c_1|^2 + \frac{m_2}{2}|c_2|^2 &= \frac{m_1}{2}|c'_1|^2 + \frac{m_2}{2}|c'_2|^2\,.
\end{aligned}$$

Das ist äquivalent dazu, dass die Geschwindigkeiten c'_1 und c'_2 gegeben sind durch

$$\begin{aligned}
c'_1 &= \frac{m_1}{m_s}c_1 + \frac{m_2}{m_s}(c_2 + \mathbf{k}')\,, \quad && m_s = m_1 + m_2\,,\\
c'_2 &= \frac{m_1}{m_s}(c_1 - \mathbf{k}') + \frac{m_2}{m_s}c_2\,, \quad && \mathbf{k}' \in \mathbb{R}^n \text{ mit } |\mathbf{k}'| = |c_1 - c_2|\,.
\end{aligned}$$

Also sind (für $n = 3$) die sechs Gleichungen für c'_1 und c'_2 durch vier Gleichungen der Impuls- und Energiebilanz gegeben. Die zwei Freiheitsgrade sind durch $\mathbf{k}' \in \partial\mathrm{B}_{|c_1 - c_2|}(0) \subset \mathbb{R}^n$ ausgedrückt.

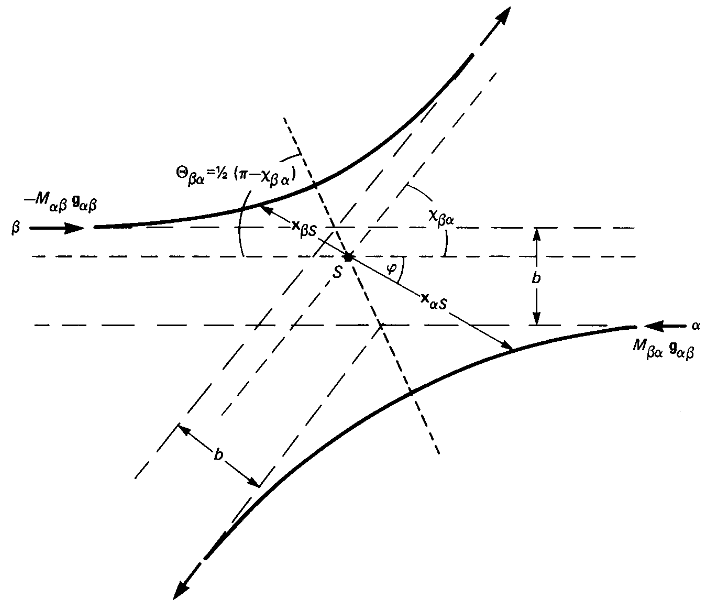

Fig. 5: "Orbits of atoms in interaction" aus I.Müller [87, Fig. 5.7]

Es ist nicht gesagt, dass sich die Teilchen wirklich treffen (siehe Fig. 5), sondern in einem gewissen Abstand sieht das so aus. Lokal kann also ein Abstoßungspotential wirksam sein.

Proof. Wir definieren den Schwerpunkt der beiden Massen durch die Gesamtmasse

$$m_s := m_1 + m_2$$

und den Impuls durch

$$m_s c_s = m_1 c_1 + m_2 c_2 = m_1 c_1' + m_2 c_2' \,,$$

also ist die Geschwindigkeit des Schwerpunkts

$$c_s := \frac{m_1}{m_s} c_1 + \frac{m_2}{m_s} c_2 = \frac{m_1}{m_s} c_1' + \frac{m_2}{m_s} c_2' \,. \tag{V2.5}$$

Physikalisch bedeutet dies, wenn man mit der Geschwindigkeit c_s an dem Stoß vorbeifliegt, sieht man die beiden Körper aufeinandertreffen mit Geschwindigkeiten $c_k - c_s$ und nach dem Stoß mit $c_k' - c_s$ auseinanderfliegen, denn aus aus der Impulsbilanz folgt $0 = m_1 c_1 + m_2 c_2 - m_s c_s = m_1 c_1' + m_2 c_2' - m_s c_s$ und daher

$$m_1(c_1 - c_s) = -m_2(c_2 - c_s) \quad \text{und} \quad m_1(c_1' - c_s) = -m_2(c_2' - c_s) \,.$$

Da gilt

$$\begin{aligned} c_1 - c_s &= (1 - \tfrac{m_1}{m_s}) c_1 - \tfrac{m_2}{m_s} c_2 = \tfrac{m_2}{m_s}(c_1 - c_2) \,, \\ c_2 - c_s &= -\tfrac{m_1}{m_s} c_1 + (1 - \tfrac{m_2}{m_s}) c_2 = -\tfrac{m_1}{m_s}(c_1 - c_2) \end{aligned} \tag{V2.6}$$

und

$$\begin{aligned} c_1' - c_s &= (1 - \tfrac{m_1}{m_s})c_1' - \tfrac{m_2}{m_s}c_2' = \tfrac{m_2}{m_s}(c_1' - c_2')\,, \\ c_2' - c_s &= -\tfrac{m_1}{m_s}c_1' + (1 - \tfrac{m_2}{m_s})c_2' = -\tfrac{m_1}{m_s}(c_1' - c_2') \end{aligned} \tag{V2.7}$$

definieren wir den ***Kollisionsvektor*** (*en:* ***collision vector***) durch

$$\mathbf{k}' = c_1' - c_2'\,.$$

Aus der Energiebilanz

$$m_1|c_1|^2 + m_2|c_2|^2 - m_s|c_s|^2 = m_1|c_1'|^2 + m_2|c_2'|^2 - m_s|c_s|^2$$

folgt unter Zuhilfenahme von (V2.6)

$$\begin{aligned} &m_1|c_1|^2 + m_2|c_2|^2 - m_s|c_s|^2 = m_1(|c_1|^2 - |c_s|^2) + m_2(|c_2|^2 - |c_s|^2) \\ &= m_1(c_1 - c_s)\bullet(c_1 + c_s) + m_2(c_2 - c_s)\bullet(c_2 + c_s) \\ &= \frac{m_1 m_2}{m_s}(c_1 - c_2)\bullet(c_1 + c_s) - \frac{m_1 m_2}{m_s}(c_1 - c_2)\bullet(c_2 + c_s) \\ &= \frac{m_1 m_2}{m_s}|c_1 - c_2|^2\,, \end{aligned}$$

und genauso mit Hilfe von (V2.7)

$$m_1|c_1'|^2 + m_2|c_2'|^2 - m_s|c_s|^2 = \frac{m_1 m_2}{m_s}|c_1' - c_2'|^2\,,$$

und daher ist nach der Energiebilanz

$$|\mathbf{k}'| = |c_1' - c_2'| = |c_1 - c_2|\,.$$

Aus der Impulserhaltung folgt mit $m_s c_s = m_1 c_1 + m_2 c_2$

$$\begin{aligned} m_s c_s &= m_1 c_1' + m_2 c_2' \\ &= (m_1 + m_2)\frac{c_1' + c_2'}{2} + (m_1 - m_2)\frac{c_1' - c_2'}{2} \\ &= m_s\frac{c_1' + c_2'}{2} + (m_1 - m_2)\frac{\mathbf{k}'}{2}\,, \end{aligned}$$

also

$$\begin{aligned} \frac{c_1' + c_2'}{2} &= c_s - \frac{m_1 - m_2}{m_s}\frac{\mathbf{k}'}{2}\,, \\ \frac{c_1' - c_2'}{2} &= \frac{\mathbf{k}'}{2}\,, \end{aligned}$$

oder

$$\begin{aligned} &c_1' = c_s + \frac{m_2}{m_s}\mathbf{k}'\,, \quad c_s = \frac{m_1}{m_s}c_1 + \frac{m_2}{m_s}c_2\,, \quad m_s = m_1 + m_2\,, \\ &c_2' = c_s - \frac{m_1}{m_s}\mathbf{k}'\,, \quad \mathbf{k}' \in \mathbb{R}^n \text{ mit } |\mathbf{k}'| = |c_1 - c_2|\,, \end{aligned}$$

was identisch mit der Behauptung ist. Wir führen noch die Probe aus. Nehmen wir diese Formeln an, so folgt daraus die Impulserhaltung

$$m_1 c_1' + m_2 c_2' = m_1 c_s + \frac{m_1 m_2}{m_s}\mathbf{k}' + m_2 c_s - \frac{m_2 m_1}{m_s}\mathbf{k}'$$
$$= (m_1 + m_2)c_s = m_s c_s = m_1 c_1 + m_2 c_2 \,,$$

und die Energieerhaltung

$$m_1|c_1'|^2 + m_2|c_2'|^2 = m_1\Big|c_s + \frac{m_2}{m_s}\mathbf{k}'\Big|^2 + m_2\Big|c_s - \frac{m_1}{m_s}\mathbf{k}'\Big|^2$$
$$= (m_1 + m_2)|c_s|^2 + \frac{m_1 m_2^2 + m_2 m_1^2}{m_s^2}|\mathbf{k}'|^2$$
$$= m_s\Big|\frac{m_1}{m_s}c_1 + \frac{m_2}{m_s}c_2\Big|^2 + \frac{m_1 m_2}{m_s}|c_2 - c_1|^2$$
$$= m_1|c_1|^2 + m_2|c_2|^2 \,.$$

Dies beendet die Probe. □

Wir beschränken uns hier auf eine Gleichung, d.h. eine Wahrscheinlichkeit f, und nehmen an, dass alle Massen m_i gleich sind. Sind dann $f(t,x,c_1)$ und $f(t,x,c_2)$ die Wahrscheinlichkeiten für Teilchen mit Geschwindigkeit c_1 und c_2, so ist (bis auf einen positiven Faktor)

$$W(c_1 - c_2, c_1' - c_2')$$

die Wahrscheinlichkeit, dass nach der Kollision zwei Geschwindigkeiten c_1' und c_2' vorhanden sind, dass heißt $f(t,x,c_1')$ und $f(t,x,c_2')$ zu betrachten sind, wobei wir 2.4 zu berücksichtigen haben, d.h. es tritt keine Reaktion auf und die Stöße verlaufen nach den Regeln der Thermodynamik. Wir haben also

$$\boxed{\begin{aligned} \mathbf{r}(t,x,c_1) = \int_{\mathbb{R}^n}\int_{\partial \mathrm{B}_{|c_1-c_2|}(0)} \Big(& f(t,x,c_1')f(t,x,c_2') \\ & - f(t,x,c_1)f(t,x,c_2)\Big)\cdot W(c_1 - c_2, \mathbf{k}')\,\mathrm{dH}^{n-1}(\mathbf{k}')\,\mathrm{d}c_2\,, \\ \text{wobei } c_1' = c_s + \frac{\mathbf{k}'}{2},\ c_2' = c_s - \frac{\mathbf{k}'}{2},\ & c_s = \frac{c_1 + c_2}{2}. \end{aligned}} \tag{V2.8}$$

Wenn wir zur Abkürzung schreiben

$$\begin{aligned} &\mathbf{r}_1 = \mathbf{r}(t,x,c_1)\,, \\ &f_1 = f(t,x,c_1),\ f_2 = f(t,x,c_2)\,, \\ &f_1' = f(t,x,c_1'),\ f_2' = f(t,x,c_2')\,, \\ &\mathbf{k} = c_1 - c_2 \ (\text{es ist } \mathbf{k}' = c_1' - c_2')\,, \end{aligned}$$

schreibt sich dies als

$$\mathbf{r}_1 = \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} \big(f_1' f_2' - f_1 f_2 \big) \cdot W(\mathbf{k}, \mathbf{k}') \, \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \,.$$

Ist dann $\varphi \in \mathscr{D}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ eine Testfunktion und $\varphi_1 = \varphi(t, x, c_1)$, so gilt

$$\begin{aligned} &\int \varphi_1 \mathbf{r}_1 \, \mathrm{d}c_1 = \int \varphi(t, x, c_1) \mathbf{r}(t, x, c_1) \, \mathrm{d}c_1 \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} \varphi_1 \big(f_1' f_2' - f_1 f_2 \big) \cdot W(\mathbf{k}, \mathbf{k}') \, \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \, \mathrm{d}c_1 \,. \end{aligned}$$

Assume that

$$\left. \begin{aligned} W(\mathbf{k}, \mathbf{k}') &= W(\mathbf{k}', \mathbf{k}) \\ W(\mathbf{k}, \mathbf{k}') &= W(-\mathbf{k}, -\mathbf{k}') \end{aligned} \right\} \text{ for } |\mathbf{k}| = |\mathbf{k}'| \,. \tag{V2.9}$$

From these symmetry properties one obtains

2.5 Theorem. Let the probabilities W satisfy (V2.9). Then we have the following symmetry properties:

(1) For any function $(c_1', c_2', c_1, c_2) \mapsto g(c_1', c_2', c_1, c_2)$

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} g(c_1', c_2', c_1, c_2) \big(f_1' f_2' - f_1 f_2 \big) W(\mathbf{k}, \mathbf{k}') \\ &\qquad\qquad \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \, \mathrm{d}c_1 \\ &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} g(c_1, c_2, c_1', c_2') \big(f_1' f_2' - f_1 f_2 \big) W(\mathbf{k}, \mathbf{k}') \\ &\qquad\qquad \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \, \mathrm{d}c_1 \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} g(c_2', c_1', c_2, c_1) \big(f_1' f_2' - f_1 f_2 \big) W(\mathbf{k}, \mathbf{k}') \\ &\qquad\qquad \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \, \mathrm{d}c_1 \,. \end{aligned}$$

(2) If the collision term $\mathbf{r}$ satisfies (V2.8), then it follows for any test function φ that

$$\begin{aligned} &\int \varphi(t, x, c_1) \mathbf{r}(t, x, c_1) \, \mathrm{d}c_1 \\ &= \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\partial \mathrm{B}_{|c_1 - c_2|}(0)} \big(\varphi_1 + \varphi_2 - \varphi_1' - \varphi_2' \big) \big(f_1' f_2' - f_1 f_2 \big) W(\mathbf{k}, \mathbf{k}') \\ &\qquad\qquad \mathrm{d}\mathrm{H}^{n-1}(\mathbf{k}') \, \mathrm{d}c_2 \, \mathrm{d}c_1 \,. \end{aligned}$$

Here $\varphi_1 = \varphi(t, x, c_1)$ and similar φ_2, φ_1', φ_2' like above.

Proof (1). In the integrals there is

$$c_1' = \frac{1}{2}(c_1 + c_2 + \mathbf{k}')\,, \quad c_2' = \frac{1}{2}(c_1 + c_2 - \mathbf{k}')\,,$$

which is due to the fact that

$$c_1' + c_2' = c_1 + c_2, \quad \mathbf{k}' = c_1' - c_2', \quad \mathbf{k} := c_1 - c_2,$$

therefore

$$c_1 = \frac{1}{2}(c_1' + c_2' + \mathbf{k})\,, \quad c_2 = \frac{1}{2}(c_1' + c_2' - \mathbf{k})\,.$$

Thus the two variables

$$s = \frac{1}{2}(c_1 + c_2), \quad k = c_1 - c_2,$$

satisfy

$$\mathrm{d}c_2\,\mathrm{d}c_1 = \mathrm{d}\mathbf{k}\,\mathrm{d}s$$

and with $r = |c_1 - c_2| = |c_1' - c_2'|$

$$\mathrm{d}\mathbf{k} = \mathrm{dH}^{n-1}(\mathbf{k})\llcorner\partial\mathrm{B}_r(0)\ \mathrm{d}r\,,$$

so that

$$\begin{aligned}&\mathrm{dH}^{n-1}(\mathbf{k}')\llcorner\partial\mathrm{B}_{|c_1-c_2|}(0)\ \mathrm{d}c_2\,\mathrm{d}c_1\\ &= \mathrm{dH}^{n-1}(\mathbf{k}')\llcorner\partial\mathrm{B}_r(0)\ \mathrm{dH}^{n-1}(\mathbf{k})\llcorner\partial\mathrm{B}_r(0)\ \mathrm{d}r\,\mathrm{d}s\,,\end{aligned}$$

which shows that the measure is symmetric in (c_1, c_2) and (c_1', c_2') and also in (c_1, c_2) and (c_2, c_1). □

Proof (2). This follows from (1). □

2.6 Property. If the collision term $\mathbf{r}$ satisfies (V2.8) with (V2.9) then

$$\int \varphi(t,x,c)\mathbf{r}(t,x,c)\,\mathrm{d}c = 0$$

for the so-called ***summational invariants***

$$\varphi(t,x,c) = 1, \quad \varphi(t,x,c) = c, \quad \varphi(t,x,c) = \frac{|c|^2}{2}.$$

Proof. This follows from 2.5(2) since $\varphi_1 + \varphi_2 = \varphi_1' + \varphi_2'$ for the summational invariants follows from the fact that the collisions satify conservation of mass, momentum, and kinetic energy. □

3 Die Chapman-Enskog Hierarchie

Wir gehen aus von der Boltzmann Gleichung (V2.1), was eine Differentialgleichung für eine Wahrscheinlichkeitsdichte

$$(t, x, c) \mapsto f(t, x, c)$$

darstellt. Wir definieren zu dieser Größe die höheren Momente.

3.1 Höhere Momente. Für $k_1, \dots, k_M \in \{0, \dots, n\}$ ist

$$F_{k_1,\dots,k_M}(t, x) := \int_{\mathbb{R}^n} m \underline{c}_{k_1} \dots \underline{c}_{k_M} f(t, x, c) \, \mathrm{d}c$$

wobei m die Partikelmasse ist und $\underline{c} := (1, c)$, also

$$\underline{c} = (c_0, c) = (c_0, c_1, \dots, c_n) \,, \quad c_0 := 1.$$

Bemerkung: Diese Definition kann für verschiedene Indices die gleiche Funktion ergeben, z.B. ist $F_{k0} = F_k$.

Wenn man nun annimmt, dass f die Boltzmann Gleichung (V2.1) erfüllt und für $|c| \to \infty$ stark genug abfällt, ist für diese höheren Momente folgende Differentialgleichung in den Variablen (t, x) erfüllt:

Gleichung für die höheren Momente:

Für $i_1, \dots, i_N \in \{0, \dots, n\}$ gilt

$$\partial_t F_{i_1,\dots,i_N} + \sum_{i=1}^{n} \partial_{x_i} F_{i_1,\dots,i_N i} = R_{i_1,\dots,i_N} + G_{i_1,\dots,i_N} \tag{V3.1}$$

$F_{k_1,\dots,k_M}$ wie in 3.1 gegeben und

$$R_{i_1,\dots,i_N}(t, x) := \int_{\mathbb{R}^n} m \underline{c}_{i_1} \cdots \underline{c}_{i_N} \mathbf{r}(t, x, c) \, \mathrm{d}c$$

$$G_{i_1,\dots,i_N}(t, x) := \sum_i \int_{\mathbb{R}^n} m \mathbf{g}_i(t, x, c) \partial_{c_i} (\underline{c}_{i_1} \cdots \underline{c}_{i_N}) f(t, x, c) \, \mathrm{d}c$$

We prove this with the help of a general test function $(t, x, c) \mapsto \varphi(t, x, c)$ and integrating the Boltzmann equation over the velocity $c \in \mathbb{R}^n$.

3.2 Theorem. For all functions $(t, x, c) \mapsto \varphi(t, x, c)$ it follows from (V2.1) that

$$\partial_t \Big(\int \varphi f \, \mathrm{d}c \Big) + \sum_i \partial_{x_i} \Big(\int c_i \varphi f \, \mathrm{d}c \Big)$$
$$= \int \varphi \mathbf{r} \, \mathrm{d}c + \int \Big(\partial_t \varphi + \sum_i c_i \partial_{x_i} \varphi + \sum_i \mathbf{g}_i \partial_{c_i} \varphi \Big) f \, \mathrm{d}c \,,$$

provided all c-integrals exist.

Proof. The only nontrivial term is

$$\int \varphi \sum_{i=1}^{n} \mathbf{g}_i \partial_{c_i} f \,\mathrm{d}c = -\int \sum_{i=1}^{n} \partial_{c_i}(\varphi \mathbf{g}_i) f \,\mathrm{d}c = -\sum_i \int \mathbf{g}_i \partial_{c_i}\varphi \cdot f \,\mathrm{d}c\,,$$

since $\operatorname{div}_c \mathbf{g} = 0$. □

We take the test function

$$\varphi(t,x,c) = m\underline{c}_{i_1} \cdots \underline{c}_{i_N}\,.$$

Then 3.2 gives the equation (V3.1) with

$$R_{i_1,\dots,i_N} = \int_{\mathbb{R}^n} \varphi \mathbf{r} \,\mathrm{d}c = \int_{\mathbb{R}^n} m\underline{c}_{i_1} \cdots \underline{c}_{i_N} \mathbf{r}(t,x,c) \,\mathrm{d}c$$

and, since φ does not depend on t and x,

$$\begin{aligned} G_{i_1,\dots,i_N} &= \int_{\mathbb{R}^n} \Big(\partial_t \varphi + \sum_i c_i \partial_{x_i} \varphi + \sum_i \mathbf{g}_i \partial_{c_i} \varphi \Big) f \,\mathrm{d}c \\ &= \int_{\mathbb{R}^n} \sum_i \mathbf{g}_i (\partial_{c_i}\varphi) f \,\mathrm{d}c = \sum_i \int_{\mathbb{R}^n} m\mathbf{g}_i(t,x,c) \partial_{c_i}(\underline{c}_{i_1} \cdots \underline{c}_{i_N}) f(t,x,c) \,\mathrm{d}c\,. \end{aligned}$$

If $\mathbf{g}$ depends linearly on c (this means affine linear) then this integral is a linear combination of the functions $F_{j_1,\dots,j_N}$ plus an absolute term. Such a form of $\mathbf{g}$ is related to the general rule (V2.4), see also 3.5.

References: See [Wikipedia: Chapman-Enskog theory] and Chapman & Cowling [81] (in an issue of 1990 with a foreword of C.Cercignani).

We see that the first moments give mass, momentum, and energy. To obtain the equations we take the 0th and the 1st moments and perform the trace of the 2nd moment. The variables under the time derivative are denoted by ϱ, ϱv, and $2e$.

3.3 Klassische Momente. Let us define (if ϱ is positive)

$$\begin{aligned} \varrho &:= \int_{\mathbb{R}^n} mf \,\mathrm{d}c = F_0\,, \\ \varrho v &:= \int_{\mathbb{R}^n} mfc \,\mathrm{d}c = (F_i)_{i=1,\dots,n}\,, \\ \Pi &:= \int_{\mathbb{R}^n} mf(c-v)(c-v)^{\mathrm{T}} \,\mathrm{d}c\,, \\ e &= \varepsilon + \frac{\varrho}{2}|v|^2 = \frac{1}{2}\sum_{i=1}^{n} F_{ii} = \int_{\mathbb{R}^n} \frac{m}{2}|c|^2 f \,\mathrm{d}c\,, \\ \varepsilon &:= \int_{\mathbb{R}^n} \frac{m}{2}|c-v|^2 f \,\mathrm{d}c\,, \\ q &:= \int_{\mathbb{R}^n} mf|c-v|^2(c-v) \,\mathrm{d}c\,, \\ \mathbf{f} &:= \int_{\mathbb{R}^n} mf\mathbf{g} \,\mathrm{d}c\,. \end{aligned}$$

Then

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v) = 0\,,\\
&\partial_t(\varrho v) + \operatorname{div}(\varrho v\, v^{\mathrm{T}} + \Pi) = \mathbf{f}\,,\\
&\partial_t e + \operatorname{div}(ev + \Pi^{\mathrm{T}}\, v + q) = g + v\bullet\mathbf{f}\,,
\end{aligned} \tag{V3.2}$$

where due to the special structure Π is symmetric with

$$\varepsilon = \frac{1}{2}\sum_{k=1}^{n}\Pi_{kk} \quad \text{and} \quad g = \int_{\mathbb{R}^n} m\mathbf{g}\bullet(c-v)f\,\mathrm{d}c \tag{V3.3}$$

are objective scalars. Here the special properties of the collision term in (V2.8) is used (see 2.6).

We remark that the differential equations in (V3.2) coincide with those in section III.2, although they are very special because of (V3.3).

Proof. The collision part is zero for the required equations. The reason is that a single collision respects conservation of mass, momentum, and energy. This results in the equations 2.6

$$R_0 = 0\,, \quad R_i = 0 \text{ for } i = 1,\dots,n, \quad \sum_{i=1}^{n} R_{ii} = 0\,.$$

Therefore it follows from (V3.1) that

$$\partial_t F_0 + \sum_i \partial_{x_i} F_i = R_0 + G_0 = 0\,,$$

since $F_{0i} = F_i$, which is the mass conservation. In the following we shall use that the definition of v implies

$$\int_{\mathbb{R}^n} f(c-v)\,\mathrm{d}c = 0\,. \tag{V3.4}$$

Now we get from (V3.1) for $k = 1,\dots,n$

$$\partial_t F_k + \sum_i \partial_{x_i} F_{ki} = R_k + G_k = G_k\,.$$

It is $F_k = \varrho v_k$ and for $i = 1,\dots,n$

$$\begin{aligned}
F_{ki} &= \int_{\mathbb{R}^n} mfc_kc_i\,\mathrm{d}c\\
&= v_kv_i\int_{\mathbb{R}^n} mf\,\mathrm{d}c + \int_{\mathbb{R}^n} mf(c_kc_i - v_kv_i)\,\mathrm{d}c\\
&= \varrho v_kv_i + \int_{\mathbb{R}^n} mf(c-v)_kv_i\,\mathrm{d}c + \int_{\mathbb{R}^n} mfv_k(c-v)_i\,\mathrm{d}c\\
&+ \int_{\mathbb{R}^n} mf(c-v)_k(c-v)_i\,\mathrm{d}c\\
&= \varrho v_kv_i + \Pi_{ki}\,,
\end{aligned}$$

where we have used (V3.4). And from (V3.1)

$$\begin{aligned} G_k &= \sum_i \int_{\mathbb{R}^n} m\mathbf{g}_i \partial_{c_i} c_k\, f\,\mathrm{d}c \\ &= \sum_i \int_{\mathbb{R}^n} m\mathbf{g}_i \delta_{i,k} f\,\mathrm{d}c = \int_{\mathbb{R}^n} m\mathbf{g}_k f\,\mathrm{d}c = \mathbf{f}_k\,. \end{aligned}$$

Thus the momentum balance is shown. For the energy we obtain from (V3.1) by taking the sum of the equation for F_{kk} for $k = 1, \dots, n$

$$\partial_t\Big(\sum_k F_{kk}\Big) + \sum_i \partial_{x_i}\Big(\sum_k F_{kki}\Big) = \sum_k (R_{kk} + G_{kk}) = \sum_k G_{kk}\,.$$

We have to compute the single terms. It is

$$\begin{aligned} 2e &:= \sum_k F_{kk} = \sum_k \int_{\mathbb{R}^n} mc_kc_k f\,\mathrm{d}c \\ &= \int_{\mathbb{R}^n} m|c|^2 f\,\mathrm{d}c = \int_{\mathbb{R}^n} m\big(|c-v|^2 + 2(c-v)\bullet v + |v|^2\big) f\,\mathrm{d}c \\ &= \int_{\mathbb{R}^n} m|c-v|^2 f\,\mathrm{d}c + 2v\bullet \int_{\mathbb{R}^n} mf(c-v)\,\mathrm{d}c + |v|^2 \int_{\mathbb{R}^n} mf\,\mathrm{d}c \\ &= 2\varepsilon + \varrho|v|^2\,, \end{aligned}$$

where we have used (V3.4). This clarifies the energy. Next we consider the energy flux

$$\begin{aligned} \sum_k F_{kki} &= \int_{\mathbb{R}^n} mf|c|^2 c_i\,\mathrm{d}c \\ &= \int_{\mathbb{R}^n} mf|c|^2\,\mathrm{d}c\, v_i + \int_{\mathbb{R}^n} mf|c|^2 (c-v)_i\,\mathrm{d}c \\ &= 2ev_i + \int_{\mathbb{R}^n} mf|c|^2(c-v)_i\,\mathrm{d}c\,. \end{aligned}$$

Now

$$\begin{aligned} |c|^2(c-v)_i &= |v|^2(c-v)_i + (|c|^2 - |v|^2)(c-v)_i \\ &= |v|^2(c-v)_i + 2(v\bullet(c-v))(c-v)_i + |c-v|^2(c-v)_i\,. \end{aligned}$$

Integrating over c the first term vanishes by (V3.4), and therefore

$$\begin{aligned} &\int_{\mathbb{R}^n} mf|c|^2(c-v)_i\,\mathrm{d}c \\ &= 2\sum_k \int_{\mathbb{R}^n} mf(c-v)_k(c-v)_i\,\mathrm{d}c\, v_k + \int_{\mathbb{R}^n} mf|c-v|^2(c-v)_i\,\mathrm{d}c \\ &= 2\sum_k \Pi_{ki}v_k + \int_{\mathbb{R}^n} mf|c-v|^2(c-v)_i\,\mathrm{d}c\,, \end{aligned}$$

therefore

$$\sum_k F_{kki} = 2ev_i + 2\sum_k \Pi_{ki}v_k + \int_{\mathbb{R}^n} mf|c-v|^2(c-v)_i \,\mathrm{d}c$$
$$= 2ev_i + 2(\Pi^{\mathrm{T}}\, v)_i + q_i\,.$$

This gives the terms of the formula in the assertion. What is left is the energy production

$$\sum_{k=1}^n G_{kk} = \sum_{k,i=1}^n \int_{\mathbb{R}^n} m\mathbf{g}_i\partial_{c_i}(c_kc_k)f\,\mathrm{d}c$$
$$= 2\sum_{k=1}^n \int_{\mathbb{R}^n} m\mathbf{g}_kc_kf\,\mathrm{d}c = 2\int_{\mathbb{R}^n} m\mathbf{g}\bullet c\,f\,\mathrm{d}c$$
$$= 2\int_{\mathbb{R}^n} m\mathbf{g}\bullet(c-v)f\,\mathrm{d}c + 2\mathbf{f}\bullet v = 2g + 2\mathbf{f}\bullet v\,,$$

if

$$g := \int_{\mathbb{R}^n} m\mathbf{g}\bullet(c-v)f\,\mathrm{d}c\,.$$

If we are an observer for which $\mathbf{g}$ is independent of c, then g vanishes. In general, g is an objective scalar. We show this by using the transformation rule (V2.4) for an acceleration

$$\mathbf{g}(t,x,c) = \ddot{X}(t^*,x^*) + 2\dot{Q}(t^*)c^* + Q(t^*)\mathbf{g}^*(t^*,x^*,c^*)\,. \tag{V3.5}$$

We obtain (not writing the arguments), since $c-v = Q(c^*-v^*)$,

$$\mathbf{g}\bullet(c-v) = Q(t^*)^{\mathrm{T}}\,(\ddot{X}(t^*,x^*) + 2\dot{Q}(t^*)v^*)\bullet(c^*-v^*)$$
$$+2(c^*-v^*)\bullet\dot{Q}(t^*)^{\mathrm{T}}\,Q(t^*)(c^*-v^*) + (Q(t^*)\mathbf{g}^*)\bullet Q(t^*)(c^*-v^*)\,.$$

The first term vanishes after integration since $Q(t^*)^{\mathrm{T}}\,(\ddot{X}(t^*,x^*)+2\dot{Q}(t^*)v^*)$ is independent of c^*, and the second term vanishes since $\dot{Q}(t^*)^{\mathrm{T}}\,Q(t^*)$ is antisymmetric. Hence

$$\int \mathbf{g}\bullet(c-v)f\,\mathrm{d}c = \int \mathbf{g}^*\bullet(c^*-v^*)f^*\,\mathrm{d}c^*\,.$$

□

Dieses Resultat verstärkt uns in der Annahme, dass die Gleichungen höherer Ordnung, die ja auch eine gewisse Realität widerspiegeln, von Nutzen in den Anwendungen sind, auf jeden Fall, wenn es sich um Gase handelt.

Transformationsverhalten

Wir lernen jetzt noch eine effiziente Methode kennen, um die Transformationsregel für Gleichungen höherer Momente zu beschreiben. Dies ist wichtig, da wir diese Gleichungen unabhängig von der Boltzmann Gleichung nehmen wollen zur Beschreibung verschiedener Materialien. Indem wir die Koordinaten $y = (t, x)$ nennen, schreibt sich die Differentialgleichung in (V3.1) auch als

$$\sum_{k=0}^{n} \partial_{y_k} F_{i_1,\dots,i_N k} = R_{i_1,\dots,i_N} + G_{i_1,\dots,i_N} \,. \tag{V3.6}$$

Wir nehmen diese Gleichungen nun für alle $i_1, \dots, i_N \in \{0, 1, \dots, n\}$ als gegeben an. Sie enthalten dann alle Differentialgleichungen für Momente der Ordnung kleiner oder gleich N. Für diese Differentialgleichungen gilt die folgende Regel bei Beobachtertransformationen.

3.4 Transformationsregel. Es gilt für $k_1, \dots, k_M \in \{0, \dots, n\}$

$$F_{k_1,\dots,k_M} \circ Y = \sum_{\overline{k}_1,\dots,\overline{k}_M=0}^{n} Y_{k_1 \prime \overline{k}_1} \cdots Y_{k_M \prime \overline{k}_M} F^*_{\overline{k}_1,\dots,\overline{k}_M} \tag{V3.7}$$

Note: This is because the indices run from 0.

Proof. Es ergibt sich aus 3.1 die folgende Identität (V3.7) für

$$\begin{bmatrix} t \\ x \end{bmatrix} = Y\left(\begin{bmatrix} t^* \\ x^* \end{bmatrix}\right) = \begin{bmatrix} T(t^*) \\ X(t^*, x^*) \end{bmatrix}, \quad \mathrm{D}Y = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix},$$

und mit dem Integrationswechsel $c = \dot{X}(t^*, x^*) + Q(t^*)c^*$, wenn wir dies schreiben als

$$\begin{bmatrix} 1 \\ c \end{bmatrix} = \underline{c} = \mathrm{D}Y c^{*\prime} = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix} \begin{bmatrix} 1 \\ c^* \end{bmatrix} = \begin{bmatrix} 1 \\ \dot{X} + Qc^* \end{bmatrix}.$$

Also ist wegen $f(t, x, c) = f^*(t^*, x^*, c^*)$

$$\begin{aligned} F_{k_1,\dots,k_N}(t, x) &= \int_{\mathbb{R}^n} m \underline{c}_{k_1} \dots \underline{c}_{k_N} f(t, x, c) \,\mathrm{d}c \\ &= \int_{\mathbb{R}^n} m \underline{c}_{k_1} \dots \underline{c}_{k_N} f^*(t^*, x^*, c^*) \,\mathrm{d}c^* \\ &= \int_{\mathbb{R}^n} m \Big(\sum_{\overline{k}_1=0}^{n} Y_{k_1 \prime \overline{k}_1} c^{*\prime}_{\overline{k}_1} \Big) \dots \Big(\sum_{\overline{k}_N=0}^{n} Y_{k_N \prime \overline{k}_N} c^{*\prime}_{\overline{k}_N} \Big) f^* \,\mathrm{d}c^* \\ &= \sum_{\overline{k}_1,\dots,\overline{k}_N=0}^{n} Y_{k_1 \prime \overline{k}_1} \cdots Y_{k_N \prime \overline{k}_N} \int_{\mathbb{R}^n} m c^{*\prime}_{\overline{k}_1} \dots c^{*\prime}_{\overline{k}_N} f^* \,\mathrm{d}c^* \\ &= \sum_{\overline{k}_1,\dots,\overline{k}_N=0}^{n} Y_{k_1 \prime \overline{k}_1} \cdots Y_{k_N \prime \overline{k}_N} F^*_{\overline{k}_1,\dots,\overline{k}_N} \,. \end{aligned} \tag{V3.8}$$

□

In den einzelnen Termen ausgeschrieben kann diese Transformationsregel sehr kompliziert aussehen, deshalb ist die allgemeine Beschreibung dieser Regel die am leichtesten merkbare. Außerdem steht diese Transformationsregel im Zusammenhang mit der generellen Formel (I5.3) in dieser Vorlesung.

3.5 Bemerkung. Die physikalischen Eigenschaften der Größen in (V3.1) sind dadurch gegeben, dass in der schwachen Version von (V3.1) die Testfunktionen

$$\zeta_{i_1\cdots i_N} \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n) \text{ für } i_1,\dots,i_N \in \{0,1,\dots,n\}$$

das Transformationsverhalten

$$\zeta^*_{\bar{i}_1\cdots\bar{i}_N} = \sum_{i_1,\dots,i_N=0}^{n} Y_{i_1},_{\bar{i}_1}\cdots Y_{i_N},_{\bar{i}_N}\zeta_{i_1\cdots i_N}\circ Y$$

für $\bar{i}_1,\dots,\bar{i}_N \in \{0,1,\dots,n\}$ besitzen.

Proof. Wir haben also das Transformationsverhalten $\zeta^* = Z^{\mathrm{T}}\,\zeta\circ Y$, wobei

$$Z_{(i_1,\dots,i_N)(\bar{i}_1,\dots,\bar{i}_N)} = Y_{i_1},_{\bar{i}_1}\cdots Y_{i_N},_{\bar{i}_N}\,.$$

Nach (I5.3) ist die Transformationsformel erfüllt, falls mit $F_{i_1\cdots i_N 0} := F_{i_1\cdots i_N}$ gilt

$$F_{i_1\cdots i_N k}\circ Y = \sum_{\bar{i}_1,\dots,\bar{i}_N,\bar{k}=0}^{n} Z_{(i_1,\dots,i_N)(\bar{i}_1,\dots,\bar{i}_N)} Y_k,_{\bar{k}} F^*_{\bar{i}_1\cdots\bar{i}_N\bar{k}} \tag{V3.9}$$

und wenn mit $g_{i_1\cdots i_N} := R_{i_1\cdots i_N} + G_{i_1\cdots i_N}$

$$\begin{aligned} g_{i_1\cdots i_N}\circ Y = & \sum_{\bar{i}_1,\dots,\bar{i}_N,\bar{k}=0}^{n} (Z_{(i_1,\dots,i_N)(\bar{i}_1,\dots,\bar{i}_N)}),_{\bar{k}} F^*_{\bar{i}_1\cdots\bar{i}_N\bar{k}} \\ & + \sum_{\bar{i}_1,\dots,\bar{i}_N}^{n} Z_{(i_1,\dots,i_N)(\bar{i}_1,\dots,\bar{i}_N)} g^*_{\bar{i}_1\cdots\bar{i}_N}\,. \end{aligned} \tag{V3.10}$$

Die Gleichung (V3.9) ist äquivalent zu (V3.7) für $M = N+1$, was schon bewiesen wurde. Es bleibt also noch (V3.10) zu beweisen. Anstatt dessen beweisen wir die Aussage mit Hilfe von Testfunktionen. □

Proof (über Testfunktionen). Since $\mathrm{div}_c\mathbf{g} = 0$ the Boltzmann equation can be written as divergence equation

$$\partial_t f + \sum_{i=1}^{n}\partial_{x_i}(c_i f) + \sum_{i=1}^{n}\partial_{c_i}(\mathbf{g}_i f) = \mathbf{r}\,.$$

If we introduce $y = (t,x)$ and $\underline{c} = (1,c)$ we can write this as

$$\sum_{k=0}^{n}\partial_{y_k}(\underline{c}_k f) + \sum_{i=1}^{n}\partial_{c_i}(\mathbf{g}_i f) = \mathbf{r}\,.$$

Clearly we can write this with test functions $(y,c) \mapsto \varphi(y,c)$ (they need not have compact support with respect to c)

$$\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^n} \big(\sum_{k=0}^{n} \partial_{y_k}\varphi \cdot \underline{c}_k f + \sum_{i=1}^{n} \partial_{c_i}\varphi \cdot \mathbf{g}_i f + \varphi \cdot \mathbf{r} \big) \,\mathrm{d}c \,\mathrm{d}y = 0\,.$$

We set

$$\varphi(y,c) = \sum_{i_1,\dots,i_N=0}^{n} \zeta_{i_1\cdots i_N}(y) m \underline{c}_{i_1} \cdots \underline{c}_{i_N}$$

and obtain, that this integral equals

$$0 = \sum_{i_1,\dots,i_N=0}^{n} \int_{\mathbb{R}^{n+1}} \Big(\sum_{k=0}^{n} \partial_{y_k}\zeta_{i_1\cdots i_N} F_{i_1\cdots i_N k} + \zeta_{i_1\cdots i_N}(G_{i_1\cdots i_N} + R_{i_1\cdots i_N}) \Big) \,\mathrm{d}y\,.$$

Wähle nun ζ^* und ζ gemäß der Formulierung im Satz. Da dann

$$\begin{aligned}\partial_{y^*_{\bar k}}\zeta^*_{\bar i_1\cdots\bar i_N} = \sum_{i_1,\dots,i_N=0}^{n} \Big(&\sum_{k=0}^{n} (\partial_{y_k}\zeta_{i_1\cdots i_N})\circ Y \cdot Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N} Y_{k{}'\bar k} \\ &+ \zeta_{i_1\cdots i_N}\circ Y \cdot \partial_{y^*_{\bar k}}(Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N}) \Big)\end{aligned}$$

wird das *-Integral zu

$$\begin{aligned}0 &= \sum_{\bar i_1,\dots,\bar i_N,\bar k=0}^{n} \int_{\mathbb{R}^{n+1}} \partial_{y^*_{\bar k}}\zeta^*_{\bar i_1\cdots\bar i_N} F^*_{\bar i_1\cdots\bar i_N\bar k} \,\mathrm{d}y^* \\ &\qquad + \sum_{\bar i_1,\dots,\bar i_N=0}^{n} \int_{\mathbb{R}^{n+1}} \zeta^*_{\bar i_1\cdots\bar i_N}(G^*_{\bar i_1\cdots\bar i_N} + R^*_{\bar i_1\cdots\bar i_N}) \,\mathrm{d}y^* \\ &= \sum_{\substack{\bar i_1,\dots,\bar i_N,\bar k,\\ i_1,\dots,i_N,k\\ =0}}^{n} \int_{\mathbb{R}^{n+1}} (\partial_{y_k}\zeta_{i_1\cdots i_N})\circ Y \cdot Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N} Y_{k{}'\bar k} F^*_{\bar i_1\cdots\bar i_N\bar k} \,\mathrm{d}y^* \\ &\qquad + \sum_{\substack{\bar i_1,\dots,\bar i_N,\bar k\\ i_1,\dots,i_N=0}}^{n} \int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y \cdot \partial_{y^*_{\bar k}}(Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N}) F^*_{\bar i_1\cdots\bar i_N\bar k} \,\mathrm{d}y^* \\ &+ \sum_{\substack{\bar i_1,\dots,\bar i_N,\\ i_1,\dots,i_N=0}}^{n} \int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y \cdot Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N}(G^*_{\bar i_1\cdots\bar i_N} + R^*_{\bar i_1\cdots\bar i_N}) \,\mathrm{d}y^*\,.\end{aligned}$$

For the first term we use

$$\sum_{\bar i_1,\dots,\bar i_N,\bar k=0}^{n} Y_{i_1{}'\bar i_1}\cdots Y_{i_N{}'\bar i_N} Y_{k{}'\bar k} F^*_{\bar i_1\cdots\bar i_N\bar k} = F_{i_1\cdots i_N k}\circ Y\,,$$

which is (V3.7) (this has been proved above), and we see that it equals to the integral

$$\begin{aligned}&\sum_{i_1,\dots,i_N,k=0}^{n} \int_{\mathbb{R}^{n+1}} (\partial_{y_k}\zeta_{i_1\cdots i_N})\circ Y \cdot F_{i_1\cdots i_N k}\circ Y \,\mathrm{d}y^* \\ &= \sum_{i_1,\dots,i_N,k=0}^{n} \int_{\mathbb{R}^{n+1}} \partial_{y_k}\zeta_{i_1\cdots i_N} F_{i_1\cdots i_N k} \,\mathrm{d}y\,.\end{aligned}$$

The last term is

$$\sum_{\substack{\bar{i}_1,\dots,\bar{i}_N,\\ i_1,\dots,i_N=0}}^{n} \int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y\cdot Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N} R^*_{\bar{i}_1\cdots\bar{i}_N}\,\mathrm{d}y^*$$
$$= \sum_{i_1,\dots,i_N=0}^{n}\int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y\cdot R_{i_1\cdots i_N}\circ Y\,\mathrm{d}y^*$$
$$= \sum_{i_1,\dots,i_N=0}^{n}\int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N} R_{i_1\cdots i_N}\,\mathrm{d}y\,,$$

since

$$Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N} R^*_{\bar{i}_1\cdots\bar{i}_N} = R_{i_1\cdots i_N}\circ Y\,,$$

which follows immediately from the definition of $R_{i_1\cdots i_N}$. Thus we have to show that

$$\sum_{\bar{i}_1,\dots,\bar{i}_N,\bar{k}=0}^{n}\int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y\cdot \partial_{y^*_{\bar{k}}}(Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N})F^*_{\bar{i}_1\cdots\bar{i}_N\bar{k}}\,\mathrm{d}y^*$$
$$+\sum_{\bar{i}_1,\dots,\bar{i}_N=0}^{n}\int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y\cdot Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N} G^*_{\bar{i}_1\cdots\bar{i}_N}\,\mathrm{d}y^*$$
$$= \int_{\mathbb{R}^{n+1}} \zeta_{i_1\cdots i_N}\circ Y\cdot G_{i_1\cdots i_N}\circ Y\,\mathrm{d}y^*\,,$$

or

$$\sum_{\bar{i}_1,\dots,\bar{i}_N=0}^{n}\Big(\sum_{\bar{k}=0}^{n}\partial_{y^*_{\bar{k}}}(Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N})F^*_{\bar{i}_1\cdots\bar{i}_N\bar{k}}$$
$$+Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N} G^*_{\bar{i}_1\cdots\bar{i}_N}\Big) = G_{i_1\cdots i_N}\circ Y\,,$$

and this is really what we have to show. Now, considering the c-integrals which give these quantities we see that it is enough to show

$$\Big(\sum_{k=1}^{n}\partial_{c_k}(\underline{c}_{i_1}\cdots\underline{c}_{i_N})\mathbf{g}_k\Big)\circ Y_B$$
$$= \sum_{\bar{i}_1,\dots,\bar{i}_N=0}^{n}\Big(\sum_{\bar{k}=0}^{n}\partial_{y^*_{\bar{k}}}(Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N})c^{*\prime}_{\bar{i}_1}\cdots c^{*\prime}_{\bar{i}_N}c^{*\prime}_{\bar{k}}$$
$$+\sum_{\bar{k}=1}^{n} Y_{i_1}{}'_{\bar{i}_1}\cdots Y_{i_N}{}'_{\bar{i}_N}\partial_{c^*_{\bar{k}}}(c^{*\prime}_{\bar{i}_1}\cdots c^{*\prime}_{\bar{i}_N})\mathbf{g}^*_{\bar{k}}\Big)\,.$$

This is satisfied, if for $m=1,\dots,N$

$$\Big(\sum_{k=1}^{n}\partial_{c_k}\underline{c}_{i_m}\cdot\mathbf{g}_k\Big)\circ Y_B$$
$$= \sum_{\bar{i}_m=0}^{n}\Big(\sum_{\bar{k}=0}^{n}\partial_{y^*_{\bar{k}}}Y_{i_m}{}'_{\bar{i}_m}\cdot c^{*\prime}_{\bar{i}_m}c^{*\prime}_{\bar{k}} + \sum_{\bar{k}=1}^{n} Y_{i_m}{}'_{\bar{i}_m}\partial_{c^*_{\bar{k}}}c^{*\prime}_{\bar{i}_m}\cdot\mathbf{g}^*_{\bar{k}}\Big)\,.$$

This is true, if for $i_m \geq 1$

$$\mathbf{g}_{i_m} \circ Y_B = \ddot{X}_{i_m} + 2 \sum_{\overline{k}=1}^{n} \dot{Q}_{i_m}{}'_{\overline{k}} c^*_{\overline{k}} + \sum_{\overline{k}=1}^{n} Q_{i_m \overline{k}} \mathbf{g}^*_{\overline{k}} \,,$$

which is equivalent to the equation (V2.4). □

Wir sehen, dass die Objektivität von Gleichungen, welche aus den Boltzmann Gleichungen abgeleitet werden, stark mit der Transformationsformel zusammenhängt, die an $\mathbf{g}$ in Analogon zur Kraft gestellt wurde. Insofern ist diese Transformationsformel für $\mathbf{g}$ eindeutig bestimmt.

4 Grad's 13-Momente Gleichung

Wir betrachten in diesem Abschnitt das System aus 13-Momenten von Grad. Es leitet sich aus der Chapman-Enskog Hierarchie her, und zwar aus der 0-ten, den 1-ten und 2-ten Momenten, sowie aus einer Spur der 3-ten Momente. Zusammen sind dies bei einer Symmetrie der 2-ten Momente

$$1 + 3 + 6 + 3 = 13 \text{ Gleichungen (bei } n = 3).$$

Wir werden dieses Differentialgleichungssystem aufstellen, und dann im nächsten Abschnitt als unabhängiges System betrachten und das Entropieprinzip für eine geeignete Entropie betrachten.

Referenzen: Wir verweisen auf den Originalartikel von Grad [108]. Desweiteren sei auf die Bücher von Müller [88] und Jou & Casas-Vázquez & Lebon [82] hingewiesen, die im Zusammenhang der "Extended Thermodynamics" erschienen sind.

Die Boltzmann Gleichung war ein Erhaltungssatz für die Wahrscheinlichkeit $(t, x, c) \mapsto f(t, x, c)$ von Partikeln mit einer Geschwindigkeit c. Daraus haben wir in 3.3 die folgenden Gleichungen für die höheren Momente

$$F_{k_1,\dots,k_M}(t,x) := \int_{\mathbb{R}^n} m c'_{k_1} \cdots c'_{k_M} f(t,x,c)\,\mathrm{d}c$$
$$\text{für } k_1,\dots,k_M \in \{0,\dots,n\}$$

sukzessive in $N \geq 0$ hergeleitet:

$$\partial_t F_{k_1,\dots,k_N} + \sum_{i=1}^{n} \partial_{x_i} F_{k_1,\dots,k_N i} = \mathbf{f}_{k_1,\dots,k_N}$$
$$\text{für } k_1,\dots,k_N \in \{0,\dots,n\}\,. \tag{V4.1}$$

Hierbei ist

$$\begin{aligned}
\mathbf{f}_{k_1,\dots,k_N} &= R_{i_1,\dots,i_N} + G_{i_1,\dots,i_N}\,, \\
R_{i_1,\dots,i_N} &:= \int_{\mathbb{R}^n} m c'_{i_1} \cdots c'_{i_N} \mathbf{r}(t,x,c)\,\mathrm{d}c\,, \\
G_{i_1,\dots,i_N} &:= \sum_i \int_{\mathbb{R}^n} m \mathbf{g}_i(t,x,c) \partial_{c_i}(c'_{i_1} \cdots c'_{i_N}) f(t,x,c)\,\mathrm{d}c\,,
\end{aligned} \tag{V4.2}$$

wobei $\mathbf{g}$ die externe Beschleunigung und $\mathbf{r}$ der Kollisionsteil der Boltzmann Gleichung ist. (Beide Funktionen, $\mathbf{g}$ und $\mathbf{r}$, hängen von c wesentlich ab!) Es sei nochmal bemerkt, dass die Gleichungen (V4.1) für N die Gleichungen für $N-1$ und kleinere Ordnungen enthalten, da $k_1, \dots, k_N$ von 0 an laufen und da wegen $c'_0 = 1$ gilt

$$F_{k_1,\dots,k_{i-1} 0 k_i,\dots,k_M} = F_{k_1,\dots,k_M}\,.$$

Mit dieser Identität ist auch zu verstehen, dass in (V4.1) die Gleichungen niedrigerer Momente dabei auch mehrfach auftreten können.

Von den unendlich vielen Gleichungen in (V4.1) betrachten wir in Grad's Theorie nur die Differentialgleichungen

$$\begin{aligned}
&\partial_t F_0 + \sum_{i=1}^n \partial_{x_i} F_i = \mathbf{f}_0\,, \\
&\partial_t F_k + \sum_{i=1}^n \partial_{x_i} F_{ki} = \mathbf{f}_k \text{ für } k = 1,\dots,n, \\
&\partial_t F_{kl} + \sum_{i=1}^n \partial_{x_i} F_{kli} = \mathbf{f}_{kl} \text{ für } k,l = 1,\dots,n, \\
&\partial_t F_{klm} + \sum_{i=1}^n \partial_{x_i} F_{klmi} = \mathbf{f}_{klm} \text{ für } k,l,m = 1,\dots,n.
\end{aligned} \tag{V4.3}$$

Hier wird von der letzten Gleichungserie nur eine Gleichung benötigt, wir bleiben zunächst aber bei den Gleichungen (V4.3). Wie in 3.3 definieren wir

$$\begin{aligned}
\varrho &:= \int_{\mathbb{R}^n} m f\,\mathrm{d}c = F_0 > 0\,, \\
\varrho v &:= \int_{\mathbb{R}^n} m f c\,\mathrm{d}c = (F_i)_{i=1,\dots,n}\,,
\end{aligned}$$

und dazu die objektiven Tensoren (siehe dazu 3.4)

$$F^0_{k_1,\dots,k_M}(t,x) := \int_{\mathbb{R}^n} m(c-v)_{k_1} \dots (c-v)_{k_M} f(t,x,c)\,\mathrm{d}c$$
$$\text{für } k_1,\dots,k_M = 1,\dots,n.$$

Damit haben wir folgende Darstellung für die Größen $F_{k_1,\dots,k_M}$: Die Differenz $F_{k_1,\dots,k_M} - F^0_{k_1,\dots,k_M}$ lässt sich als Linearkombination von Termen schreiben, die zu einer geringeren Ordnung gehören. Mit obiger Definition von ϱ und v gilt das folgende

4.1 Lemma. Definiere $F^1_{k_1,\dots,k_M} := F_{k_1,\dots,k_M} - F^0_{k_1,\dots,k_M}$, also

$$F_{k_1,\dots,k_M} = F^1_{k_1,\dots,k_M} + F^0_{k_1,\dots,k_M}\,.$$

Dann gilt für $k,l,m = 1,\dots,n$

$$\begin{aligned}
&F^1_k = \varrho v_k\,, \quad F^0_k = 0\,,\\
&F^1_{kl} = \varrho v_k v_l\,, \quad F^0_{kl} \text{ ist der Drucktensor,}\\
&F^1_{klm} = \varrho v_k v_l v_m + (F^0_{kl} v_m + F^0_{lm} v_k + F^0_{km} v_l)\\
&F^1_{klmi} = \varrho v_k v_l v_m v_i +\\
&+(F^0_{kl} v_m v_i + F^0_{lm} v_k v_i + F^0_{km} v_l v_i + F^0_{mi} v_k v_l + F^0_{li} v_k v_m + F^0_{ki} v_l v_m)\\
&+(F^0_{klm} v_i + F^0_{lmi} v_k + F^0_{kmi} v_l + F^0_{kli} v_m)\,.
\end{aligned}$$

Es sind also die Funktionen mit oberem Index 1 Polynome in v mit objektiven Koeffizientenfunktionen und diese sind die Funktionen mit oberem Index 0.

Proof. Die Darstellung von F_k ergibt sich aus obiger Definition. Zur Darstellung von F_{kl} gilt für $c \in \mathbb{R}^n$

$$c_k c_l = v_k v_l + v_k (c-v)_l + (c-v)_k v_l + (c-v)_k (c-v)_l\,,$$

also ergibt Integration über c

$$\begin{aligned}
F_{kl} &= \int_{\mathbb{R}^n} m c_k c_l f \,\mathrm{d}c\\
&= v_k v_l \int_{\mathbb{R}^n} m f \,\mathrm{d}c + v_k \int_{\mathbb{R}^n} m (c-v)_l f \,\mathrm{d}c + v_l \int_{\mathbb{R}^n} m (c-v)_k f \,\mathrm{d}c\\
&+ \int_{\mathbb{R}^n} m (c-v)_k (c-v)_l f \,\mathrm{d}c\\
&= \varrho v_k v_l + \int_{\mathbb{R}^n} m (c-v)_k (c-v)_l f \,\mathrm{d}c\\
&= \varrho v_k v_l + F^0_{kl}\,.
\end{aligned}$$

Für die Darstellung von F_{klm} bemerke, dass

$$\begin{aligned}
&c_k c_l c_m - c_k c_l v_m = c_k c_l (c-v)_m\\
&= \big(v_k v_l + v_k (c-v)_l + (c-v)_k v_l + (c-v)_k (c-v)_l\big)(c-v)_m\\
&= v_k v_l (c-v)_m + v_k (c-v)_l (c-v)_m + v_l (c-v)_k (c-v)_m\\
&+(c-v)_k (c-v)_l (c-v)_m\,,
\end{aligned}$$

also ergibt die Integration über c ähnlich wie vorher

$$\begin{aligned}
&F_{klm} - F_{kl}v_m = \int_{\mathbb{R}^n} mc_k c_l (c-v)_m f \,\mathrm{d}c \\
&= v_k v_l \int_{\mathbb{R}^n} m(c-v)_m f \,\mathrm{d}c \\
&+ v_k \int_{\mathbb{R}^n} m(c-v)_l (c-v)_m f \,\mathrm{d}c + v_l \int_{\mathbb{R}^n} m(c-v)_k (c-v)_m f \,\mathrm{d}c \\
&+ \int_{\mathbb{R}^n} m(c-v)_k (c-v)_l (c-v)_m f \,\mathrm{d}c \\
&= v_k F^0_{lm} + v_l F^0_{km} + F^0_{klm}
\end{aligned}$$

und damit

$$\begin{aligned}
&F^1_{klm} := F_{klm} - F^0_{klm} = F_{kl}v_m + F^0_{lm}v_k + F^0_{km}v_l \\
&= \varrho v_k v_l v_m + (F^0_{kl}v_m + F^0_{lm}v_k + F^0_{km}v_l)\,.
\end{aligned}$$

Schließlich gilt für die Darstellung von F_{klmi} wegen

$$\begin{aligned}
&c_k c_l c_m - (c-v)_k (c-v)_l (c-v)_m \\
&= v_k v_l v_m + v_k v_l (c-v)_m + v_k v_m (c-v)_l + v_l v_m (c-v)_k \\
&+ v_k (c-v)_l (c-v)_m + v_l (c-v)_k (c-v)_m + v_m (c-v)_k (c-v)_l
\end{aligned}$$

die Identität

$$\begin{aligned}
&c_k c_l c_m c_i - c_k c_l c_m v_i = c_k c_l c_m (c-v)_i \\
&= v_k v_l v_m (c-v)_i \\
&+ v_k v_l (c-v)_m (c-v)_i + v_k v_m (c-v)_l (c-v)_i + v_l v_m (c-v)_k (c-v)_i \\
&+ v_k (c-v)_l (c-v)_m (c-v)_i \\
&+ v_l (c-v)_k (c-v)_m (c-v)_i \\
&+ v_m (c-v)_k (c-v)_l (c-v)_i \\
&+ (c-v)_k (c-v)_l (c-v)_m (c-v)_i
\end{aligned}$$

und daraus durch Integration (wobei das erste Integral wegfällt),

$$\begin{aligned}
&F_{klmi} - F_{klm}v_i = \int_{\mathbb{R}^n} mc_k c_l c_m (c-v)_i f \,\mathrm{d}c \\
&= v_k v_l F^0_{mi} + v_k v_m F^0_{li} + v_l v_m F^0_{ki} + v_k F^0_{lmi} + v_l F^0_{kmi} + v_m F^0_{kli} + F^0_{klmi}\,.
\end{aligned}$$

Und damit ist

$$\begin{aligned}
&F^1_{klmi} := F_{klmi} - F^0_{klmi} \\
&= F_{klm}v_i + F^0_{mi}v_k v_l + F^0_{li}v_k v_m + F^0_{ki}v_l v_m + F^0_{lmi}v_k + F^0_{kmi}v_l + F^0_{kli}v_m \\
&= (\varrho v_k v_l v_m + F^0_{kl}v_m + F^0_{lm}v_k + F^0_{km}v_l + F^0_{klm})v_i \\
&+ F^0_{mi}v_k v_l + F^0_{li}v_k v_m + F^0_{ki}v_l v_m + F^0_{lmi}v_k + F^0_{kmi}v_l + F^0_{kli}v_m\,.
\end{aligned}$$

Das ist die Behauptung. □

Die Differentialgleichungen (V4.3) werden also mit 4.1 zu

$$\begin{aligned}
&\partial_t\varrho + \sum_{i=1}^{n} \partial_{x_i}(\varrho v_i) = \mathbf{f}_0\,, \\
&\partial_t(\varrho v_k) + \sum_{i=1}^{n} \partial_{x_i}(\varrho v_k v_i + F^0_{ki}) = \mathbf{f}_k\,, \\
&\partial_t(\varrho v_k v_l + F^0_{kl}) + \sum_{i=1}^{n} \partial_{x_i}(F^1_{kli} + F^0_{kli}) = \mathbf{f}_{kl}\,, \\
&\partial_t(F^1_{klm} + F^0_{klm}) + \sum_{i=1}^{n} \partial_{x_i}(F^1_{klmi} + F^0_{klmi}) = \mathbf{f}_{klm}
\end{aligned} \tag{V4.4}$$

für $k, l, m = 1, \ldots, n$, wobei die Größen mit Index 1 in 4.1 definiert sind. Es besteht natürlich die Möglichkeit, die Differentialgleichungen auch als Gleichungen für $\dot{\varrho}$, $\dot{v}$, $\dot{F}^0_{kl}$, $\dot{F}^0_{klm}$ zu schreiben, siehe [108, (5.17)]. Hierbei ist $\dot{h} = \partial_t h + v \bullet \nabla h$.

Wir haben es also mit den "abgeschnittenen" Gleichungen der unendlichen Hierarchie in Abschnitt 3 zu tun. Dieses endliche Gleichungssystem ist unterbestimmt, da es keine konstitutiven Gleichungen gibt, z.B. für den objektiven 4-Tensor F^0_{klmi}. Die Frage ist also, wie das System "abzuschließen" ist. Das reduzierte Gleichungssystem kommt dadurch zustande, dass die Spur

$$F^0_{klm} = \frac{1}{n+2}\big(S_m\delta_{kl} + S_k\delta_{lm} + S_l\delta_{km}\big)$$

mit einem Wärmefluss S betrachtet wird, siehe [108, Section 5]. Hierdurch wird das System auf 13 Gleichungen reduziert.

An Weiterem wird noch gearbeitet.

VI Speed of light

Die Geschwindigkeit des Lichtes wurde mit

$$c = 2.99792458 \cdot 10^8 \frac{m}{s}$$

gemessen (c celeritas (*lat*)), d.h. dies ist der Wert im “Vakuum”, d.h. ohne die Störung von irgendeiner Materie. Sie liegt also bei ca. 300000 Kilometer pro Sekunde. Die Lichtgeschwindigkeit in einem Medium ist kleiner. Das mag man sich mit der Partikelvorstellung am besten klarmachen. Photonen stoßen in einem Medium mit den vorhandenen Partikeln zusammen und werden so gebremst. Die Messergebnisse sagen, dass in bodennaher Luft die Lichtgeschwindigkeit ca. $299710 \frac{km}{s}$ beträgt, in Wasser ca. $225000 \frac{km}{s}$ und in Gläsern mit hoher optischer Dichte ca. $160000 \frac{km}{s}$.

[**Wikipedia: Lichtgeschwindigkeit**]**:** Seit 1983 wird das Meter über diejenige Entfernung definiert, die Licht im 299792458-ten Bruchteil einer Sekunde zurücklegt. Präzise Entfernungsmessungen werden heute direkt auf die Lichtgeschwindigkeit bezogen, z.B. bei Laserentfernungsmessern oder beim GPS (Global Positioning System).

Das Licht braucht also ca. 8 Minuten von der Sonne zur Erde, denn die Entfernung der Erde zur Sonne ist ca. 149,6 Millionen Kilometer, genauer zwischen 147,1 Mkm und 152,1 Mkm [Wikipedia: Sonne]. Bei einer Entfernung der Erde von durchschnittlich 149,6 Mkm von der Sonne ist die Geschwindigkeit der Erde (bei in Ruhe befindlicher Sonne) etwa

$$\frac{2\pi \cdot 149,6\ Mkm}{365,25\ dies} = \frac{2\pi \cdot 149,6 \cdot 10^6}{365,25 \cdot 86400} \frac{km}{s} = 29,79 \frac{km}{s} \approx 30 \frac{km}{s}\ .$$

Das ergibt einmal eine Bewegung von einer weit entfernten Lichtquelle weg und nach einem halben Jahr auf diese Lichtquelle zu, also insgesamt 0,02% der Lichtgeschwindigkeit. In Fig. 1, siehe [Wikipedia: Äther (Physik)], wurde dies in Verbindung mit einem “Äther” gebracht. Allerdings wurde experimentell immer dieselbe Lichtgeschwindigkeit gemessen, was ein Problem darstellte. In [Wikipedia: Michelson-Morley-Experiment] heißt es: “Diese Problematik konnte erst durch die Spezielle Relativitätstheorie gelöst werden, in der auf ein bevorzugtes Bezugssystem wie den Äther verzichtet wird.

Deswegen gilt es als eines der bedeutendsten Experimente in der Geschichte der Physik." Dies hat das physikalische Weltbild grundlegend verändert. Tests der speziellen Relativitätstheorie werden bis heute durchgeführt. Sie waren für die Entwicklung und Akzeptanz der Theorie von entscheidender Bedeutung, wobei moderne Experimente in Übereinstimmung mit der Theorie sind. "Die Stärke der Theorie liegt vielmehr darin, dass sie die einzige

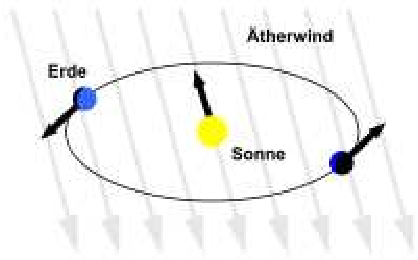

Fig. 1: "Wenn elektromagnetische Wellen an einen ruhenden Äther gebunden wären, müsste man die Eigenbewegung von Erde und Sonne als Ätherwind messen können" aus [Wikipedia: Michelson-Morley-Experiment].

ist, die mehrere grundverschiedene Experimente widerspruchsfrei erklären kann. Mögliche Abweichungen, die im Gültigkeitsbereich der speziellen Relativitätstheorie liegen, können nur noch im experimentell schwer zugänglichen Bereich der Planck-Skala oder im Neutrino-Sektor liegen." Soweit der Abschnit aus [Wikipedia: Tests der speziellen Relativitätstheorie].

Die auf dem EINSTEINschen Relativitätsprinzip (das wir kurz Relativitätsprinzip nennen werden) basierende Mechanik heißt *relativistische Mechanik*. In dem Grenzfall, in dem die Geschwindigkeiten der bewegten Körper klein gegenüber der Lichtgeschwindigkeit sind, kann der Einfluß der endlichen Wirkungsgeschwindigkeit auf die Bewegung vernachlässigt werden. Dann geht die relativistische Mechanik in die gewöhnliche Mechanik über, die die Annahme einer sofortigen Wirkungsausbreitung enthält. Diese gewöhnliche Mechanik wird auch als NEWTONsche oder klassische Mechanik bezeichnet. Der Grenzübergang von der relativistischen zur klassischen Mechanik erfolgt formal dadurch, daß wir in den Formeln der ersteren $c \to \infty$ gehen lassen.

Fig. 2: Aus Landau & Lifschitz [84, II Seite 2]

Dass die Physik, in der mit endlicher Lichtgeschwindigkeit gearbeitet wird, anders ist als die klassische Physik, drückt sich darin aus, dass statt der Newton'schen Beobachtertransformationen nun allgemeinere Transformationen erforderlich sind. Dabei werden insbesondere Lorentz-Transformationen als linere Transformationen benutzt. Dies hat Auswirkungen auf physikalische Gesetze, sie müssen objektiv sein und das hängt von den zugrundegelegten Transformationen ab. Nun enthalten die Lorentz-Transformationen die Zahl

c, wobei man die klassische Physik erhält, wenn man den Limes $\mathbf{c} \to \infty$ betrachtet. Nun ist es aber so, dass die Lorentz-Transformationen nicht die gesamte nichtklassische Physik repräsentieren können, sie generieren nur lineare Beobachtertransformationen (siehe [23, Section I.4]). Deswegen haben wir es mit allgemeineren Transformationen zu tun, und in der Tat werden die Erhaltungssätze in einer Form präsentiert, die invariant unter beliebigen Beobachtertransformationen sind.

1 Observer transformations

Wir bezeichnen die Koordinaten, die ein Beobachter in der ***Raumzeit*** $\mathbb{R}^4$ benutzt, mit $y \in \mathbb{R}^4$. Darüberhinaus hat jeder Beobachter $\mathcal{B}$ eine Matrix $\mathrm{G} : \mathbb{R}^4 \to \mathbb{R}^{4\times 4}$. Sind $\mathcal{B}$ und $\mathcal{B}^*$ zwei Beobachter, dann ist die Beobachtertransformationen $y = Y(y^*)$ definiert durch folgende Transformationsformel

$$\mathrm{G}\circ Y = \mathrm{D}Y\,\mathrm{G}^*\,(\mathrm{D}Y)^{\mathrm{T}} \ , \qquad \text{(VI1.1)}$$

was in Koordinaten geschrieben heißt

$$\mathrm{G}_{\alpha\beta}\circ Y = \sum_{\gamma,\delta=0}^{3} Y_{\alpha\,'\gamma} Y_{\beta\,'\delta} \mathrm{G}^*_{\gamma\delta} \text{ für } \alpha, \beta = 0, 1, \dots, 3.$$

Die Matrizen G und G^* sind also die Matrizen für die beiden Beobachter $\mathcal{B}$ und $\mathcal{B}^*$, und sie sind Bestandteil der physikalischen Gesetze, die die Beobachter aufstellen. Wir nehmen ohne Einschränkung an, dass sie symmetrisch sind. Und wie bisher nehmen wir an, dass die Determinante von Beobachtertransformationen gleich 1 ist.

Im Grunde liegt der Unterschied zur klassischen Physik darin, dass für relativistischen Beobachtertransformationen die Matrix G invertierbar ist wie die Standardmatrix

$$\mathrm{G}_{\mathbf{c}} := \begin{bmatrix} -\dfrac{1}{\mathbf{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix},$$

wobei $\mathbf{c} > 0$ die wesentliche Konstante ist. Im Limes $\mathbf{c} \to \infty$ wir erhalten die klassische Matrix (siehe Aufgabe 6.1)

$$\mathrm{G}_{\infty} := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}.$$

In Standardbeispielen wird im relativistischen Fall immer $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ gewählt, das heißt im Lorentzfall:

1.1 Lorentzfall. Beim ***Lorentzfall*** meinen wir einen Beobachter mit Matrix $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und Variablen $y = (t, x) \in \mathbb{R}^4$, wobei t die Zeitvariable ist und wobei $x = (x_1, x_2, x_3)$ die Raumvariablen sind. *Bemerkung:* Diese Definition betrifft nur einen einzigen Beobachter, es ist hier nicht von Beobachtertransformationen die Rede.

1.2 Notation. A contravariant m-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m}$ is defined by

$$T_{k_1\cdots k_m}\circ Y = \sum_{\overline{k}_1,\ldots,\overline{k}_m\geq 0} Y_{k_1}{}'_{\overline{k}_1}\cdots Y_{k_m}{}'_{\overline{k}_m} T^*_{\overline{k}_1\cdots\overline{k}_m}\,, \tag{VI1.2}$$

and the definition of a covariant m-tensor $T = (T_{k_1\cdots k_m})_{k_1,\ldots,k_m}$

$$T^*_{\overline{k}_1\cdots\overline{k}_m} = \sum_{k_1,\ldots,k_m\geq 0} Y_{k_1}{}'_{\overline{k}_1}\cdots Y_{k_m}{}'_{\overline{k}_m} T_{k_1\cdots k_m}\circ Y\,. \tag{VI1.3}$$

Here $y = Y(y^*)$ is the observer transformation.

A 4-vector $\underline{q}$ is

$$\begin{aligned}&\text{contravariant if} && \underline{q}\circ Y = \mathrm{D}Y\,\underline{q}^*\,,\\ &\text{covariant if} && \underline{q}^* = \mathrm{D}Y^{\mathrm{T}}\,\underline{q}\circ Y\,,\end{aligned}$$

and a 4-matrix $\underline{M}$ is

$$\begin{aligned}&\text{contravariant if} && \underline{M}\circ Y = \mathrm{D}Y\,\underline{M}^*\,\mathrm{D}Y^{\mathrm{T}}\,,\\ &\text{covariant if} && \underline{M}^* = \mathrm{D}Y^{\mathrm{T}}\,\underline{M}\circ Y\mathrm{D}Y\,.\end{aligned}$$

It is clear that G eine contravariante Matrix.

2 Maxwell equations

Die Maxwell-Gleichungen sind im Allgemeinen gegeben durch eine Erhaltungsgleichung, das Ampére'sche "Durchflutungsgesetz" (*en:* "Ampére's circuital law"). Sie wird durch eine konstitutive Annahme vervollständigt, für die das Faraday'sche "Induktionsgesetz" (*en:* "Faraday's law of induction") gilt. Dieser Abschnitt gliedert sich somit in

- "Ampére's circuital law": $\mathrm{div}_y \mathfrak{H} = \mathfrak{j}$
  In Lorentz frame:
  $$\mathfrak{H} = \begin{bmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{bmatrix}, \quad \mathfrak{j} = \begin{bmatrix} \boldsymbol{\varrho} \\ \mathbf{j}_1 \\ \mathbf{j}_2 \\ \mathbf{j}_3 \end{bmatrix},$$
  H magnetische Feldstärke, D elektrische Flussdichte,
  $\mathbf{j}$ elektrische Stromdichte ,
  $$\mathrm{div}_x D = \boldsymbol{\varrho}\,, \qquad -\partial_t D + \mathrm{rot}_x H = \mathbf{j}$$ [1])
- Consequence: $\mathrm{div}_y \mathfrak{j} = 0$
- Electrical quantities: $\mathfrak{H} = \frac{1}{\mu_0} \mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}} - \mathfrak{P}\,, \quad \varepsilon_0 \mu_0 = \frac{1}{\mathbf{c}^2}$
  In Lorentz frame:
  $$\mathfrak{E} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$
  E elektrische Feldstärke, B magnetische Flussdichte.
  $$D = \varepsilon_0 E + P\,, \quad H = \frac{1}{\mu_0} B - M$$
- "Faraday's law of induction": $\mathfrak{E}_{ik\,'l} + \mathfrak{E}_{kl\,'i} + \mathfrak{E}_{li\,'k} = 0$
  In Lorentz frame:
  $$\mathrm{div}_x B = 0\,, \qquad \partial_t B + \mathrm{rot}_x E = 0$$

Die eingerahmten Gleichungen sind die "Maxwell-Hertz'schen Gleichungen".

Referenzen: [Wikipedia: Maxwell-Gleichungen] und die englische Seite [Wikipedia: Maxwell's equations], sowie [Wikipedia: Electromagnetism], also [Wikipedia: Mathematical descriptions of the electromagnetic field].

[1] We write "rot" instead of "curl"

Die Erhaltungsgleichung von Ampére ist das

Durchflutungsgesetz (*en:* ***Ampére's circuital law***):

$$\mathrm{div}_y \mathfrak{H} = \mathfrak{j}\,,$$

$$\mathfrak{H} : \mathbb{R}^4 \to \mathbb{R}^4 \text{ schiefsymmetrisch,} \tag{VI2.1}$$

und die Testfunktionen ζ sind ein kovarianter Vektor: $\zeta^* = \mathrm{D}Y^{\mathrm{T}} \zeta \circ Y$ mit der Beobachtertransformation Y

d.h. für Testfunktionen $y \mapsto \zeta(y) \in \mathbb{R}^4$ ist

$$\int_{\mathbb{R}^4} \Big(\sum_{kl} \partial_l \zeta_k \cdot \mathfrak{H}_{kl} + \sum_k \zeta_k \, \mathfrak{j}_k \Big) \, \mathrm{dL}^4 = 0\,. \tag{VI2.2}$$

Die Schiefsymmetrie von $\mathfrak{H}$ hat folgende Konsequenz:

2.1 Lemma. Es folgt

$$\mathrm{div}_y \mathfrak{j} = 0$$

und diese Gleichung ist eine skalare Gleichung, d.h. es gilt

$$\int_{\mathbb{R}^4} \sum_k \partial_{y_k} \eta \cdot \mathfrak{j}_k \, \mathrm{dL}^4 = 0$$

für Testfunktionen η, welche sich gemäß $\eta^* = \eta \circ Y$ transformieren.

Proof. Wir setzen $\zeta_k := \partial_k \eta$ mit einer skalaren Funktion η (aus $\eta^* = \eta \circ Y$ folgt dann die an ζ geforderte Transformation). Dann ist

$$\begin{aligned} 0 &= \int_{\mathbb{R}^4} \Big(\sum_{kl} \partial_l \zeta_k \cdot \mathfrak{H}_{kl} + \sum_k \zeta_k \, \mathfrak{j}_k \Big) \, \mathrm{dL}^4 \\ &= \int_{\mathbb{R}^4} \Big(\sum_{kl} \partial_{lk} \eta \cdot \mathfrak{H}_{kl} + \sum_k \partial_k \eta \cdot \mathfrak{j}_k \Big) \, \mathrm{dL}^4 \,. \end{aligned}$$

Da $(\mathfrak{H}_{kl})_{kl}$ eine schiefsymmetrische und $(\partial_{lk}\eta)_{kl}$ eine symmetrische Matrix ist, verschwindet der erste Summand. Wir haben also

$$\int_{\mathbb{R}^4} \Big(\sum_k \partial_k \eta \cdot \mathfrak{j}_k \Big) \, \mathrm{dL}^4 = 0\,,$$

die Behauptung. □

In (VI2.1) wurde eine Aussage über die physikalische Bedeutung gemacht, nämlich dass die Testfunktion sich wie ein kovarianter Vektor verhält. Es folgt nach (I5.8), dass dies erfüllt ist, wenn für alle Beobachtertransformationen Y gilt:

$$\mathfrak{H}_{kl} \circ Y = \sum_{\overline{k},\overline{l} \geq 0} Y_{k'\overline{k}} Y_{l'\overline{l}} \, \mathfrak{H}^*_{\overline{k}\overline{l}} \,, \tag{VI2.3}$$

$$\mathfrak{j}_k \circ Y = \sum_{p,q \geq 0} Y_{k'pq} \mathfrak{H}^*_{pq} + \sum_{l \geq 0} Y_{k'l} \, \mathfrak{j}^*_l \,,$$

und da $Y_{k'pq}$ symmetrisch in p und q ist, wohingegen $\mathfrak{H}^*_{pq}$ in p und q asymmetrisch ist, verschwindet dieser Term, also ist

$$\mathfrak{j}_k \circ Y = \sum_{l \geq 0} Y_{k'\overline{k}} \, \mathfrak{j}^*_{\overline{k}} \,. \tag{VI2.4}$$

Die Bedeutung von $\mathfrak{H}$ und $\mathfrak{j}$ wird klarer, wenn man die Gleichungen im Lorentzfall hinschreibt. Zur Definition des Lorentzfalls siehe 1.1.

2.2 Speziell. Betrachte den Lorentz Fall, siehe 1.1, also $y = (t, x)$. Die Antisymmetrie von $\mathfrak{H}$ kann geschrieben werden als

$$\mathfrak{H} = \begin{bmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{bmatrix}, \quad \mathfrak{j} = \begin{bmatrix} \boldsymbol{\varrho} \\ \mathbf{j}_1 \\ \mathbf{j}_2 \\ \mathbf{j}_3 \end{bmatrix}.$$

Dann ist die Differentialgleichung in (VI2.1) äquivalent zu

$$\begin{gathered} \mathrm{div}_x D = \boldsymbol{\varrho} \,, \\ -\partial_t D + \mathrm{rot}_x H = \mathbf{j} \end{gathered} \tag{VI2.5}$$

H magnetische Feldstärke,
D elektrische Flussdichte,
$\mathbf{j}$ elektrische Stromdichte
$\boldsymbol{\varrho}$ Ladungsdichte

(Diese Gleichungen werden in der Literatur, z.B. [84, §30], auch als die “zweite Gruppe der Maxwell-Gleichungen” bezeichnet, und in [49, Drittes Buch VII §1] als die “erste Hauptgleichung des elektromagnetischen Feldes”.)
Folgerung: Aus diesen Gleichungen folgt (siehe 2.1)

$$\partial_t \boldsymbol{\varrho} + \mathrm{div}_x \mathbf{j} = 0 \,. \tag{VI2.6}$$

Wir definieren noch für jeden 3-Vektor q

$$\mathcal{R}(q) := \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix}, \tag{VI2.7}$$

d.h. $\mathcal{R}(q)$ ist diejenige Matrix, die

$$\mathcal{R}(q)z = z \times q \text{ für alle } z \in \mathbb{R}^3 \tag{VI2.8}$$

und für eine Vektorfeld $q:\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathrm{div}_x \mathcal{R}(q) = \mathrm{rot}_x q \tag{VI2.9}$$

erfüllt. Man kann nun die generellen Transformationsformeln in (VI2.3) für $\mathfrak{H}$ und $\mathfrak{j}$ umschreiben und erhält für den Lorentzfall die entsprechenden Transformationsregeln für D, H, $\boldsymbol{\varrho}$ und $\mathbf{j}$, siehe dazu 3.1.

Proof. Die Koordinaten sind $y = (t, x)$. Daher ist

$$\begin{aligned}
\mathrm{div}_y \mathfrak{H} &= \mathrm{div}_y \begin{bmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{bmatrix} \\
&= \partial_t \begin{bmatrix} 0 \\ -D_1 \\ -D_2 \\ -D_3 \end{bmatrix} + \partial_{x_1} \begin{bmatrix} D_1 \\ 0 \\ -H_3 \\ H_2 \end{bmatrix} + \partial_{x_2} \begin{bmatrix} D_2 \\ H_3 \\ 0 \\ -H_1 \end{bmatrix} + \partial_{x_3} \begin{bmatrix} D_3 \\ -H_2 \\ H_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \partial_{x_1} D_1 + \partial_{x_2} D_2 + \partial_{x_3} D_3 \\ -\partial_t D_1 + \partial_{x_2} H_3 - \partial_{x_3} H_2 \\ -\partial_t D_2 - \partial_{x_1} H_3 + \partial_{x_3} H_1 \\ -\partial_t D_3 + \partial_{x_1} H_2 - \partial_{x_2} H_1 \end{bmatrix} = \begin{bmatrix} \mathrm{div}_x D \\ -\partial_t D + \mathrm{rot}_x H \end{bmatrix}
\end{aligned}$$

(siehe auch (VI2.9)), und $\mathrm{div}_y \mathfrak{j} = \partial_t \boldsymbol{\varrho} + \mathrm{div}_x \mathbf{j}$. □

Proof der Folgerung. Man kann die letzte Gleichung natürlich auch aus den Differentialgleichungen in (t, x) herleiten. Es ist

$$\partial_t \boldsymbol{\varrho} = \partial_t \,\mathrm{div}_x D = \mathrm{div}_x(\partial_t D) = \mathrm{div}_x(\mathrm{rot}_x H - \mathbf{j}) = -\,\mathrm{div}_x \mathbf{j}\,,$$

da $\mathrm{div}_x \,\mathrm{rot}_x = 0$. □

Die Differentialgleichung (VI2.1) gilt im Allgemeinen natürlich auch im Distributionssinn, d.h. für $\mathfrak{H}_{kl}$ und $\mathfrak{j}_k$ in $\mathscr{D}'(\mathbb{R}^4)$ gilt das Gesetz (VI2.1) von Ampére in der gleichen Form

$$\boxed{\mathrm{div}_y \mathfrak{H} = \mathfrak{j} \quad \text{in } \mathscr{D}'(\mathbb{R}^4; \mathbb{R}^4)\,.} \tag{VI2.10}$$

Wir geben einige wichtige Beispiele an, und zwar betrachten wir die Situation in 2.2. Wir behandeln das elektrische Feld D bei einer Punktladung und dann das magnetische Feld H um einen elektrischen Leiter.

2.3 Beispiel. Wir betrachten jetzt distributionelle Lösungen von (VI2.5), und zwar zunächst nur einen einzelnen Beobachter. Der betrachte den stationären Fall, also $D \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}^3)$ und $\boldsymbol{\varrho} \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^3; \mathbb{R})$ zeitunabhängig sowie $H = 0$ und $\mathbf{j} = 0$. Sie seien Lösungen von

$$\operatorname{div}_x[D] = \boldsymbol{\varrho} \quad \text{in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^3; \mathbb{R}), \tag{VI2.11}$$

und trivialerweise der zweiten Differentialgleichung von (VI2.5). Gegeben sei nun die Punktladung mit Ladung $\mathrm{Q} \in \mathbb{R}$ im Punkte $0 \in \mathbb{R}^3$. Dann ist

$$D(x) := \frac{\mathrm{Q}}{4\pi} \frac{x}{|x|^3}, \quad \boldsymbol{\varrho} := \mathrm{Q}\boldsymbol{\mu}_0, \quad \langle \zeta, \boldsymbol{\mu}_0 \rangle := \int_{\mathbb{R}} \zeta(t, 0)\, \mathrm{d}t,$$

eine Distributionslösung von (VI2.11).
Zusatz: Auch das Faraday'sche Induktionsgesetz ist erfüllt (ohne Polarisation und Magnetisierung) da $\operatorname{rot}_x[D] = 0$.
Bemerkung: $\boldsymbol{\mu}_0$ ist also das eindimensionale Maß auf $\mathbb{R} \times \{0\} \subset \mathbb{R}^4$.

Für einen anderen Beobachter ist dieselbe Ladung natürlich in Bewegung, und deshalb kann dann $\mathfrak{H}$, also D und H, mit Hilfe von (VI2.3) berechnet werden, siehe dazu 3.2.

Proof. Es ist im $\mathbb{R}^3$

$$\varphi(x) := \frac{\mathrm{Q}}{4\pi|x|}, \quad \nabla_x\varphi = -\frac{\mathrm{Q}}{4\pi} \frac{x}{|x|^3}, \quad -\Delta_x[\varphi] = \mathrm{Q}\boldsymbol{\delta}_0,$$

so dass also $[D] = \nabla_x[-\varphi] = -[\nabla_x\varphi]$ in $\mathbb{R} \times \mathbb{R}^3$ und damit

$$\operatorname{div}_x[D] = \operatorname{div}_x\nabla_x[-\varphi] = -\Delta_x[\varphi] = \mathrm{Q}\boldsymbol{\mu}_0$$

in $\mathscr{D}'(\mathbb{R} \times \mathbb{R}^3; \mathbb{R})$. □

Proof Zusatz. Es ist $[D] = \nabla_x[-\varphi]$ und damit ist E, mit $[D] = \varepsilon_0[E]$, ein Gradient (siehe 2.13). Alle anderen Größen sind 0. □

2.4 Superposition. Es gilt hier das ***Gesetz der Superposition*** (*en:* ***superposition principle***). Sind Ladungen $\mathrm{Q}(x_\alpha)$ an den Punkten x_α gegeben, so ist die stationäre Lösung von (VI2.11) gegeben durch

$$D(x) = \sum_\alpha \frac{\mathrm{Q}(x_\alpha)}{4\pi} \frac{x - x_\alpha}{|x - x_\alpha|^3}, \quad \boldsymbol{\varrho} = \sum_\alpha \mathrm{Q}(x_\alpha)\boldsymbol{\mu}_{x_\alpha}.$$

Beachte, dass D nur in L^1_{loc} liegt. Bei einer kontinuierlichen Ladungsverteilung über eine Menge $\Lambda \subset \mathbb{R}^3$ mit einem Maß λ auf Λ erhält man

$$D(x) = \int_\Lambda \frac{\mathrm{Q}(x')}{4\pi} \frac{x - x'}{|x - x'|^3}\, \mathrm{d}\lambda(x'), \quad \boldsymbol{\varrho} = \mathrm{Q}\lambda.$$

Es ist

$$[D] = -\nabla_x[\varphi]\,, \quad \varphi(x) := \int_\Lambda \frac{\mathrm{Q}(x')}{4\pi|x-x'|}\,\mathrm{d}\lambda(x')\,.$$

In allen Fällen gilt $\operatorname{div}_x[D] = \boldsymbol{\varrho}$.

Wir betrachten nun das Magnetfeld um einen Strom in einem Leiter.

2.5 Beispiel. Wir betrachten distributionelle Lösungen von (VI2.5), und zwar zunächst nur einen einzelnen Beobachter. Der betrachte den stationären Fall, also sei $H \in L^1_{loc}(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$ und $\mathbf{j} \in \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)$ zeitunabhängig und Lösung von

$$\operatorname{rot}_x[H] = \mathbf{j} \quad \text{in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3)\,. \tag{VI2.12}$$

Ein Beispiel für eine solche Distribution tritt dann auf, wenn wir einen elektrischen Leiter als eindimensionales Objekt Γ im $\mathbb{R}^3$ auffassen. Ist dieser Leiter eine Gerade, also etwa

$$\Gamma := \{s\mathbf{e}_3\,;\; s\in\mathbb{R}\}\,, \quad \langle\,\zeta\,,\boldsymbol{\mu}_\Gamma\,\rangle := \int_\mathbb{R}\int_\mathbb{R} \zeta(t,s\mathbf{e}_3)\,\mathrm{d}s\,\mathrm{d}t\,,$$

und ist $\mathrm{I} \in \mathbb{R}$ ein konstanter "Strom" in Richtung $\mathbf{e}_3$, so ist

$$H(x) = \frac{\mathrm{I}}{2\pi}\frac{x_1\mathbf{e}_2 - x_2\mathbf{e}_1}{x_1^2+x_2^2}\,, \quad \mathbf{j} = \mathrm{I}\,\mathbf{e}_3\boldsymbol{\mu}_\Gamma\,,$$

eine Lösung von $\operatorname{rot}_x[H] = \mathbf{j}$. Es gilt auch $\operatorname{div}_x\mathbf{j} = 0$.
Bemerkung: Über D und $\mathbf{j}$ wird hier nichts gesagt.
Zusatz: Auch das Faraday'sche Induktionsgesetz ist erfüllt (ohne Polarisation und Magnetisierung), denn $\operatorname{div}_x[H] = 0$.

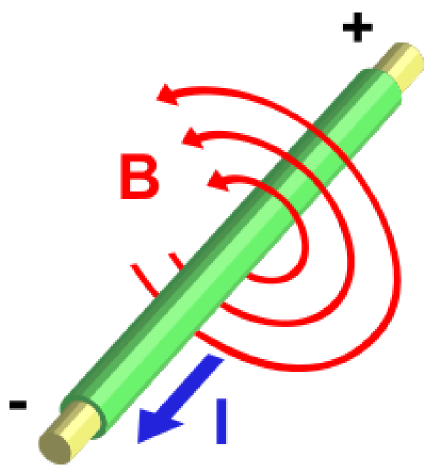

Fig. 3: Aus Wikipedia

Für einen anderen Beobachter ist der Leiter natürlich in Bewegung, und deshalb kann dann H und auch D, also $\mathfrak{H}$, mit Hilfe von (VI2.3) berechnet werden, siehe dazu 3.4.

Proof. Ohne Einschränkung sei $\mathrm{I} = 1$. Dann ist also

$$\mathbf{j} = \mathbf{e}_3 \boldsymbol{\mu}_\Gamma \,, \quad H(x) = \frac{(-x_2, +x_1, 0)}{2\pi(x_1^2 + x_2^2)} = \frac{\mathbf{e}_3 \times x}{2\pi |(x_1, x_2)|^2}$$

für $x \neq 0$, somit

$$H(x) = \big(\partial_{x_2}\varphi(x_1,x_2), -\partial_{x_1}\varphi(x_1,x_2), 0\big) \,,$$
$$\varphi(x_1,x_2) := \frac{1}{2\pi}\log\frac{1}{\sqrt{x_1^2 + x_2^2}} \,.$$

Nun ist φ die zweidimensionale Fundamentallösung des negativen Laplace-operators, d.h. es gilt

$$-\Delta_{(x_1,x_2)}[\varphi] = \boldsymbol{\delta}_0 \text{ in } \mathcal{D}'(\mathbb{R}^2) \quad \text{somit} \quad -\Delta_{(x_1,x_2,x_3)}[\varphi] = \boldsymbol{\mu}_\Gamma \text{ in } \mathcal{D}'(\mathbb{R}^3) \,.$$

Also gilt in $\mathcal{D}'(\mathbb{R}^3)$

$$[H] = \partial_{x_2}[\varphi]\,\mathbf{e}_1 - \partial_{x_1}[\varphi]\,\mathbf{e}_2 \,,$$
$$\mathrm{rot}_x[H] = (\partial_{x_1}[H_2] - \partial_{x_2}[H_1])\mathbf{e}_3 = -\Delta_x[\varphi]\mathbf{e}_3 = \mathbf{e}_3\boldsymbol{\mu}_\Gamma = \mathbf{j} \,.$$

Das ist die Behauptung an H. Da $\mathbf{j} = \mathbf{e}_3\boldsymbol{\mu}_\Gamma$ und Γ die Gerade in Richtung $\mathbf{e}_3$ ist, folgt $\mathrm{div}_x \mathbf{j} = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. □

Proof des Zusatzes. Es ist

$$[H] = \partial_2[\varphi]\,\mathbf{e}_1 - \partial_1[\varphi]\,\mathbf{e}_2 = \mathrm{rot}_x(-[\varphi]\mathbf{e}_3) \,,$$

und damit, mit $B = \mu_0 H$, ist B eine Rotation (siehe 2.13). □

Wie die Felder für eine bewegte Ladung aussehen, wird im Abschnitt 3 untersucht.

Für $\mathfrak{H}$ mögen die folgenden konstitutiven Gleichungen gelten, welche die Einführung der elektrischen Größen $\mathfrak{E}$ und der magnetischen Größen $\mathfrak{P}$ erfordert,

$$\boxed{\mathfrak{H} = \frac{1}{\mu_0}\mathrm{G}\,\mathfrak{E}\,\mathrm{G}^{\mathrm{T}} - \mathfrak{P} \,, \qquad \mathfrak{E},\, \mathfrak{P} \text{ schiefsymmetrisch,}} \tag{VI2.13}$$

in Komponentenschreibweise

$$\mathfrak{H}_{kl} = \frac{1}{\mu_0}\sum_{\overline{k},\overline{l}\geq 0} \mathrm{G}_{k\overline{k}}\mathrm{G}_{l\overline{l}}\,\mathfrak{E}_{\overline{k}\overline{l}} - \mathfrak{P}_{kl} \,.$$

Die Transformationsformeln (VI2.3) schreiben sich wie folgt um.

2.6 Lemma. Ist (VI2.13) erfüllt mit einem objektiven Skalar μ_0 und hat $\mathfrak{P}$ dieselbe Transformationformel wie $\mathfrak{H}$ (d.h. ist es ein kontravarianter Tensor), so folgt

$$\mathfrak{E}^* = \mathrm{D}Y^{\mathrm{T}}\, \mathfrak{E}{\circ}Y\, \mathrm{D}Y\,, \tag{VI2.14}$$

also ist $\mathfrak{E}$ ein kovarianter Tensor.

Proof. Es ist $\mathfrak{E}$ eingeführt durch $\mathfrak{H} + \mathfrak{P} = \frac{1}{\mu_0}\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}}$, also gilt

$$\begin{aligned}(\mathfrak{H} + \mathfrak{P}){\circ}Y &= \frac{1}{\mu_0{\circ}Y}\mathrm{G}{\circ}Y\, \mathfrak{E}{\circ}Y\, (\mathrm{G}{\circ}Y)^{\mathrm{T}} \\ &= \frac{1}{\mu_0{\circ}Y}\mathrm{D}Y\mathrm{G}^*\big(\mathrm{D}Y^{\mathrm{T}}\mathfrak{E}{\circ}Y\mathrm{D}Y\big)\mathrm{G}^{*\mathrm{T}}\mathrm{D}Y^{\mathrm{T}}\,.\end{aligned}$$

Da $\mathfrak{H}$ und $\mathfrak{P}$ kontravariante Tensoren sind, ist die linke Seite gleich

$$\begin{aligned}(\mathfrak{H} + \mathfrak{P}){\circ}Y &= \mathrm{D}Y(\mathfrak{H}^* + \mathfrak{P}^*)\mathrm{D}Y^{\mathrm{T}} \\ &= \frac{1}{\mu_0^*}\mathrm{D}Y\mathrm{G}^*\mathfrak{E}^*\mathrm{G}^{*\mathrm{T}}\mathrm{D}Y^{\mathrm{T}}\,,\end{aligned}$$

also muss $\mathfrak{E}^* = \mathrm{D}Y^{\mathrm{T}}\mathfrak{E}{\circ}Y\mathrm{D}Y$ sein. □

Die Bedeutung von $\mathfrak{E}$ und $\mathfrak{P}$ wird wiederum klar, wenn man die Gleichungen im Lorentzfall hinschreibt.

2.7 Speziell. Betrachte den Lorentz-Fall, also ist $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und die Koordinaten sind $y = (t, x) \in \mathbb{R} \times \mathbb{R}^3$. Dann schreibe

$$\mathfrak{E} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}, \quad \mathfrak{P} = \begin{bmatrix} 0 & -P_1 & -P_2 & -P_3 \\ P_1 & 0 & M_3 & -M_2 \\ P_2 & -M_3 & 0 & M_1 \\ P_3 & M_2 & -M_1 & 0 \end{bmatrix}$$

und $\mathfrak{H}$ wie in 2.2, so ist die Darstellung (VI2.13) äquivalent zu

$$D = \varepsilon_0 E + P\,, \qquad H = \frac{1}{\mu_0}B - M$$

$$\varepsilon_0\mu_0 = \frac{1}{\mathbf{c}^2}$$

E elektrische Feldstärke

B magnetische Flussdichte

$\mu_0 =$ const Permeabilität (in Vakuum)

$\varepsilon_0 =$ const Permitivität (in Vakuum)

P elektrische Polarisation

M Magnetisierung

Proof. Es ist

$$\begin{bmatrix} 0 & D^{\mathrm{T}} \\ -D & \mathcal{R}(H) \end{bmatrix} + \begin{bmatrix} 0 & -P^{\mathrm{T}} \\ P & \mathcal{R}(M) \end{bmatrix} = \mathfrak{H} + \mathfrak{P} = \frac{1}{\mu_0} \mathrm{G}_{\mathbf{c}} \mathfrak{E} \mathrm{G}_{\mathbf{c}}{}^{\mathrm{T}}$$

$$= \frac{1}{\mu_0} \begin{bmatrix} -\frac{1}{\mathbf{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \begin{bmatrix} 0 & -E^{\mathrm{T}} \\ E & \mathcal{R}(B) \end{bmatrix} \begin{bmatrix} -\frac{1}{\mathbf{c}^2} & 0 \\ 0 & \mathrm{Id} \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} 0 & \frac{1}{\mathbf{c}^2} E^{\mathrm{T}} \\ -\frac{1}{\mathbf{c}^2} E & \mathcal{R}(B) \end{bmatrix},$$

also folgt die Behauptung, denn $\varepsilon_0 \mu_0 \mathbf{c}^2 = 1$. □

The constants ε_0 and μ_0 are measured as:

electric permittivity (in vacuum or free space):

$$\varepsilon_0 = 8.854187817 \cdot 10^{-12} \frac{F}{m} \ (F\text{=farad}),$$

$$\Big(F = \frac{As}{V} = \frac{(As)^2}{kg} \Big(\frac{s}{m}\Big)^2,\ A\text{=ampere}\Big),$$

magnetic permeability (in vacuum or free space):

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{H}{m} \ (H\text{=henry}), \tag{VI2.15}$$

$$\Big(\frac{H}{m} = \frac{N}{A^2},\ N\text{=newton},\ N = \frac{kg\, m}{s^2}\Big),$$

speed of light (in vacuum or free space):

$$\mathbf{c} = c_0 = 299792.458 \frac{km}{s} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$

References: See the page [Wikipedia: Permeability (electromagnetism)] and [Wikipedia: Vacuum permittivity] and for a list of units see [130, Electromagnetism].

Die Maxwell-Gleichungen werden komplettiert durch das Faraday'sche "Induktionsgesetz" (*en:* "Faraday's law of induction"). (Die Gleichungen dieses Gesetzes werden in der Literatur, z.B. [84, §26 (26,5)], als die "erste Gruppe der Maxwell-Gleichungen" bezeichnet, und z.B. in [49, Drittes Buch VI §1] als die "zweite Hauptgleichung des elektromagnetischen Feldes".) Es gilt das

Induktionsgesetz (*en:* ***Faraday's law of induction***):

$$\mathfrak{E}_{ik\,'l} + \mathfrak{E}_{kl\,'i} + \mathfrak{E}_{li\,'k} = 0 \text{ für } i,k,l = 0,\ldots,3 \tag{VI2.16}$$

Da $\mathfrak{E}$ schiefsymmetrisch, nur für verschiedene Indizes nichttrivial, d.h. für $\{i,k,l\}$ gleich $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$, $\{1,2,3\}$

Wir zeigen nun, dass dieses Gesetz bei Beobachterwechsel gleich bleibt.

2.8 Theorem. Let $F:\mathbb{R}^4 \to \mathbb{R}^{4\times 4}$ be a field satifying

$$\begin{aligned} F_{ik\,'l} + F_{kl\,'i} + F_{li\,'k} = 0 \text{ for } i,k,l = 0,\dots,3, \\ F_{kl} + F_{lk} = 0 \text{ for } k,l = 0,\dots,3. \end{aligned} \tag{VI2.17}$$

If it transforms like

$$F^*_{ij} = \sum_{k,l} Y_{k\,'i} Y_{l\,'j} F_{kl} \circ Y\,,$$

i.e. if it is a covariant tensor, then (VI2.17) is objective.

Proof. It is

$$\begin{aligned}
&F^*_{ik\,'l} + F^*_{kl\,'i} + F^*_{li\,'k} \\
&= \Big(\sum_{\bar i,\bar k} Y_{\bar i\,'i} Y_{\bar k\,'k} F_{\bar i\bar k}\circ Y\Big)_{'l} + \Big(\sum_{\bar k,\bar l} Y_{\bar k\,'k} Y_{\bar l\,'l} F_{\bar k\bar l}\circ Y\Big)_{'i} \\
&\qquad + \Big(\sum_{\bar l,\bar i} Y_{\bar l\,'l} Y_{\bar i\,'i} F_{\bar l\bar i}\circ Y\Big)_{'k} \\
&= \sum_{\bar i,\bar k} (Y_{\bar i\,'i} Y_{\bar k\,'k})_{'l} F_{\bar i\bar k}\circ Y + \sum_{\bar k,\bar l} (Y_{\bar k\,'k} Y_{\bar l\,'l})_{'i} F_{\bar k\bar l}\circ Y \\
&\qquad + \sum_{\bar l,\bar i} (Y_{\bar l\,'l} Y_{\bar i\,'i})_{'k} F_{\bar l\bar i}\circ Y \\
&\quad + \sum_{\bar i,\bar k,\bar l} Y_{\bar i\,'i} Y_{\bar k\,'k} Y_{\bar l\,'l} F_{\bar i\bar k\,'\bar l}\circ Y + \sum_{\bar k,\bar l,\bar i} Y_{\bar k\,'k} Y_{\bar l\,'l} Y_{\bar i\,'i} F_{\bar k\bar l\,'\bar i}\circ Y \\
&\qquad + \sum_{\bar l,\bar i,\bar k} Y_{\bar l\,'l} Y_{\bar i\,'i} Y_{\bar k\,'k} F_{\bar l\bar i\,'\bar k}\circ Y \\
&= \sum_{\bar i,\bar k} \Big((Y_{\bar i\,'i} Y_{\bar k\,'k})_{'l} + (Y_{\bar i\,'k} Y_{\bar k\,'l})_{'i} + (Y_{\bar i\,'l} Y_{\bar k\,'i})_{'k} \Big) F_{\bar i\bar k}\circ Y \\
&+ \sum_{\bar i,\bar k,\bar l} Y_{\bar i\,'i} Y_{\bar k\,'k} Y_{\bar l\,'l} \big(F_{\bar i\bar k\,'\bar l} + F_{\bar k\bar l\,'\bar i} + F_{\bar l\bar i\,'\bar k} \big)\circ Y \\
&= \sum_{\bar i,\bar k,\bar l} Y_{\bar i\,'i} Y_{\bar k\,'k} Y_{\bar l\,'l} \big(F_{\bar i\bar k\,'\bar l} + F_{\bar k\bar l\,'\bar i} + F_{\bar l\bar i\,'\bar k} \big)\circ Y\,,
\end{aligned}$$

since F is antisymmetric and

$$\begin{aligned}
&(Y_{\bar i\,'i} Y_{\bar k\,'k})_{'l} + (Y_{\bar i\,'k} Y_{\bar k\,'l})_{'i} + (Y_{\bar i\,'l} Y_{\bar k\,'i})_{'k} \\
&= Y_{\bar i\,'il} Y_{\bar k\,'k} + Y_{\bar i\,'i} Y_{\bar k\,'kl} + Y_{\bar i\,'ki} Y_{\bar k\,'l} + Y_{\bar i\,'k} Y_{\bar k\,'li} + Y_{\bar i\,'lk} Y_{\bar k\,'i} + Y_{\bar i\,'l} Y_{\bar k\,'ik}
\end{aligned}$$

symmetric in $\bar i$ and $\bar k$. □

2.9 Speziell. Betrachte den Lorentz-Fall, also $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und die Koordinaten sind $y = (t, x)$. Dann ist das Induktionsgesetz (VI2.16) äquivalent zu

$$\mathrm{div}_x B = 0\,,$$
$$\partial_t B + \mathrm{rot}_x E = 0$$

E elektrische Feldstärke,
B magnetische Flussdichte.

Proof. Es sind nur vier Gleichungen des Induktionsgesetzes nichttrivial und unabhängig voneinander, und zwar für (i, k, l) gleich $(0, 1, 2)$, $(0, 1, 3)$, $(0, 2, 3)$, $(1, 2, 3)$. Wir erhalten

$$\begin{aligned}
(0,1,2): &\quad -E_{1'2} + B_{3'0} + E_{2'1} = 0\,,\\
(0,1,3): &\quad -E_{1'3} - B_{2'0} + E_{3'1} = 0\,,\\
(0,2,3): &\quad -E_{2'3} + B_{1'0} + E_{3'2} = 0\,,\\
(1,2,3): &\quad B_{3'3} + B_{1'1} + B_{2'2} = 0\,.
\end{aligned}$$

□

Insgesamt sind die elektrischen Maxwell-Hertz-Gleichungen im Lorentzfall gleich

Maxwell-Gleichungen im Lorentzfall:

$$\begin{gathered}
\mathrm{div}_x D = \boldsymbol{\varrho}\,, \quad -\partial_t D + \mathrm{rot}_x H = \mathbf{j}\,,\\
D = \varepsilon_0 E + P\,, \quad H = \frac{1}{\mu_0} B - M\,, \quad \varepsilon_0 \mu_0 = \frac{1}{\mathbf{c}^2}\,,\\
\mathrm{div}_x B = 0\,, \quad \partial_t B + \mathrm{rot}_x E = 0
\end{gathered} \tag{VI2.18}$$

Folgerung: $\partial_t \boldsymbol{\varrho} + \mathrm{div}_x \mathbf{j} = 0\,.$

Es gibt eine Methode, das Faraday'sche Induktionsgesetz zu erfüllen, und die ist $\mathfrak{E}$ durch Ableitungen auszudrücken. Dies folgt aus den Voraussetzungen von Faraday, indem man das Lemma von Poincaré anwendet,

2.10 Lemma von Poincaré. [2] "Auf einem einfach zusammenhängenden Gebiet $\mathcal{U}$ im $\mathbb{R}^N$ ist jede ***geschlossene*** Differentialform, d.h. ω mit $\mathrm{d}\omega = 0$, eine ***exakte*** Form, d.h. es gibt eine Differentialform λ mit $\mathrm{d}\lambda = \omega$."
Hinweis: λ ist nicht eindeutig definiert.
Wir nehmen nun $N = 4$ und es sind die folgenden Beispiele von Bedeutung.

[2] Wir setzen elementare Kenntnisse über Differentialformen voraus.

(1) Es sei $\underline{F} = (F_k)_{k\geq 0} : \mathcal{U} \to \mathbb{R}^4$ mit $\mathrm{d}\Big(\sum_{k\geq 0} F_k \mathrm{d}y_k\Big) = 0$. Dann gibt es eine Funktion $\phi : \mathcal{U} \to \mathbb{R}$ mit $\underline{F} = -\underline{\nabla}\phi$, d.h.

$$\mathrm{d}\phi = -\sum_{k\geq 0} F_k \mathrm{d}y_k\,.$$

(2) Es sei $\mathfrak{E} = (E_{kl})_{k,l\geq 0} : \mathcal{U} \to \mathbb{R}^{4\times 4}$ mit $\mathrm{d}\Big(\sum_{k,l\geq 0} E_{kl} \mathrm{d}y_k \wedge \mathrm{d}y_l\Big) = 0$ und $E_{kl} + E_{lk} = 0$ für alle $k, l \geq 0$. Dann gibt es ein $\underline{A} = (A_k)_{k\geq 0} : \mathcal{U} \to \mathbb{R}^4$ mit

$$\mathrm{d}\Big(\sum_{k\geq 0} A_k \mathrm{d}y_k\Big) = \frac{1}{2} \sum_{k,l\geq 0} E_{kl} \mathrm{d}y_k \wedge \mathrm{d}y_l\,,$$

d.h. $E_{kl} = \partial_k A_l - \partial_l A_k$ für $k, l \geq 0$.

Proof Hinweis. Jedes λ' mit $\mathrm{d}\lambda' = \omega$ ist ebenfalls eine Lösung. Also gilt: Ist μ' mit $\mathrm{d}\mu' = 0$ so ist $\lambda' := \lambda + \mu'$ eine weitere Lösung. Es sei verwiesen auf [21, Poincaré Lemma]. Siehe auch [Wikipedia: Poincaré-Lemma]. □

Proof (1). Die Vorausstzung ist

$$0 = \mathrm{d}\Big(\sum_{k\geq 0} F_k \mathrm{d}y_k\Big) = \sum_{k,l\geq 0} \partial_l F_k \mathrm{d}y_l \wedge \mathrm{d}y_k = \sum_{k<l} (\partial_l F_k - \partial_k F_l) \mathrm{d}y_l \wedge \mathrm{d}y_k\,,$$

also $\partial_l F_k - \partial_k F_l = 0$ für $k < l$. As Folgerung haben wir

$$-\sum_{k\geq 0} F_k \mathrm{d}y_k = \mathrm{d}\phi = \sum_{k\geq 0} \partial_k \phi \mathrm{d}y_k$$

also $-F_k = \partial_k \phi$, oder $\underline{F} = -\underline{\nabla}\phi$. □

Proof (2). Die Voraussetzung sagt, dass

$$\begin{aligned} 0 &= \mathrm{d}\Big(\sum_{k,l\geq 0} E_{kl}\, \mathrm{d}y_k \wedge \mathrm{d}y_l\Big) = \sum_{j,k,l\geq 0} \partial_j E_{kl}\, \mathrm{d}y_j \wedge \mathrm{d}y_k \wedge \mathrm{d}y_l \\ &= \frac{1}{3} \sum_{j,k,l\geq 0} (\partial_j E_{kl} + \partial_k E_{lj} + \partial_l E_{jk})\, \mathrm{d}y_j \wedge \mathrm{d}y_k \wedge \mathrm{d}y_l \\ &= 2 \sum_{j<k<l} (\partial_j E_{kl} + \partial_k E_{lj} + \partial_l E_{jk})\, \mathrm{d}y_j \wedge \mathrm{d}y_k \wedge \mathrm{d}y_l\,, \end{aligned}$$

also $\partial_j E_{kl} + \partial_k E_{lj} + \partial_l E_{jk} = 0$ für $0 \leq j < k < l \leq 3$. Und die Konklusion besagt, dass

$$\begin{aligned} 2\sum_{k<l} E_{kl}\, \mathrm{d}y_k \wedge \mathrm{d}y_l &= \sum_{k,l\geq 0} E_{kl}\, \mathrm{d}y_k \wedge \mathrm{d}y_l = 2\mathrm{d}\big(\sum_{k\geq 0} A_k\, \mathrm{d}y_k\big) \\ &= 2\sum_{k,l\geq 0} \partial_l A_k\, \mathrm{d}y_l \wedge \mathrm{d}y_k = 2\sum_{k<l} (\partial_k A_l - \partial_l A_k)\, \mathrm{d}y_k \wedge \mathrm{d}y_l\,, \end{aligned}$$

was $E_{kl} = \partial_k A_l - \partial_l A_k$ für $k < l$ also auch für $k, l \geq 0$ bedeutet. □

2.11 4-Vektorpotential. Die Voraussetzung (VI2.16) impliziert nach dem Lemma von Poincaré, dass

$$\mathfrak{E}_{ik} = \underline{A}_{k\,'i} - \underline{A}_{i\,'k} \text{ für } i,k = 0,\ldots,3 \tag{VI2.19}$$

mit $\underline{A}:\mathbb{R}^4 \to \mathbb{R}^4$, Wenn $\underline{A}$ gegeben ist, so erfüllt $\underline{A}^* = \mathrm{D}Y^{\mathrm{T}}\underline{A}{\circ}Y$ ebenfalls die Eigenschaft (VI2.19) mit $\mathfrak{E}^*$. *Warnings:* Die Existenz von $\underline{A}$ ist nur für ein einfach zusammenhängendes Gebiet in der Raumzeit gesichert. Und: Falls $\underline{A}$ durch eine Integralformel mittels $\mathfrak{E}$ dargestellt wird, führt die Nichteindeutigkeit von $\underline{A}$ auch dazu, dass $\underline{A}^*$ nicht durch dieselbe Integralformel, natürlich auf $\mathfrak{E}^*$ angewandt, dargestellt wird.

Also ist im Folgenden immer vorausgesetzt, dass das Gebiet einfach zusammenhängend ist, was z.B. erfüllt ist, wenn der ganze Raum betrachtet wird. Die Darstellung (VI2.19) ist in der Marixschreibweise

$$\boxed{\mathfrak{E} = (\mathrm{D}\underline{A})^{\mathrm{T}} - \mathrm{D}\underline{A}\,.} \tag{VI2.20}$$

Proof. Die Faraday'sche Voraussetzung (VI2.16) an $\mathfrak{E}$ ist äquivalent, siehe den Beweis von 2.10(2), zu der Voraussetzung in 2.10(2). Die Konklusion von 2.10(2) sagt aus, dass $\mathfrak{E}_{kl} = \partial_k \underline{A}_l - \partial_l \underline{A}_k = \underline{A}_{l\,'k} - \underline{A}_{k\,'l}$. Aus dieser Darstellung folgt natürlich, dass $\mathfrak{E}$ antisymmetrisch ist und das Faraday'sche Induktionsgesetz (VI2.16) erfüllt. Wenn wir dann

$$\underline{A}^*_k = \textstyle\sum_{\overline{k}} Y_{\overline{k}\,'k} \underline{A}_{\overline{k}} \circ Y \tag{VI2.21}$$

definieren, also durch die Transformationsformel für kovariante Vektoren $\underline{A}$, so folgt durch Bildung der Ableitungen

$$\underline{A}^*_{k\,'i} = \sum_{\overline{k}} Y_{\overline{k}\,'k} \sum_{\overline{i}} Y_{\overline{i}\,'i} \underline{A}_{\overline{k}\,'\overline{i}} \circ Y + \sum_{\overline{k}} Y_{\overline{k}\,'ki} \underline{A}_{\overline{k}} \circ Y\,.$$

Da $Y_{\overline{k}\,'ki}$ symmetrisch in k und i ist, folgt, wenn $\mathfrak{E}^*_{ik} := \underline{A}^*_{k\,'i} - \underline{A}^*_{i\,'k}$,

$$\mathfrak{E}^*_{ik} = \underline{A}^*_{k\,'i} - \underline{A}^*_{i\,'k} = \sum_{\overline{k},\overline{i}} Y_{\overline{k}\,'k} Y_{\overline{i}\,'i} (\underline{A}_{\overline{k}\,'\overline{i}} - \underline{A}_{\overline{i}\,'\overline{k}}) \circ Y = \sum_{\overline{k},\overline{i}} Y_{\overline{k}\,'k} Y_{\overline{i}\,'i} \mathfrak{E}_{\overline{i}\,'\overline{k}} \circ Y\,,$$

was die Transformationsformel für $\mathfrak{E}$ ist. □

Das Vektorpotential ist, wie beim Lemma von Poincaré schon gesagt, nicht eindeutig definiert, es gilt die

2.12 Gauge invariance. Ist f irgendeine skalare Funktion und wird in 2.11 das Vektorpotential $\underline{A}$ durch

$$\underline{A}' := \underline{A} + \underline{\nabla} f$$

ersetzt, so ändert sich dadurch $\mathfrak{E}$ nicht. *Bemerkung:* Ist f ein objektiver Skalar, so ist $\underline{\nabla} f$ kovariant, also mit $\underline{A}$ auch $\underline{A}'$ ein kovarianter Vektor.

Proof. Folgt unmittelbar aus (VI2.19), siehe auch Aufgabe 6.5. □

2.13 Speziell. Im Lorentz Fall 2.7 gilt mit $\underline{A} =: (-\Phi, A_1, A_2, A_3)$

$$E_i = -\partial_{x_i}\Phi - \partial_t A_i \text{ für } i = 1,2,3,$$
$$B = \mathrm{rot}_x A \text{ wobei } A = (A_1, A_2, A_3)\,,$$

also

$$\boxed{E = -\nabla_x\Phi - \partial_t A\,, \quad B = \mathrm{rot}_x A\,.}$$

Die Eichinvarianz lautet dann, dass für eine skalare Funktion f gilt, dass

$$\Phi' := \Phi - \partial_t f \quad \text{und} \quad A' := A + \nabla_x f$$

dieselben Felder E und B ergeben.

Proof. Dies folgt als Spezialfall aus der Eichinvarianz 2.12, kann aber auch diekt eingesehen werden:

$$-E = \nabla_x\Phi + \partial_t A = \nabla_x\Phi' + \nabla_x\partial_t f + \partial_t A' - \partial_t\nabla_x f = \nabla_x\Phi' + \partial_t A'\,,$$
$$B = \mathrm{rot}_x A = \mathrm{rot}_x(A' - \nabla_x f) = \mathrm{rot}_x A' - \mathrm{rot}_x\nabla_x f = \mathrm{rot}_x A'\,,$$

□

Damit erhalten wir im allgemeinen Fall

2.14 Maxwell Gleichungen ohne Polarisation. In Abwesenheit von $\mathfrak{P}$ gilt

$$\sum_{j\geq 0}\partial_j\Big(\sum_{k\neq l}(\mathrm{G}_{ik}\mathrm{G}_{jl} - \mathrm{G}_{il}\mathrm{G}_{jk})\partial_l\underline{A}_k\Big) = \mu_0\,\mathfrak{j}_i$$

für alle i. Das sind also die gesamten Maxwell-Gleichungen ohne Polarisation und Magnetisierung, d.h. $\mathfrak{P} = 0$.

Proof. Es gilt, da $\mathfrak{P} = 0$ ist, das Ampére'sche Durchflutungsgesetz

$$\mathfrak{j} = \underline{\mathrm{div}}\mathfrak{H} = \frac{1}{\mu_0}\underline{\mathrm{div}}(\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}}) = \frac{1}{\mu_0}\underline{\mathrm{div}}\big(G(\mathrm{D}\underline{A} - \mathrm{D}\underline{A}^{\mathrm{T}})\mathrm{G}^{\mathrm{T}}\big)$$

und andererseits ist das Faraday'sche Induktionsgesetz wegen der Einführung des 4er-Vektorpotentials $\underline{A}$ erfüllt. □

Die Abwesenheit von $\mathfrak{P}$ bedeutet, dass die Gleichung

$$\mathfrak{H} = \frac{1}{\mu_0}\mathrm{G}\mathfrak{E}\mathrm{G}^{\mathrm{T}}$$

erfüllt ist. Im Lorentz-Fall $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und $y = (t, x)$ folgt damit, dass $\underline{\mathrm{div}}\mathfrak{H} = \mathfrak{j}$ äquivalent ist zu

$$\boxed{\begin{aligned} \mathrm{div} E &= \frac{\varrho}{\varepsilon_0}\,, \\ -\frac{1}{\mathbf{c}^2}\partial_t E + \mathrm{rot} B &= \mu_0\,\mathbf{j}\,. \end{aligned}} \tag{VI2.22}$$

Daraus folgt, siehe [48, (6.10) and (6.11)], dass die Maxwell-Gleichungen zu den folgenden Differentialgleichungen äquivalent sind.

2.15 Speziell. Betrachte den Lorentz-Fall, also $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und $y = (t, x)$. Dann sind die Maxwell-Gleichungen ohne Polarisation 2.14 äquivalent zu

$$-\Delta\Phi - \partial_t\,\mathrm{div} A = \frac{\varrho}{\varepsilon_0}\,,$$
$$\frac{1}{\mathbf{c}^2}\partial_t^2 A - \Delta A + \nabla\Big(\frac{1}{\mathbf{c}^2}\partial_t\Phi + \mathrm{div} A\Big) = \mu_0\,\mathbf{j}\,.$$

Proof. Wenn $\mathfrak{P} = 0$ ist, also keine Polarisation P und keine Magnetisierung M präsent ist, so sind sie Maxwell-Gleichungen äquivalent zu (VI2.22) und den Gleichungen in 2.13

$$E = -\nabla\Phi - \partial_t A\,, \quad B = \mathrm{rot} A\,.$$

Einsetzen dieser Gleichungen in (VI2.22) ergibt die Behauptung, wobei die Gleichung $\mathrm{rot}\,\mathrm{rot} A = -\Delta A + \nabla\,\mathrm{div} A$ gilt, siehe Übung 6.4. □

We bring this system in a simpler version by using the gauge invariance. This is the Lorenz gauge.[3]

Referenzen: For the Lorenz gauge see [Wikipedia: Lorenz gauge condition]., sowie Jackson [48, 6.3 Gauge Transformations, Lorenz Gauge, Coulomb Gauge]. Es sei auch verwiesen auf das Originalpaper von Lorenz [115].

2.16 Lorenz condition (L. Lorenz 1867). Assume the Lorentz frame $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ and $y = (t, x)$. Then by a certain gauge transformation one can assume[4]

$$\frac{1}{\mathbf{c}^2}\partial_t\Phi + \mathrm{div}_x A = 0\,. \tag{VI2.23}$$

Remark: This condition is observer independent if written $\underline{\mathrm{div}}(\mathrm{G}\underline{A}) = 0$.

Proof. If $\underline{A}$ is the original quantity we switch to $\underline{A}' = \underline{A} + \underline{\nabla} f$. We want that $\underline{\mathrm{div}}(\mathrm{G}\underline{A}') = 0$ which means

$$\underline{\mathrm{div}}(-\mathrm{G}\underline{\nabla} f) = \underline{\mathrm{div}}(\mathrm{G}\underline{A})\,.$$

[3] Two persons, the Danish physicist and mathematician Ludwig Lorenz (1829-1891) and the Dutch physicist Hendrik Antoon Lorentz (1853-1928)

[4] see [84, §46], specially the correct second comment in this paragraph

Given $\underline{A}$ this differential equation has a solution f. The solution f is unique under certain given boundary values in spacetime (that is, the correct initial and boundary values). Hence the new vector $\underline{A}'$ has the property $\underline{\operatorname{div}}(\mathrm{G}\underline{A}') = 0$. In the Lorentz case one has

$$\underline{\operatorname{div}}(-\mathrm{G}\nabla f) = \frac{1}{\mathbf{c}^2}\partial_t^2 f - \Delta f$$

and with $\underline{A}' = (-\Phi', A_1', A_2', A_3')$

$$0 = \underline{\operatorname{div}}(\mathrm{G}\underline{A}') = \frac{1}{\mathbf{c}^2}\partial_t \Phi' + \operatorname{div} A' \,.$$

□

Proof Remark. By (VI2.21) $\underline{A}$ is a covariant vector and therefore $\mathrm{G}\underline{A}$ a contravariant vector so that $\underline{\operatorname{div}}(\mathrm{G}\underline{A})$ is an objevtive scalar. □

With this we obtain the wave equations in electrodynamics.

2.17 Electromagnetic waves. By choosing the Lorenz condition 2.16 the Maxwell equations without polarization 2.15 read

$$\begin{aligned}
&\frac{1}{\mathbf{c}^2}\partial_t^2 \Phi - \Delta\Phi = \frac{\boldsymbol{\varrho}}{\varepsilon_0}\,,\\
&\frac{1}{\mathbf{c}^2}\partial_t^2 A - \Delta A = \mu_0\,\mathbf{j}\,,\\
&\frac{1}{\mathbf{c}^2}\partial_t \Phi + \operatorname{div} A = 0\,.
\end{aligned}$$

These are two wave equations with a differential equation coupling them.

This you will find in [48, (6.14)–(6.16)].

Proof. Insert the identity (VI2.23) into the equations in 2.15. □

In the general case where $\mathrm{G} = \text{const}$ these equations read as follows.

2.18 General electromagnetic waves. If $\mathrm{G} = \text{const}$ then Maxwell equations without polarization 2.15 read

$$\sum_{j,l}\partial_{jl}\Big(\mathrm{G}_{jl}\Big(\sum_k \mathrm{G}_{ik}\underline{A}_k\Big)\Big) = \mu_0\,\underline{\mathrm{j}}_i \quad \text{für alle } i \geq 0,$$

$$\sum_j \partial_j\Big(\sum_k \mathrm{G}_{jk}\underline{A}_k\Big) = 0\,.$$

Proof. Nach 2.14 ist, da G konstant ist,

$$\begin{aligned}
\mu_0\,\underline{\mathrm{j}}_i &= \sum_j \partial_j\Big(\sum_{k,l}(\mathrm{G}_{ik}\mathrm{G}_{jl} - \mathrm{G}_{il}\mathrm{G}_{jk})\partial_l\underline{A}_k\Big)\\
&= \sum_{j,l}\partial_{jl}\Big(\mathrm{G}_{jl}\sum_k \mathrm{G}_{ik}\underline{A}_k\Big) - \sum_l \partial_l\Big(\mathrm{G}_{il}\underbrace{\sum_j \partial_j\Big(\sum_k \mathrm{G}_{jk}\underline{A}_k\Big)}_{=0}\Big)\,,
\end{aligned}$$

where the second summand is zero. □

Remark: Es wird noch die Aussendung von Licht behandelt.

3 Moving charges

Hier untersuchen wir, wie die Felder bei zeitabhängigen Ladungen aussehen, wobei wir die Maxwell-Gleichungen für Distributionen, siehe (VI2.10), benutzen müssen, wie das auch schon in den Beispielen einer Punktladung in 2.3 und eines Leiters in 2.5 der Fall war. Wie dort betrachten wir den Lorentz-Fall, also ist $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und $y = (t, x) \in \mathbb{R} \times \mathbb{R}^3$ sind die Koordinaten. Die Distrubutionslösungen sind dann die entsprechenden Lösungen von (VI2.5).

Referenzen: Für eine gleichförmige bewegte Ladung siehe [84, Kap. V §38], für das Feld einer allgemein bewegten Ladung siehe [84, Kap. VIII]. ....

Aus den generellen Transformationsformeln in (VI2.3) für $\mathfrak{H}$ und $\mathfrak{j}$ folgen folgende Transformationsformeln für D, H, $\boldsymbol{\varrho}$ und $\mathbf{j}$, und zwar im Spezialfall einer linearen Beobachtertransformation.

3.1 Spezielle Transformationsformel. Wir betrachten zwei Beobachter, die im Lorentz-Fall wie in 2.2 sind. Dann gilt für die Transformationsformeln (VI2.3) und (VI2.4), wenn die Beobachtertransformation Y speziell eine Lorentz-Transformation mit $\mathrm{D}Y = \mathbf{L}_{\mathbf{c}}(V, Q)$ ist, wobei V und Q Konstanten sind,

$$\begin{aligned} D\circ Y &= \gamma \mathbf{B}(V)^{-1} Q D^* - \frac{\gamma}{\mathbf{c}^2}\mathbf{B}(V)(V\times QH^*)\,, \\ H\circ Y &= \gamma\, V\times(\mathbf{B}(V)QD^*) + \gamma \mathbf{B}(V)^{-1}QH^*\,, \end{aligned} \tag{VI3.1}$$

sowie

$$\begin{aligned} \boldsymbol{\varrho}\circ Y &= \gamma\boldsymbol{\varrho}^* + \frac{\gamma}{\mathbf{c}^2}V\bullet(Q\mathbf{j}^*)\,, \\ \mathbf{j}\circ Y &= \gamma\boldsymbol{\varrho}^* V + \mathbf{B}(V)Q\mathbf{j}^*\,. \end{aligned} \tag{VI3.2}$$

Beachte: Diese Formeln gelten nur für Funktionen, die Dichten des Lebesgue-Maßes in der Raumzeit sind.

Proof. Die Transformationsformeln (VI2.3) und (VI2.4) besagen

$$\begin{bmatrix} 0 & D^{\mathrm{T}} \\ -D & \mathcal{R}(H) \end{bmatrix}\circ Y = \begin{bmatrix} \gamma & \frac{\gamma}{\mathbf{c}^2}V^{\mathrm{T}}Q \\ \gamma V & \mathbf{B}(V)Q \end{bmatrix} \begin{bmatrix} 0 & D^{*\mathrm{T}} \\ -D^* & \mathcal{R}(H^*) \end{bmatrix} \begin{bmatrix} \gamma & \gamma V^{\mathrm{T}} \\ \frac{\gamma}{\mathbf{c}^2}Q^{\mathrm{T}}V & Q^{\mathrm{T}}\mathbf{B}(V) \end{bmatrix}$$

und

$$\begin{bmatrix} \boldsymbol{\varrho} \\ \mathbf{j} \end{bmatrix}\circ Y = \begin{bmatrix} \gamma & \frac{\gamma}{\mathbf{c}^2}V^{\mathrm{T}}Q \\ \gamma V & \mathbf{B}(V)Q \end{bmatrix} \begin{bmatrix} \boldsymbol{\varrho}^* \\ \mathbf{j}^* \end{bmatrix}.$$

Ausrechnen ergibt nun wegen $\mathbf{B}(V) - \frac{\gamma}{\mathbf{c}^2}VV^{\mathrm{T}} = \mathbf{B}(V)^{-1}$ und wenn wir $B := \mathbf{B}(V)$ schreiben

$$\begin{aligned} D\circ Y &= \gamma B^{-1}QD^* - \frac{\gamma}{\mathbf{c}^2}BQ\mathcal{R}(H^*)Q^{\mathrm{T}}V\,, \\ \mathcal{R}(H)\circ Y &= \gamma\big(V\otimes(BQD^*) - (BQD^*)\otimes V\big) + BQ\mathcal{R}(H^*)Q^{\mathrm{T}}B, \end{aligned}$$

sowie die behaupteten Formeln (VI3.2) für ϱ and $\mathbf{j}$. Nun gilt für $z_1, z_2 \in \mathbb{R}^3$ unter Benutzung von (VI2.8) und wegen der Symmetrie der Matrix B

$$\begin{aligned} &z_2 \bullet \big(BQ\mathcal{R}(H^*)Q^{\mathrm{T}}B^{\mathrm{T}}\big)z_1 = (Q^{\mathrm{T}}Bz_2)\bullet\big(\mathcal{R}(H^*)Q^{\mathrm{T}}Bz_1\big) \\ &= (Q^{\mathrm{T}}Bz_2)\bullet\big((Q^{\mathrm{T}}Bz_1)\times H^*\big) \\ &= H^*\bullet\big((Q^{\mathrm{T}}Bz_2)\times(Q^{\mathrm{T}}Bz_1)\big) = (QH^*)\bullet\big((Bz_2)\times(Bz_1)\big) \end{aligned} \tag{VI3.3}$$

Since this is equal

$$= (Bz_2)\bullet\big((Bz_1)\times(QH^*)\big) = z_2\bullet B\big((Bz_1)\times(QH^*)\big)$$

we obtain

$$BQ\mathcal{R}(H^*)Q^{\mathrm{T}}B^{\mathrm{T}}z_1 = B\big((Bz_1)\times QH^*\big)\,.$$

Now $z_1 := B^{-\mathrm{T}}V$ gives

$$BQ\mathcal{R}(H^*)Q^{\mathrm{T}}V = B\big(V\times\big(QH^*\big)\big)$$

and we obtain for the field D

$$D\circ Y = \gamma B^{-1}QD^* - \frac{\gamma}{\mathbf{c}^2}B(V\times QH^*)$$

This is the first statement. Now to the above equation for $\mathcal{R}(H)$. Using (VI2.8) we see that for $z_1, z_2 \in \mathbb{R}^3$

$$\begin{aligned} &(H\circ Y)\bullet(z_2\times z_1) = z_2\bullet(z_1\times(H\circ Y)) = z_2\bullet(\mathcal{R}(H)\circ Y)z_1 \\ &= \gamma z_2\bullet\big(V\otimes(BQD^*) - (BQD^*)\otimes V\big)z_1 + z_2\bullet(BQ\mathcal{R}(H^*)Q^{\mathrm{T}}B^{\mathrm{T}})z_1\,. \end{aligned}$$

The first term equals

$$\begin{aligned} &= \gamma\big(z_2\bullet V\cdot z_1\bullet(BQD^*) - z_2\bullet(BQD^*)\cdot z_1\bullet V\big) \\ &= \gamma(V\times(BQD^*))\bullet(z_2\times z_1)\,, \end{aligned}$$

hence it is a vector times $z_2\times z_1$. The second term equals, as computed in (VI3.3),

$$\begin{aligned} &= (QH^*)\bullet\big((Bz_2)\times(Bz_1)\big) = \gamma(QH^*)\bullet B^{-1}(z_2\times z_1) \\ &= (\gamma B^{-1}QH^*)\bullet(z_2\times z_1) \end{aligned}$$

since (see the exercise 6.8)

$$(\mathbf{B}(V)z_2)\times(\mathbf{B}(V)z_1) = \gamma\mathbf{B}(V)^{-1}(z_2\times z_1)\,,$$

hence the second term of is also a vector times $z_2\times z_1$. Thus we have proved that

$$H\circ Y = \gamma(V\times(BQD^*)) + \gamma B^{-1}QH^*$$

This is the second statement. □

Wir geben nun vor, dass der *-Beobachter die Situation von 2.3 vorfindet, also die Ladung Q im Punkt 0 beobachtet, d.h.

$$D^*(t^*,x^*) = \frac{\mathrm{Q}}{4\pi}\frac{x^*}{|x^*|^3}\,, \quad H^* = 0\,,$$
$$\boldsymbol{\varrho}^* := \mathrm{Q}\,\boldsymbol{\mu}_{\Gamma^*}\,, \quad \mathbf{j}^* = 0\,,$$

mit der Punktladung

$$\Gamma^* = \{(t^*,0)\,;\ t^*\in\mathbb{R}\}\,, \quad \xi^*(t^*) = 0 \text{ für } t^*\in\mathbb{R}$$

wobei $\boldsymbol{\mu}_{\Gamma^*}$ das eindimensionale Lebesgue-Maß auf $\mathbb{R}\times\{0\}$ ist. Ein sich bewegenden Beobachter sieht die Punktladung, wie sie sich bewegt. Wir betrachten beide Beobachter im Lorentz-Fall, und die Beobachtertransformation Y sei linear und bestehe aus einer Lorentz-Transformation $\mathrm{D}Y = \mathbf{L}_{\mathbf{c}}(V,Q)$ wie in 3.1, das heißt

$$\begin{aligned} t &= \gamma t^* + \tfrac{\gamma}{\mathbf{c}^2}V^{\mathrm{T}}Qx^*\,,\\ x &= \gamma t^* V + \mathbf{B}(V)Qx^*\,. \end{aligned} \tag{VI3.4}$$

Deshalb ist die Ladung jetzt am Punkt $\xi(t)$ wenn $(t,\xi(t)) = Y(t^*,\xi^*(t^*))$ wobei $\xi^*(t^*)=0$, also ist $\xi(t)=tV$. Eine beliebige Bewegung $t\mapsto\xi(t)\in\mathbb{R}^3$ wird im Beispiel von Liénhard-Wiechert 3.6 weiter unten behandelt.

3.2 Gleichförmig bewegte Ladung. Es sei $V\in\mathbb{R}^3$ und die Ladung Q befinde sich an einer sich bewegenden Stelle $\xi(t)=tV$. Dann ist eine Lösung in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^3)$ von

$$\mathrm{div}_x[D] = \mathrm{Q}\boldsymbol{\mu}_\Gamma\,,$$
$$-\partial_t[D] + \mathrm{rot}_x[H] = \mathrm{Q}V\boldsymbol{\mu}_\Gamma\,,$$

wobei D und H gegeben sind durch

$$D(t,x) := \frac{\gamma\mathrm{Q}}{4\pi}\frac{x-tV}{|\mathbf{B}(V)(x-tV)|^3}\,,$$
$$H(t,x) := V\times D(t,x)\,.$$

Hier ist $\Gamma := \{(t,x)\,;\ x=\xi(t)\}$ und das Maß $\boldsymbol{\mu}_\Gamma$ ist

$$\langle\,\zeta\,,\,\boldsymbol{\mu}_\Gamma\,\rangle := \int_{\mathbb{R}}\zeta(t,\xi(t))\,\mathrm{d}t\,.$$

Zusatz: Auch das Faraday'sche Induktionsgesetz ist erfüllt (ohne Magnetisierung und Polarization).

Bei einer Bewegung auf die Ladung zu oder von der Ladung weg tritt somit ein magnetisches Feld H auf.

Proof 1. Version. Die Transformationsformeln in 3.1 liefern, d.h. (VI3.1) mit $H^* = 0$,

$$D\circ Y = \gamma \mathbf{B}(V)^{-1}QD^* , \quad H\circ Y = \gamma\, V\times(\mathbf{B}(V)QD^*)\,, \tag{VI3.5}$$

es sind also beide Größen D und H nichtnull. Nun ist wegen (VI3.4)

$$\begin{aligned} x - tV &= \mathbf{B}(V)Qx^* - \frac{\gamma}{\mathbf{c}^2}(V^{\mathrm{T}}Qx^*)V \\ &= \big(\mathbf{B}(V) - \frac{\gamma}{\mathbf{c}^2}VV^{\mathrm{T}}\big)Qx^* = \mathbf{B}(V)^{-1}Qx^* \end{aligned}$$

since $\mathbf{B}(V)^{-1} = \mathbf{B}(V) - \frac{\gamma}{\mathbf{c}^2}VV^{\mathrm{T}}$. Also erhalten wir für D

$$D(t,x) = \gamma\mathbf{B}(V)^{-1}QD^*(x^*) = \frac{\mathrm{Q}\gamma}{4\pi|x^*|^3}\mathbf{B}(V)^{-1}Qx^* = \frac{\mathrm{Q}\gamma}{4\pi}\frac{x-tV}{|x^*|^3}$$

und $|x^*| = |Qx^*|$ sowie $Qx^* = \mathbf{B}(V)(x-tV)$, also

$$D(t,x) = \frac{\mathrm{Q}\gamma}{4\pi}\frac{x-tV}{|\mathbf{B}(V)(x-tV)|^3}\,.$$

Jetzt zum Vektorfeld H. Nach (VI3.5) gilt

$$\begin{aligned} H\circ Y &= \gamma V\times(\mathbf{B}(V)QD^*) \\ &= V\times\mathbf{B}(V)^2(\gamma\mathbf{B}(V)^{-1}QD^*) = V\times\mathbf{B}(V)^2(D\circ Y) \end{aligned}$$

or

$$H = V\times\mathbf{B}(V)^2 D = V\times(\mathrm{Id} + (\gamma^2-1)\widehat{V}\widehat{V}^{\mathrm{T}})D = V\times D\,.$$

Nun zur rechten Seite. Wäre j^* eine glatte Funktion, so gilt nach (VI2.4) die Identität $\mathrm{j}\circ Y = \mathrm{D}Y\mathrm{j}^*$, wobei $y = Y(y^*)$ die Beobachtertransformation ist. Daraus folgt für Testfunktionen ζ, die kovariante Vektoren $\zeta^* = \mathrm{D}Y^{\mathrm{T}}\zeta\circ Y$ sind,

$$\zeta^*\bullet\mathrm{j}^* = (\mathrm{D}Y^{\mathrm{T}}\zeta\circ Y)\bullet\mathrm{j}^* = (\zeta\circ Y)\bullet(\mathrm{D}Y\mathrm{j}^*) = (\zeta\bullet\mathrm{j})\circ Y\,,$$

und damit

$$\begin{aligned} \langle\,\zeta^*\,,\,[\mathrm{j}^*]\,\rangle_{\mathscr{D}(\mathbb{R}^4)} &= \int \zeta^*\bullet\mathrm{j}^*\,\mathrm{dL}^4(y^*) \\ &= \int(\zeta\bullet\mathrm{j})\circ Y\,\mathrm{dL}^4(y^*) = \int\zeta\bullet\mathrm{j}\,\mathrm{dL}^4(y) = \langle\,\zeta\,,\,[\mathrm{j}]\,\rangle_{\mathscr{D}(\mathbb{R}^4)}\,. \end{aligned}$$

Also gilt das auch für $j^*\in\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^3)$, das heißt für kovariantes ζ und da auch $j\in\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^3)$

$$\langle\,\zeta^*\,,\,\mathrm{j}^*\,\rangle_{\mathscr{D}(\mathbb{R}^4)} = \langle\,\zeta\,,\,\mathrm{j}\,\rangle_{\mathscr{D}(\mathbb{R}^4)}\,. \tag{VI3.6}$$

Diese Gleichung ersetzt (VI2.4), d.h. ist allgemeiner. Nun gilt

$$\mathrm{j}^* = \begin{bmatrix} q \\ 0 \end{bmatrix} \boldsymbol{\mu}_0 \quad \text{und} \quad \mathrm{D}Y \begin{bmatrix} q \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma & \frac{\gamma}{\mathbf{c}^2} V^{\mathrm{T}} Q \\ \gamma V & \mathbf{B}(V)Q \end{bmatrix} \begin{bmatrix} q \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma q \\ \gamma q V \end{bmatrix}$$

also

$$\begin{aligned} \langle \zeta^* , \mathrm{j}^* \rangle_{\mathscr{D}(\mathbb{R}^4)} &= \int_{\mathbb{R}} \zeta^*(t^*,0)\bullet \begin{bmatrix} q \\ 0 \end{bmatrix} \mathrm{d}t^* \\ &= \int_{\mathbb{R}} \zeta\circ Y(t^*,0)\bullet \mathrm{D}Y \begin{bmatrix} q \\ 0 \end{bmatrix} \mathrm{d}t^* = \int_{\mathbb{R}} \zeta\circ Y(t^*,0)\bullet \begin{bmatrix} \gamma q \\ \gamma q V \end{bmatrix} \mathrm{d}t^* \\ &= \int_{\mathbb{R}} \zeta(t,tV)\bullet \begin{bmatrix} q \\ qV \end{bmatrix} \mathrm{d}t \quad (\text{wegen } t = \gamma t^* \text{ für } x^* = 0) \\ &= \int \zeta\bullet \begin{bmatrix} q \\ qV \end{bmatrix} \mathrm{d}\boldsymbol{\mu}_\Gamma = \left\langle \zeta , \begin{bmatrix} q \\ qV \end{bmatrix} \boldsymbol{\mu}_\Gamma \right\rangle_{\mathscr{D}(\mathbb{R}^4)} = \langle \zeta , \mathrm{j} \rangle_{\mathscr{D}(\mathbb{R}^4)} \,, \end{aligned}$$

wenn

$$\mathrm{j} := \begin{bmatrix} q \\ qV \end{bmatrix} \boldsymbol{\mu}_\Gamma \,.$$

Da die Gleichungen invariant unter Beobachterwechsel sind, ist damit die Behauptung bewiesen. □

Proof 2. Version. Wir können auch, als Probe zum ersten Beweis, die Differentialgleichungen explizit nachprüfen. Es gilt außerhalb $\{(t,x)\,;\ x = tV\}$ für das angegebene

$$D(t,x) = \widetilde{D}(x - tV)\,, \quad \widetilde{x} := x - tV\,, \quad \widetilde{D}(\widetilde{x}) := \frac{\mathrm{Q}\gamma}{4\pi} \frac{\widetilde{x}}{|\mathbf{B}(V)\widetilde{x}|^3}\,.$$

Dann gilt mit $B := \mathbf{B}(V)$

$$\begin{aligned} \mathrm{div}_x D &= \frac{\mathrm{Q}\gamma}{4\pi} \sum_{i\geq 1} \partial_{\widetilde{x}_i} \frac{\widetilde{x}_i}{|B\widetilde{x}|^3} = \frac{\mathrm{Q}\gamma}{4\pi} \sum_{i\geq 1} \Big(\frac{1}{|B\widetilde{x}|^3} - \frac{3\widetilde{x}_i}{|B\widetilde{x}|^4} \partial_{\widetilde{x}_i} |B\widetilde{x}| \Big) \\ &= \frac{\mathrm{Q}\gamma}{4\pi} \sum_{i\geq 1} \Big(\frac{1}{|B\widetilde{x}|^3} - \frac{3\widetilde{x}_i}{|B\widetilde{x}|^5} \sum_{j\geq 1} (B\widetilde{x})_j B_{ji} \Big) \\ &= \frac{\mathrm{Q}\gamma}{4\pi} \Big(\frac{3}{|B\widetilde{x}|^3} - \frac{3}{|B\widetilde{x}|^5} \sum_{j\geq 1} |(B\widetilde{x})_j|^2 \Big) = 0\,. \end{aligned}$$

Nun zur Differentialgleichung $\mathrm{div}_x[D] = \mathrm{Q}\boldsymbol{\mu}_\Gamma$ im ganzen Raum. Für skalare

Testfunktionen ζ ist mit $K_\varepsilon := \{\widetilde{x}\,;\ |B\widetilde{x}| < \varepsilon\}$

$$\begin{aligned}
&\langle\, \zeta\,,\ \mathrm{div}_x[D]\,\rangle = -\langle\, \nabla_x\zeta\,,\,[D]\,\rangle = -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \nabla_x\zeta(t,x)\bullet D(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}^3} \nabla_{\widetilde{x}}\zeta(t,\widetilde{x}+tV)\bullet\widetilde{D}(\widetilde{x})\,\mathrm{d}\widetilde{x}\,\mathrm{d}t\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\partial K_\varepsilon} \zeta(t,\widetilde{x}+tV)\nu_{K_\varepsilon}(\widetilde{x})\bullet\widetilde{D}(\widetilde{x})\,\mathrm{dH}^2(\widetilde{x})\,\mathrm{d}t\\
&= \frac{\mathrm{Q}\gamma}{4\pi}\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{\partial K_\varepsilon} \zeta(t,\widetilde{x}+tV)\frac{\nu_{K_\varepsilon}(\widetilde{x})\bullet\widetilde{x}}{|B\widetilde{x}|^3}\,\mathrm{dH}^2(\widetilde{x})\,\mathrm{d}t\\
&= \frac{\mathrm{Q}\gamma}{4\pi}\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\partial K_\varepsilon} \Big(\zeta(t,\widetilde{x}+tV)\frac{\widetilde{x}}{\varepsilon}\Big)\bullet\nu_{K_\varepsilon}(\widetilde{x})\,\mathrm{dH}^2(\widetilde{x})\,\mathrm{d}t\\
&= \frac{\mathrm{Q}\gamma}{4\pi}\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{\partial K_1} (\zeta(t,\varepsilon z+tV)z)\bullet\nu_{K_1}(z)\,\mathrm{dH}^2(z)\,\mathrm{d}t \quad (\text{mit } \widetilde{x}=\varepsilon z)\\
&= \frac{\mathrm{Q}\gamma}{4\pi}\int_{\mathbb{R}}\zeta(t,tV)\Big(\int_{\partial K_1} z\bullet\nu_{K_1}(z)\,\mathrm{dH}^2(z)\Big)\,\mathrm{d}t\\
&= \mathrm{Q}\int_{\mathbb{R}}\zeta(t,tV)\,\mathrm{d}t = \int \zeta\,\mathrm{Q}\,\mathrm{d}\boldsymbol{\mu}_\Gamma = \langle\,\zeta\,,\,\mathrm{Q}\boldsymbol{\mu}_\Gamma\,\rangle\ ,
\end{aligned}$$

da

$$\int_{\partial K_1} z\bullet\nu_{D_1}(z)\,\mathrm{dH}^2(z) = \int_{K_1}\mathrm{div}\,z\,\mathrm{dL}^3(z) = 3\mathrm{L}^3(K_1) = 3\frac{4\pi}{3}\cdot 1\cdot 1\cdot\frac{1}{\gamma} = \frac{4\pi}{\gamma}\,,$$

denn K_1 ist ein Ellipsoid mit den Hauptachsen 1, 1, und $\frac{1}{\gamma}$.

Jetzt zum Vektorfeld H. Da D von der Zeit abhängt und $-\partial_t[D]+\mathrm{rot}_x[H]$ außerhalb der Singularität 0 ergeben muss, folgt dass $H\neq 0$. In der Tat ist $H = V\times D$, und wir berechnen $-\partial_t[D]+\mathrm{rot}_x[H]$ in $\mathscr{D}'(\mathbb{R}^4)$ für dieses H. Für eine vektorwertige Testfunktion ζ definiere

$$\widetilde{\zeta}(t,\widetilde{x}) := \zeta(t,x) \text{ für } \widetilde{x} = x - tV$$

so dass

$$\partial_t\zeta = \partial_t\widetilde{\zeta} - (V\bullet\nabla_{\widetilde{x}})\widetilde{\zeta}\,,\quad \nabla_x\zeta = \nabla_{\widetilde{x}}\widetilde{\zeta}\,.$$

Damit folgt

$$\begin{aligned}
&\langle\,\zeta\,,\,-\partial_t[D]+\mathrm{rot}_x[H]\,\rangle = \langle\,\partial_t\zeta\,,\,[D]\,\rangle + \langle\,\mathrm{rot}_x\zeta\,,\,[H]\,\rangle\\
&= \int_{\mathbb{R}^4}(\partial_t\zeta\bullet D + \mathrm{rot}_x\zeta\bullet(V\times D))\,\mathrm{d}(t,x)\\
&= \int_{\mathbb{R}^4}(\partial_t\zeta - V\times\mathrm{rot}_x\zeta)\bullet D\,\mathrm{d}(t,x)\\
&= \int_{\mathbb{R}^4}(\partial_t\widetilde{\zeta} - (V\bullet\nabla_{\widetilde{x}})\widetilde{\zeta} - V\times\mathrm{rot}_{\widetilde{x}}\widetilde{\zeta})\bullet\widetilde{D}\,\mathrm{d}(t,\widetilde{x})\,.
\end{aligned}$$

By a general rule, see exercise 6.3, we have

$$(V\bullet\nabla_{\widetilde{x}})\widetilde{\zeta} + V\times\mathrm{rot}_{\widetilde{x}}\widetilde{\zeta} = \sum_{i\geq 1} V_i\nabla_{\widetilde{x}}\widetilde{\zeta}_i = \nabla_{\widetilde{x}}(V\bullet\widetilde{\zeta})$$

since $V = \mathrm{const}$. Therefore the integral is

$$\begin{aligned} &= \int_{\mathbb{R}^4} (\partial_t\widetilde{\zeta} - \nabla_{\widetilde{x}}(V\bullet\widetilde{\zeta}))\bullet\widetilde{D}\,\mathrm{d}(t,\widetilde{x}) \\ &= \int_{\mathbb{R}^3} \Big(\underbrace{\int_{\mathbb{R}} \partial_t\widetilde{\zeta}\,\mathrm{d}t}_{=0} \Big)\bullet\widetilde{D}\,\mathrm{d}\widetilde{x} - \Big\langle \nabla_{\widetilde{x}}(V\bullet\zeta)\,,\,[\widetilde{D}] \Big\rangle \\ &= \Big\langle V\bullet\zeta\,,\,\nabla_{\widetilde{x}}[\widetilde{D}] \Big\rangle = \langle V\bullet\zeta\,,\,\mathrm{Q}\boldsymbol{\mu}_\Gamma\rangle = \langle \zeta\,,\,\mathrm{Q}V\boldsymbol{\mu}_\Gamma\rangle\,. \end{aligned}$$

□

Proof Zusatz. Nach der 1.Version des Beweises ist das Problem mittels einer Beobachtertransformation auf 2.3 zurückgeführt. Da das Faraday'sche Induktionsprizip beobachterunabhängig ist, muss es, da es in 2.3 gilt, auch hier erfüllt sein. □

Wir berechnen nun die Felder um einen Strom in einem Leiter. In 2.5 the magnetic field H has been computed. But there is also a corresponding independent electric displacement field D.

3.3 Beispiel. Sei Q eine konstante Ladungsdichte auf dem Leiter Γ von 2.5. Dann ist eine zugehörige elektrische Flussdichte D, die

$$\mathrm{div}_x[D] = \boldsymbol{\varrho} := \mathrm{Q}\,\boldsymbol{\mu}_\Gamma \quad \text{in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^3;\mathbb{R}^3) \tag{VI3.7}$$

erfüllt, gegeben durch

$$D(x) = \frac{\mathrm{Q}}{2\pi}\frac{(x_1,x_2,0)}{x_1^2+x_2^2} = \frac{\mathrm{Q}}{2\pi}\frac{x_1\mathbf{e}_1+x_2\mathbf{e}_2}{|x_1\mathbf{e}_1+x_2\mathbf{e}_2|^2} \quad \text{für } x_1^2+x_2^2>0.$$

Da die Lösung stationär ist, gilt natürlich $\partial_t\boldsymbol{\varrho} = 0$. *Zusatz:* Auch das Faraday'sche Induktionsgesetz ist erfüllt (ohne Magnetisierung und Polarization), denn $\mathrm{rot}_x[D] = 0$.

Da die Situation stationär ist, sind die in 2.5 und 3.3 vorkommenden I und Q nicht gekoppelt, es gilt ja hier auch $\partial_t\boldsymbol{\varrho} = 0$ und $\mathrm{div}_x\mathbf{j} = 0$.

Proof. Zur Differentialgleichung (VI3.7) im ganzen Raum ist zu sagen, dass der Beweis ähnlich verläuft wie für H im Beweis von 2.5. Wenn wir φ wie dort definieren, ist für $x \neq 0$

$$D(x) = -\mathrm{Q}(\partial_{x_1}\varphi(x_1,x_2), \partial_{x_2}\varphi(x_1,x_2), 0)\,.$$

Dann ist für $\zeta \in \mathscr{D}(\mathbb{R}^3)$

$$\begin{aligned}\langle \zeta\,,\ \mathrm{div}_x[D]\,\rangle &= -\mathrm{Q}\,\langle \zeta\,,\, (\partial_{x_1}[\partial_{x_1}\varphi] + \partial_{x_2}[\partial_{x_2}\varphi])\,\rangle \\ &= \mathrm{Q}\big(\langle \partial_{x_1}\zeta\,,\,[\partial_{x_1}\varphi]\,\rangle + \langle \partial_{x_2}\zeta\,,\,[\partial_{x_2}\varphi]\,\rangle\big) \\ &= \mathrm{Q}\int_{\mathbb{R}^3}\Big(\sum_{i=1,2}\partial_{x_i}\zeta(x_1,x_2,x_3)\cdot\partial_{x_i}\varphi(x_1,x_2)\Big)\,\mathrm{dL}^3(x) \\ &= \mathrm{Q}\int_{\mathbb{R}}\langle \nabla_{\mathbb{R}^2}\zeta(\bullet,x_3)\,,\,[\nabla_{\mathbb{R}^2}\varphi]\,\rangle\,\mathrm{dL}^1(x_3) \\ &= -\mathrm{Q}\int_{\mathbb{R}}\langle \zeta(\bullet,x_3)\,,\,\Delta_{\mathbb{R}^2}[\varphi]\,\rangle\,\mathrm{dL}^3(x_3) \\ &= \mathrm{Q}\int_{\mathbb{R}}\zeta(0,0,x_3)\,\mathrm{dL}^3(x_3) = \langle \zeta\,,\,\mathrm{Q}\boldsymbol{\mu}_\Gamma\,\rangle\,.\end{aligned}$$

□

Wir betrachten nun einen bewegten Leiter. Der *-Beobachter finde die Situation von 2.5 vor mit der Vervollständigung in 3.3, also

$$D^*(t^*,x^*) = \frac{\mathrm{Q}}{2\pi}\frac{(x_1^*,x_2^*,0)}{|x_1^*|^2+|x_2^*|^2}\,, \quad \boldsymbol{\varrho}^* = \mathrm{Q}\,\boldsymbol{\mu}_{\Gamma^*}$$
$$H^*(t^*,x^*) = \frac{\mathrm{I}}{2\pi}\frac{(-x_2^*,x_1^*,0)}{|x_1^*|^2+|x_2^+|^2}\,, \quad \mathbf{j}^* = \mathrm{I}\,\mathbf{e}_3\boldsymbol{\mu}_{\Gamma^*}\,,$$

wobei

$$\Gamma^* = \{(t^*,s\mathbf{e}_3)\,;\ t^*,s\in\mathbb{R}\}\,,$$
$$\langle \zeta\,,\,\boldsymbol{\mu}_{\Gamma^*}\,\rangle := \int_{\mathbb{R}}\int_{\mathbb{R}}\zeta\big(t^*,s\mathbf{e}_3\big)\,\mathrm{d}s\,\mathrm{d}t^*\,.$$

Zu dem bewegten Beobachter bestehe eine Lorentz-Transformation

$$(t,x) = Y(t^*,x^*) := \mathbf{L_c}(V,Q)(t^*,x^*)\,,\ \text{also } \mathrm{D}Y = \mathbf{L_c}(V,Q)$$

wie in 3.1. In dieser Situation erhalten wir folgende Gleichungen.

3.4 Bewegter Leiter. Betrachte obige Situation von 2.5 und 3.3. Sei der Leiter nun mit Geschwindigkeit $V\in\mathbb{R}^3$ bewegt, also

$$\Gamma := Y(\Gamma^*) = \{(t,tV+s\mathbf{B}(V)^{-1}Q\mathbf{e}_3)\,;\ s,t\in\mathbb{R}\}\,.$$

Die Differentialgleichungen lauten in $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^3)$

$$\mathrm{div}_x[D] = \boldsymbol{\varrho} := \big(\mathrm{Q} + \frac{\mathrm{I}}{\mathbf{c}^2}V\bullet Q\mathbf{e}_3\big)\boldsymbol{\mu}_\Gamma\,,$$
$$-\partial_t[D] + \mathrm{rot}_x[H] = \mathbf{j} := \big(\mathrm{Q}V + \frac{\mathrm{I}}{\gamma}\mathbf{B}(V)Q\mathbf{e}_3\big)\boldsymbol{\mu}_\Gamma\,,$$

wobei

$$\langle \zeta\,,\,\boldsymbol{\mu}_\Gamma\,\rangle := \int_{\mathbb{R}}\int_{\mathbb{R}}\zeta\big(t,tV+s\mathbf{B}(V)^{-1}Q\mathbf{e}_3\big)\,\mathrm{d}s\,\mathrm{d}t^*\,.$$

Die Lösung D und H dieses Problems findet sich in (VI3.10).

Wir betrachten nun die Spezialfälle, dass $Q = \mathrm{Id}$ und V einerseits parallel und andererseits orthogonal zum unbewegten Leiter $\Gamma^* = \{(t^*, s\mathbf{e}_3)\,;\ s \in \mathbb{R}\}$ liegt.

(1) Im Fall $V \,||\, \mathbf{e}_3$ ist

$$D(t,x) = \big(\mathrm{Q} - \frac{\mathrm{I}}{\mathbf{c}^2} V \bullet \mathbf{e}_3\big) D_0(x)\,,$$
$$H(t,x) = \big(\mathrm{Q}\, V \bullet \mathbf{e}_3 + \mathrm{I}\,\big)\, \mathbf{e}_3 \times D_0(x)\,.$$

(2) Im Fall $V \perp \mathbf{e}_3$ ist

$$D(t,x) = \big(\mathrm{Q}\,\mathrm{Id} - \frac{\mathrm{I}}{\mathbf{c}^2}\, \mathbf{e}_3 \otimes V\big)\, D_0(x - tV)$$
$$H(t,x) = \mathrm{B}(V)^{-1}\Big(\big(\mathrm{Q}\, V + \mathrm{I}\, \mathbf{e}_3\big) \times \big(\mathrm{B}(V) D_0(x - tV)\big)\Big)\,.$$

Dabei ist

$$D_0(z) := \frac{\gamma \mathbb{I}_2(z)}{2\pi |\mathrm{B}(V)\mathbb{I}_2(z)|^2}\,,$$

wobei $\mathbb{I}_2(z) := z - z \bullet \mathbf{e}_3\, \mathbf{e}_3$ die Projektion auf die Ebene ist, die senkrecht zum Leiter steht. *Im Falle* (1)*:* Es gilt $\mathrm{B}(V)\mathbb{I}_2 = \mathbb{I}_2$ und darüberhinaus $\mathbb{I}_2(x - tV) = \mathbb{I}_2(x)$, also $D_0(x - tV) = D_0(x)$.

The function D_0 satifies

$$-\Delta[D_0] = \mathrm{H}^1 \llcorner \Gamma_0\,, \quad \Gamma_0 := \mathbb{R}\mathbf{e}_3 = \{s\mathbf{e}_3 \in \mathbb{R}^3\,;\ s \in \mathbb{R}\}\,.$$

Proof by reduction to 2.5. We prove this by an observer transformation to the stationary case in 2.5 and 3.3. Using the abbreviation $B := \mathbf{B}(V)$ this is

$$\begin{bmatrix} t \\ x \end{bmatrix} = \mathbf{L_c}(V, \mathrm{Id}) \begin{bmatrix} t^* \\ x^* \end{bmatrix}, \quad \mathrm{D}Y = \mathbf{L_c}(V, \mathrm{Id}) = \begin{bmatrix} \gamma & \frac{\gamma}{\mathbf{c}^2} V^{\mathrm{T}} \\ \gamma V & B \end{bmatrix},$$

hence

$$t = \gamma t^* + \frac{\gamma}{\mathbf{c}^2} V^{\mathrm{T}} x^*\,, \quad x = \gamma t^* V + B x^*\,, \tag{VI3.8}$$

which in particular implies (see exercise 6.7 for $B - \frac{\gamma}{\mathbf{c}^2} V V^{\mathrm{T}} = B^{-1}$)

$$x - tV = (B - \frac{\gamma}{\mathbf{c}^2} V V^{\mathrm{T}}) x^* \quad \text{therefore} \quad x^* = B(x - tV)\,. \tag{VI3.9}$$

With this transformation we apply the transformation rules in (VI3.1)

$$D \circ Y = \gamma B^{-1} D^* - \frac{\gamma}{\mathbf{c}^2} B(V \times H^*)\,,$$
$$H \circ Y = \gamma\, V \times (B D^*) + \gamma B^{-1} H^*\,,$$

where labeled with a star D^* and H^* are the quantities in 3.3 and 2.5,

$$D^*(x^*) = \frac{\mathrm{Q}}{2\pi}\frac{(x_1^*, x_2^*, 0)}{x_1^{*2} + x_2^{*2}} = \frac{\mathrm{Q}}{2\pi}\frac{\mathbb{I}_2 x^*}{|\mathbb{I}_2 x^*|^2}\,,$$

$$H^*(x^*) = \frac{\mathrm{I}}{2\pi}\frac{(-x_2^*, x_1^*, 0)}{x_1^{*2} + x_2^{*2}} = \frac{\mathrm{I}}{2\pi}\frac{\mathbf{e}_3\times(x_1^*, x_2^*, 0)}{x_1^{*2} + x_2^{*2}} = \frac{\mathrm{I}}{2\pi}\mathbf{e}_3\times\frac{\mathbb{I}_2 x^*}{|\mathbb{I}_2 x^*|^2}\,.$$

With

$$D_0(z) := \frac{\gamma}{2\pi}\frac{B^{-1}\mathbb{I}_2 Bz}{|\mathbb{I}_2 Bz|^2}$$

the fields D and H become for $x^* = B(x - tV)$

$$\begin{aligned}
D(t,x) &= B^{-1}(\gamma D^*(x^*)) - \frac{1}{\mathbf{c}^2}B\big(V\times(\gamma H^*(x^*))\big)\\
&= \mathrm{Q}\,D_0(x-tV) - \frac{\mathrm{I}}{\mathbf{c}^2}B\big(V\times\big(\mathbf{e}_3\times(B\,D_0(x-tV))\big)\big)\,,\\
H(t,x) &= V\times\big(B(\gamma D^*(x^*))\big) + B^{-1}(\gamma H^*(x^*))\\
&= \mathrm{Q}\,V\times(B^2 D_0(x-tV)) + \mathrm{I}\,B^{-1}\big(\mathbf{e}_3\times(B\,D_0(x-tV))\big)\,,
\end{aligned}$$

therefore we have the formulas

$$\boxed{\begin{aligned}
D(t,x) &= \mathrm{Q}\,D_0(x-tV)\\
&\qquad - \frac{\mathrm{I}}{\mathbf{c}^2}\mathbf{B}(V)\big(V\times\big(\mathbf{e}_3\times(\mathbf{B}(V)\,D_0(x-tV))\big)\big)\\
H(t,x) &= \mathrm{Q}\,V\times(\mathbf{B}(V)^2 D_0(x-tV))\\
&\qquad + \mathrm{I}\,\mathbf{B}(V)^{-1}\big(\mathbf{e}_3\times(\mathbf{B}(V)\,D_0(x-tV))\big)\,.
\end{aligned}} \tag{VI3.10}$$

We consider two cases. In both cases

$$\mathbb{I}_2 B^k = B^k \mathbb{I}_2\,,$$

so that D_0 above coincides with the definition in the statement. □

Proof of right side. We prove this by an observer transformation of Lorentz type $y = \mathbf{L_c}(V,Q)y^*$, which is

$$t = \gamma t^* + \frac{\gamma}{\mathbf{c}^2}V^{\mathrm{T}}Qx^*\,,\quad x = \gamma t^* V + \mathbf{B}(V)Qx^*\,. \tag{VI3.11}$$

For the right-hand side of the equation $\underline{\mathrm{div}}\mathfrak{H} = \mathfrak{j}$ we have proved in (VI3.6) the following identity for $\mathfrak{j}$ with covariant test functions

$$\langle\,\zeta^*\,,\,\mathfrak{j}^*\,\rangle_{\mathscr{D}(\mathbb{R}^4)} = \langle\,\zeta\,,\,\mathfrak{j}\,\rangle_{\mathscr{D}(\mathbb{R}^4)}\qquad \text{if } \zeta^* = \mathrm{D}Y^{\mathrm{T}}\zeta\circ Y\,. \tag{VI3.12}$$

Now with $\mathrm{Q}, \mathrm{I} \in \mathbb{R}$ and $\Gamma^* := \{(t^*, x^*) \in \mathbb{R}^4\,;\ x^* = s^*\mathbf{e}_3\,,\ s^* \in \mathbb{R}\}$

$$\mathfrak{j}^* = \begin{bmatrix}\mathrm{Q}\\ \mathrm{I}\,\mathbf{e}_3\end{bmatrix}\mu_{\Gamma^*}\,,\quad \langle\,\zeta\,,\,\mu_{\Gamma^*}\,\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}}\zeta(t^*, s\mathbf{e}_3)\,\mathrm{d}s\,\mathrm{d}t^*\,,$$

hence

$$\langle \zeta^* , \mathrm{j}^* \rangle_{\mathscr{D}(\mathbb{R}^4)} = \int_{\mathbb{R}} \int_{\mathbb{R}} \zeta^*(t^*, s^*\mathbf{e}_3) \bullet \begin{bmatrix} \mathrm{Q} \\ \mathrm{I}\,\mathbf{e}_3 \end{bmatrix} \mathrm{d}s\,\mathrm{d}t^*$$
$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \zeta\big(Y(t^*, s\mathbf{e}_3)\big) \bullet \big(\mathrm{D}Y \begin{bmatrix} \mathrm{Q} \\ \mathrm{I}\,\mathbf{e}_3 \end{bmatrix} \big)\,\mathrm{d}t^* \Big)\,\mathrm{d}s\,.$$

Now, $(t, x) = Y(t^*, s\mathbf{e}_3)$ is equivalent to

$$t = \gamma t^* + \frac{\gamma s}{\mathbf{c}^2} V \bullet Q\mathbf{e}_3\,,$$
$$x = tV + s\big(Q\mathbf{e}_3 - \frac{\gamma s}{\mathbf{c}^2} V \bullet Q\mathbf{e}_3\, V\big) = tV + s\mathbf{B}(V)^{-1}Q\mathbf{e}_3\,,$$

hence $\mathrm{d}t = \gamma\,\mathrm{d}t^*$ for given s. Thus the above integral equals

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \zeta\big(t, tV + s\mathbf{B}(V)^{-1}Q\mathbf{e}_3\big) \bullet \big(\frac{1}{\gamma}\mathrm{D}Y \begin{bmatrix} \mathrm{Q} \\ \mathrm{I}\,\mathbf{e}_3 \end{bmatrix} \big)\,\mathrm{d}s\,\mathrm{d}t = \langle \zeta , \mathrm{j} \rangle_{\mathscr{D}(\mathbb{R}^4)}\,,$$

if

$$\Gamma := Y(\Gamma^*) = \{(t, tV + s\mathbf{B}(V)^{-1}Q\mathbf{e}_3)\,;\ s, t \in \mathbb{R}\}\,,$$
$$\langle \eta , \boldsymbol{\mu}_\Gamma \rangle := \int_{\mathbb{R}} \int_{\mathbb{R}} \eta\big(t, tV + s\mathbf{B}(V)^{-1}Q\mathbf{e}_3\big)\,\mathrm{d}s\,\mathrm{d}t\,,$$
$$\mathrm{j} := \frac{1}{\gamma}\mathrm{D}Y \begin{bmatrix} \mathrm{Q} \\ \mathrm{I}\,\mathbf{e}_3 \end{bmatrix} \boldsymbol{\mu}_\Gamma = \begin{bmatrix} \mathrm{Q} + \frac{\mathrm{I}}{\mathbf{c}^2} V \bullet Q\mathbf{e}_3 \\ \mathrm{Q}V + \frac{\mathrm{I}}{\gamma}\mathbf{B}(V)Q\mathbf{e}_3 \end{bmatrix} \boldsymbol{\mu}_\Gamma\,.$$

□

Proof case (1). We let $V = V_3\mathbf{e}_3$. Then $B\mathbb{I}_2 = \mathbb{I}_2 B = \mathbb{I}_2$ and $\mathbb{I}_2(x - tV) = \mathbb{I}_2 x$ hence

$$D_0(x - tV) = D_0(x) = \frac{\gamma}{2\pi} \frac{\mathbb{I}_2 x}{|\mathbb{I}_2 x|^2}\,,$$

which means $D_0(x) \in \mathbb{I}_2(\mathbb{R}^3)$. This implies $B^k D_0(x) = D_0(x)$. Therefore

$$H = \mathrm{Q}\, V \times D_0(x) + \mathrm{I}\, B^{-1}(\mathbf{e}_3 \times D_0(x))\,.$$

and by exercise 6.8 and since $V = V \bullet \mathbf{e}_3\, \mathbf{e}_3$ and $B\widehat{V} = \gamma\widehat{V}$

$$B^{-1}(\mathbf{e}_3 \times D_0(x)) = \frac{1}{\gamma}(B\mathbf{e}_3) \times B\, D_0(x)) = \mathbf{e}_3 \times D_0(x)\,,$$

which gives the stated expression for H. By a general formula

$$B\big(V \times (\mathbf{e}_3 \times D_0(x))\big) = B\big(V \bullet D_0(x)\, \mathbf{e}_3 - V \bullet \mathbf{e}_3\, D_0(x)\big) = -V \bullet \mathbf{e}_3\, D_0(x)$$

since $V \bullet D_0(x) = 0$. This gives the expression for D. □

Proof case (2). We let $V = V_1\mathbf{e}_1 + V_2\mathbf{e}_2$.

.... □

Proof directly. .... □

Referenzen: Für eine gleichförmige bewegte Ladung siehe [84, Kap. V §38], für das Feld allgemeiner bewegter Ladungen siehe [84, Kap. VIII]. ——weitere referenzen—— For the Liénhard-Wiechert potential siehe J.D.Jackson [48, 12.11 and 14.1].
Siehe auch [Wikipedia: Liénhard-Wiechert potential].

Wir kommen nun zurück zu der Ladung in einem Punkte. Das Liénhard-Wiechert Potential ist eine Lösung des elektromagnetischen Systems in 2.17. Wir benötigen hier eine Version des Fundamentallemmas für die Wellengleichung, wobei wir von einem sich bewegenden Punkt $t \mapsto \xi(t)$ ausgehen, dessen Geschwindigkeit $\mathbf{c}$ nicht überschreitet, der also

$$|\dot{\xi}(t)| < \mathbf{c} \tag{VI3.13}$$

erfüllt.

3.5 Theorem. Sei $t \mapsto \xi(t)$ mit (VI3.13) und $t \mapsto m(t)$ eine gegebene Funktion. Dann ist eine Lösung ϕ von

$$\frac{1}{\mathbf{c}^2}\partial_t^2[\phi] - \Delta[\phi] = m\boldsymbol{\mu}_\xi$$

with $\phi(t,x) \to 0$ for $|x| \to \infty$ gegeben durch

$$\boxed{\begin{gathered}\phi(t,x) := \frac{m(s)}{4\pi\Big(1 - \frac{\xi'(s)}{\mathbf{c}}\bullet\frac{x-\xi(s)}{|x-\xi(s)|}\Big)}\frac{1}{|x-\xi(s)|} \\ \text{with} \quad t = s + \frac{|x-\xi(s)|}{\mathbf{c}}\,.\end{gathered}}$$

Diese Lösung ist der Limes von glatten Lösungen zu rechten Seiten, die im Distributionssinn gegen $m\boldsymbol{\mu}_\xi$ konvergieren. *Definition:* Das Maß $\boldsymbol{\mu}_\xi$ ist definiert durch

$$\langle\, \zeta\,,\, \boldsymbol{\mu}_\xi \,\rangle := \int_{\mathbb{R}} \zeta(t,\xi(t))\,\mathrm{d}t$$

Proof. Der Beweis findet sich in [21, Relativistisches Schwerefeld]. Es wird dabei vorausgesetzt, dass man glatte Funktionen ρ_ϵ hat, die für $\epsilon \to 0$ im Distributionssinn konvergieren, also $[\rho_\epsilon] \to m\boldsymbol{\mu}_\xi$. Die Lösungen ϕ_ϵ von

$$\frac{1}{\mathbf{c}^2}\partial_t^2[\phi_\epsilon] - \Delta[\phi_\epsilon] = \rho_\epsilon$$

konvergieren dann im Distributionssinn gegen ϕ. Hierbei wird wesentlich benutzt, dass $t \mapsto \xi(t)$ die Eigenschaft (VI3.13) hat. □

Es folgt

3.6 Liénhard-Wiechert Potential (1898). Sei $t \mapsto \xi(t)$ die Position des geladenen Teilchens, sei $\mathrm{Q} \in \mathbb{R}$ dessen Ladung und $V(t) := \dot{\xi}(t)$ dessen Geschwindigkeit, die (VI3.13) erfüllt. Dann ist die Lösung $(-\Phi, A)$ von

$$\frac{1}{\mathbf{c}^2}\partial_t^2[\Phi] - \Delta[\Phi] = \frac{\mathrm{Q}}{\varepsilon_0}\boldsymbol{\mu}_\xi\,,$$
$$\frac{1}{\mathbf{c}^2}\partial_t^2[A] - \Delta[A] = \mu_0 \mathrm{Q} V \boldsymbol{\mu}_\xi\,,$$
$$\frac{1}{\mathbf{c}^2}\partial_t[\Phi] + \mathrm{div}[A] = 0\,,$$

in $\mathscr{D}'(\mathbb{R}^4)$ gegeben durch

$$\Phi(t,x) = \frac{\mathrm{Q}}{4\pi\varepsilon_0\Big(1 - \frac{V(s)}{\mathbf{c}}\bullet\mathbf{n}(s,x)\Big)}\frac{1}{|x-\xi(s)|}\,,$$
$$A(t,x) = \frac{\mu_0\mathrm{Q}}{4\pi\Big(1 - \frac{V(s)}{\mathbf{c}}\bullet\mathbf{n}(s,x)\Big)}\frac{V(s)}{|x-\xi(s)|}\,,$$
$$\text{mit} \quad t = s + \frac{|x-\xi(s)|}{\mathbf{c}}\,, \quad \mathbf{n}(s,x) := \frac{x-\xi(s)}{|x-\xi(s)|}\,.$$

Remark: In literature $s = t_{ret}$ is the "retardet time" (depending on x).

Proof. Definiere Distributionen $\boldsymbol{\varrho} := \mathrm{Q}\boldsymbol{\mu}_\xi$ und $\mathbf{j} := \mathrm{Q}V\boldsymbol{\mu}_\xi$, so dass also

$$\partial_t\boldsymbol{\varrho} + \mathrm{div}_x\mathbf{j} = 0 \text{ in } \mathscr{D}'(\mathbb{R}^4)\,,$$

denn für Testfunktionen ζ

$$-\langle \zeta\,,\, \partial_t\boldsymbol{\varrho} + \mathrm{div}_x\mathbf{j}\rangle_{\mathscr{D}(\mathbb{R}^4)} = \langle \partial_t\zeta\,,\, \boldsymbol{\varrho}\rangle_{\mathscr{D}(\mathbb{R}^4)} + \langle \nabla_x\zeta\,,\, \mathbf{j}\rangle_{\mathscr{D}(\mathbb{R}^4)}$$
$$= \big\langle \partial_t\zeta\,,\, \mathrm{Q}\boldsymbol{\mu}_\xi\big\rangle + \big\langle \nabla_x\zeta\,,\, \mathrm{Q}V\boldsymbol{\mu}_\xi\big\rangle = \int_{\mathbb{R}} \mathrm{Q}\big(\partial_t\zeta + \dot{\xi}\bullet\nabla_x\zeta\big)(t,\xi(t))\,\mathrm{d}(t) = 0\,.$$

Definiere als Approximation für $\epsilon > 0$ Funktionen

$$\boldsymbol{\varrho}_\epsilon := \varphi_\epsilon * \boldsymbol{\varrho}\,, \quad \mathbf{j}_\epsilon := \varphi_\epsilon * \mathbf{j}\,.$$

Hierbei ist $\varphi_\epsilon(y) := \frac{1}{\epsilon^4}\varphi(\frac{y}{\epsilon})$ und $\varphi \in C_0^\infty(\mathbb{R}^4)$. Dann folgt

$$\partial_t\boldsymbol{\varrho}_\epsilon + \mathrm{div}\mathbf{j}_\varepsilon = \partial_t(\varphi_\epsilon * \boldsymbol{\varrho}) + \mathrm{div}(\varphi_\epsilon * \mathbf{j})$$
$$= \varphi_\epsilon * \big(\partial_t\boldsymbol{\varrho} + \mathrm{div}\mathbf{j}\big) = 0\,.$$

Zu den glatten Daten $\boldsymbol{\varrho}_\epsilon$ und $\mathbf{j}_\epsilon$ gibt es nach dem Theorem 3.5 Funktionen Φ_ϵ und $A_\epsilon = (A_{\epsilon 1}, A_{\epsilon 2}, A_{\epsilon 3})$ mit

$$\frac{1}{\mathbf{c}^2}\partial_t^2\Phi_\epsilon - \Delta\Phi_\epsilon = \frac{\boldsymbol{\varrho}_\epsilon}{\varepsilon_0}\,, \quad \frac{1}{\mathbf{c}^2}\partial_t^2 A_\epsilon - \Delta A_\epsilon = \mu_0\,\mathbf{j}_\epsilon\,,$$

also ist, wenn F die Fundamentallösung von $\frac{1}{\mathbf{c}^2}\partial_t F - \Delta F$ ist,

$$\Phi_\epsilon(y) = \frac{1}{\varepsilon_0}\,\langle\, \boldsymbol{\varrho}_\epsilon(y-\bullet)\,,\, F\,\rangle\;, \qquad A_\epsilon(y) = \mu_0\,\langle\, \mathbf{j}_\epsilon(y-\bullet)\,,\, F\,\rangle\;.$$

Daraus folgt

$$\begin{aligned}
&\frac{1}{\mathbf{c}^2}\partial_t\Phi_\varepsilon(t,x) + \operatorname{div}A_\varepsilon(t,x)\\
&= \frac{1}{\varepsilon_0\mathbf{c}^2}\,\langle\, \partial_t\boldsymbol{\varrho}_\epsilon((t,x)-\bullet)\,,\, F\,\rangle + \mu_0\,\langle\, \operatorname{div}\mathbf{j}_\epsilon((t,x)-\bullet)\,,\, F\,\rangle\\
&= \mu_0\,\langle\, \partial_t\boldsymbol{\varrho}_\epsilon((t,x)-\bullet) + \operatorname{div}\mathbf{j}_\epsilon((t,x)-\bullet)\,,\, F\,\rangle = 0\,,
\end{aligned}$$

also ist auch die Lorenz-Bedingung erfüllt. Nach 3.5 konvergieren diese ϵ-Lösungen gegen die gewünschte Lösung mit der Lorenz-Bedingung. Es ist also

$$\begin{aligned}
\Phi(t,x) &= \frac{\mathrm{Q}}{4\pi\varepsilon_0\Big(1 - \frac{\xi'(s)}{\mathbf{c}}\bullet\frac{x-\xi(s)}{|x-\xi(s)|}\Big)}\,\frac{1}{\big|x-\xi(s)\big|}\\
A(t,x) &= \frac{\mu_0\mathrm{Q}\xi'(s)}{4\pi\Big(1 - \frac{\xi'(s)}{\mathbf{c}}\bullet\frac{x-\xi(s)}{|x-\xi(s)|}\Big)}\,\frac{1}{\big|x-\xi(s)\big|}\;,
\end{aligned}$$

wobei $s = \widehat{s}(t,x)$ mit $t = s + \frac{1}{\mathbf{c}}|x-\xi(s)|$. □

Folgerung: Die Lösungen für E und B sind ....

4 The limit towards MHD

"Magnetohydrodynamics (MHD) couples Maxwells equations of electromagnetism with hydrodynamics to describe the macroscopic behavior of conducting fluids such as plasmas." [Introduction_to_MHD.pdf]

References: The equations of MHD you find in Priest [59, 2 The Basis Equations of MHD]. "The basic theory of magnetohydrodynamics (MHD) is summarised by Cowling in his book on the subject." See Cowling [104].

Wir betrachten zunächst den Limes $\mathbf{c} \to \infty$ für Lösungen der Elektrodynamik, und zwar untersuchen wir nur den elektrischen Fall ohne Magnetisierung und erhalten so den elektrischen Teil der Magnetohydrodynamik. Außerdem benutzen wir als Approximation des klassischen Falles nur den Lorentz-Fall (*en:* Lorentz frame), d.h. $\mathrm{G} = \mathrm{G}_{\mathbf{c}}$ und die Koordinaten $y = (t, x)$. Es gilt die fundamentale Gleichung

$$\varepsilon_0 \mu_0 = \frac{1}{\mathbf{c}^2} \to 0\,,$$

die im Limes gegen 0 konvergiert, wobei μ_0 im Limes positiv bleibt. Also wird der Fall betrachtet, dass auch $\varepsilon_0 \to 0$. Da E einen Grenzwert besitzt, folgt somit $D = \varepsilon_0 E \to 0$. Außerdem ist $B = \mu_0 H$, wobei also die Konstante μ_0 im Limes bleibt. Daraus folgt, siehe 4.1, dass

$$\boxed{\varrho^{el} := \frac{\boldsymbol{\varrho}}{\varepsilon_0} \quad \text{und} \quad \mathrm{j}^{el} := \mu_0 \mathbf{j}} \tag{VI4.1}$$

einen Limes besitzen. Wir erhalten also Folgendes im Limes $\mathbf{c} \to \infty$.

4.1 Ampére's circuital law. Es gilt im Limes

$$\mathrm{div}_x E = \varrho^{el}\,, \quad \mathrm{rot}_x B = \mathrm{j}^{el}\,.$$

und es ist $\mathrm{div}_x \mathrm{j}^{el} = 0$. *Hinweis:* Während ϱ^{el} und j^{el} in (VI4.1) die Größen für $\mathbf{c} < \infty$ sind, sind hier die Größen im Limes $\mathbf{c} = \infty$ zu verstehen.

Proof. Wegen $D = \varepsilon_0 E$ und $B = \mu_0 H$ lauten die Gleichungen

$$\mathrm{div}_x(\varepsilon_0 E) = \boldsymbol{\varrho}\,,$$
$$-\partial_t(\varepsilon_0 E) + \mathrm{rot}_x\Big(\frac{1}{\mu_0} B\Big) = \mathbf{j}$$

Dann ist mit ϱ^{el} und j^{el} von (VI4.1)

$$\mathrm{div}_x E = \frac{\boldsymbol{\varrho}}{\varepsilon_0} =: \varrho^{el}\,,$$
$$\mathrm{j}^{el} = \mu_0 \mathbf{j} = -\partial_t(\mu_0 \varepsilon_0 E) + \mathrm{rot}_x B = \mathcal{O}\Big(\frac{1}{\mathbf{c}^2}\Big) + \mathrm{rot}_x B\,,$$

und damit im Limes die behaupteten Differentialgleichungen für E und B. Außerdem gilt

$$\mu_0 \mathrm{j} = \begin{bmatrix} \mu_0 \boldsymbol{\varrho} \\ \mu_0 \mathbf{j} \end{bmatrix} = \begin{bmatrix} \mu_0 \varepsilon_0 \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathbf{c}^2} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} ,$$

also

$$0 = \mu_0 \,\mathrm{div}_y \mathrm{j} = \frac{1}{\mathbf{c}^2} \partial_t \varrho^{el} + \mathrm{div}_x \mathrm{j}^{el} ,$$

also im Limes $\mathrm{div}_x \mathrm{j}^{el} = 0$. Die folgt auch aus der bewiesenen Differentialgleichung für B

$$\mathrm{div}_x \mathrm{j}^{el} = \mathrm{div}_x \,\mathrm{rot}_x B = 0$$

wegen $\mathrm{div}_x \mathrm{rot}_x = 0$. □

The Faraday's law in the original version is used.

4.2 Faraday's induction law. These laws are

$$\mathrm{div}_x B = 0 , \quad \partial_t B + \mathrm{rot}_x E = 0 ,$$

and they stay in the classical limit. Also the equivalence of these equations with the 4-vector field $\underline{A} = (-\Phi, A)$ satisfying

$$E = -\nabla_x \Phi - \partial_t A , \quad B = \mathrm{rot}_x A$$

in a simply connected domain, remains the same in the limit. *Remark:* The Lorenz condition becomes $\mathrm{div}_x A = 0$.

Proof Remark. By 2.16 the Lorenz condition is $\frac{1}{\mathbf{c}^2} \partial_t \Phi + \mathrm{div}_x A = 0$. Since Φ and A have limits we get in the limit $\mathbf{c} \to \infty$ that $\mathrm{div}_x A = 0$. □

Altogether Maxwell equations become

Maxwell equations in MHD:

$$\mathrm{div}_x E = \varrho^{el} , \quad \mathrm{rot}_x B = \mathrm{j}^{el}$$

$$\mathrm{div}_x B = 0 , \quad \partial_t B + \mathrm{rot}_x E = 0 \qquad \text{(VI4.2)}$$

E electric field, B magnetic flux density,

ϱ^{el} electric charge, $\mathrm{j}^{el} = \mu_0 \mathbf{j}$ electric current.

In Termen von $\underline{A} = (-\Phi, A)$ we obtain the result in 6.9.

Referenzen: See Davidson [33, 2 The Governing Equations of Electrodynamics].

As transformation rule we take a Lorentz transformation as in 3.1 and insert the definition of E and B. One obtains

4.3 Transformation rule. The transformation rules for E and B and for ϱ^{el} and j^{el} are, if Y is a Newton transformation and writing $V = \dot{X}$,

$$E\circ Y = QE^* - V\times QB^*\,, \qquad \varrho^{el}\circ Y = \varrho^{el*} + V\bullet Q\,\mathrm{j}^{el*} + (Q^{\mathrm{T}}\dot{Q})\mathbin{\vdots}\mathcal{R}(B^*)\,,$$
$$B\circ Y = QB^*\,, \qquad \mathrm{j}^{el} = Q\,\mathrm{j}^{el*}\,.$$

This we show, if $V = \mathrm{const}$, as a consequence of 3.1 where in the limit the Lorentz transformation becomes a Galilei transformation.

Proof by 3.1. The equations in 3.1

$$D\circ Y = \gamma\mathbf{B}(V)^{-1}QD^* - \frac{\gamma}{\mathbf{c}^2}\mathbf{B}(V)(V\times QH^*)\,,$$
$$H\circ Y = \gamma\, V\times(\mathbf{B}(V)QD^*) + \gamma\mathbf{B}(V)^{-1}QH^*\,,$$

become with $D = \varepsilon_0 E$ and $B = \mu_0 H$

$$\varepsilon_0 E\circ Y = \varepsilon_0\gamma\mathbf{B}(V)^{-1}QE^* - \frac{\gamma}{\mu_0\mathbf{c}^2}\mathbf{B}(V)(V\times QB^*)\,,$$
$$\frac{1}{\mu_0}B\circ Y = \varepsilon_0\gamma\, V\times(\mathbf{B}(V)QE^*) + \frac{1}{\mu_0}\gamma\mathbf{B}(V)^{-1}QB^*\,,$$

also wegen $\varepsilon_0\mu_0\mathbf{c}^2 = 1$

$$E\circ Y = \gamma\mathbf{B}(V)^{-1}QE^* - \gamma\mathbf{B}(V)(V\times QB^*)\,,$$
$$B\circ Y = \frac{\gamma}{\mathbf{c}^2}V\times(\mathbf{B}(V)QE^*) + \gamma\mathbf{B}(V)^{-1}QB^*\,.$$

Nun können wir zum Limes $\mathbf{c}\to\infty$ übergehen, wobei natürlich gilt, dass $\gamma\to 1$ und $\mathbf{B}(V)\to\mathrm{Id}$. Weiter gilt in 3.1

$$\begin{aligned}\boldsymbol{\varrho}\circ Y &= \gamma\boldsymbol{\varrho}^* + \frac{\gamma}{\mathbf{c}^2}V\bullet(Q\mathbf{j}^*)\,,\\ \mathbf{j}\circ Y &= \gamma\boldsymbol{\varrho}^*V + \mathbf{B}(V)Q\mathbf{j}^*\,.\end{aligned} \tag{VI4.3}$$

Wenn man nun $\boldsymbol{\varrho} = \varepsilon_0\varrho^{el}$ und $\mathbf{j} = \frac{1}{\mu_0}\mathrm{j}^{el}$ schreibt, erhält man

$$\varrho^{el}\circ Y = \gamma\varrho^{el*} + \frac{\gamma}{\varepsilon_0\mu_0\mathbf{c}^2}V\bullet(Q\mathrm{j}^{el*})\,,$$
$$\mathrm{j}^{el}\circ Y = \varepsilon_0\mu_0\gamma\varrho^{el*}V + \mathbf{B}(V)Q\mathrm{j}^{el*}\,,$$

und daraus mit $\varepsilon_0\mu_0\mathbf{c}^2 = 1$ im Limes die Behauptung. □

Proof gneral case. Wir haben kein Analogon zu 3.1 für variables V bewiesen, das wird noch nachgetragen, ggf. in [21]. Eigentlich kommt (VI4.3) aus der Transformationsformel (VI2.4), die lautet $\mathrm{j}\circ Y = \mathrm{D}Y\mathrm{j}^*$ und im Lorentz-Fall, also für $\mathrm{D}Y = \mathrm{const}$,

$$\begin{bmatrix}\boldsymbol{\varrho}\\ \mathbf{j}\end{bmatrix}\circ Y = \mathrm{D}Y\begin{bmatrix}\boldsymbol{\varrho}^*\\ \mathbf{j}^*\end{bmatrix} = \begin{bmatrix}\gamma & \frac{\gamma}{\mathbf{c}^2}V^{\mathrm{T}}Q\\ \gamma V & \mathbf{B}(V)Q\end{bmatrix}\begin{bmatrix}\boldsymbol{\varrho}^*\\ \mathbf{j}^*\end{bmatrix},$$

oder mit (VI4.1), d.h. $\boldsymbol{\varrho} = \varepsilon_0 \varrho^{el}$ und $\mathbf{j} = \frac{1}{\mu_0}\mathrm{j}^{el}$,

$$\frac{1}{\mu_0}\begin{bmatrix} \frac{1}{\mathbf{c}^2}\varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} \circ Y = \begin{bmatrix} \varepsilon_0 \varrho^{el} \\ \frac{1}{\mu_0}\mathrm{j}^{el} \end{bmatrix} \circ Y = \mathrm{D}Y \begin{bmatrix} \varepsilon_0 \varrho^{el*} \\ \frac{1}{\mu_0}\mathrm{j}^{el*} \end{bmatrix} = \frac{1}{\mu_0}\mathrm{D}Y \begin{bmatrix} \frac{1}{\mathbf{c}^2}\varrho^{el*} \\ \mathrm{j}^{el*} \end{bmatrix}.$$

Indem wir diese Gleichung mit der Matrix

$$\mu_0 \begin{bmatrix} \mathbf{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}$$

von links multiplizieren, also

$$\begin{bmatrix} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} \circ Y = \begin{bmatrix} \mathbf{c}^2 & 0 \\ 0 & \mathrm{Id} \end{bmatrix} \mathrm{D}Y \begin{bmatrix} \frac{1}{\mathbf{c}^2}\varrho^{el*} \\ \mathrm{j}^{el*} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma V^{\mathrm{T}} Q \\ \frac{\gamma}{\mathbf{c}^2} V & \mathbf{B}(V)Q \end{bmatrix} \begin{bmatrix} \varrho^{el*} \\ \mathrm{j}^{el*} \end{bmatrix},$$

ergibt sich daraus im Limes $\mathbf{c} \to \infty$

$$\begin{bmatrix} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} \circ Y = Z \begin{bmatrix} \varrho^{el*} \\ \mathrm{j}^{el*} \end{bmatrix}, \quad Z := \begin{bmatrix} 1 & V^{\mathrm{T}} Q \\ 0 & Q \end{bmatrix}. \tag{VI4.4}$$

Da V und Q konstant sind, ist diese Identität mit der Behauptung identisch.

Wir werden nun beliebige Newton-Transformationen Y zulassen und definieren Z mit Hilfe von $V = \dot{X}$ und $Q = \mathrm{D}_{x^*} X$. Es folgt ein Beweis des Satzes, der sich auf das Ergebnis in (I5.11) stützt. Hierzu schreiben wir die Ampére'schen Differentialgleichungen der MHD als System

$$\mathrm{div}_x \begin{bmatrix} E \\ \mathcal{R}(B) \end{bmatrix} = \begin{bmatrix} \mathrm{div}_x E \\ \mathrm{div}_x \mathcal{R}(B) \end{bmatrix} = \begin{bmatrix} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} \tag{VI4.5}$$

oder als schwache Gleichung für Testfunktionen (φ, ζ)

$$\int_{\mathbb{R}^4} \Big((\partial_t \begin{bmatrix} \varphi \\ \zeta \end{bmatrix}) \bullet 0 + (\mathrm{D}_x \begin{bmatrix} \varphi \\ \zeta \end{bmatrix}) \mathbin{:} \begin{bmatrix} E \\ \mathcal{R}(B) \end{bmatrix} + \begin{bmatrix} \varphi \\ \zeta \end{bmatrix} \bullet \begin{bmatrix} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix} \Big)\, \mathrm{dL}^4 = 0$$

mit der Maßgabe, dass die Testfunktionen mit der Regel

$$\begin{bmatrix} \varphi^* \\ \zeta^* \end{bmatrix} = Z^{\mathrm{T}} \begin{bmatrix} \varphi \\ \zeta \end{bmatrix} \circ Y$$

transformieren. Nach (I5.11) ist also (VI4.5) objektiv, falls die folgenden Transformationsregeln (es ist $J = \det \mathrm{D}_{x^*} X = 1$ und $u = 0$)

$$q_i \circ Y = \sum_j Q_{ij} Z q_j^*, \quad \mathbf{r} \circ Y = \sum_j Z_{,x_j^*} q_j^* + Z\mathbf{r}^*$$
$$q_i := \begin{bmatrix} E_i \\ (\mathcal{R}(B)_{ki})_k \end{bmatrix}, \quad \mathbf{r} := \begin{bmatrix} \varrho^{el} \\ \mathrm{j}^{el} \end{bmatrix}, \quad Z := \begin{bmatrix} 1 & V^{\mathrm{T}} Q \\ 0 & Q \end{bmatrix}$$

gelten. Since

$$Z_{\prime x_j^*} = \begin{bmatrix} 0 & V_{\prime x_j^*}{}^{\mathrm{T}} Q \\ 0 & 0 \end{bmatrix}, \quad V_{\prime x_j^*}{}^{\mathrm{T}} Q \mathbf{e}_k = \sum_i V_{i\,\prime x_j^*} Q_{ik} = \sum_i \dot{Q}_{ij} Q_{ik} = (Q^{\mathrm{T}} \dot{Q})_{kj}$$

we get

$$\varrho^{el} \circ Y = (Q^{\mathrm{T}} \dot{Q}) \mathbin{\vdots} \mathcal{R}(B^*) + \varrho^{el*} + V^{\mathrm{T}} Q \mathrm{j}^{el*}, \quad \mathrm{j}^{el} \circ Y = Q \mathrm{j}^{el*},$$

and the assertions for E and $\mathcal{R}(B)$. □

Soweit der Limes in dem elektrischen Teil. Darüberhinaus verlangt die MHD die Kopplung mit den üblichen Gleichungen von Masse und Impuls, und diese Kopplung geschieht in dem Kraftterm. Da wir hier im klassischen Fall sind, besteht die Verbindung in der

4.4 Newton- und Lorentz-Kraft. Es wird angenommen, dass der Kraftterm gegeben ist durch

$$\mathbf{f} = \mathbf{f}_{fict} + \mathfrak{g} \varrho \nabla \boldsymbol{\phi} + \mathbf{j} \times B .$$

Dabei ist $\mathbf{r} = 0$ und $\mathbf{f}_{fict}$ ist die Scheinkraft, die vom Beobachter abhängig ist, und als objektive Vektoren

- $\mathfrak{g} \varrho \nabla \boldsymbol{\phi}$ die ***Newton-Kraft***, die von der Gravitationsgleichung $-\Delta \boldsymbol{\phi} = \varrho$ induziert wird.
- $\mathbf{j} \times B$ die ***Lorenz-Kraft***, die von dem Ampére'schen Durchflutungsgesetz unter Weglassen der Polarisation induziert wird.

Da hier Übergangskoeffizienten $\mathfrak{g}$ und 1 benötigt werden, beweisen wir zunächst folgendes allgemeine Theorem.

4.5 Dimensionstheorem. Ein allgemeines System, das erfüllt ist für Testfunktionen, die einen kovarianten Vektor darstellen, ist im klassischen Fall (und auch im Lorentz-Fall)

$$\partial_t \big(\square \, [*] \big) + \mathrm{div}_x \big(\square \left[\frac{* \, m}{s} \right] \big) = \square \left[\frac{*}{s} \right],$$

$$\partial_t \big(\square \left[\frac{* \, m}{s} \right] \big) + \mathrm{div}_x \big(\square \left[\frac{* \, m^2}{s^2} \right] \big) = \square \left[\frac{* \, m}{s^2} \right].$$

Da ∂_t mit der Dimension $\left[\frac{1}{s} \right]$ und div_x mit der Dimension $\left[\frac{1}{m} \right]$ versehen wird, ist das System konsistent bei jeder festen Wahl von $[*]$. Wird in dem System eine Größe □ mit Maßeinheiten angegeben, so sind damit die Maßeinheiten aller Größen □ festgelegt.

Beim Masse-Impulssystem

$$\begin{aligned}\partial_t\big(\varrho\,[*]\big) + \mathrm{div}_x\big(\varrho v\left[\frac{*\,m}{s}\right]\big) &= \mathbf{r}\left[\frac{*}{s}\right],\\ \partial_t\big(\varrho v\left[\frac{*\,m}{s}\right]\big) + \mathrm{div}_x\big((\varrho v v^{\mathrm{T}} + \Pi)\left[\frac{*\,m^2}{s^2}\right]\big) &= \mathbf{f}\left[\frac{*\,m}{s^2}\right]\end{aligned} \tag{VI4.6}$$

setzen wir

$$* = \frac{kg}{m^3}\,, \quad \text{also} \quad \varrho\left[\frac{kg}{m^3}\right], \quad v\left[\frac{m}{s}\right],$$

und damit ist gegeben

$$\mathbf{r}\left[\frac{kg}{m^3 s}\right], \quad \Pi = (p\mathrm{Id} - S)\left[\frac{kg}{m\,s^2}\right], \quad \mathbf{f}\left[\frac{kg}{m^2 s^2}\right].$$

Bei Ampére's Durchflutungsgesetz

$$\begin{aligned}\partial_t\big(0\,[*]\big) + \mathrm{div}_x\big(D\left[\frac{*\,m}{s}\right]\big) &= \boldsymbol{\varrho}\left[\frac{*}{s}\right],\\ \partial_t\big(-D\left[\frac{*\,m}{s}\right]\big) + \mathrm{div}_x\big(\mathcal{R}(H)\left[\frac{*\,m^2}{s^2}\right]\big) &= \mathbf{j}\left[\frac{*\,m}{s^2}\right]\end{aligned} \tag{VI4.7}$$

setzen wir

$$\boldsymbol{\varrho}\left[\frac{As}{m^3}\right], \quad \text{also} \quad * = \frac{As^2}{m^3}\,,$$

und damit ist gegeben

$$D\left[\frac{As}{m^2}\right], \quad H\left[\frac{A}{m}\right], \quad \mathbf{j}\left[\frac{A}{m^2}\right].$$

Der einzige Unterschied bei den Massendichten besteht darin, dass bei Ampére As (Ampére-Sekunden) statt bei Masse-Impuls kg benutzt werden.

Wenden wir dies auf die MHD-Gleichungen an, so ergibt sich, wenn wir μ_0 aus (VI2.15) nehmen,

$$\begin{aligned}B = \mu_0 H \text{ hat das Maß } &\frac{kg\,m}{A^2 s^2}\cdot\frac{A}{m} = \frac{kg}{A\,s^2}\\ \mathbf{j}\times B \text{ hat das Maß } &\frac{A}{m^2}\cdot\frac{kg}{A\,s^2} = \frac{kg}{m^2\,s^2},\end{aligned}$$

also hat $\mathbf{j}\times B$ dasselbe Maß wie die Kraftdichte $\mathbf{f}$ in der Masse-Impuls Gleichung, deshalb ist der Faktor 1 zulässig. Nun zu der Schwerkraft. Die Gravitationsgleichung ist

$$\partial_t\big(0\,[*]\big) + \mathrm{div}_x\big(-\nabla_x\boldsymbol{\phi}\left[\frac{*\,m}{s}\right]\big) = \varrho\left[\frac{*}{s}\right] \tag{VI4.8}$$

und von oben haben wir

$$\varrho\left[\frac{kg}{m^3}\right], \quad \text{also} \quad * = \frac{kg\,s}{m^3}\,, \quad \text{somit} \quad \nabla_x\boldsymbol{\phi}\left[\frac{kg}{m^2}\right],$$

und damit hat

$$\varrho\nabla_x\boldsymbol{\phi} \text{ das Maß } \frac{kg^2}{m^5}\,,$$

also hat, mit $\mathfrak{g} = 4\pi G$ von (I3.13),

$$\mathfrak{g}\varrho\nabla_x\boldsymbol{\phi} \text{ das Maß } \frac{m^3}{kg\,s^2}\cdot\frac{kg^2}{m^5} = \frac{kg}{m^2s^2}$$

das gleiche Maß wie wie die Kraftdichte $\mathbf{f}$ in der Masse-Impuls Gleichung, daher ist der Faktor $\mathfrak{g}$ der richtige. Dies rechtfertigt die Aussage in 4.4. Mit den Masse-Impuls Gleichungen von Abschnitt II.3 sind die Magnetohydrodynamik-Gleichungen im Fall $\mathbf{r} = 0$ gleich

Magnetohydrodynamics:

$$\begin{gathered}
\partial_t\varrho + \operatorname{div}_x(\varrho v) = 0\,,\\
\partial_t(\varrho v) + \operatorname{div}_x(\varrho v v^{\mathrm{T}} + \Pi) = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} + \mathbf{j}\times B + \mathbf{f}_{fict}\,,\\
-\Delta\boldsymbol{\phi} = \varrho\,,\\
\operatorname{rot}_x B = \mu_0\mathbf{j}\,,\quad \operatorname{div}_x B = 0\,,\\
\operatorname{div}_x E = \varrho^{el}\,,\quad \partial_t B + \operatorname{rot}_x E = 0
\end{gathered} \tag{VI4.9}$$

$\mathbf{j}$ satisfies e.g. Ohm's law

Es fehlt Energiegleichung und Entropieprinzip

Zum Ohm'schen Gesetz siehe den nächsten Abschnitt. Dann lassen sich nach 5.2 die Gleichungen der MHD mit Ohm'schen Gesetz schreiben als

$$\begin{gathered}
\partial_t\varrho + \operatorname{div}_x(\varrho v) = 0\,,\\
\partial_t(\varrho v) + \operatorname{div}_x(\varrho v v^{\mathrm{T}} + \Pi) = \mathfrak{g}\varrho\nabla\boldsymbol{\phi} + \frac{1}{\mu_0}(\operatorname{rot}_x B)\times B + \mathbf{f}_{fict}\,,\\
-\Delta\boldsymbol{\phi} = \varrho\,,\\
\partial_t B - \operatorname{rot}_x(v\times B) - \frac{1}{\mu_0\sigma_{Ohm}}\Delta_x B = 0\,,\quad \operatorname{div}_x B = 0\,.
\end{gathered} \tag{VI4.10}$$

Die Gleichung $\operatorname{div}_x E = \varrho^{el}$ und auch $\operatorname{rot}_x B = \mu_0\mathbf{j}$ bleibt hierbei unberücksichtigt. Zusammen mit dem Ohm'schen Gesetz $\mathbf{j} = \sigma_{Ohm}(E + v\times B)$, was ja in (VI4.10) auch nicht als Gleichung auftritt, lassen sich aus B und v die Größen E, ϱ^{el} und $\mu_0\mathbf{j}$ berechnen.

Für die Gleichungen der Magentohydrodynamik gibt es im Wesentlichen zwei Anwendungen, einmal ist dies die Untersuchung der Materieströme am Rande der heißen Sonne, zum anderen die Kernverschmelzung, die auf der Erde international betrieben wird.

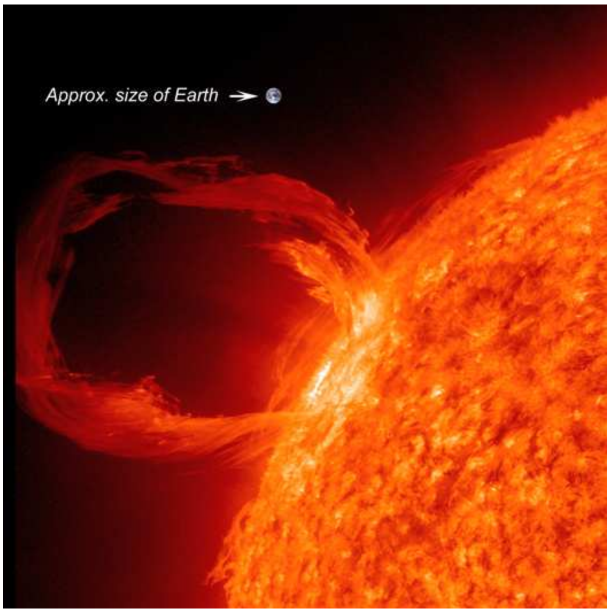

Fig. 4: Die Sonne nach 'Ingrid Science'

Im Weltall ist natürlich die Massenanziehung, so wie sie in den Gleichungen formuliert ist, wesentlich.

Auf der Erdoberfläche wird $\mathfrak{g}\nabla\boldsymbol{\phi} \approx \mathfrak{g}\nabla\boldsymbol{\phi}_{Erde} \approx -\vec{g}_{Erde} = -g_{Erde}e$ genommen (siehe Ende des Abschnittes I.4), wobei e die nach außen zeigende Einheitsnormale und g_{Erde} der Betrag der Erdanziehungskraft ist. Dies deshalb, weil $\boldsymbol{\phi} \approx \boldsymbol{\phi}_{Erde}$, wobei $\boldsymbol{\phi}_{Erde}$ das nur durch die Erde hervorgerufene Anziehungspotential ist. Außerdem ist oft $\mathbf{r} = 0$, und $\mathbf{f}_{fict}$ ist vernachlässigbar (oder ein Teil ist in der Anziehungskraft der Erde enthalten, die ja ohnehin nur als Konstante genommen wird). Diese Gleichungen werden zum Beispiel gebraucht bei der Berechnung der Lösung in einem ***Tokamak***, siehe [Wikipedia: Tokamak] "Das Wort ist eine Transliteration des russischen токамак, eine Abkürzung für тороидалнаја камера в магнитныч катусчкач (**to**roidalnaja **ka**mera w **ma**gnitnych **k**atuschkach) übersetzt '**To**roidale **Ka**mmer in **Mag**netspulen'. Auch verweist die Silbe ток auf Strom und damit den Stromfluss im Plasma, die entscheidende Besonderheit dieses Einschlusskonzepts."

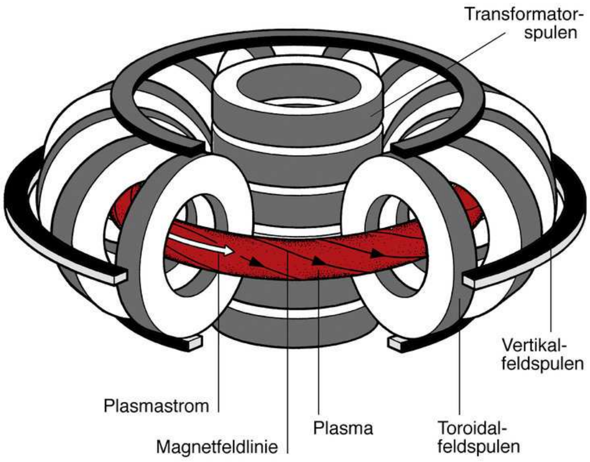

Fig. 5: "Das Spulensystem und Magnetfeld eines Tokamaks" ©IPP

"Das Max-Planck-Institut für Plasmaphysik ist das einzige Fusionszentrum weltweit, das beide Experimentiertypen untersucht - in Garching den Tokamak ASDEX Upgrade, in Greifswald den Stellarator Wendelstein 7-X. Dies ermöglicht den direkten Vergleich." IPP [www.ipp.mpg.de]

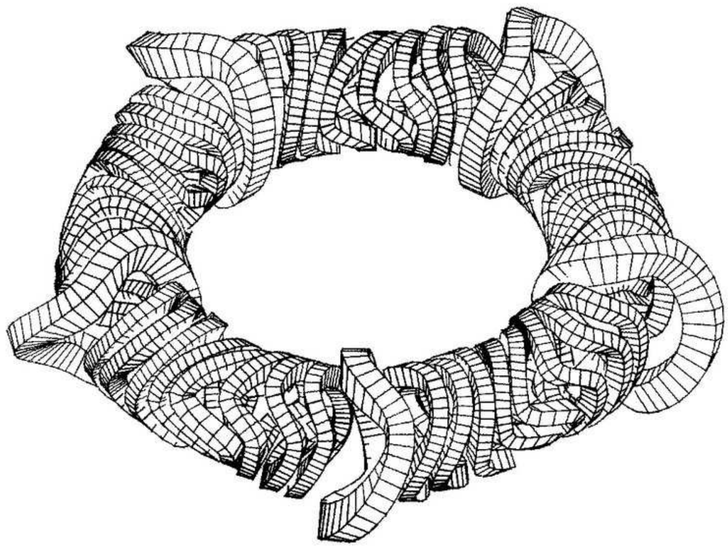

Fig. 6: "Spulensystem des ersten optimierten Stellarators" IPP

5 Ohm's law

Referenzen: Zur Geschichte siehe [Wikipedia: Ohmsches Gesetz], and see [Wikipedia: Ohm's Law]. See Davidson [33, 2.2 Ohm's Law and the Volumetric Lorentz force].

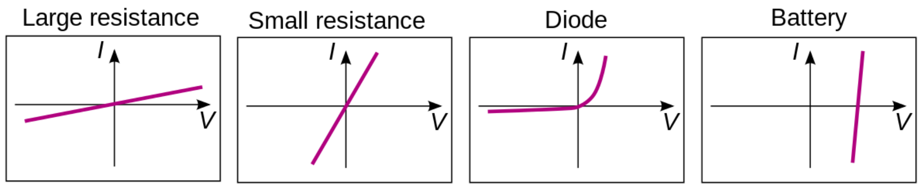

Fig. 7: "The IV curves of four devices: Two resistors, a diode, and a battery. The two resistors follow Ohm's law: The plot is a straight line through the origin. The other two devices do not follow Ohm's law." Aus Wikipedia

Für eine ruhende Probe ist das Ohm'sche Gesetz

$$\mathrm{j} = \sigma_{Ohm} E \quad \big(\text{in Schaltkreisnotation } I = \frac{U}{R}\big),$$

ein Gesetz, das z.B. bei einem Strom in Leitungen mit einer Konstanten σ_{Ohm} erfüllt ist, wobei der Kehrwert den ***Widerstand*** (*en:* ***Resistance***) darstellt (in Fig. 7 ist V die Spannung U). Da aber nur $E + v \times B$ (siehe 5.1), wobei v eine Geschwindigkeit ist, wie j ein objektiver Vektor ist, ist

$$\mathrm{j} = \sigma_{Ohm}(E + v \times B)\,, \tag{VI5.1}$$

wobei σ_{Ohm} ein objektiver Skalar ist, eine gültige konstitutive Gleichung für alle Beobachter nach Abschnitt II.4. Es ist v die Geschwindigkeit, mit der sich die Probe bewegt, und diese Geschwindigkeit ist auch vom Beobachter abhängig.

5.1 Lemma. Wenn v eine Geschwindigkeit ist,

$$\text{sind} \quad E + v \times B\,, \quad B\,, \quad \mathrm{j}^{el} \quad \text{objektive Vektoren,}$$
$$\text{und} \quad \varrho^{el} - v \bullet \mathrm{j}^{el} - \mathrm{D}v \mathbin{:} \mathcal{R}(B) \quad \text{ist objektiver Skalar.}$$

Proof. Dass B und $\mathbf{j}^{el*} = \mu_0 \mathbf{j}$ objektive Vektoren sind, wurde in 4.3 gezeigt. Ist v eine Geschwindigkeit, also $v \circ Y = V + Qv^*$, so folgt aus 4.3, dass

$$\begin{aligned}(E + v \times B) \circ Y &= QE^* - V \times QB^* + (V + Qv^*) \times QB^* \\ &= QE^* + (Qv^*) \times QB^* = Q(E^* + v^* \times B^*)\,,\end{aligned}$$

also ist $E + v \times B$ ein objektiver Vektor. Nun ist nach 4.3

$$\varrho^{el} \circ Y = \varrho^{el*} + V \bullet Q\,\mathbf{j}^{el*} + (Q^{\mathrm{T}} \dot{Q}) \mathbin{:} \mathcal{R}(B^*)$$

und es gilt $(v\bullet \mathrm{j}^{el})\circ Y = V\bullet \mathrm{j}^{el} + (Qv^*)\bullet(Q\mathrm{j}^{el*}) = V\bullet Q\mathrm{j}^{el*} + v^*\bullet \mathrm{j}^{el*}$, sowie wegen $\mathrm{D}v\circ Y = \dot{Q}Q^{\mathrm{T}} + Q\mathrm{D}v^* Q^{\mathrm{T}}$ und $\mathcal{R}(B)\circ Y = Q\mathcal{R}(B^*)Q^{\mathrm{T}}$ gilt insgesamt, dass $\varrho^{el} - v\bullet \mathrm{j}^{el} - \mathrm{D}v : \mathcal{R}(B)$ ein objektiver Skalar ist. □

Ohm's law implies that the formula $\partial_t B + \mathrm{rot}_x E = 0$ as part of Maxwell equations can be written in terms of B only.

5.2 Evolution equation for *B*. The law of Ohm implies

$$\partial_t B - \mathrm{rot}_x(v\times B) - \frac{1}{\mu_0\sigma_{Ohm}}\Delta_x B = 0$$

in Maxwell equations.

Proof. Write Ohm's law (VI5.1) as

$$E + v\times B = \frac{1}{\sigma_{Ohm}}\mathrm{j} = \frac{1}{\mu_0\sigma_{Ohm}}\,\mathrm{rot}_x B$$

by Ampéres law $\mathrm{rot}_x B = \mu_0 \mathrm{j}$. Then from Faraday's law

$$\partial_t B = -\,\mathrm{rot}_x E = \mathrm{rot}_x(v\times B) - \frac{1}{\mu_0\sigma_{Ohm}}\,\mathrm{rot}_x\,\mathrm{rot}_x B\,,$$

that is

$$\partial_t B - \mathrm{rot}_x(v\times B) + \frac{1}{\mu_0\sigma_{Ohm}}\,\mathrm{rot}_x\,\mathrm{rot}_x B = 0\,.$$

Since by 6.4 we have $\mathrm{rot}_x\,\mathrm{rot}_x B = -\Delta_x B + \nabla_x\,\mathrm{div}_x B$ and $\mathrm{div}_x B = 0$ also by Faraday's law. Therefore we get the assertion. □

---

will be continued

....

6 Exercises

Observer transformations

6.1 Klassische Beobachter. Für eine klassische (siehe Abschnitt II.1) Beobachtertransformation gilt $\mathrm{G}_\infty = \mathrm{D}Y\mathrm{G}_\infty \mathrm{D}Y^{\mathrm{T}}$.

Solution. Wir können die Ableitung von Y schreiben als

$$\mathrm{D}Y = \begin{bmatrix} 1 & 0 \\ V & Q \end{bmatrix},$$

woraus die Behauptung folgt. □

Maxwell equations

6.2 Konstanten. Bestimme Konstanten $\varepsilon^{ijk} \in \{-1,0,1\}$, so dass für jedes Vektorfeld $B:\mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{rot} B = \sum_{i,j,k=1,2,3} \varepsilon^{ijk} \partial_i B_j \mathbf{e}_k \,.$$

Hinweis: Es ist $\varepsilon^{123} = 1$ und $\left(\varepsilon^{ijk}\right)_{i,j,k=1,2,3}$ antisymmetrisch in allen zwei Argumenten.

6.3 Lemma. For $v,w:\mathbb{R}^3 \to \mathbb{R}^3$

$$v \times \operatorname{rot} w + (v \bullet \nabla) w = (\nabla w) v \ \Big(= \sum_i v_i \nabla w_i \Big)\,,$$

although $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a} \bullet \vec{b})\vec{c} = (\vec{a} \bullet \vec{c})\vec{b}$ for vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$.
Remark: ∇w is defined as in the Remark to I.1.2(5).

Solution. We compute

$$v \times \operatorname{rot} w = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} \partial_2 w_3 - \partial_3 w_2 \\ \partial_3 w_1 - \partial_1 w_3 \\ \partial_1 w_2 - \partial_2 w_1 \end{bmatrix} = \begin{bmatrix} v_2(\partial_1 w_2 - \partial_2 w_1) - v_3(\partial_3 w_1 - \partial_1 w_3) \\ v_3(\partial_2 w_3 - \partial_3 w_2) - v_1(\partial_1 w_2 - \partial_2 w_1) \\ v_1(\partial_3 w_1 - \partial_1 w_3) - v_2(\partial_2 w_3 - \partial_3 w_2) \end{bmatrix}$$

$$= -(v \bullet \nabla) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} v_1 \partial_1 w_1 + v_2 \partial_1 w_2 + v_3 \partial_1 w_3 \\ v_2 \partial_2 w_2 + v_3 \partial_2 w_3 + v_1 \partial_2 w_1 \\ v_3 \partial_3 w_3 + v_1 \partial_3 w_1 + v_2 \partial_3 w_2 \end{bmatrix} = -(v \bullet \nabla) w + \sum_i v_i \nabla w_i \,,$$

hence the assertion. For vectors $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a} \bullet \vec{b})\vec{c} = (\vec{b} \otimes \vec{a})\vec{c} = \vec{b}\,\vec{a}^{\mathrm{T}}\vec{c}$ would be the corresponding formula. □

6.4 Lemma. For $u:\mathbb{R}^3 \to \mathbb{R}^3$

$$\operatorname{rot} \operatorname{rot} u = -\Delta u + \nabla \operatorname{div} u \,,$$

and for any test function $\zeta:\mathbb{R}^3 \to \mathbb{R}^3$

$$\int_{\mathbb{R}^3} \operatorname{rot} \zeta \bullet \operatorname{rot} u \, \mathrm{dL}^3 = \int_{\mathbb{R}^3} \big(\nabla \zeta \,\mathbin{\bullet\!\!\bullet}\, \nabla u - \operatorname{div} \zeta \cdot \operatorname{div} u \big) \, \mathrm{dL}^3 \,.$$

Solution. The differential equation follows from the integral relation with integration by parts. For the integrand one computes

$$\operatorname{rot} \zeta \bullet \operatorname{rot} u = \sum_{i<j} (\partial_j \zeta_i - \partial_i \zeta_j)(\partial_j u_i - \partial_i u_j) = \frac{1}{2} \sum_{i,j} (\partial_j \zeta_i - \partial_i \zeta_j)(\partial_j u_i - \partial_i u_j)$$

$$= \frac{1}{2} \sum_{i,j} (\partial_j \zeta_i \partial_j u_i + \partial_i \zeta_j \partial_i u_j - \partial_j \zeta_i \partial_i u_j - \partial_i \zeta_j \partial_j u_i)$$

$$= \sum_{i,j} \partial_j \zeta_i \partial_j u_i - \sum_{i,j} \partial_j \zeta_i \partial_i u_j = \nabla \zeta \,\mathbin{\bullet\!\!\bullet}\, \nabla u - \sum_{i,j} \partial_j \zeta_i \partial_i u_j$$

and

$$\sum_{i,j} \int_{\mathbb{R}^3} \partial_j \zeta_i \partial_i u_j \, \mathrm{dL}^3 = \sum_{i,j} \int_{\mathbb{R}^3} \partial_i \partial_j \zeta_i \cdot u_j \, \mathrm{dL}^3$$

$$= \sum_{i,j} \int_{\mathbb{R}^3} \partial_j \partial_i \zeta_i \cdot u_j \, \mathrm{dL}^3 = \sum_{i,j} \int_{\mathbb{R}^3} \partial_i \zeta_i \partial_j u_j \, \mathrm{dL}^3 = \int_{\mathbb{R}^3} \operatorname{div} \zeta \cdot \operatorname{div} u \, \mathrm{dL}^3 \,,$$

which gives the result. □

6.5 Eichinvarianz. Beweise mit Hilfe von Differentialformen die Eichinvarianz 2.12.

Solution. Für $E = \sum_{k,l\geq 0} E_{kl} \mathrm{d}y_k \wedge \mathrm{d}y_l$ ist die Voraussetzung $\mathrm{d}E = 0$, siehe den Beweis von 2.10(2). Daraus folgt nach dem Poincaré-Lemma, dass es ein $A = \sum_{k\geq 0} A_k \mathrm{d}y_k$ gibt mit $\mathrm{d}A = E$. Gibt es jetzt noch ein anderes A' mit $\mathrm{d}A' = E$, so folgt $\mathrm{d}(A' - A) = 0$. Also gibt es nach dem Poincaré-Lemma eine Funktion f mit $\mathrm{d}f = A' - A$. Daher ist $A' = A + \mathrm{d}f$ wie in 2.12 gesagt. □

Moving charges

Now some properties about the boost operator.

6.6 Iterated boost operator. For any $V \in \mathbb{R}^3$ with $V \neq 0$ and $k \in \mathbb{Z}$

$$\mathbf{B}(V)^k = \mathrm{Id} + (\gamma^k - 1)\widehat{V}\widehat{V}^{\mathrm{T}} .$$

Solution. Let $V \neq 0$ and let $\widehat{V}$ be the unit vector in direction V and define

$$B_a := \mathrm{Id} + (a-1)\widehat{V}\widehat{V}^{\mathrm{T}} \quad \text{for } a \in \mathbb{R}.$$

Then

$$\begin{aligned} &B_a B_b = (\mathrm{Id} + (a-1)\widehat{V}\widehat{V}^{\mathrm{T}})(\mathrm{Id} + (b-1)\widehat{V}\widehat{V}^{\mathrm{T}}) \\ &= \mathrm{Id} + ((a-1) + (b-1) + (a-1)(b-1))\widehat{V}\widehat{V}^{\mathrm{T}} = \mathrm{Id} + (ab-1)\widehat{V}\widehat{V}^{\mathrm{T}} = B_{ab} , \end{aligned}$$

that is, $B_a B_b = B_{ab}$. Then inductively $B_a^k = B_{a^k}$ for $k \in \mathbb{N}$ and $B_a^{-1} = B_{1/a}$. □

6.7 Inverse boost operator. For any $V \in \mathbb{R}^3$

$$\mathbf{B}(V) - \frac{\gamma}{\mathbf{c}^2} V V^{\mathrm{T}} = \mathbf{B}(V)^{-1} .$$

Solution. We have for $V \neq 0$

$$\mathbf{B}(V) - \frac{\gamma}{\mathbf{c}^2} V V^{\mathrm{T}} = \mathrm{Id} + \big(\gamma - 1 - \frac{\gamma |V|^2}{\mathbf{c}^2}\big)\widehat{V}\widehat{V}^{\mathrm{T}} = \mathrm{Id} + \big(\frac{1}{\gamma} - 1\big)\widehat{V}\widehat{V}^{\mathrm{T}}$$

using exercise 6.6. □

6.8 Boost operator and cross product. For all $z_1, z_2 \in \mathbb{R}^3$

$$(\mathbf{B}(V) z_2) \times (\mathbf{B}(V) z_1) = \gamma \mathbf{B}(V)^{-1} (z_2 \times z_1) .$$

Solution. In section II.2 we had defined $\mathbf{B}(V) = \mathrm{Id} + (\gamma - 1)\widehat{V}\widehat{V}^{\mathrm{T}}$. And here we start with the general well known formula

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \bullet \vec{c})\, \vec{b} - (\vec{b} \bullet \vec{c})\, \vec{a}$$

which implies for $V \neq 0$ letting $\widehat{V}$ the unit vector in the direction of V, and for $z := z_2 \times z_1$ with $z_1, z_2 \in \mathbb{R}^3$

$$(\widehat{V} \times z) \times \widehat{V} = z - z \bullet \widehat{V}\ \widehat{V} \quad \text{and} \quad z \times \widehat{V} = (z_2 \bullet \widehat{V}) z_1 - (z_1 \bullet \widehat{V}) z_2 ,$$

hence

$$z_2 \times z_1 - (z_2 \times z_1) \bullet \widehat{V}\ \widehat{V} = ((z_1 \bullet \widehat{V}) z_2 - (z_2 \bullet \widehat{V}) z_1) \times \widehat{V} .$$

Then we compute

$$
\begin{aligned}
&(\mathbf{B}(V)z_2)\times(\mathbf{B}(V)z_1) = \big(z_2+(\gamma-1)(\widehat{V}\bullet z_2)\widehat{V}\big)\times\big(z_1+(\gamma-1)(\widehat{V}\bullet z_1)\widehat{V}\big)\\
&= z_2\times z_1 + (\gamma-1)\big((\widehat{V}\bullet z_2)\widehat{V}\times z_1 + (\widehat{V}\bullet z_1)z_2\times\widehat{V}\big)\\
&= z_2\times z_1 + (\gamma-1)\big((z_1\bullet\widehat{V})z_2 - (z_2\bullet\widehat{V})z_1\big)\times\widehat{V}\\
&= z_2\times z_1 + (\gamma-1)\big(z_2\times z_1 - (z_2\times z_1)\bullet\widehat{V}\ \widehat{V}\big)\\
&= \gamma(z_2\times z_1) - (\gamma-1)\widehat{V}\bullet(z_2\times z_1)\widehat{V}\\
&= \gamma\big((z_2\times z_1) + (\frac{1}{\gamma}-1)\widehat{V}\bullet(z_2\times z_1)\widehat{V}\big)(z_2\times z_1)\\
&= \gamma\big(\mathrm{Id} + (\frac{1}{\gamma}-1)\widehat{V}\widehat{V}^{\mathrm{T}}\big)(z_2\times z_1) = \gamma\mathbf{B}(V)^{-1}(z_2\times z_1)\,,
\end{aligned}
$$

where $\mathbf{B}(V)^{-1}$ is from 6.6. □

The limit towards MHD

6.9 Maxwell equations in MHD. Under Lorenz conditions we obtain

$$-\Delta\Phi = \varrho^{el}\,,\quad -\Delta A = \mathbf{j}^{el}\,,\quad \mathrm{div}_x A = 0\,.$$

Solution. This follows from 2.17 in the limit $\mathbf{c}\to\infty$. It follows also from (VI4.2), since

$$
\begin{aligned}
\mathrm{div}_x E &= -\,\mathrm{div}_x(\nabla_x\Phi + \partial_t A) = -\Delta\Phi - \partial_t\,\mathrm{div}_x A = -\Delta\Phi\,,\\
\mathrm{rot}_x B &= \mathrm{rot}_x\,\mathrm{rot}_x A = -\Delta A + \nabla_x\,\mathrm{div}_x A = -\Delta A\,.
\end{aligned}
$$

see 6.4. □

Figures

Fig. II1 ”” Fig. II2 ”Classical observers” Fig. IV3 “Tides on the surface of the Earth” Copyright by H.W.Alt & G.Witterstein

Preliminary Version:
Fig. I1 “Gas and solid” ???????
Fig. I2 from A.H. Zemanian [79, Preface]
Fig. I4 from [Wikipedia: Atmosphere of earth]
Fig. I5 from A.H. Zemanian [79, Introduction]
Fig. I8 from [Wikipedia: Gravity of Earth]
Fig. I16 “Interface of water and air”
Fig. I19 “Parabolic shape of the surface”
Fig. I20 “Wasserbehälter auf rotierender Oberfläche” from [vlex.physik.uni-oldenburg.de/download/mqc_rotierendesfluid.pdf]
Fig. I21 *Left:* from [Wikipedia: Sonne]. *Right:* 2.1.2013 (Photo by Thorsten Edelmann) from Sterne und Weltraum 3|2013
Fig. II4 “Geschichtetes Material” from the book [4] by Eck & Garcke & Knabner
Fig. III3 Thermometer from “Grimsehl’s Lehrbuch der Physik”.
Fig. IV1 “Tide” from [Wikipedia: Tide] [www.wattwandern-johann.de/watt-lernen/ebbe-und-flut/]
Fig. IV2 “Distribution of Tidal Phases” from [Wikipedia: Tide]
Fig. IV3 “Tides on the surface of the Earth” Copyright by H.W.Alt & G.Witterstein
Fig. IV4 “Tide on Earth” from [Wikipedia: Gezeiten]
Fig. IV5 “Der Zusammenhang zwischen Masse, Stoffmenge, Volumen und Teilchenanzahl” from [Wikipedia: Molare Masse] author Johannes Schneider
Fig. IV12
Fig. IV20
Fig. IV19
Fig. IV24 “A waterspout near the Florida Keys in 1969” from [Wikipedia: Tornado] author Dr. Joseph Golden, NOAA, Image ID: wea00308, Historic NWS Collection
Fig. IV35

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